

Asymptotics of solutions to the Navier-Stokes system in exterior domains

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Abstract

We consider the incompressible Navier-Stokes equations with the Dirichlet boundary condition in an exterior domain of \mathbb{R}^n with $n \geq 2$. We compare the long-time behaviour of solutions to this initial-boundary value problem with the long-time behaviour of solutions of the analogous Cauchy problem in the whole space \mathbb{R}^n . We find that the long-time asymptotics of solutions to both problems coincide either in the case of small initial data in the weak L^n -space or for a certain class of large initial data.

1 Introduction

Large time behaviour of solutions.

It is well-known that the large time behaviour of solutions to the initial-value problem for the Navier-Stokes equations considered either in the whole space \mathbb{R}^n with $n \geq 2$ or in an exterior domain depends on the integrability properties of the initial conditions. In the finite energy case, that is when the initial velocity is square integrable, the L^2 -norm of the corresponding solution tends to zero as time goes to infinity, see *e.g.* [22, 27, 28, 20] for the problem in the whole space \mathbb{R}^n and [4] for analogous results in exterior domains. In such a case, the nonlinear effects are negligible for large values of time and the asymptotics of the solutions is determined by the corresponding Stokes semigroup, *cf. e.g.* [20, Thm. 2].

On the other hand, when the initial velocity is not square integrable, a solution of the initial value problem for the Navier-Stokes equations in \mathbb{R}^n can be constructed in a so-called scaling invariant space (*e.g.* in a homogeneous Besov space or in a weak L^n -space) under a suitable smallness assumption on the initial data, see the review article by Cannone [6] and the book of Lemarié-Rieusset [24] for more details. In particular, if the initial velocity is small and homogeneous of degree -1 , the corresponding solution is self-similar and such self-similar solutions describe the large time behaviour of a large class of solutions of the Navier-Stokes system in \mathbb{R}^n , see *e.g.* [26, 21] and [24, Ch. 23]. Here, the asymptotic of solutions is no longer determined by the Stokes semi-group due to the fact that the viscosity term and the bilinear term are in exact balance in the sense that none of them dominates the large time behaviour.

Finally, it should be noted that, for certain initial data, the large time behaviour of the corresponding solutions can be much more involved. Indeed, the authors of [8] noticed a chaotic behaviour of some solutions of the Navier-Stokes system, namely, the sequence $\{u(t_n, x)\}$ may exhibit different asymptotic properties as $t_n \rightarrow \infty$, depending on a choice of the sequence $\{t_n\}$.

Mathematics Subject Classification: 35Q30, 76D05, 35Q35.

Lamb-Oseen vortex in two dimensions.

Now, we focus for a moment on the self-similar large time behaviour of solutions of the two dimensional Navier-Stokes system. As an important and physically relevant case, let us recall that a velocity in \mathbb{R}^2 corresponding to an integrable vorticity is not square integrable in general. This is an immediate consequence of the Biot-Savart law (see *e.g.* [17]). The large time asymptotic of solutions of the Navier-Stokes system in \mathbb{R}^2 supplemented with such initial conditions is well-understood. To recall this result, let us introduce the Lamb-Oseen vortex

$$\Theta(t, x) = \frac{x^\perp}{2\pi|x|^2} \left(1 - e^{-\frac{|x|^2}{4t}}\right) \quad \text{with } x^\perp = (x_2, -x_1),$$

which is an explicit self-similar solution of the system in \mathbb{R}^2 corresponding to the initial velocity $\Theta_0(x) = \frac{x^\perp}{2\pi|x|^2}$ and the initial vorticity $\text{curl } \Theta_0 = \delta_0$ (the Dirac mass). The Lamb-Oseen vortex Θ is known to characterize the large time behaviour of the solutions of the Navier-Stokes system in \mathbb{R}^2 supplemented with the initial datum u_0 satisfying $\omega_0 = \text{curl } u_0 \in L^1(\mathbb{R}^2)$ in the following sense

$$\lim_{t \rightarrow \infty} t^{\frac{1}{2} - \frac{1}{p}} \|u(t) - m\Theta(t)\|_{L^p} = 0 \quad \text{for all } p \in (2, \infty],$$

where $m = \int_{\mathbb{R}^2} \omega_0 dx$. This result is due to Giga & Kambe [16] if $\|\omega_0\|_{L^1}$ is small, to Carpio [7] in the case when $\int_{\mathbb{R}^2} \omega_0 dx$ is small, and to Gallay & Wayne [14] in the general case (together with the higher order terms in the asymptotic expansion of solutions).

Such general asymptotic results are not known for problems in two dimensional exterior domains. It is not even clear whether the hypothesis on the integrability of the initial vorticity is relevant. Here, no L^1 -bound for the vorticity is known because of the absence of any reasonable boundary condition for the vorticity. Thus, to overcome this technical problem in our unpublished manuscript [19], we assumed that the initial velocity behaves for large values of $|x|$ like a multiple of $\frac{x^\perp}{|x|^2}$. More precisely, we showed in [19] that if $u_0(x) = w_0(x) + \alpha \frac{x^\perp}{|x|^2}$ with $w_0 \in L^2(\Omega)$ and $\alpha \in \mathbb{R}$, then there exists $\alpha_0 = \alpha_0(w_0, \Omega) > 0$ such that for all $|\alpha| \leq \alpha_0$ we have that

$$\lim_{t \rightarrow \infty} t^{\frac{1}{2} - \frac{1}{p}} \|u(t) - \alpha\Theta(t)\|_{L^p(\Omega)} = 0$$

for each $p \in (2, \infty)$. Comparing with the analogous result in the full plane case, we have an additional smallness condition on the parameter α , which is due to the fact that neither L^2 -estimates for the velocity nor L^p -estimates for the vorticity are known for the problem in an exterior domain (because the velocity is not square-integrable and because of the absence of boundary conditions for the vorticity). Consequently, we have proved in [19] that the large time asymptotics of the solutions of the problem in an exterior domain is the same as in the full plane case and is given by the Lamb-Oseen vortex. This unpublished result is now a particular case of Theorem 5.2, below (see Remark 5.4). Our result from [19] has been recently improved by Gallay & Maekawa [13], where the smallness constant α_0 can be chosen independently of w_0 if one imposes the additional minor assumption that $w_0 \in L^q(\Omega)$ for some $q < 2$.

More general slowly decaying initial conditions.

Even though the most physically interesting case in two dimensions occurs when $u_0(x) \simeq C \frac{x^\perp}{|x|^2}$ at infinity, one could consider other behaviours of u_0 at infinity. For example, one could assume that the initial velocity behaves at infinity like an arbitrary divergence-free homogeneous vector field of degree -1 . More generally, one can consider initial velocities from the

Marcinkiewicz (weak L^2) space $L^{2,\infty}(\Omega)$; however, in such a general setting and due to the chaotic behaviour observed in [8], we may not actually have any clear asymptotics of solutions for large values of times.

To avoid such a difficulty, instead of looking for the asymptotic behaviour of solutions to the problem in an exterior domain directly, we will compare the solutions of the Navier-Stokes system in an exterior domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) to the solutions of the Navier-Stokes system in the whole space \mathbb{R}^n and we show that the solutions to both problems behave, as $t \rightarrow \infty$, in the same way provided that their initial data are “comparable at infinity” (meaning that the difference is slightly better than $L^{n,\infty}$ at infinity, see the definition below). This approach allows us to remove the obstacle from the problem under considerations and to reduce the study of the Navier-Stokes system in an exterior domain to the study of this system in the whole space \mathbb{R}^n (at least as far as large time asymptotics are concerned).

We discuss now what kind of initial conditions can be used in our approach.

Definition 1.1. *Let v_0 be a divergence free vector field defined on \mathbb{R}^n and let u_0 be a divergence free vector field defined on an exterior domain $\Omega \subset \mathbb{R}^n$. Assume moreover that u_0 is tangent to the boundary of Ω , i.e. $u \cdot \nu = 0$ on the boundary, where ν is the normal vector to the boundary. We say that u_0 and v_0 are comparable at infinity if $u_0 - v_0|_{\Omega}$ belongs to $L^{q_0}(\Omega)$ for some $q_0 \in (1, n]$.*

Given $v_0 \in L^{n,\infty}(\mathbb{R}^n)$ a divergence free vector field, examples of vector fields u_0 comparable at infinity are $u_0 = \mathbb{P}_{\Omega}v_0|_{\Omega}$ and also the vector field obtained from v_0 by the truncation procedure described in Section 2 (which in dimension two corresponds to cutting-off the stream function). Here, \mathbb{P}_{Ω} is the Leray projector associated to Ω , i.e. the L^2 -orthogonal projection on the space of vector fields which are divergence free and tangent to the boundary of Ω . Note also that if v_0 is the extension of u_0 to \mathbb{R}^n with zero values on $\mathbb{R}^n \setminus \Omega$, then u_0 and v_0 are comparable at infinity. Indeed, in this case, we have that $u_0 = \mathbb{P}_{\Omega}v_0|_{\Omega}$, see below for more details.

Results of this work.

Now, we briefly present the main results of this work. Let v be a solution of the Navier-Stokes equations in \mathbb{R}^n with initial velocity v_0 :

$$\partial_t v - \Delta v + v \cdot \nabla v = -\nabla p \quad \text{in } \mathbb{R}^n, \quad v|_{t=0} = v_0,$$

and let u be a solution of the Navier-Stokes equations in the exterior domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) with initial velocity u_0 and with the homogeneous Dirichlet boundary conditions:

$$(1.1) \quad \partial_t u - \Delta u + u \cdot \nabla u = -\nabla q \quad \text{in } \Omega, \quad u|_{t=0} = u_0, \quad u|_{\partial\Omega} = 0.$$

The main result of this paper, formulated in Theorem 3.2 below, states that if u_0 and v_0 are comparable at infinity and small in the norm of the space $L^{n,\infty}$, then u and v have the same large time behaviour in the following sense

$$(1.2) \quad \lim_{t \rightarrow \infty} t^{\frac{1}{2} - \frac{n}{2p}} \|u(t) - v(t)\|_{L^p(\Omega)} = 0$$

for all $n < p < \infty$.

Remark 1.2. The decay rate in (1.2) corresponds to the optimal decay of solutions measured in L^p -norms. Indeed, if $v_0 \in L^{n,\infty}(\mathbb{R}^n)$ is small and homogeneous of degree -1 , then the corresponding solution is self-similar: $v(x, t) = t^{-1/2}v(xt^{-1/2}, 1)$ for all $x \in \mathbb{R}^n$ and $t > 0$. Hence, changing variables, we obtain $t^{\frac{1}{2}-\frac{n}{2p}} \|v(t)\|_{L^p(\Omega)} = \|v(1)\|_{L^p(\Omega)}$ for all $t > 0$.

Next, we can extend this large time asymptotics result to a class of large initial data. In dimension $n \geq 3$, we can replace the smallness of u_0, v_0 in the Marcinkiewicz space $L^{n,\infty}(\Omega)$ by the smallness of both quantities

$$\limsup_{\lambda \rightarrow 0} \lambda \text{mes}\{x \in \Omega : |u_0(x)| > \lambda\}^{\frac{1}{n}} \quad \text{and} \quad \limsup_{\lambda \rightarrow 0} \lambda \text{mes}\{x \in \mathbb{R}^n : |v_0(x)| > \lambda\}^{\frac{1}{n}}$$

in order to show in Theorem 4.1 below that the limit (1.2) is again valid. However, with this improved smallness assumption, the uniqueness of the solutions is no longer ensured and our large time asymptotics result holds true for some weak solutions, only.

