On the large time behavior of two-dimensional vortex dynamics

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Abstract. In this paper we prove two results regarding the large-time behavior of vortex dynamics in the full plane. In the first result we show that the total integral of vorticity is confined in a region of diameter growing at most like the square-root of time. In the second result we show that if a dynamic rescaling of the absolute value of vorticity with spatial scale growing linearly with time converges weakly, then it must converge to a discrete sum of Dirac masses. This last result extends in scope a previous result by the authors, valid for nonnegative initial vorticity on a half-plane

Key words: Vorticity, confinement, incompressible flow, ideal flow.

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Contents

1. Introduction

Incompressible, ideal fluid flow can be described in terms of the behavior of vorticity, the curl of the fluid velocity. This is especially useful in two space dimensions, as in this case (the scalar) vorticity is conserved along particle trajectories. The equations of fluid dynamics can then be recast as the transport of an active scalar with vorticity as the dynamic variable. In this context, the problem of describing the large time behavior of vorticity is a very natural one, and it is the broad subject we address in the present work.

Let ω_0 be a compactly supported function in $L^p(\mathbb{R}^2)$, with $p > 2$, and let $\omega = \omega(x,t)$ be the vorticity associated to a weak solution of the incompressible two-dimensional Euler equations in the full plane, with initial vorticity ω_0 (see [5] and references therein for the existence of such a solution). In vorticity form, the Euler equations may be written as an active scalar transport equation:

$$\begin{cases} \omega_t + (K * \omega) \cdot \nabla\omega = 0, \\ \omega(x,0) = \omega_0, \end{cases} \tag{1.1}$$

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with K the Biot-Savart vector kernel for the full plane, given by

$$K(x) = K(x_1, x_2) = \frac{1}{2\pi|x|^2}(-x_2, x_1) = \frac{x^\perp}{2\pi|x|^2}. \tag{1.2}$$

We are interested in obtaining information on the behavior of the solution $\omega(\cdot, t)$ as $t \to \infty$, particularly with regards to the spatial distribution of vorticity. We consider a self-similar rescaling of vorticity of the form:

$$\widetilde{\omega}_\alpha(x, t) \equiv t^{2\alpha}\omega(t^\alpha x, t),$$

with $\alpha \in (0, 1]$. This scaling preserves the integral of vorticity and its L^1 norm. The large time behavior of $\widetilde{\omega}_\alpha$ carries information on the distribution of vorticity, focusing on a certain asymptotic scale determined by the parameter α. The purpose of this note is to prove two results. The first result is that, for any initial data ω_0 and parameter $\alpha > 1/2$, we have $\widetilde{\omega}_\alpha \rightharpoonup m\delta_0$, where $m = \int \omega_0$ and δ_0 is the Dirac measure at the origin. The second result is a generalization of a previous result by the authors. The main result in [3] can be reformulated as stating that, if: (i) the initial vorticity ω_0 is odd with respect to the horizontal axis, (ii) its restriction to the upper half-plane has a distinguished sign and (iii) $\alpha = 1$, then the hypothesis that $|\widetilde{\omega}_1(x, t)| \rightharpoonup \mu$, where μ is a measure (which must be supported in $\{|x_1| \le M\} \times \{x_2 = 0\}$ for confinement reasons), implies that μ must consist of an at most countable sum of Dirac masses whose supports may only accumulate at the origin. Our second result in this article is to remove conditions (i) and (ii) on ω_0, keeping the same conclusion.

In 1994, C. Marchioro proved that the solution $\omega = \omega(x, t)$ of equation (1.1) with an initial vorticity bounded and nonnegative, with support contained in a disk of radius $R_0 > 0$ centered at the origin, has support contained in a disk of radius $(R_0^a + bt)^{1/a}$, with $a = 3$, for some constant $b > 0$, see [7]. This first result in confinement of vorticity has been improved and extended in several ways. The exponent $1/3$ has been improved to $1/4+$ by P. Serfati, see [12] and independently by Iftimie, Sideris and Gamblin, see [4]. Other extensions and improvements include unbounded initial vorticity [6, 2], flows in exterior domains [8], slightly viscous flows [9] and axisymmetric flows, [10, 11]. Confinement results basically control the rate at which vorticity is spreading. The present work is an attempt to go beyond controlling this rate, actually describing the way in which vorticity is spreading.