In dimension two, the statement of our large data asymptotic result is more involved and we refer the reader to Theorem 5.2 for its precise formulation. Here, we only point out the following special case. The limit in (1.2) holds true for the problem in an exterior two dimensional domain under the following hypothesis. The initial velocity u_0 for the exterior domain problem can be decomposed in the form $u_0 = \tilde{u}_0 + w_0$, where $\tilde{u}_0 \in L^2(\Omega)$ is divergence free and tangent to the boundary (and arbitrarily large) and $w_0 \in L^{2,\infty}(\Omega) \cap B_{4,4}^{-1/2}(\Omega)$ verifies the smallness assumptions $\|w_0\|_{L^{2,\infty}(\Omega)} < \varepsilon_1(\Omega)$ and $\|w_0\|_{B_{4,4}^{-1/2}(\Omega)} < \varepsilon_2(\tilde{u}_0)$ for some small positive constants ε_1 and ε_2 . The initial velocity v_0 for the problem in \mathbb{R}^2 can be decomposed in a similar manner $v_0 = \tilde{U}_0 + W_0$ with similar conditions on \tilde{U}_0 and W_0 . Moreover, w_0 is assumed to be comparable at infinity to W_0 .

We mention now two applications of our results. The first one is that if the initial velocity for the exterior domain problem is small in $L^{n,\infty}(\Omega)$ and comparable at infinity to a velocity field homogeneous of degree -1 , then the large time behaviour of the solution is a self-similar behaviour. The second one is that there exists an initial data for the exterior domain problem such that the corresponding solution exhibits a chaotic behaviour at infinity. Indeed, it suffices to consider v_0 the example of initial data in \mathbb{R}^n exhibited in [8] and set $u_0 = \mathbb{P}_\Omega v_0|_\Omega$ as initial velocity for the exterior domain problem.

The remainder of this paper is organized in the following manner. In the next section, we introduce the notation, we recall the decay estimates for the Stokes semigroup, and we show some preliminary technical lemmas concerning initial data comparable at infinity. In Section 3, we prove our asymptotic result in the case of small initial data. The case of large data in dimension $n \geq 3$ is considered in Section 4. Section 5 deals with the asymptotic behaviour of solutions with large initial conditions in dimension two.

2 Preliminaries

Notation

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be an exterior domain with a smooth boundary Γ , and choose $R > 0$ such that $\mathbb{R}^n \setminus \Omega \subset B_{R/2}$. Here, we set $B_{R/2} = B(0, R/2)$ for the ball of radius $R/2$ centered at the origin. We denote by \mathbb{P}_Ω the Leray projector in Ω , *i.e.* the L^2 orthogonal projection from $L^2(\Omega)$ on the subspace of divergence free vector fields which are tangent to the boundary Γ . It is well-known that \mathbb{P}_Ω extends to a bounded operator on every $L^p(\Omega)$, $1 < p < \infty$. We denote by

$L_\sigma^p(\Omega)$ the closure of the set of smooth, divergence-free, and compactly supported vector fields $C_c^\infty(\Omega)$ with respect to the usual L^p -norm. The space $L_\sigma^p(\Omega)$ can also be viewed as the image of $L^p(\Omega)$ by \mathbb{P}_Ω . In a similar way, we write that $u \in L_\sigma^{p,\infty}(\Omega)$ when $u \in L^{p,\infty}(\Omega)^n$, $\operatorname{div} u = 0$ in Ω , and $u \cdot \nu = 0$ on the boundary Γ , where $L^{p,\infty}(\Omega)$ denotes the usual weak L^p -space (the Marcinkiewicz space) and ν is the normal vector to the boundary $\partial\Omega$. In this work, we use systematically a fixed cut-off function $f \in C^\infty(\mathbb{R}^n, \mathbb{R}_+)$ such that $f(x) = 0$ for $|x| < R/2$ and $f(x) = 1$ for $|x| > 2R/3$.

Stokes semigroup

The stationary Stokes operator $A = -\mathbb{P}_\Omega \Delta$ generates an analytic semigroup of linear operators $\{e^{-tA}\}_{t \geq 0}$ on $L_\sigma^p(\Omega)$ for each $1 < p < \infty$, cf. [15]. If v_0 is divergence free and tangent to the boundary of Ω , then $v(t) = e^{-tA}v_0$ is the solution of the following linear boundary value problem

$$\begin{aligned} \partial_t v - \Delta v + \nabla p &= 0, & \operatorname{div} v &= 0 & \text{for } t > 0, \quad x \in \Omega, \\ v(t, x) &= 0 & & & \text{for } t > 0, \quad x \in \Gamma, \\ v(0, x) &= v_0(x) & & & \text{for } x \in \Omega. \end{aligned}$$

The Stokes semigroup $\{e^{-tA}\}_{t \geq 0}$ satisfies the following L^p -decay estimates.

Proposition 2.1. *Assume that $1 < q < \infty$.*

Let $q \leq p \leq \infty$. There exists $K_1 = K_1(\Omega, p, q) > 0$ such that for every $v_0 \in L_\sigma^q(\Omega)$

$$(2.1) \quad \|e^{-tA}v_0\|_{L^p(\Omega)} \leq K_1 t^{\frac{n}{2p} - \frac{n}{2q}} \|v_0\|_{L^q(\Omega)} \quad \text{for all } t > 0.$$

If, in addition, we assume that $q < p \leq \infty$, then for every $v_0 \in L_\sigma^{q,\infty}(\Omega)$ we also have that

$$(2.2) \quad \|e^{-tA}v_0\|_{L^p(\Omega)} \leq K_1 t^{\frac{n}{2p} - \frac{n}{2q}} \|v_0\|_{L^{q,\infty}(\Omega)} \quad \text{for all } t > 0.$$

There exists $K_2 = K_2(\Omega, q) > 0$ such that for every $v_0 \in L_\sigma^{q,\infty}(\Omega)$ we have the inequality

$$(2.3) \quad \|e^{-tA}v_0\|_{L^{q,\infty}(\Omega)} \leq K_2 \|v_0\|_{L^{q,\infty}(\Omega)} \quad \text{for all } t > 0.$$

There exists $K_3 = K_3(\Omega, q) > 0$ such that for every $v_0 \in L_\sigma^q(\Omega)$ we have the inequality

$$(2.4) \quad \|Ae^{-tA}v_0\|_{L^q(\Omega)} \leq K_3 t^{-1} \|v_0\|_{L^q(\Omega)} \quad \text{for all } t > 0.$$

Finally, if $n \leq q \leq p < \infty$ then there exists $K_4 = K_4(\Omega, p, q) > 0$ such that for every matrix valued function $F \in L^q(\Omega; M_n(\mathbb{R}))$ we have that

$$(2.5) \quad \|e^{-tA}\mathbb{P}_\Omega \operatorname{div} F\|_{L^p(\Omega)} \leq K_4 t^{-\frac{1}{2} + \frac{n}{2p} - \frac{n}{2q}} \|F\|_{L^q(\Omega)} \quad \text{for all } t > 0,$$

with the divergence operator div computed along the rows of the matrix F .

Estimates (2.1)–(2.3) were proved in [9, 10, 23, 25]. Relation (2.4) is a consequence of the fact that e^{-tA} is an analytic semigroup, see [15].

The following corollary contains a minor improvement of the decay estimate (2.1).

Corollary 2.2. *Assume that $1 < q < \infty$ and let $v_0 \in L_\sigma^q(\Omega)$. Then for every $p \in [q, \infty)$*

$$\lim_{t \rightarrow \infty} t^{\frac{n}{2q} - \frac{n}{2p}} \|e^{-tA}v_0\|_{L^p(\Omega)} = 0.$$

Proof. The validity of this limit is clear when the initial datum v_0 is smooth and compactly supported. To show it for all $v_0 \in L_\sigma^q(\Omega)$, it suffices to use a standard density argument combined with estimate (2.1). \square

Initial data comparable at infinity

Before we compare the large time behaviour of the Navier-Stokes equations in exterior domains with the large time behaviour of the Navier-Stokes equations in the whole \mathbb{R}^n , we have to clarify the issue of the initial data. More precisely, given a divergence-free initial datum v_0 on \mathbb{R}^n , we would like to construct an initial datum u_0 for the problem in the exterior domain which is comparable at infinity with v_0 . The simplest approach which consists in taking the restriction of v_0 to Ω , *i.e.* $u_0 = v_0|_{\Omega}$, does not work because in general this restriction is not tangent to the boundary Γ . Thus, in order to obtain a vector field which is divergence free and tangent to the boundary, the most natural way is to define u_0 by applying the Leray projection to the restriction of v_0 to Ω :

$$(2.6) \quad u_0 = \mathbb{P}_{\Omega}(v_0|_{\Omega}).$$

Vice versa, given a velocity field u_0 on Ω which is divergence free and tangent to the boundary, we can construct a velocity field v_0 in \mathbb{R}^n , simply by extending u_0 with zero values outside Ω . Here, the divergence free condition is preserved across the boundary because u_0 is tangent to the boundary. Hence, with this choice of v_0 we clearly have relation (2.6).

Unfortunately, defining u_0 as in (2.6) is not practical from the point of view of estimates, because $\mathbb{P}_{\Omega}(v_0|_{\Omega})$ does not verify the Dirichlet boundary condition. Instead, we will use a cutoff procedure that we detail now.

First, we need to prove a technical result.

Lemma 2.3. *Let $1 < p < \infty$ and $v \in L^p(\mathbb{R}^n)$ be a vector field. There exists a unique matrix valued function ψ such that*

$$(2.7) \quad \psi \in W_{\text{loc}}^{1,p}(\mathbb{R}^n), \quad \nabla \psi \in L^p(\mathbb{R}^n), \quad \int_{B_R} \psi = 0 \quad \text{and} \quad \Delta \psi_{ij} = \partial_j v_i - \partial_i v_j$$

for all i, j . Moreover, there exists a constant $C = C(p, R)$ such that

$$(2.8) \quad \|\psi\|_{L^p(B_R)} + \|\nabla \psi\|_{L^p(\mathbb{R}^n)} \leq C \|v\|_{L^p(\mathbb{R}^n)}.$$

Proof. We start by proving the uniqueness of ψ . Let ψ_1 and ψ_2 verify (2.7). Then $\nabla(\psi_1 - \psi_2)$ is harmonic and belongs to $L^p(\mathbb{R}^n)$, therefore it must vanish so $\psi_1 - \psi_2 = \mathcal{C}$ for some constant matrix \mathcal{C} . But $\int_{B_R} \mathcal{C} = \int_{B_R} \psi_1 - \int_{B_R} \psi_2 = 0$, hence $\mathcal{C} = 0$.

We show now the existence of ψ . The operators $\partial_i \partial_j \Delta^{-1}$ are bounded on $L^p(\mathbb{R}^n)$ as products of the Riesz transforms. Therefore $w_{ij} = \nabla \Delta^{-1}(\partial_j v_i - \partial_i v_j)$ is well defined and belongs to $L^p(\mathbb{R}^n)$. Now, if φ is a divergence free, smooth and compactly supported vector field then we clearly have that

$$\int_{\mathbb{R}^n} w_{ij} \cdot \varphi = \int_{\mathbb{R}^n} \nabla \Delta^{-1}(\partial_j v_i - \partial_i v_j) \cdot \varphi = \int_{\mathbb{R}^n} (v_i \partial_j - v_j \partial_i) \Delta^{-1} \operatorname{div} \varphi = 0.$$

We infer from [12, Lemma III.1.1 and Corollary II.4.1]) that there exists $\psi_{ij} \in W_{\text{loc}}^{1,p}(\mathbb{R}^n)$ such that $w_{ij} = \nabla \psi_{ij}$. Adding a constant if necessary, we can also assume that $\int_{B_R} \psi_{ij} = 0$. Finally, we remark that $\Delta \psi_{ij} = \operatorname{div} \nabla \psi_{ij} = \operatorname{div} w_{ij} = \operatorname{div} \nabla \Delta^{-1}(\partial_j v_i - \partial_i v_j) = \Delta \Delta^{-1}(\partial_j v_i - \partial_i v_j) = \partial_j v_i - \partial_i v_j$. Therefore ψ has all properties stated in (2.7). Moreover, the bound $\|\nabla \psi\|_{L^p(\mathbb{R}^n)} \leq C \|w\|_{L^p(\mathbb{R}^n)} \leq C \|v\|_{L^p(\mathbb{R}^n)}$ is obvious. Finally, since ψ has vanishing mean on the ball B_R , we can apply the Poincaré inequality to obtain that $\|\psi\|_{L^p(B_R)} \leq C \|\nabla \psi\|_{L^p(B_R)} \leq C \|v\|_{L^p(\mathbb{R}^n)}$. This completes the proof. \square

It is well-known that $L^{p,\infty}$ can be defined by the real interpolation method: $L^{p,\infty}_\sigma = (L^{p_0}_\sigma, L^{p_1}_\sigma)_{\theta,\infty}$, where $p_0 < p < p_1$ and $1/p = (1-\theta)/p_0 + \theta/p_1$. Since the application $v \mapsto \psi$ is obviously linear, the theory of the real interpolation implies that ψ is well defined for $v \in L^{p,\infty}(\mathbb{R}^n)$ and relation (2.8) remains true in weak L^p -spaces:

$$(2.9) \quad \|\psi\|_{L^{p,\infty}(B_R)} + \|\nabla\psi\|_{L^{p,\infty}(\mathbb{R}^n)} \leq C \|v\|_{L^{p,\infty}(\mathbb{R}^n)}$$

for all $v \in L^{p,\infty}(\mathbb{R}^n)$.