If the initial vorticity does not have a distinguished sign, the best confinement one may expect in general is at the rate $a = 1$, see [4]. In fact, [4] contains the construction of an example of smooth, compactly supported vorticity for which the support grows precisely in a linear fashion. This means that the self-similar scale of interest is $\alpha = 1$, and the time asymptotic behavior of $|\widetilde{\omega}_1|$ is what would give a reasonably complete description of vorticity scattering in this case.

The remainder of this article is divided into three sections. In the next section we discuss the result on the asymptotic behavior of $\widetilde{\omega}_\alpha$. The following section contains the result on $|\widetilde{\omega}_1|$. The final section contains a few comments on this work.

2. Confinement of the net vorticity

Let $\omega_0 \in L^p_c(\mathbb{R}^2)$, for some $p > 2$ and consider $\omega = \omega(x,t)$ a solution of (1.1) with initial data ω_0. Our basic problem is to describe the spatial distribution of the vorticity $\omega(\cdot,t)$ for large t. For $\alpha \in (0,1]$ we introduce the rescaled vorticity:

$$\widetilde{\omega}_\alpha = \widetilde{\omega}_\alpha(x,t) \equiv t^{2\alpha}\omega(t^\alpha x, t).$$

Clearly, if ω_0 is single-signed, the known results on confinement tell us that, for any $\alpha > 1/4$, the support of $\widetilde{\omega}_\alpha$ is contained in a disk centered at the origin whose radius vanishes as $t \to \infty$. What happens when the vorticity is allowed to change sign?

Let $\widetilde{u}_\alpha \equiv K * \widetilde{\omega}_\alpha$, with K given by (1.2). It is a straightforward calculation to verify that $\widetilde{\omega}_\alpha$ and $\widetilde{u}_\alpha$ satisfy the equation

$$\frac{\partial \widetilde{\omega}_\alpha}{\partial t} - \frac{\alpha}{t}\,\mathrm{div}\,(x\widetilde{\omega}_\alpha) + \frac{1}{t^{2\alpha}}\,\mathrm{div}\,(\widetilde{u}_\alpha\widetilde{\omega}_\alpha) = 0. \tag{2.1}$$

We are now ready to state and prove our first result.

Theorem 2.1. *Let $\alpha > 1/2$ and set $m = \int \omega_0(x)\,dx$. Then $\widetilde{\omega}_\alpha(\cdot,t) \rightharpoonup m\delta_0$ weak-$*$ in $\mathcal{BM}(\mathbb{R}^2)$ as $t \to \infty$.*

Proof. We will begin by considering the linear part of the evolution equation (2.1) with initial condition at $t = 1$:

$$\begin{cases} \dfrac{\partial f}{\partial t} - \dfrac{\alpha}{t}\,\mathrm{div}\,(xf) = 0 \\[2ex] f(x,1) = g(x). \end{cases}$$

The solution f is given by the (multiplicative) semigroup $f(x,t) = S_t[g](x) \equiv t^{2\alpha}g(t^\alpha x)$, interpreted in the sense of distributions. We then write (2.1) as an inhomogeneous version of this linear equation, with source term given by

$$h(x,t) \equiv -\frac{1}{t^{2\alpha}}\,\mathrm{div}\,(\widetilde{u}_\alpha\widetilde{\omega}_\alpha).$$

With this we can write the solution $\widetilde{\omega}_\alpha$ of (2.1), with initial data $\widetilde{\omega}_\alpha(x,1) = \omega(x,1) \equiv g(x)$, using Duhamel's formula:

$$\widetilde{\omega}_\alpha(x,t) = S_t[g](x) + \int_1^t S_{t/s}[h](x,s)\,\mathrm{d}s. \tag{2.2}$$