Observe next that if v is also divergence free, then

$$v = \operatorname{div} \psi$$

where ψ is defined in Lemma 2.3 and the divergence of the matrix ψ is computed along its rows. Remark also that the divergence of a skew-symmetric matrix is a divergence free vector field.

Now, let f be a smooth cut-off function from \mathbb{R}^n to \mathbb{R}_+ such that f vanishes for $|x| < R/2$ and $f(x) = 1$ for $|x| > 2R/3$. In this work, we consider initial data for the exterior domain problem obtained as the cut-off of the matrix ψ_0 associated to v_0 as in Lemma 2.3:

$$\bar{v}_0 = \operatorname{div}(f\psi_0).$$

Since ψ_0 is skew-symmetric, we have $\operatorname{div} \bar{v}_0 = 0$. Moreover, from the localization property of the cut-off f , we infer that \bar{v}_0 vanishes on the boundary Γ and $\bar{v}_0 = v_0$ for $|x| > R$. We also have the following lemma.

Lemma 2.4. *The mapping $v_0 \mapsto \bar{v}_0$ is bounded from $L^p_\sigma(\mathbb{R}^n)$ into $L^p_\sigma(\Omega)$ for each $1 < p < \infty$ as well as from $L^{n,\infty}_\sigma(\mathbb{R}^n)$ into $L^{n,\infty}_\sigma(\Omega)$.*

Proof. Since $\bar{v}_0 = fv_0 + \psi_0 \nabla f$, where ∇f is supported in the ball B_R , we may use inequality (2.8) to obtain the estimate

$$\|\bar{v}_0\|_{L^p(\Omega)} \leq \|v_0\|_{L^p(\mathbb{R}^n)} + \|\nabla f\|_{L^\infty} \|\psi_0\|_{L^p(B_R)} \leq C \|v_0\|_{L^p(\mathbb{R}^n)}.$$

Therefore, the operator $v_0 \mapsto \bar{v}_0$ is linear and bounded from $L^p_\sigma(\mathbb{R}^n)$ into $L^p_\sigma(\Omega)$ for each $1 < p < \infty$. By interpolation, it is also bounded from $L^{n,\infty}_\sigma(\mathbb{R}^n)$ into $L^{n,\infty}_\sigma(\Omega)$. \square

Let us prove that both \bar{v}_0 and u_0 chosen as in (2.6) are comparable at infinity to v_0 .

Lemma 2.5. *Let $v_0 \in L^{n,\infty}(\mathbb{R}^n)$ be divergence free and construct the matrix ψ_0 from v_0 as in Lemma 2.3. Then, all three vector fields*

$$v_0|_\Omega, \quad \mathbb{P}_\Omega(v_0|_\Omega), \quad \operatorname{div}(f\psi_0) \quad \text{are comparable at infinity.}$$

More precisely, the differences

$$v_0|_\Omega - \mathbb{P}_\Omega(v_0|_\Omega) \quad \text{and} \quad v_0|_\Omega - \operatorname{div}(f\psi_0) \quad \text{belong to} \quad L^q(\Omega)$$

for each $q \in (1, n)$.

Proof. To show that $v_0 - \operatorname{div}(f\psi_0) \in L^q(\Omega)$ for each $q \in (1, n)$, we observe that $v_0 - \operatorname{div}(f\psi_0) = 0$ for $|x| > R$ (due to properties of the cut-off f) and that $v_0 - \operatorname{div}(f\psi_0) \in L^{n,\infty}(B_R)$ (see relation (2.9)). Next, it suffices to use the imbedding $L^{n,\infty}(B_R) \subset L^q(B_R)$ for each $q \in (1, n)$.

Finally, using that the Leray projector is bounded in L^q , we have that $\mathbb{P}_\Omega(v_0 - \operatorname{div}(f\psi_0)) \in L^q(\Omega)$ for all $q \in (1, n)$. But $\operatorname{div}(f\psi_0)$ is divergence free and vanishes on the boundary, so $\mathbb{P}_\Omega \operatorname{div}(f\psi_0) = \operatorname{div}(f\psi_0)$. We infer that $\operatorname{div}(f\psi_0) - \mathbb{P}_\Omega v_0 \in L^q(\Omega)$ for all $q \in (1, n)$. This completes the proof. \square

We conclude this section by recalling a stability result concerning the large time behaviour of solutions of the Navier-Stokes system in an exterior domain.

Theorem 2.6. *Let $u_0, \tilde{u}_0 \in L_\sigma^{n,\infty}(\Omega)$. There exists $\varepsilon > 0$ such that if $\|u_0\|_{L^{n,\infty}(\Omega)} < \varepsilon$ and $\|\tilde{u}_0\|_{L^{n,\infty}(\Omega)} < \varepsilon$, then the global small solutions u and \tilde{u} of the exterior Navier-Stokes problem (1.1) with initial data u_0 and \tilde{u}_0 verify the following stability result. We have*

$$t^{\frac{1}{2} - \frac{n}{2p}} \|u(t) - \tilde{u}(t)\|_{L^p(\Omega)} \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad \forall p \in (n, \infty)$$

if and only if

$$t^{\frac{1}{2} - \frac{n}{2p}} \|e^{-tA}(u_0 - \tilde{u}_0)\|_{L^p(\Omega)} \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad \forall p \in (n, \infty).$$

In this theorem, the global existence of the solutions was proved by Kozono & Yamazaki [23] and the stability part is shown in [2].

Thus, in particular, as long as the initial datum $v_0 \in L^{n,\infty}(\mathbb{R}^n)$ is small, Lemma 2.5 combined with Corollary 2.2 and Theorem 2.6 imply that the Navier-Stokes solutions in the exterior domain Ω , supplemented with the initial data $\mathbb{P}_\Omega(v_0|_\Omega)$ and $\operatorname{div}(f\psi_0)$, have the same large time asymptotics.

An analogous result on the large time behaviour of solutions of semilinear parabolic equations with a scaling property was proved in [21] in the case of the whole space \mathbb{R}^n .

3 Asymptotics for small data in weak L^n -spaces

Statement of the results

Now we show that, for certain small initial conditions in the whole space \mathbb{R}^n and in the exterior domain Ω , the corresponding solutions of the Navier-Stokes system have the same large time asymptotics.

Let $v_0 \in L_\sigma^{n,\infty}(\mathbb{R}^n)$ be sufficiently small and denote by v the unique global-in-time solution of the Navier-Stokes equations in \mathbb{R}^n with initial velocity v_0 :

$$(3.1) \quad \partial_t v - \Delta v + v \cdot \nabla v = -\nabla p \quad \text{for } x \in \mathbb{R}^n,$$

$$(3.2) \quad \operatorname{div} v = 0,$$

$$(3.3) \quad v(0, \cdot) = v_0.$$

We denote by ψ_0 the skew-symmetric matrix constructed as in Lemma 2.3. In particular, $v_0 = \operatorname{div} \psi_0$ and $\int_{B_R} \psi_0(x) dx = 0$. Then, the vector field

$$(3.4) \quad \bar{v}_0 = \operatorname{div}(f\psi_0),$$

where f is our fixed cut-off function, is divergence free and vanishes on $\Gamma = \partial\Omega$. Moreover, if v_0 is small in $L_\sigma^{n,\infty}(\mathbb{R}^n)$, then $u_0 = \bar{v}_0|_\Omega$ is also small in $L_\sigma^{n,\infty}(\Omega)$, as an immediate consequence of Lemma 2.4.

We are going to compare the large time behaviour of the solution $v = v(t, x)$ of the whole space problem (3.1)-(3.3) with the solution $u = u(t, x)$ of the following exterior Navier-Stokes problem

$$(3.5) \quad \partial_t u - \Delta u + u \cdot \nabla u = -\nabla q \quad \text{for } x \in \Omega,$$

$$(3.6) \quad \operatorname{div} u = 0,$$

$$(3.7) \quad u(0, \cdot) = u_0 \equiv \bar{v}_0|_\Omega,$$

where \bar{v}_0 is defined in (3.4).

If $\|v_0\|_{L^{n,\infty}(\mathbb{R}^n)}$ is sufficiently small, say $\|v_0\|_{L^{n,\infty}(\mathbb{R}^n)} \leq \varepsilon_1$ for suitable small $\varepsilon_1 > 0$, then a solution $v = v(x, t)$ of the Cauchy problem (3.1)-(3.3) exists globally-in-time and for every $n < p < \infty$ it satisfies the following bounds

$$(3.8) \quad \begin{aligned} \|v(t)\|_{L^p(\mathbb{R}^n)} &\leq C\varepsilon_1 t^{\frac{n}{2p}-\frac{1}{2}}, & \|\nabla v(t)\|_{L^p(\mathbb{R}^n)} &\leq C\varepsilon_1 t^{\frac{n}{2p}-1}, \\ \|\partial_t v(t)\|_{L^p(\mathbb{R}^n)} &\leq C\varepsilon_1 t^{\frac{n}{2p}-\frac{3}{2}}, & \|\Delta v(t)\|_{L^p(\mathbb{R}^n)} &\leq C\varepsilon_1 t^{\frac{n}{2p}-\frac{3}{2}}, \end{aligned}$$

for all $t > 0$ (see *e.g.* [18]). The constant C depends on p but not on ε_1 . Moreover, since by Lemma 2.4, there exists a constant $C > 0$ such that $\|u_0\|_{L^p(\Omega)} = \|\bar{v}_0\|_{L^p(\Omega)} \leq C\varepsilon_1$, for sufficiently small $\varepsilon_1 > 0$ there exists a global-in-time solution $u = u(x, t)$ of the exterior problem (3.5)-(3.7) which satisfies the estimate

$$\|u(t)\|_{L^p(\Omega)} \leq C\varepsilon_1 t^{\frac{n}{2p}-\frac{1}{2}}$$

for some constant $C = C(p) > 0$ and for all $t > 0$ (see [23]).

In the main theorem of this section, we show that both solutions u and v have the same large time behaviour.

Theorem 3.1. *There exists a constant $\varepsilon_1 > 0$ with the following property. Let $v_0 \in L_\sigma^{n,\infty}(\mathbb{R}^n)$ with $\|v_0\|_{L^{n,\infty}(\mathbb{R}^n)} \leq \varepsilon_1$ and define $u_0 = \bar{v}_0|_\Omega$ obtained from v_0 via the cut-off procedure (3.4). Then, there exist $v = v(t, x)$ and $u = u(t, x)$ unique global-in-time solutions of the Cauchy problem (3.1)-(3.3) and of the exterior problem (3.5)-(3.7), respectively. Moreover, for every $n < p < \infty$ we have*

$$\lim_{t \rightarrow \infty} t^{\frac{1}{2}-\frac{n}{2p}} \|u(t) - v(t)\|_{L^p(\Omega)} = 0.$$

We might want to have a similar result for other initial conditions u_0 . This is easily obtained via the stability result stated in Theorem 2.6. For example, the same result holds true if we choose $u_0 = \mathbb{P}_\Omega(v_0|_\Omega)$ instead of $u_0 = \bar{v}_0|_\Omega$. Indeed, by Lemma 2.5 and the decay estimates for the Stokes operator given in (2.1), we have that

$$\lim_{t \rightarrow \infty} t^{\frac{1}{2}-\frac{n}{2p}} \|e^{-tA}(\mathbb{P}_\Omega(v_0|_\Omega) - \operatorname{div}(f\psi_0))\|_{L^p(\Omega)} = 0$$

for every $p > n$. Moreover, if $\|v_0\|_{L^{n,\infty}(\mathbb{R}^n)} \leq \varepsilon$, by the continuity of the Leray projector, we obtain that $\|\mathbb{P}_\Omega(v_0|_\Omega)\|_{L^{n,\infty}(\Omega)} \leq C\varepsilon$. Hence, if $\varepsilon > 0$ is sufficiently small, it suffices to apply Theorem 2.6.

Conversely, we might want to fix u_0 and to construct v_0 instead of the other way round. This is also made possible by our results. Given $u_0 \in L_\sigma^{n,\infty}(\Omega)$, we may choose v_0 to be the

extension of u_0 to \mathbb{R}^n with zero values outside Ω (or any other extension that preserves the divergence free condition and the smallness of the $L^{n,\infty}$ -norm). Indeed, for such an extension we have that $u_0 = \mathbb{P}_\Omega(v_0|_\Omega)$ and the above applies provided that u_0 is sufficiently small in $L^{n,\infty}$.

In fact, this property can be applied to all initial data comparable at infinity. More precisely, we have the following result.

Theorem 3.2. *Let $v_0 \in L_\sigma^{n,\infty}(\mathbb{R}^n)$ and $u_0 \in L_\sigma^{n,\infty}(\Omega)$ be comparable at infinity. There exists $\varepsilon_1 > 0$ such that if $\|v_0\|_{L^{n,\infty}(\mathbb{R}^n)} \leq \varepsilon_1$ and $\|u_0\|_{L^{n,\infty}(\Omega)} \leq \varepsilon_1$, then for every $p \in (n, \infty)$, the corresponding solutions of the Cauchy problem (3.1)-(3.3) and the exterior problem (3.5)-(3.7), respectively, satisfy $\lim_{t \rightarrow \infty} t^{\frac{1}{2} - \frac{n}{2p}} \|u(t) - v(t)\|_{L^p(\Omega)} = 0$.*

Proof. Let us denote by \tilde{u} the solution of the Navier-Stokes equations on Ω with initial velocity $\tilde{u}_0 = \bar{v}_0|_\Omega$, where \bar{v}_0 is constructed in (3.4). Using Lemma 2.4 we have $\|\tilde{u}_0\|_{L^{n,\infty}(\Omega)} < C\varepsilon$. Hence, if $\varepsilon_1 > 0$ is sufficiently small, by Theorem 3.1, we obtain that $\lim_{t \rightarrow \infty} t^{\frac{1}{2} - \frac{n}{2p}} \|\tilde{u}(t) - v(t)\|_{L^p(\Omega)} = 0$ for each $p \in (n, \infty)$.

Next, we use the decomposition

$$u_0 - \tilde{u}_0 = \mathbb{P}_\Omega(u_0 - \tilde{u}_0) = \mathbb{P}_\Omega(u_0 - v_0|_\Omega) + \mathbb{P}_\Omega(v_0|_\Omega - \tilde{u}_0).$$

Since, by Definition 1.1, we have that $u_0 - v_0 \in L^{q_0}(\Omega)$ for some $q_0 \in (1, n]$, we obtain $\mathbb{P}_\Omega(u_0 - v_0|_\Omega) \in L^{q_0}(\Omega)$. Moreover, recalling that $\tilde{u}_0 = \bar{v}_0|_\Omega$ we have that $\mathbb{P}_\Omega(v_0|_\Omega - \tilde{u}_0) \in L^q(\Omega)$ for every $q \in (1, n)$. Hence, either by decay estimate (2.1) or by Corollary 2.2 (if $q_0 = n$), we get that $\lim_{t \rightarrow \infty} t^{\frac{1}{2} - \frac{n}{2p}} \|e^{-tA}(u_0 - \tilde{u}_0)\|_{L^p(\Omega)} = 0$. Thus, if $\varepsilon_1 > 0$ is sufficiently small, we may apply Theorem 2.6 to show $\lim_{t \rightarrow \infty} t^{\frac{1}{2} - \frac{n}{2p}} \|u(t) - \tilde{u}(t)\|_{L^p(\Omega)} = 0$. The triangle inequality completes the proof of Theorem 3.2. \square

Proof of Theorem 3.1

The remainder of this section is devoted to the proof of Theorem 3.1. Here, we use the auxiliary vector field $\bar{v} = \operatorname{div}(f\psi)$, where $\psi = \psi(x, t)$ is the skew-symmetric matrix obtained from the solution $v = v(x, t)$ as in Lemma 2.3. From (3.8) we infer that

$$(3.9) \quad t^{\frac{1}{2} - \frac{n}{2p}} \|\bar{v}(t)\|_{L^p(\Omega)} \leq C t^{\frac{1}{2} - \frac{n}{2p}} \|v(t)\|_{L^p(\mathbb{R}^n)} \leq C(p)\varepsilon_1$$

for all $t > 0$, each $p > n$, and some constant $C(p) > 0$.

To prove Theorem 3.1, it suffices to show that for each $p \in (n, \infty)$ we have that

$$(3.10) \quad \lim_{t \rightarrow \infty} t^{\frac{1}{2} - \frac{n}{2p}} \|w(t)\|_{L^p(\Omega)} = 0, \quad \text{where } w = u - \bar{v}.$$

Indeed, Theorem 3.1 will be a direct consequence of the inequality

$$\|u - v\|_{L^p(\Omega)} \leq \|w\|_{L^p(\Omega)} + \|v - \bar{v}\|_{L^p(\Omega)},$$

where the difference $v - \bar{v}$ is compactly supported, so $\|v - \bar{v}\|_{L^p(\Omega)} \leq C \|v - \bar{v}\|_{L^r(\Omega)}$ for all $r > p$. Thus, using the L^r -decay estimate of v given in (3.8) together with (3.9) we obtain for $r > p$

$$t^{\frac{1}{2} - \frac{n}{2p}} \|v(t) - \bar{v}(t)\|_{L^p(\Omega)} \leq C t^{\frac{1}{2} - \frac{n}{2p}} (\|v(t)\|_{L^r(\Omega)} + \|\bar{v}(t)\|_{L^r(\Omega)}) \leq C t^{\frac{n}{2r} - \frac{n}{2p}} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

To prove the limit in (3.10), we note that the vector field $w = u - \bar{v}$ verifies the following system

$$(3.11) \quad \partial_t w - \Delta w + w \cdot \nabla u + \bar{v} \cdot \nabla w = -\nabla(q - fp) - F \quad \text{in } \Omega, \quad w|_{t=0} = 0,$$

where the forcing term F is given by the following expression

$$(3.12) \quad F = p\nabla f + \partial_t \psi \nabla f + (f\Delta v - \Delta \bar{v}) + (\bar{v} \cdot \nabla \bar{v} - f v \cdot \nabla v) \equiv F_1 + F_2 + F_3 + F_4.$$

Moreover, w is divergence free and vanishes on the boundary.

In the following proposition, we state that (3.10) is a consequence of the decay properties of F .

Proposition 3.3. *If $\varepsilon_1 > 0$ is sufficiently small and if for each $p \in (n, \infty)$*

$$(3.13) \quad \lim_{t \rightarrow \infty} t^{\frac{1}{2} - \frac{n}{2p}} \left\| \int_0^t e^{-(t-s)A} \mathbb{P}_\Omega F(s) ds \right\|_{L^p(\Omega)} = 0,$$

then for each $p \in (n, \infty)$ we have that $\lim_{t \rightarrow \infty} t^{\frac{1}{2} - \frac{n}{2p}} \|w(t)\|_{L^p(\Omega)} = 0$.

Proof. Our reasoning is inspired by the proof of the stability Theorem 2.6 from [2].

Because of the interpolation inequality

$$\|\cdot\|_{L^{p_2}(\Omega)} \leq \|\cdot\|_{L^{p_1}(\Omega)}^\alpha \|\cdot\|_{L^{p_3}(\Omega)}^{1-\alpha}, \quad \text{where } \alpha = \frac{p_1 p_3 - p_1 p_2}{p_2 p_3 - p_1 p_2} \quad \text{and } p_1 < p_2 < p_3,$$

and the fact that $t^{\frac{1}{2} - \frac{n}{2p}} \|w(t)\|_{L^p(\Omega)}$ is bounded, we remark that it suffices to get the limit of the L^p -norm of $w(t)$ for only one $p \in (n, \infty)$. Let us consider $p > 2n$.

Applying the Leray projector \mathbb{P}_Ω to (3.11) and using the Duhamel formula we obtain that

$$(3.14) \quad w(t) = - \int_0^t e^{-(t-s)A} \mathbb{P}_\Omega \operatorname{div}(w \otimes u + \bar{v} \otimes w)(s) ds - \int_0^t e^{-(t-s)A} \mathbb{P}_\Omega F(s) ds.$$

The function

$$h(t) \equiv t^{\frac{1}{2} - \frac{n}{2p}} \|w(t)\|_{L^p(\Omega)}$$

is bounded on $(0, \infty)$ due to estimates (3.9) and (3.8). Hence, using the decay estimate (2.5) with $q = p/2 \geq n$, we may bound

$$\begin{aligned} & \left\| \int_0^t e^{-(t-s)A} \mathbb{P}_\Omega \operatorname{div}(w \otimes u + \bar{v} \otimes w)(s) ds \right\|_{L^p(\Omega)} \\ & \leq C \int_0^t (t-s)^{-\frac{1}{2} - \frac{n}{2p}} \|(w \otimes u + \bar{v} \otimes w)(s)\|_{L^{\frac{p}{2}}(\Omega)} ds \\ & \leq C \int_0^t (t-s)^{-\frac{1}{2} - \frac{n}{2p}} \|w(s)\|_{L^p(\Omega)} (\|u(s)\|_{L^p(\Omega)} + \|\bar{v}(s)\|_{L^p(\Omega)}) ds \\ & \leq C\varepsilon_1 \int_0^t (t-s)^{-\frac{1}{2} - \frac{n}{2p}} s^{-1 + \frac{n}{p}} h(s) ds \\ & = C\varepsilon_1 t^{-\frac{1}{2} + \frac{n}{2p}} \int_0^1 (1-\tau)^{-\frac{1}{2} - \frac{n}{2p}} \tau^{-1 + \frac{n}{p}} h(t\tau) d\tau. \end{aligned}$$

Thus, computing the L^p -norm in (3.14) we have

$$h(t) \leq C\varepsilon_1 \int_0^1 (1-\tau)^{-\frac{1}{2}-\frac{n}{2p}} \tau^{-1+\frac{n}{p}} h(t\tau) d\tau + t^{\frac{1}{2}-\frac{n}{2p}} \left\| \int_0^t e^{-(t-s)A} \mathbb{P}_\Omega F(s) ds \right\|_{L^p(\Omega)}.$$

Now, we apply the $\limsup_{t \rightarrow \infty}$ to both sides of the previous inequality. By the Lebesgue dominated convergence theorem and (3.13), we infer that

$$\begin{aligned} \limsup_{t \rightarrow \infty} h(t) &\leq C\varepsilon_1 \limsup_{t \rightarrow \infty} h(t) \int_0^1 (1-\tau)^{-\frac{1}{2}-\frac{n}{2p}} \tau^{-1+\frac{n}{p}} d\tau \\ &\quad + \limsup_{t \rightarrow \infty} t^{\frac{1}{2}-\frac{n}{2p}} \left\| \int_0^t e^{-(t-s)A} \mathbb{P}_\Omega F(s) ds \right\|_{L^p(\Omega)} \\ &= C\varepsilon_1 \limsup_{t \rightarrow \infty} h(t). \end{aligned}$$

Choosing $\varepsilon_1 > 0$ such that $C\varepsilon_1 < 1$, we conclude that $\limsup_{t \rightarrow \infty} h(t) = 0$. This completes the proof of Proposition 3.3. \square