(In the integral above the semigroup is acting in the spatial variable only.) Of course (2.2) must be interpreted in the sense of distributions. We now turn to the analysis of each term in (2.2). Let $\varphi \in C^\infty_c(\mathbb{R}^2)$. We then have:

$$\int_{\mathbb{R}^2} \varphi(x)\widetilde{\omega}_\alpha(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^2} \varphi\left(\frac{y}{t^\alpha}\right) g(y)\,\mathrm{d}y + \int_1^t \int_{\mathbb{R}^2} \varphi\left(\frac{s^\alpha y}{t^\alpha}\right) h(y,s)\,\mathrm{d}y\,\mathrm{d}s \equiv I_1 + I_2.$$

First note that, as $t \to \infty$,

$$I_1 \to \left(\int_{\mathbb{R}^2} g(y)\,\mathrm{d}y\right)\varphi(0),$$

by the Lebesgue Dominated Convergence Theorem. Next, recall that the total integral of vorticity is conserved and hence the proof will be concluded once we establish that $I_2 \to 0$. We compute directly, integrating by parts and using the relation between $\widetilde{u}_\alpha$ and $\widetilde{\omega}_\alpha$:

$$\begin{aligned} I_2 &= -\int_1^t \int_{\mathbb{R}^2} \varphi\left(\frac{s^\alpha y}{t^\alpha}\right) \frac{1}{s^{2\alpha}} \operatorname{div}\,(\widetilde{u}_\alpha \widetilde{\omega}_\alpha)(y,s)\,\mathrm{d}y\,\mathrm{d}s \\ &= \int_1^t \int_{\mathbb{R}^2} \frac{1}{s^\alpha t^\alpha} \nabla\varphi\left(\frac{s^\alpha y}{t^\alpha}\right) \cdot (\widetilde{u}_\alpha \widetilde{\omega}_\alpha)(y,s)\,\mathrm{d}y\,\mathrm{d}s \\ &= \int_1^t \frac{1}{s^\alpha t^\alpha} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \nabla\varphi\left(\frac{s^\alpha y}{t^\alpha}\right) \cdot K(y-z)\widetilde{\omega}_\alpha(z,s)\widetilde{\omega}_\alpha(y,s)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}s. \end{aligned}$$

We now use the antisymmetry of the Biot-Savart kernel K to obtain:

$$I_2 = \frac{1}{2}\int_1^t \frac{1}{s^\alpha t^\alpha} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} H_\varphi(s,t,z,y)\widetilde{\omega}_\alpha(z,s)\widetilde{\omega}_\alpha(y,s)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}s,$$

where

$$H_\varphi(s,t,z,y) \equiv \left(\nabla\varphi\left(\frac{s^\alpha y}{t^\alpha}\right) - \nabla\varphi\left(\frac{s^\alpha z}{t^\alpha}\right)\right) \cdot K(y-z).$$

Let us observe that

$$|H_\varphi| \leq \frac{s^\alpha}{t^\alpha}\|D^2\varphi\|_{L^\infty}|y-z||K(y-z)| \leq C(\varphi)\frac{s^\alpha}{t^\alpha}.$$

Hence we arrive finally at

$$|I_2| \leq C(\varphi)\left(\int_{\mathbb{R}^2}|\omega_0|\right)^2 \frac{t-1}{t^{2\alpha}},$$

which clearly converges to 0 as $t \to \infty$ as long as $2\alpha > 1$. This concludes the proof. $\square$

Remark 2.1. The particular way in which we use the antisymmetry of the Biot-Savart kernel together with the bilinearization of the nonlinearity of the Euler equations is due to J.-M. Delort, who used it in his proof of existence of weak solutions for 2D Euler with vortex sheet initial data, see [1].

Remark 2.2. This result does not say anything new if the initial vorticity has a distinguished sign. As we mentioned in the introduction, if the vorticity has a distinguished sign, the support of vorticity is contained in a ball whose radius grows like $\mathcal{O}(t^\alpha)$, with $1/4 < \alpha$. From that, Theorem 2.1 follows immediately.