To establish (3.13), we use the decomposition of F in (3.12) to write

$$\int_0^t e^{-(t-s)A} \mathbb{P}_\Omega F(s) ds = \sum_{j=1}^4 \int_0^t e^{-(t-s)A} \mathbb{P}_\Omega F_j(s) ds \equiv \sum_{j=1}^4 I_j,$$

and we show that

$$\lim_{t \rightarrow \infty} t^{\frac{1}{2}-\frac{n}{2p}} \|I_j(t)\|_{L^p(\Omega)} = 0 \quad \text{for each } j \in \{1, 2, 3, 4\} \quad \text{and for every } p \in (n, \infty).$$

As $\text{supp}(\nabla f) \subset B_R$, we note that $\bar{v} = v$ outside the ball B_R . Using also the fact that $f = 1$ outside $B_{2R/3}$, we obtain immediately (cf. (3.12)) that the terms F_1, \dots, F_4 are compactly supported in the ball B_R . In order to estimate each term I_j , we prove the following lemma:

Lemma 3.4. *Let $q \in (1, p]$, $r \in [q, \infty)$, $0 \leq a < b \leq t$ and consider a sufficiently regular function $g(t, x)$ supported in $\mathbb{R}_+ \times B_R$. There exists a constant $C = C(R, p, q, r) > 0$ such that*

$$\left\| \int_a^b e^{-(t-s)A} \mathbb{P}_\Omega g(s) ds \right\|_{L^p(\Omega)} \leq C \int_a^b (t-s)^{\frac{n}{2p}-\frac{n}{2q}} \|g(s)\|_{L^r(B_R)} ds.$$

Proof. Using the decay estimate (2.1) and recalling that the Leray projector \mathbb{P}_Ω is bounded on $L^q(\Omega)$ for all $1 < q < \infty$, we may estimate

$$\begin{aligned} \left\| \int_a^b e^{-(t-s)A} \mathbb{P}_\Omega g(s) ds \right\|_{L^p(\Omega)} &\leq C \int_a^b (t-s)^{\frac{n}{2p}-\frac{n}{2q}} \|g(s)\|_{L^q(\Omega)} ds \\ &\leq C \int_a^b (t-s)^{\frac{n}{2p}-\frac{n}{2q}} \|g(s)\|_{L^r(B_R)} ds. \end{aligned}$$

\square

In the remainder of this section, we use exponents p, q, r satisfying

$$\frac{n}{2} < q \leq p \quad \text{and} \quad q \leq r < \infty.$$

The values of q and r will be conveniently chosen later on and can possibly change from a term to another. Sometimes, we may use the notation r_1 or r_2 instead of r , when different additional assumptions on r are imposed on r_1 and r_2 and if there is a possibility of confusion. We always assume that $q \leq r_1, r_2 < \infty$. Observe also that, under these assumptions, we have that $\frac{n}{2p} - \frac{n}{2q} > -1$.

Estimate of I_1 . Using Lemma 3.4 we may bound

$$t^{\frac{1}{2} - \frac{n}{2p}} \|I_1(t)\|_{L^p(\Omega)} \leq C t^{\frac{1}{2} - \frac{n}{2p}} \int_0^t (t-s)^{\frac{n}{2p} - \frac{n}{2q}} \|p(s)\|_{L^r(\mathbb{R}^n)} ds.$$

It is well-known that the pressure term in (3.1) can be expressed as $p = -\sum_{i,j} \partial_i \partial_j \Delta^{-1}(v_i v_j)$, where the operator $\partial_i \partial_j \Delta^{-1}$ is bounded on $L^\alpha(\mathbb{R}^n)$ for every $1 < \alpha < \infty$. Thus, using the Hölder inequality we obtain the estimate

$$\|p(s)\|_{L^r(\mathbb{R}^n)} \leq C \|v(s)\|_{L^{2r}(\mathbb{R}^n)}^2 \leq C s^{\frac{n}{2r} - 1}$$

which leads to the inequality

$$t^{\frac{1}{2} - \frac{n}{2p}} \|I_1(t)\|_{L^p(\Omega)} \leq C t^{\frac{1}{2} - \frac{n}{2q} + \frac{n}{2r}}.$$

Clearly, the right-hand side goes to 0 as $t \rightarrow \infty$ if $q < n$ and r is sufficiently large.

Estimate of I_2 . We decompose

$$\begin{aligned} I_2 &= \int_0^t e^{-(t-s)A} \mathbb{P}_\Omega[\partial_s \psi(s) \nabla f] ds \\ &= \int_0^{\frac{t}{2}} e^{-(t-s)A} \mathbb{P}_\Omega[\partial_s \psi(s) \nabla f] ds + \int_{\frac{t}{2}}^t e^{-(t-s)A} \mathbb{P}_\Omega[\partial_s \psi(s) \nabla f] ds \\ &\equiv I_{21} + I_{22}. \end{aligned}$$

Using Lemma 3.4 we have that

$$t^{\frac{1}{2} - \frac{n}{2p}} \|I_{22}(t)\|_{L^p(\Omega)} \leq C t^{\frac{1}{2} - \frac{n}{2p}} \int_{\frac{t}{2}}^t (t-s)^{\frac{n}{2p} - \frac{n}{2q}} \|\partial_s \psi(s)\|_{L^r(B_R)} ds.$$

Since ψ has zero average on the ball B_R , so does $\partial_t \psi$. By the Poincaré inequality applied on B_R and recalling (3.8), we have that

$$\|\partial_s \psi(s)\|_{L^r(B_R)} \leq C \|\partial_s \nabla \psi(s)\|_{L^r(B_R)} \leq C \|\partial_s v(s)\|_{L^r(\mathbb{R}^n)} \leq C s^{\frac{n}{2r} - \frac{3}{2}},$$

where we have used (2.8) applied to $\partial_t \psi$. Thus

$$t^{\frac{1}{2} - \frac{n}{2p}} \|I_{22}(t)\|_{L^p(\Omega)} \leq C t^{\frac{1}{2} - \frac{n}{2p}} \int_{\frac{t}{2}}^t (t-s)^{\frac{n}{2p} - \frac{n}{2q}} s^{\frac{n}{2r} - \frac{3}{2}} ds \leq C t^{\frac{n}{2r} - \frac{n}{2q}} \rightarrow \infty \quad \text{as } t \rightarrow \infty$$

provided that $q < r$.

To bound I_{21} , we integrate by parts

$$(3.15) \quad \begin{aligned} I_{21} &= \int_0^{\frac{t}{2}} e^{-(t-s)A} \partial_s \mathbb{P}_\Omega[\psi(s) \nabla f] ds \\ &= e^{-\frac{t}{2}A} \mathbb{P}_\Omega[\psi(t/2) \nabla f] - e^{-tA} \mathbb{P}_\Omega[\psi_0 \nabla f] + \int_0^{\frac{t}{2}} A e^{-(t-s)A} \mathbb{P}_\Omega[\psi(s) \nabla f] ds. \end{aligned}$$

The L^p -norm of the first term on the right-hand side is easily bounded by

$$\begin{aligned} \left\| e^{-\frac{t}{2}A} \mathbb{P}_\Omega[\psi(t/2) \nabla f] \right\|_{L^p(\Omega)} &\leq C \|\psi(t/2)\|_{L^p(B_R)} \leq C \|\psi(t/2)\|_{L^r(B_R)} \\ &\leq C \|\nabla \psi(t/2)\|_{L^r(B_R)} \leq C \|v(t/2)\|_{L^r(\mathbb{R}^n)} \leq C t^{\frac{n}{2r} - \frac{1}{2}}. \end{aligned}$$

The second right-hand side term of (3.15) can be estimated as follows

$$\left\| e^{-tA} \mathbb{P}_\Omega[\psi_0 \nabla f] \right\|_{L^p(\Omega)} \leq C t^{\frac{n}{2p} - \frac{n}{2q}} \|\psi_0\|_{L^q(B_R)}.$$

Concerning the last term in (3.15), we first note that

$$\begin{aligned} \|\mathbb{P}_\Omega[\psi(s) \nabla f]\|_{L^p(\Omega)} &\leq C \|\psi(s)\|_{L^p(B_R)} \leq C \|\psi(s)\|_{L^r(B_R)} \\ &\leq C \|\nabla \psi(s)\|_{L^r(B_R)} \leq C \|v(s)\|_{L^r(\mathbb{R}^n)} \leq C s^{\frac{n}{2r} - \frac{1}{2}}. \end{aligned}$$

The Stokes semigroup estimates from Proposition 2.1 give

$$\begin{aligned} \left\| \int_0^{\frac{t}{2}} A e^{-(t-s)A} \mathbb{P}_\Omega[\psi(s) \nabla f] ds \right\|_{L^p(\Omega)} &\leq C \int_0^{\frac{t}{2}} (t-s)^{-1} \|\mathbb{P}_\Omega[\psi(s) \nabla f]\|_{L^p(\Omega)} ds \\ &\leq C \int_0^{\frac{t}{2}} (t-s)^{-1} s^{\frac{n}{2r} - \frac{1}{2}} ds = C t^{\frac{n}{2r} - \frac{1}{2}}. \end{aligned}$$

We conclude from the previous estimates that

$$t^{\frac{1}{2} - \frac{n}{2p}} \|I_2(t)\|_{L^p(\Omega)} \leq t^{\frac{1}{2} - \frac{n}{2p}} \|I_{21}(t)\|_{L^p(\Omega)} + t^{\frac{1}{2} - \frac{n}{2p}} \|I_{22}(t)\|_{L^p(\Omega)} \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

provided that $q < n$ and $r > p$.

Estimate of I_4 . We have that $F_4 = \bar{v} \cdot \nabla \bar{v} - f v \cdot \nabla v = \bar{v} \cdot (\nabla \bar{v} - f \nabla v) + f(\bar{v} - v) \cdot \nabla v$, where both terms $\bar{v} \cdot (\nabla \bar{v} - f \nabla v)$ and $f(\bar{v} - v) \cdot \nabla v$ are supported in B_R . Now, we decompose

$$I_4 = \int_0^t e^{-(t-s)A} \mathbb{P}_\Omega[f(\bar{v} - v) \cdot \nabla v](s) ds + \int_0^t e^{-(t-s)A} \mathbb{P}_\Omega[\bar{v} \cdot (\nabla \bar{v} - f \nabla v)](s) ds \equiv I_{41} + I_{42}.$$

First, we use Lemma 3.4 to bound

$$\|I_{41}\|_{L^p(\Omega)} \leq C \int_0^t (t-s)^{\frac{n}{2p} - \frac{n}{2q}} \|(\bar{v} - v) \cdot \nabla v(s)\|_{L^{r_1}(B_R)} ds.$$

Lemma 2.4 and relation (3.8) imply that

$$\|(\bar{v} - v) \cdot \nabla v\|_{L^{r_1}(\mathbb{R}^n)} \leq C \|v\|_{L^{2r_1}(\mathbb{R}^n)} \|\nabla v\|_{L^{2r_1}(\mathbb{R}^n)} \leq C t^{\frac{n}{2r_1} - \frac{3}{2}}$$

so

$$\|I_{41}\|_{L^p(\Omega)} \leq C \int_0^t (t-s)^{\frac{n}{2p}-\frac{n}{2q}} s^{\frac{n}{2r_1}-\frac{3}{2}} ds \leq Ct^{\frac{n}{2p}-\frac{n}{2q}+\frac{n}{2r_1}-\frac{1}{2}}.$$

Assuming that $\frac{n}{2r_1} - \frac{3}{2} > -1$ (i.e. $r_1 < n$) and $q < r_1$, it follows that

$$t^{\frac{1}{2}-\frac{n}{2p}} \|I_{41}\|_{L^p(\Omega)} \leq Ct^{-\frac{n}{2q}+\frac{n}{2r_1}} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Using again Lemma 3.4 we bound

$$\|I_{42}\|_{L^p(\Omega)} \leq C \int_0^t (t-s)^{\frac{n}{2p}-\frac{n}{2q}} \|\bar{v} \cdot (\nabla \bar{v} - f \nabla v)(s)\|_{L^{r_2}(B_R)} ds.$$

Since $\nabla \bar{v} = f \nabla v + v \otimes \nabla f + \nabla \psi \nabla f + \psi \nabla^2 f$, by relation (2.8), we can further estimate

$$\begin{aligned} \|\bar{v} \cdot (\nabla \bar{v} - f \nabla v)\|_{L^{r_2}(B_R)} &\leq \|\bar{v}\|_{L^{2r_2}(\mathbb{R}^n)} \|v \otimes \nabla f + \nabla \psi \nabla f + \psi \nabla^2 f\|_{L^{2r_2}(\mathbb{R}^n)} \\ &\leq C \|\bar{v}\|_{L^{2r_2}(\mathbb{R}^n)} (\|v\|_{L^{2r_2}(\mathbb{R}^n)} + \|\psi\|_{L^{2r_2}(B_R)}) \\ &\leq C \|v\|_{L^{2r_2}(\mathbb{R}^n)}^2 \leq Ct^{\frac{n}{2r_2}-1}. \end{aligned}$$

Thus, we infer that

$$\|I_{42}\|_{L^p(\Omega)} \leq C \int_0^t (t-s)^{\frac{n}{2p}-\frac{n}{2q}} s^{\frac{n}{2r_2}-1} ds \leq Ct^{\frac{n}{2p}-\frac{n}{2q}+\frac{n}{2r_2}},$$

and therefore $t^{\frac{1}{2}-\frac{n}{2p}} \|I_{42}\|_{L^p(\Omega)} \leq Ct^{\frac{1}{2}-\frac{n}{2q}+\frac{n}{2r_2}} \rightarrow 0$ as $t \rightarrow \infty$ provided that $q < n$ and r_2 is sufficiently large.

Estimate of I_3 . Finally, we deal with the remaining term I_3 . Since

$$F_3 = f \Delta v - \Delta \operatorname{div}(f \psi) = f \Delta v - \Delta(fv) - \Delta(\psi \nabla f) = -v \Delta f - 2 \nabla f \nabla v - \Delta(\psi \nabla f),$$

we obtain

$$\begin{aligned} I_3 &= \int_0^t e^{-(t-s)A} \mathbb{P}_\Omega F_3(s) ds \\ &= - \int_0^t e^{-(t-s)A} \mathbb{P}_\Omega (v \Delta f)(s) ds - 2 \int_0^t e^{-(t-s)A} \mathbb{P}_\Omega (\nabla f \nabla v)(s) ds - \int_0^t e^{-(t-s)A} \mathbb{P}_\Omega \Delta(\psi \nabla f)(s) ds \\ &\equiv I_{31} + I_{32} + I_{33}. \end{aligned}$$

To bound the middle term, it suffices to use Lemma 3.4 in the following way

$$\begin{aligned} t^{\frac{1}{2}-\frac{n}{2p}} \|I_{32}(t)\|_{L^p(\Omega)} &\leq Ct^{\frac{1}{2}-\frac{n}{2p}} \int_0^t (t-s)^{\frac{n}{2p}-\frac{n}{2q}} \|\nabla v(s)\|_{L^r(B_R)} ds \\ &\leq Ct^{\frac{1}{2}-\frac{n}{2p}} \int_0^t (t-s)^{\frac{n}{2p}-\frac{n}{2q}} s^{\frac{n}{2r}-1} ds \\ &\leq Ct^{\frac{1}{2}-\frac{n}{2q}+\frac{n}{2r}} \rightarrow 0 \quad \text{as } t \rightarrow \infty, \end{aligned}$$

provided that $q < n$ and r is sufficiently large.

Now, we are going to estimate I_{33} . It is well-known that for every vector field w (not necessarily neither divergence free nor tangent to the boundary), the quantity $w - \mathbb{P}_\Omega w$ is a gradient: $w - \mathbb{P}_\Omega w = \nabla g$ for some scalar function g . It is also known that the Leray projection \mathbb{P}_Ω of such a vector field vanishes. Therefore, $\mathbb{P}_\Omega \Delta w - \mathbb{P}_\Omega \Delta \mathbb{P}_\Omega w = \mathbb{P}_\Omega \Delta(w - \mathbb{P}_\Omega w) = \mathbb{P}_\Omega \Delta \nabla g = \mathbb{P}_\Omega \nabla \Delta g = 0$ so that $\mathbb{P}_\Omega \Delta w = \mathbb{P}_\Omega \Delta \mathbb{P}_\Omega w = A \mathbb{P}_\Omega w$. Using this observation we may write

$$\begin{aligned} I_{33} &= - \int_0^t e^{-(t-s)A} \mathbb{P}_\Omega \Delta(\psi \nabla f)(s) ds \\ &= \int_0^{t-\frac{1}{t}} A e^{-(t-s)A} \mathbb{P}_\Omega(\psi \nabla f)(s) ds - \int_{t-\frac{1}{t}}^t e^{-(t-s)A} \mathbb{P}_\Omega \Delta(\psi \nabla f)(s) ds \\ &\equiv I_{331} + I_{332}. \end{aligned}$$

We apply Lemma 3.4 with $p = q$ and recall relation (2.8) to bound

$$\begin{aligned} \|I_{331}\|_{L^p(\Omega)} &\leq \int_0^{t-\frac{1}{t}} (t-s)^{-1} \|\psi(s)\|_{L^r(B_R)} ds \\ &= \int_0^{t-\frac{1}{t}} (t-s)^{-1} \|v(s)\|_{L^r(B_R)} ds \\ &\leq \int_0^{t-\frac{1}{t}} (t-s)^{-1} s^{\frac{n}{2r}-\frac{1}{2}} ds \leq C t^{\frac{n}{2r}-\frac{1}{2}} \ln(2+t). \end{aligned}$$

We conclude that $t^{\frac{1}{2}-\frac{n}{2p}} \|I_{331}\|_{L^p(\Omega)} \leq C t^{\frac{n}{2r}-\frac{n}{2p}} \ln(2+t) \rightarrow 0$ as $t \rightarrow \infty$ provided that $r > p$.

To estimate I_{332} , we use again Lemma 3.4 with $p = q$:

$$\|I_{332}\|_{L^p(\Omega)} \leq \int_{t-\frac{1}{t}}^t \|\Delta(\psi \nabla f)(s)\|_{L^r(B_R)} ds.$$

Clearly, we have

$$\begin{aligned} \|\Delta(\psi \nabla f)(s)\|_{L^r(B_R)} &\leq C(\|\psi\|_{L^r(B_R)} + \|\nabla \psi\|_{L^r(B_R)} + \|\Delta \psi\|_{L^r(B_R)}) \\ &\leq C(\|v\|_{L^r(\Omega)} + \|\nabla v\|_{L^r(\Omega)}) \leq C(s^{\frac{n}{2r}-\frac{1}{2}} + s^{\frac{n}{2r}-1}) \leq C s^{\frac{n}{2r}-\frac{1}{2}} \end{aligned}$$

for all $s \in (t-1/t, t)$ (notice that we may assume $t \geq 2$). Therefore,

$$t^{\frac{1}{2}-\frac{n}{2p}} \|I_{332}\|_{L^p(\Omega)} \leq t^{\frac{1}{2}-\frac{n}{2p}} \int_{t-\frac{1}{t}}^t s^{\frac{n}{2r}-\frac{1}{2}} ds \leq C t^{\frac{1}{2}-\frac{n}{2p}} t^{\frac{n}{2r}-\frac{3}{2}} = C t^{\frac{n}{2r}-\frac{n}{2p}-1} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

It remains to estimate the term I_{31} which we decompose in the following way

$$\begin{aligned} I_{31} &= - \int_0^t e^{-(t-s)A} \mathbb{P}_\Omega(v \Delta f)(s) ds \\ &= \int_0^t e^{-(t-s)A} \mathbb{P}_\Omega(v \Delta(1-f))(s) ds \\ &= \int_0^{\frac{1}{t}} e^{-(t-s)A} \mathbb{P}_\Omega(v \Delta(1-f))(s) ds + \int_{\frac{1}{t}}^t e^{-(t-s)A} \mathbb{P}_\Omega(v \Delta(1-f))(s) ds \end{aligned}$$

$$\begin{aligned}
&= \int_0^{\frac{1}{t}} e^{-(t-s)A} \mathbb{P}_\Omega(v\Delta(1-f))(s) ds + \int_{\frac{1}{t}}^t e^{-(t-s)A} \mathbb{P}_\Omega\Delta(v(1-f))(s) ds \\
&\quad + 2 \int_{\frac{1}{t}}^t e^{-(t-s)A} \mathbb{P}_\Omega(\nabla v \cdot \nabla f)(s) ds - \int_{\frac{1}{t}}^t e^{-(t-s)A} \mathbb{P}_\Omega(\Delta v(1-f))(s) ds \\
&\equiv I_{311} + I_{312} + I_{313} + I_{314}.
\end{aligned}$$

As the function $1-f$ is supported on the ball B_R , the terms I_{312} and I_{313} can be treated in the same way as the integrals I_{33} and I_{32} . Next, we use Lemma 3.4 to deal with I_{311} :

$$\begin{aligned}
t^{\frac{1}{2}-\frac{n}{2p}} \|I_{311}\|_{L^p(\Omega)} &\leq Ct^{\frac{1}{2}-\frac{n}{2p}} \int_0^{\frac{1}{t}} (t-s)^{\frac{n}{2p}-\frac{n}{2q}} \|v(s)\|_{L^r(\mathbb{R}^n)} ds \\
&\leq Ct^{\frac{1}{2}-\frac{n}{2p}} \int_0^{\frac{1}{t}} (t-s)^{\frac{n}{2p}-\frac{n}{2q}} s^{\frac{n}{2r}-\frac{1}{2}} ds \leq Ct^{-\frac{n}{2q}-\frac{n}{2r}} \rightarrow 0
\end{aligned}$$

as $t \rightarrow \infty$. Moreover, we study I_{314} in a similar manner

$$\begin{aligned}
t^{\frac{1}{2}-\frac{n}{2p}} \|I_{314}\|_{L^p(\Omega)} &\leq Ct^{\frac{1}{2}-\frac{n}{2p}} \int_{\frac{1}{t}}^t (t-s)^{\frac{n}{2p}-\frac{n}{2q}} \|\Delta v(s)\|_{L^r(\mathbb{R}^n)} ds \\
&\leq Ct^{\frac{1}{2}-\frac{n}{2p}} \int_{\frac{1}{t}}^t (t-s)^{\frac{n}{2p}-\frac{n}{2q}} s^{\frac{n}{2r}-\frac{3}{2}} ds \leq Ct^{1-\frac{n}{2q}-\frac{n}{2r}} \rightarrow 0
\end{aligned}$$

as $t \rightarrow \infty$ provided that $r > n$ and q is sufficiently close to $\frac{n}{2}$.

This completes the proof of Theorem 3.1.

4 Asymptotics of solutions for large data and $n \geq 3$

In the case $n \geq 3$, Theorem 3.2 can be extended to a certain class of large initial data as follows.

Theorem 4.1. *Suppose that $n \geq 3$. There exists $\varepsilon > 0$ such that if $u_0 \in L^{n,\infty}(\Omega)$ is divergence free, tangent to the boundary, and such that*

$$\limsup_{\lambda \rightarrow 0} \lambda \text{mes}\{x \in \Omega : |u_0(x)| > \lambda\}^{\frac{1}{n}} < \varepsilon,$$

then there exists a global-in-time solution u of the Navier-Stokes equations on the exterior domain Ω with initial velocity u_0 and such that

$$(4.1) \quad \limsup_{t \rightarrow \infty} t^{\frac{1}{2}-\frac{n}{2p}} \|u(t)\|_{L^p(\Omega)} < \infty.$$

Moreover, let $v_0 \in L^{n,\infty}(\mathbb{R}^n)$ be divergence free and such that

$$\limsup_{\lambda \rightarrow 0} \lambda \text{mes}\{x \in \mathbb{R}^n : |v_0(x)| > \lambda\}^{\frac{1}{n}} < \varepsilon.$$

Then the Navier-Stokes equations on \mathbb{R}^n with initial velocity v_0 admit a global-in-time solution v which satisfies a similar bound to the one in (4.1). Finally, if u_0 and v_0 are comparable at infinity (cf. Definition 1.1), then the corresponding solutions u and v have the same asymptotic behaviour as $t \rightarrow \infty$ in our usual sense: $\lim_{t \rightarrow \infty} t^{\frac{1}{2}-\frac{n}{2p}} \|u(t) - v(t)\|_{L^p(\Omega)} = 0$ for each $p \in (n, \infty)$.