Remark 2.3. What new information is contained in the conclusion of Theorem 2.1? Imagine that we are given initial vorticity $\omega_0 = \omega_0^+ - \omega_0^-$, which are the positive and negative parts of the initial vorticity. Let $\omega = \omega^+ - \omega^-$ be the solution of 2D Euler with initial vorticity ω_0. Due to the nature of vortex dynamics, both ω^+ and ω^- are time-dependent rearrangements of ω_0^+ and ω_0^- respectively, and hence their integrals, which we may call m^+ and m^-, are constant in time. Additionally, a consequence of Theorem 2.1 is that the integral of vorticity in a ball of radius t^α converges to $m^+ - m^-$, for any $\alpha > 1/2$. This is weak confinement of the *imbalance* between the positive and negative parts of vorticity in a ball of sublinear radius. This weak confinement is consistent with the conjectural picture that the only way for the support of vorticity to grow fast is through the shedding of vortex pairs.

3. Vortex scattering

Let $\omega = \omega(x,t)$ be a solution of the incompressible 2D Euler equations (1.1) with initial vorticity $\omega_0 \in L^p_c(\mathbb{R}^2)$, for some $p > 2$. The simplest picture consistent with what is known regarding large-time vortex dynamics would have ω_0 scattering into a confined part, which would remain near the center of motion for all time, plus a number of soliton-like vortex pairs, travelling with roughly constant speed. One illustration of this behavior was provided in [4] involving the interaction of two mirror-image vortex pairs, scattering in opposite directions. This example suggests that the scale of linear growth in time is of particular interest for vortex dynamics. In the notation of the previous section, this means concentrating in the behavior of

$$\widetilde{\omega} = \widetilde{\omega}(x,t) \equiv \widetilde{\omega}_1(x,t) = t^2\omega(tx,t).$$

The main result in the previous section implies that $\widetilde{\omega} \rightharpoonup m\delta_0$ when $t \to \infty$, but the weak convergence completely ignores the scattering of vortex pairs, due to their linear-scale self-cancellation. We propose that the large-time behavior of $|\widetilde{\omega}|$ provides a useful rough picture of vortex scattering.

First note that $|\widetilde{\omega}(\cdot,t)|$ is a bounded one-parameter family in $L^1(\mathbb{R}^2)$. Since the velocity $K * \omega$ is *a priori* globally bounded, the family $|\widetilde{\omega}(\cdot,t)|$ has its support contained in a single disk D. One can therefore extract a sequence of times $t_k \to \infty$ such that $|\widetilde{\omega}(\cdot,t_k)| \rightharpoonup \mu$, for some measure $\mu \in \mathcal{B}M_+(D)$. It follows from Theorem 2.1 that $\mu \geq |m|\delta_0$, where $m = \int \omega_0$. Indeed, if φ is a nonnegative test function,

$$\langle |m|\delta_0, \varphi\rangle = \lim_{k\to\infty} \left| \int \varphi(x)\widetilde{\omega}(x,t_k)dx \right| \leq \lim_{k\to\infty} \int \varphi(x)|\widetilde{\omega}(x,t_k)|dx = \langle \mu, \varphi\rangle.$$

Our purpose is to obtain more information about the measure μ. The result we will present is a generalization of a previous result by the authors, see [3], which described the structure of the measure μ in the situation of half-plane vortex scattering, and under an important restriction, which we will have to impose in the present context as well. In [3], we introduced the terminology *asymptotic velocity density* for any measure μ which is a limit of $|\widetilde{\omega}(\cdot,t_k)|$ for some sequence $t_k \to \infty$. In fact, due to a sign restriction, the explicit use of the absolute value in the definition of asymptotic velocity densities was not needed in [3], and because scattering in the half-plane is a one-dimensional affair, the density μ in the previous article was a measure on the real line, describing the asymptotic density only of the relevant component of velocity. Our result about the structure of μ in [3] and the result we will present here only applies to initial vorticities which have a *unique* asymptotic velocity density, i.e. those initial vorticities for which $|\widetilde{\omega}|(\cdot,t)$ converges weakly to a measure μ, rather than being merely weakly compact.

Theorem 3.1. *Suppose that the initial vorticity $\omega_0 \in L^p_c(\mathbb{R}^2)$, $p > 2$ has a unique asymptotic velocity density μ. Then μ must be of the form:*

$$\mu = \sum_{i=1}^{\infty} m_i \ \delta_{\alpha_i}$$

where:

(a) *$\alpha_i \neq \alpha_j$ if $i \neq j$ and $\alpha_i \to 0$ as $i \to \infty$;*
(b) *the masses m_i are nonnegative and verify $\sum_{i=1}^{\infty} m_i = \|\omega_0\|_{L^1}$;*
(c) *for all i, $|\alpha_i| \in [0, M]$, where $M = \|u\|_{L^\infty([0,\infty)\times\mathbb{R}^2)}$;*
(d) *there exists a constant $D > 0$, depending solely on p, such that, for all i with $m_i \neq 0$ we have*

$$|\alpha_i| \leq D \|\omega_0\|_{L^p}^{\frac{p'}{2}} \ m_i^{1-\frac{p'}{2}}.$$

Remark 3.1. In the statement above, the masses m_i are allowed to vanish only to include the case when the limit measure contains a finite number of Diracs. For notational convenience, in the case when there are only a finite number of Dirac masses, we artificially added a countable number of Dirac masses with zero masses and positions converging to 0.

Proof. The proof we will present here has much in common with the special case done in [3], so that we will concentrate on the aspects of the proof which differ from the original case, briefly outlining the remainder.

We first note that, since ω is transported by the velocity u, the same holds for $|\omega|$. This means that $|\omega|$ satisfies, in the weak sense, the equation

$$\partial_t|\omega| + \operatorname{div}(u|\omega|) = 0.$$

The equation for the absolute value of the rescaled vorticity is then given by

$$\partial_t|\widetilde{\omega}(y,t)| - \frac{1}{t}\operatorname{div}\big[y|\widetilde{\omega}(y,t)|\big] + \frac{1}{t^2}\operatorname{div}\big[\widetilde{u}(y,t)|\widetilde{\omega}(y,t)|\big] = 0,$$

where $\widetilde{u}(y,t)$ denotes the rescaled velocity $\widetilde{u}(y,t) = tu(ty,t)$.

Let us take the product with a test function $\varphi \in C^1(\mathbb{R}^2)$ and integrate in space:

$$\partial_t \int |\widetilde{\omega}(y,t)|\,\varphi(y)\,\mathrm{d}y = -\frac{1}{t}\int |\widetilde{\omega}(y,t)|\,y\cdot\nabla\varphi(y)\,\mathrm{d}y + \frac{1}{t^2}\int |\widetilde{\omega}(y,t)|\,\widetilde{u}(y,t)\cdot\nabla\varphi(y)\,\mathrm{d}y. \tag{3.1}$$

We now recall the following argument that was used in [3]. The left-hand side of (3.1), when integrated from 1 to t, is uniformly bounded in t. By hypothesis, we know that

$$\lim_{t\to\infty}\int |\widetilde{\omega}(y,t)|\,y\cdot\nabla\varphi(y)\,\mathrm{d}y = \big\langle y\mu, \nabla\varphi\big\rangle,$$

so that the integral from 1 to t of the first term on the right-hand side of (3.1) behaves like $\big\langle y\mu,\nabla\varphi\big\rangle \log t$. As for the third term, it is not difficult to see that it is $\mathcal{O}(1/t)$. The dominant part of the third term must balance the logarithmic blow-up in time of the second term. This argument implies, adapting [3, Lemma 3.3] to the present situation, that the following inequality must hold:

$$\limsup_{t\to\infty}\Big(\frac{1}{t}\int |\widetilde{\omega}(y,t)|\,\widetilde{u}(y,t)\cdot\nabla\varphi(y)\,\mathrm{d}y\Big) \geq \langle y\mu,\nabla\varphi\rangle. \tag{3.2}$$

A straightforward adaptation of [3, Lemma 3.1] to compactly supported nonnegative finite measures on the plane yields a decomposition of μ into the sum of a discrete part plus a continuous part, i.e.:

$$\mu = \nu + \sum_{i=1}^{\infty} m_i\delta_{\alpha_i}. \tag{3.3}$$

On the other hand, it was also proved in [3, Proposition 3.1] a key estimate that in the present case reads

$$\limsup_{t\to\infty}\Big|\frac{1}{t}\int |\widetilde{\omega}(y,t)|\,\widetilde{u}(y,t)\cdot\nabla\varphi(y)\,\mathrm{d}y\Big| \leq D\|\omega_0\|_{L^p}^{\frac{p'}{2}}\sum_{i=1}^{\infty} m_i^{2-\frac{p'}{2}}|\nabla\varphi(\alpha_i)| \tag{3.4}$$