Remark 4.2. Even though the initial velocity u_0 is not small in $L^{n,\infty}(\Omega)$ and is not square-integrable, it is not surprising that a weak solution exists globally. Our result is reminiscent of the result in [5] on the global-in-time existence of weak solutions to the Navier-Stokes system that holds true for initial data in $L^p_\sigma(\mathbb{R}^n)$ with $2 < p < n$.

Proof of Theorem 4.1. We borrow a method from [3]. Using [3, Lemma 4.2] we decompose $u_0 = z_0 + w_0$ where $z_0 \in L^2(\Omega)$ and $\|w_0\|_{L^{n,\infty}(\Omega)} \leq \varepsilon$. Note that even though the statement of that lemma is given for $n = 3$, the proof goes through for $n \geq 3$. Let w solve the Navier-Stokes equations on the exterior domain Ω with initial velocity w_0 and set $z = u - w$. Then z solves

$$\partial_t z - \Delta z + u \cdot \nabla z + z \cdot \nabla w = -\nabla p'.$$

We multiply this equation by z and integrate in space to obtain after an integration by parts:

$$\partial_t \|z\|_{L^2(\Omega)}^2 + 2 \|\nabla z\|_{L^2(\Omega)}^2 = 2 \int_{\Omega} z \cdot \nabla z \cdot w \leq C \|z\|_{L^{\frac{2n}{n-2},2}(\Omega)} \|\nabla z\|_{L^2(\Omega)} \|w\|_{L^{n,\infty}(\Omega)}$$

where we used the Hölder inequality for Lorentz spaces. Next, we recall the following Sobolev inequality in the Lorentz spaces

$$(4.2) \quad \|z\|_{L^{\frac{2n}{n-2},2}(\Omega)} \leq C \|\nabla z\|_{L^2(\Omega)}.$$

Indeed, in the case of functions defined on \mathbb{R}^n , this relation follows from the Young inequality for convolution in Lorentz spaces and from the observation that $(-\Delta)^{-\frac{1}{2}}$ is a convolution operator with a function homogeneous of degree $1 - n$ which therefore belongs to $L^{\frac{n}{n-1},\infty}(\mathbb{R}^n)$. For z vanishing on $\partial\Omega$, we can extend it with zero values outside Ω . Denoting by Ez this extension and applying the inequality known in \mathbb{R}^n for Ez , we get (4.2) for z defined in the exterior domain. Consequently, we infer that

$$\partial_t \|z\|_{L^2(\Omega)}^2 + 2 \|\nabla z\|_{L^2(\Omega)}^2 \leq C \|\nabla z\|_{L^2(\Omega)}^2 \|w\|_{L^{n,\infty}(\Omega)}.$$

Recall now that w is a small solution. Therefore, if $\|w_0\|_{L^{n,\infty}(\Omega)}$ is sufficiently small, *i.e.* if $\varepsilon > 0$ is sufficiently small, then we have that $C\|w(t)\|_{L^{n,\infty}(\Omega)} \leq 1$ for all t . Thus,

$$\partial_t \|z\|_{L^2(\Omega)}^2 + \|\nabla z\|_{L^2(\Omega)}^2 \leq 0,$$

so that $Ez \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^n)) \cap L^2(\mathbb{R}_+; \dot{H}^1(\mathbb{R}^n))$. These are of course just *a priori* estimates, but a standard approximation procedure gives us the existence of a solution verifying these estimates. Since $Ez \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^n)) \cap L^2(\mathbb{R}_+; \dot{H}^1(\mathbb{R}^n))$, we infer that $Ez \in L^4(\mathbb{R}_+; \dot{H}^{\frac{1}{2}}(\mathbb{R}^n))$. Therefore, for every $\gamma > 0$, the set of times t where $\|Ez(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^n)} \geq \gamma$ has finite measure. Since $\dot{H}^{\frac{1}{2}}(\mathbb{R}^n) \subset L^{n,\infty}(\mathbb{R}^n)$, we infer that the set of times where $\|z(t)\|_{L^{n,\infty}(\Omega)} \geq \gamma$ has finite measure.

A similar argument can be performed in the case of the solution v to the Navier-Stokes problem in the whole space \mathbb{R}^n . Indeed, we may decompose $v_0 = Z_0 + W_0$ where both Z_0 and W_0 are divergence free, such that $Z_0 \in L^2(\mathbb{R}^n)$ and $\|W_0\|_{L^{n,\infty}(\mathbb{R}^n)} \leq \varepsilon$ ([3, Lemma 4.2]). We denote by W the solution of the Navier-Stokes equations on \mathbb{R}^n with initial data W_0 and set $Z = v - W$. Similar estimates as for z show that, for every $\gamma > 0$, the set of times where $\|Z(t)\|_{L^{n,\infty}(\mathbb{R}^n)} \geq \gamma$ has finite measure.

We conclude that there is a time $T > 0$ where we have both inequalities $\|Z(T)\|_{L^{n,\infty}(\mathbb{R}^n)} \leq \varepsilon$ and $\|z(T)\|_{L^{n,\infty}(\Omega)} \leq \varepsilon$. Since by the theory of small solutions we also have that $\|W(t)\|_{L^{n,\infty}(\mathbb{R}^n)} \leq C\varepsilon$ and $\|w(t)\|_{L^{n,\infty}(\Omega)} \leq C\varepsilon$ for all $t \geq 0$, we infer that $\|v(T)\|_{L^{n,\infty}(\mathbb{R}^n)} \leq C'\varepsilon$ and $\|u(T)\|_{L^{n,\infty}(\Omega)} \leq C'\varepsilon$ with $C' = C + 1$. We choose ε sufficiently small such that the small data solutions, the stability result from Proposition 2.6, and the asymptotic result Theorem 3.2 can be applied starting from both times $t = 0$ and $t = T$. In particular, we can change if necessary the values of u and v starting from time T with the values of the small solutions that can be constructed starting from $u(T)$ and $v(T)$.

Since $u(T) - w(T) \in L^2(\Omega)$, the stability result stated in Theorem 2.6 applied from time $t = T$ and the decay estimates of the Stokes operator given in Proposition 2.1 imply that for all $p \in (n, \infty)$

$$\lim_{t \rightarrow \infty} t^{\frac{1}{2} - \frac{n}{2p}} \|u(t) - w(t)\|_{L^p(\Omega)} = 0.$$

Similarly, for all $p \in (n, \infty)$ we have that

$$\lim_{t \rightarrow \infty} t^{\frac{1}{2} - \frac{n}{2p}} \|v(t) - W(t)\|_{L^p(\mathbb{R}^n)} = 0.$$

Moreover, $w_0 - W_0 = (u_0 - v_0) - (z_0 - Z_0)$ with $u_0 - v_0 \in L^{q_0}(\Omega)$ for some $q_0 \in (1, n]$ and $z_0 - Z_0 \in L^2(\Omega)$ so $w_0 - W_0 \in L^{q_0}(\Omega) + L^2(\Omega)$. We cannot apply directly Theorem 3.2 to w and W because w_0 and W_0 are not comparable at infinity in the sense of Definition 1.1: we have $w_0 - W_0 \in L^{q_0} + L^2$ instead of $w_0 - W_0 \in L^q$ for some $q \in (1, n]$. Nevertheless, the proof of Theorem 3.2 goes through in this case and we obtain that for all $p \in (n, \infty)$ we have

$$\lim_{t \rightarrow \infty} t^{\frac{1}{2} - \frac{n}{2p}} \|w(t) - W(t)\|_{L^p(\Omega)} = 0.$$

Putting together the three previous relations completes the proof. \square

5 Asymptotics for large data and $n = 2$

We assume throughout this section that $n = 2$. In this case we are not able to show a result as general as in dimension $n \geq 3$, because $L^2(\mathbb{R}^2)$ is a critical space (*i.e.* invariant with respect to the scaling of the Navier-Stokes equations). Here, we decompose the initial velocity as follows

$$u_0 = \tilde{u}_0 + w_0,$$

where $\tilde{u}_0 \in L^2_\sigma(\Omega)$ is arbitrarily large and w_0 is small in $L^{2,\infty}(\Omega)$. Let us recall now the following classical result on the L^2 -decay of weak solutions of problem (1.1).

Theorem 5.1 (Borchers & Miyakawa [4, Thm. 1.2]). *For every $\tilde{u}_0 \in L^2_\sigma(\Omega)$ there is a unique weak solution $\tilde{u} \in L^\infty((0, \infty); L^2(\Omega)) \cap L^2_{loc}([0, \infty); H^1(\Omega))$ of the initial-boundary value problem (1.1) with \tilde{u}_0 as initial velocity. This solution satisfies $\lim_{t \rightarrow \infty} \|\tilde{u}(t)\|_{L^2} = 0$.*

The aim of this section is to show the following theorem.

Theorem 5.2. *Let $\tilde{u}_0 \in L^2_\sigma(\Omega)$ be arbitrary and let T_ε be a time such that $\|\tilde{u}(T_\varepsilon)\|_{L^{2,\infty}(\Omega)} < \varepsilon/3$, where \tilde{u} is the unique global-in-time solution of the Navier-Stokes equations on Ω with initial velocity \tilde{u}_0 and $\varepsilon > 0$ is the small constant from Theorem 3.2 (such a time T_ε exists thanks to*

Theorem 5.1 and to the imbedding $L^2(\Omega) \subset L^{2,\infty}(\Omega)$). Let $w_0 \in L^{2,\infty}(\Omega)$ be divergence free, tangent to the boundary, and such that

$$\|w_0\|_{L^{2,\infty}(\Omega)} < \frac{\varepsilon}{3}.$$

There exists a constant $K = K(\varepsilon)$ such that if the following additional smallness condition holds true:

$$(5.1) \quad \|e^{-tA}w_0\|_{L^4((0,T_\varepsilon)\times\Omega)} \leq \frac{1}{K \|\tilde{u}_0\|_{L^2(\Omega)} e^{K\|\tilde{u}_0\|_{L^2(\Omega)}^4}}$$

then the Navier-Stokes equations on the exterior domain Ω with the initial data $u_0 = \tilde{u}_0 + w_0$ has a unique global solution u . This solution verifies that $\sup_{t>0} t^{\frac{1}{2}-\frac{1}{p}} \|u(t)\|_{L^p(\Omega)} < \infty$.

Moreover, let $v_0 \in L^{2,\infty}(\mathbb{R}^2)$ be such that $\|v_0\|_{L^{2,\infty}(\mathbb{R}^2)} < \varepsilon$ and such that w_0 and v_0 are comparable at infinity. Let v be the unique global solution of the Navier-Stokes equations on \mathbb{R}^2 with initial data v_0 . Then u and v have the same asymptotic behaviour as $t \rightarrow \infty$ in the sense of relation (1.2).

Remark 5.3. If $\|\tilde{u}_0\|_{L^2(\Omega)}$ is sufficiently small, then the initial data u_0 is small in $L^{2,\infty}(\Omega)$ and we can directly apply Theorem 3.2 to reach the desired conclusion without the need to assume that $e^{-tA}w_0 \in L^4_{loc}(\mathbb{R}_+; L^4(\Omega))$ and condition (5.1).

Remark 5.4. The dependence on the parameter T_ε in the smallness condition (5.1) is not explicit. Nevertheless, it can be made explicit if $w_0 \in L^a(\Omega)$ for some $a > 2$. Indeed, for $w_0 \in L^a(\Omega)$, the decay estimates from Proposition 2.1 imply that $\|e^{-tA}w_0\|_{L^4(\Omega)} \leq C \|w_0\|_{L^a(\Omega)} t^{\frac{1}{4}-\frac{1}{a}}$ (note that we can assume that $a \leq 4$ because we also have that $w_0 \in L^{2,\infty}(\Omega)$ so, by interpolation, w_0 belongs to all intermediate spaces $L^b(\Omega)$ for all $2 < b \leq a$). Therefore $\|e^{-tA}w_0\|_{L^4((0,T_\varepsilon)\times\Omega)} \leq C \|w_0\|_{L^a(\Omega)} T_\varepsilon^{\frac{1}{2}-\frac{1}{a}}$. We conclude that the smallness condition (5.1) is implied by the following smallness condition

$$\|w_0\|_{L^a(\Omega)} \leq \frac{1}{CT_\varepsilon^{\frac{1}{2}-\frac{1}{a}} \|\tilde{u}_0\|_{L^2(\Omega)} e^{C\|\tilde{u}_0\|_{L^2(\Omega)}^4}}$$

for some large constant C . In particular, the result in [19] is a special case of Theorem 5.2.

Remark 5.5. The condition (5.1) requires in particular that $e^{-tA}w_0 \in L^4_{loc}([0, \infty); L^4(\Omega))$. The optimal assumption required on w_0 in order to have this property is that w_0 belongs to the inhomogeneous Besov space $B_{4,4}^{-\frac{1}{2}}(\Omega)$ (see [1, 11, 24]). Therefore, Theorem 5.2 can be reformulated in the following way: for every $\tilde{u}_0 \in L^2(\Omega)$ which is divergence free and tangent to the boundary, there exists $\varepsilon_3 = \varepsilon_3(\Omega) > 0$ and $\varepsilon_4 = \varepsilon_4(\tilde{u}_0) > 0$ such that if $w_0 \in L^{2,\infty}(\Omega) \cap B_{4,4}^{-\frac{1}{2}}(\Omega)$, $\|w_0\|_{L^{2,\infty}(\Omega)} < \varepsilon_3$, and $\|w_0\|_{B_{4,4}^{-\frac{1}{2}}(\Omega)} < \varepsilon_4$ then the conclusion of Theorem 5.2 holds true.

Proof of Theorem 5.2. The global existence and the uniqueness of u is proved in [23, Theorem 4]. However, the bound for $t^{\frac{1}{2}-\frac{1}{p}} \|u(t)\|_{L^p(\Omega)}$ proved in [23] is only local in time, *i.e.* it is only shown that $\sup_{0 < t < T} t^{\frac{1}{2}-\frac{1}{p}} \|u(t)\|_{L^p(\Omega)} \leq C(T)$ for all $T < \infty$. Here we will show that this bound is also global, *i.e.* we can have $T = \infty$.

Let $\bar{w} = e^{-tA}w_0$ and $z = u - \tilde{u} - \bar{w}$. Then, the vector field z is divergence free, vanishes on the boundary, and verifies the following system

$$\partial_t z - \Delta z + (z + \bar{w} + \tilde{u}) \cdot \nabla (z + \bar{w} + \tilde{u}) - \tilde{u} \cdot \nabla \tilde{u} = -\nabla \bar{p},$$

where $\bar{p} = p - p_{\tilde{u}} - p_{\bar{w}}$, the scalar function $p_{\tilde{u}}$ is the pressure corresponding to \tilde{u} , and $p_{\bar{w}}$ is the pressure from the Stokes problem satisfied by \bar{w} . We multiply the above relation by z and integrate over Ω to obtain

$$\begin{aligned} \frac{1}{2} \partial_t \|z\|_{L^2(\Omega)}^2 + \|\nabla z\|_{L^2(\Omega)}^2 &= \int_{\Omega} \tilde{u} \cdot \nabla \tilde{u} \cdot z - \int_{\Omega} (z + \bar{w} + \tilde{u}) \cdot \nabla (z + \bar{w} + \tilde{u}) \cdot z \\ &= \int_{\Omega} (z + \bar{w} + \tilde{u}) \cdot \nabla z \cdot \bar{w} + \int_{\Omega} (z + \bar{w}) \cdot \nabla z \cdot \tilde{u} \\ &\leq C \|\nabla z\|_{L^2(\Omega)} (\|\bar{w}\|_{L^4(\Omega)}^2 + \|\bar{w}\|_{L^4(\Omega)} \|\tilde{u}\|_{L^4(\Omega)} \\ &\quad + \|\bar{w}\|_{L^4(\Omega)} \|z\|_{L^4(\Omega)} + \|\tilde{u}\|_{L^4(\Omega)} \|z\|_{L^4(\Omega)}). \end{aligned}$$

Next, we use the bound $\|z\|_{L^4(\Omega)} \leq C \|z\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla z\|_{L^2(\Omega)}^{\frac{1}{2}}$ and we apply the Young inequality:

$$\begin{aligned} \partial_t \|z\|_{L^2(\Omega)}^2 + \|\nabla z\|_{L^2(\Omega)}^2 &\leq C \|\bar{w}\|_{L^4(\Omega)}^2 (\|\bar{w}\|_{L^4(\Omega)}^2 + \|\tilde{u}\|_{L^4(\Omega)}^2) \\ &\quad + C \|z\|_{L^2(\Omega)}^2 (\|\bar{w}\|_{L^4(\Omega)}^4 + \|\tilde{u}\|_{L^4(\Omega)}^4). \end{aligned}$$

The Gronwall and Hölder inequalities imply that

$$\|z(t)\|_{L^2(\Omega)}^2 \leq C \|\bar{w}\|_{L^4((0,t) \times \Omega)}^2 \left(\|\bar{w}\|_{L^4((0,t) \times \Omega)}^2 + \|\tilde{u}\|_{L^4((0,t) \times \Omega)}^2 \right) e^{C(\|\bar{w}\|_{L^4((0,t) \times \Omega)}^4 + \|\tilde{u}\|_{L^4((0,t) \times \Omega)}^4)}.$$

Since \tilde{u} is a solution of the Navier-Stokes equations, the classical energy estimate reads

$$\|\tilde{u}(t)\|_{L^2(\Omega)} + 2 \int_0^t \|\nabla \tilde{u}\|_{L^2(\Omega)}^2 \leq \|\tilde{u}_0\|_{L^2(\Omega)}.$$

Using the inequality $\|\tilde{u}\|_{L^4(\Omega)} \leq C \|\tilde{u}\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla \tilde{u}\|_{L^2(\Omega)}^{\frac{1}{2}}$, we infer that $\|\tilde{u}\|_{L^4((0,t) \times \Omega)} \leq C \|\tilde{u}_0\|_{L^2(\Omega)}$ for all $t > 0$. Hence, we have

$$(5.2) \quad \|z(T_\varepsilon)\|_{L^2(\Omega)}^2 \leq C_1 \|\bar{w}\|_{L^4((0,T_\varepsilon) \times \Omega)}^2 \left(\|\bar{w}\|_{L^4((0,T_\varepsilon) \times \Omega)}^2 + \|\tilde{u}_0\|_{L^2(\Omega)}^2 \right) e^{C_1(\|\bar{w}\|_{L^4((0,T_\varepsilon) \times \Omega)}^4 + \|\tilde{u}_0\|_{L^2(\Omega)}^4)}$$

for some constant C_1 .

If $\|\tilde{u}_0\|_{L^2(\Omega)}$ is small, then $\|u_0\|_{L^{2,\infty}(\Omega)}$ is also small and the conclusion follows after applying Theorem 3.2. More precisely, there exists $\eta = \eta(\varepsilon)$ such that if $\|\tilde{u}_0\|_{L^2(\Omega)} \leq \eta$ then $\|u_0\|_{L^{2,\infty}(\Omega)} \leq \frac{\varepsilon}{3}$. Therefore if $\|\tilde{u}_0\|_{L^2(\Omega)} \leq \eta$ then $\|u_0\|_{L^{2,\infty}(\Omega)} \leq \varepsilon$ and the desired conclusions follow at once from Theorem 3.2.

We assume now that $\|\tilde{u}_0\|_{L^2(\Omega)} > \eta$ and that relation (5.1) holds true for some constant K to be determined later. We impose first that K verifies the condition $1 < K\eta^2 e^{K\eta^4}$. Then $\|\tilde{u}_0\|_{L^2(\Omega)} > \eta$ and (5.1) imply that $\|\bar{w}\|_{L^4((0,T_\varepsilon) \times \Omega)} \leq \|\tilde{u}_0\|_{L^2(\Omega)}$. We infer from (5.2) and (5.1) that

$$\|z(T_\varepsilon)\|_{L^2(\Omega)}^2 \leq 2C_1 \|\bar{w}\|_{L^4((0,T_\varepsilon) \times \Omega)}^2 \|\tilde{u}_0\|_{L^2(\Omega)}^2 e^{2C_1 \|\tilde{u}_0\|_{L^2(\Omega)}^4} \leq \frac{2C_1}{K^2} e^{2(C_1 - K) \|\tilde{u}_0\|_{L^2(\Omega)}^4} \leq \frac{2C_1}{K^2}$$

if we further assume that $K \geq C_1$. Therefore, if K is chosen sufficiently large then we can conclude that $\|z(T_\varepsilon)\|_{L^{2,\infty}(\Omega)} < \frac{\varepsilon}{3}$ implying that

$$\|u(T_\varepsilon)\|_{L^{2,\infty}(\Omega)} \leq \|\tilde{u}(T_\varepsilon)\|_{L^{2,\infty}(\Omega)} + \|\bar{w}(T_\varepsilon)\|_{L^{2,\infty}(\Omega)} + \|z(T_\varepsilon)\|_{L^{2,\infty}(\Omega)} < \varepsilon.$$

We have reached a time where the solution u becomes small in $L^{2,\infty}(\Omega)$. We wish to conclude as in the proof of the corresponding result in dimension $n \geq 3$ stated in Theorem 4.1. If we look at that proof, we notice that we need to show that $u(T_\varepsilon) - w(T_\varepsilon) \in L^2(\Omega)$ where w denotes the Navier-Stokes solution on the exterior domain with initial data w_0 . Since $\tilde{u}(T_\varepsilon) \in L^2(\Omega)$ and $z(T_\varepsilon) \in L^2(\Omega)$ (as shown above) it suffices to prove that $w(T_\varepsilon) - \bar{w}(T_\varepsilon) \in L^2(\Omega)$. But this is a particular case of the estimates obtained above. Indeed, if we set $\tilde{u} = 0$ then $u = w$ and $w(T_\varepsilon) - \bar{w}(T_\varepsilon) = z(T_\varepsilon) \in L^2(\Omega)$.

The end of the proof is similar to the end of the proof of Theorem 4.1. On one hand, we use that $u(T_\varepsilon) - w(T_\varepsilon) \in L^2(\Omega)$ and we apply the stability result from Theorem 2.6 starting from time T_ε to deduce that u and w have the same asymptotic behaviour as $t \rightarrow \infty$. On the other hand we use Theorem 3.2 to say that w and v have the same asymptotic behaviour as $t \rightarrow \infty$. We conclude that u and v have the same asymptotic behaviour as $t \rightarrow \infty$. \square

Remark 5.6. It is of course possible to add a large square-integrable part to v_0 . Assuming similar hypothesis as for the data in the exterior domain, more precisely as in relation (5.1), we can show exactly in the same manner the existence of a unique global solution which has the same asymptotic behaviour as u at infinity. We do not follow this way in order to have a simpler statement for Theorem 5.2.

Acknowledgments. The authors would like to express their gratitude to Thierry Gallay for his helpful remarks which allowed them to simplify and to generalize the previous version of this work.

The work of the second author was partially supported by the NCN grant No. N N201 418839 and the Foundation for Polish Science operated within the Innovative Economy Operational Programme 2007-2013 funded by European Regional Development Fund (Ph.D. Programme: Mathematical Methods in Natural Sciences). The third author is partially supported by the Project ‘‘Instabilities in Hydrodynamics’’ financed by Paris city hall (program ‘‘Emergences’’) and the Fondation Sciences Mathématiques de Paris.

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