where $\sum_{i=1}^{\infty} m_i\delta_{\alpha_i}$ is the discrete part in the decomposition (3.3). The proof of [3, Proposition 3.1] valid in the case of the half-plane can be adapted in a straightforward manner to the full plane context due to the fact that the key estimate in the original proof is the inequality below, which relates the rescaled velocity to the rescaled vorticity:

$$|\widetilde{u}(x,t)| \leq \int \frac{C}{|x-y|}\,|\widetilde{\omega}(y,t)|\,\mathrm{d}y,$$

and this inequality holds in the case of the full space as well.

It follows from (3.2) and (3.4) that

$$\langle y\mu,\nabla\varphi\rangle \leq D\|\omega_0\|_{L^p}^{\frac{p'}{2}}\sum_{i=1}^{\infty} m_i^{2-\frac{p'}{2}}|\nabla\varphi(\alpha_i)|.$$

Substituting φ by $-\varphi$ we obtain

$$|\langle y\mu,\nabla\varphi\rangle| \leq D\|\omega_0\|_{L^p}^{\frac{p'}{2}}\sum_{i=1}^{\infty} m_i^{2-\frac{p'}{2}}|\nabla\varphi(\alpha_i)|. \tag{3.5}$$

Next we will use (3.5) to deduce that

$$|\alpha_i| \leq D\|\omega_0\|_{L^p}^{\frac{p'}{2}} m_i^{1-\frac{p'}{2}}. \tag{3.6}$$

To this end, let us fix $i_0\in\mathbb{N}$ and choose $\varphi\in C_c^\infty(\mathbb{R}^2)$ such that $\nabla\varphi(0)=\alpha_{i_0}$. Define $\varphi_\varepsilon(x)=\varepsilon\varphi\big(\frac{x-\alpha_{i_0}}{\varepsilon}\big)$ and use it as test function in (3.5) to obtain

$$\big|\langle y\mu,\nabla\varphi\big(\frac{y-\alpha_{i_0}}{\varepsilon}\big)\rangle\big| \leq D\|\omega_0\|_{L^p}^{\frac{p'}{2}}\sum_{i=1}^{\infty} m_i^{2-\frac{p'}{2}}\big|\nabla\varphi\big(\frac{\alpha_i-\alpha_{i_0}}{\varepsilon}\big)\big|. \tag{3.7}$$

The series on the right-hand side converges uniformly for $\varepsilon > 0$ and hence, when $\varepsilon \to 0$, it converges to

$$D\|\omega_0\|_{L^p}^{\frac{p'}{2}} m_{i_0}^{2-\frac{p'}{2}} |\alpha_{i_0}|.$$

As for the left-hand side, first we note that the functions $\nabla\varphi\big(\frac{y-\alpha_{i_0}}{\varepsilon}\big)$ converge pointwise to $\alpha_{i_0}\chi_{\{\alpha_{i_0}\}}$ (which does not vanish μ-almost everywhere, since μ attaches positive mass to α_{i_0}). Also, these functions are bounded uniformly with respect to ε and have supports contained in a single disk. The Lebesgue Dominated Convergence Theorem therefore implies that

$$\Big\langle y\mu, \nabla\varphi\big(\frac{y-\alpha_{i_0}}{\varepsilon}\big)\Big\rangle \to \Big\langle y\mu, \alpha_{i_0}\chi_{\{\alpha_{i_0}\}}\Big\rangle = |\alpha_{i_0}|^2 m_{i_0}$$

as $\varepsilon \to 0$. Putting these arguments together yields (3.6) in the limit, as $\varepsilon \to 0$.

We just proved part (d) of Theorem 3.1. Part (a) also follows at once by remarking that we have $m_i \to 0$ so, by (3.6), $\alpha_i \to 0$ as $i \to \infty$ too. Part (c) is a trivial consequence of the fact that the support of the vorticity is transported by the flow of u. Finally, part (b) is a direct consequence of the nonnegativity of the measure μ and also from the conservation of the L^1 norm of $|\widetilde{\omega}|$, once we established that the continuous part of μ vanishes.

We now go to the last part of the argument, i.e. the proof that the continuous part of the measure μ vanishes. Here is where the present proof requires a more substantial modification of the original one.

Let D be a strip of the form $D = \{c \le ay_1 + by_2 \le d\}$ disjoint with the set $A \equiv \{0\}\bigcup_{i\ge1}\{\alpha_i\}$. We prove that the measure μ must necessarily vanish in the interior of such a strip. First, since $0 \notin D$ we must have that $cd > 0$. We assume without loss of generality that $c, d > 0$. Let $[c', d']$ a subinterval of (c, d) and choose a smooth function $h \in C^\infty(\mathbb{R})$ such that $h' \in C_c^\infty(c, d)$, $h' \ge 0$ and $h'(s) = 1/s$ for all $s \in [c', d']$. Choose now $\varphi(y_1, y_2) = h(ay_1 + by_2)$ as test function in (3.5). Since $\operatorname{supp}\varphi \subset D$ we have that $\operatorname{supp}\varphi \cap A = \emptyset$, which implies in turn that the right-hand side of (3.5) vanishes for this choice of test function. Therefore the left-hand side must vanish too:

$$0 = \Big\langle y\mu, \nabla\big(h(ay_1+by_2)\big)\Big\rangle = \Big\langle \mu, (ay_1+by_2)h'(ay_1+by_2)\Big\rangle. \tag{3.8}$$

The function $y \mapsto (ay_1 + by_2)h'(ay_1 + by_2)$ is nonnegative and it is equal to 1 on the strip $\{c' \le ay_1 + by_2 \le d'\}$. Since the measure μ is nonnegative too, we deduce from (3.8) that μ vanishes on the strip $\{c' \le ay_1 + by_2 \le d'\}$. Also, since $[c', d']$ was an arbitrary subinterval of (c, d), we finally deduce that μ vanishes in the interior of the strip D.

In order to conclude the proof of Theorem 3.1, we only need to show that the measure μ vanishes in the neighborhood of each point of A^c. Let $y_0 \in A^c$. Since the only possible accumulation point of the set A is 0, there exists a line $\{ay_1 + by_2 = c\}$ passing through y_0 and which does not cross A. A continuity argument using again that the points α_i can only accumulate at $\{0\}$ shows that there exists a strip $\{c - \varepsilon \le ay_1 + by_2 \le c + \varepsilon\}$ disjoint of A. But we proved that the measure μ must vanish on such a strip. This implies that μ vanishes in the neighborhood of y_0 and this completes the proof of Theorem 3.1.

□

4. Final comments

First we observe that Theorem 2.1 does not require that the initial vorticity ω_0 belong to L^p. The argument works just as well if the initial vorticity is a bounded (signed) Radon measure, as long as the existence of a (global in time) weak solution is provided. The estimate itself only depends on the total mass of the initial vorticity.

We note also that Theorem 3.1 draws a much stronger conclusion than Theorem 2.1, but it relies on the hypothesis that the initial vorticity $\omega_0 \in L^p_c$, with $p > 2$ have a unique asymptotic velocity density. This hypothesis clearly deserves further scrutiny, and we have provided a lengthy discussion in the special case of the half-plane in [3]. The discussion in the present case is similar, so we have omitted it.

One natural question arising from this work is the role of the critical exponent $\alpha = 1/2$ in Theorem 2.1. This exponent is far from the known critical exponent $\alpha = 1/4$ for the vorticity confinement in the distinguished sign case, see [4, 7, 12]. In the vortex confinement literature, the critical exponent $\alpha = 1/2$ appears naturally when one does not have *a priori* control over moments of vorticity, see [8], whereas the sharper estimates are obtained when using the conserved moments of vorticity. Using just the moment of inertia one obtains critical exponent $\alpha = 1/3$, in the case of the full plane, see [7], and in the case of the exterior of a disk, see [8]. Using both the moment of inertia and the center of vorticity, we obtain, in the case of the full plane, the critical exponent $\alpha = 1/4$, see [4, 12]. It is therefore reasonable to expect that we might improve the condition on α in Theorem 2.1 by using the conserved moments of vorticity, but this would require a new approach.

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