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A UNIQUENESS RESULT FOR THE NAVIER–STOKES EQUATIONS WITH VANISHING VERTICAL VISCOSITY*

DRAGOŞ IFTIMIE†

Abstract. Chemin et al. [M2AN Math. Model. Numer. Anal., 34 (2000), pp. 315–335.] considered the three-dimensional Navier–Stokes equations with vanishing vertical viscosity. Assuming that the initial velocity is square-integrable in the horizontal direction and H^s in the vertical direction, they prove existence of solutions for $s > 1/2$ and uniqueness of solutions for $s > 3/2$. Here, we close the gap between existence and uniqueness, proving uniqueness of solutions for $s > 1/2$. Standard techniques are used.

Key words. Navier–Stokes equations, Sobolev spaces

AMS subject classifications. 35Q30, 35Q35, 76D03, 76D05

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Introduction. Chemin, Desjardins, Gallagher, and Grenier [2] considered the following anisotropic Navier–Stokes equations:

$$(NS_h)\begin{cases} \partial_t v - \nu(\partial_1^2+\partial_2^2)v - \nu_{_V}\partial_3^2 v + v\cdot\nabla v = -\nabla p \quad \text{for} \quad (t,x)\in(0,\infty)\times\mathbb{R}^3, \\ \operatorname{div} v = 0 \quad \text{for} \quad (t,x)\in[0,\infty)\times\mathbb{R}^3, \\ v\big|_{t=0} = v_0, \end{cases}$$

where $v(t,\cdot):\mathbb{R}^3\to\mathbb{R}^3$ is an incompressible velocity field, p is the pressure, and the constants $\nu > 0$ and $\nu_{_V}\geq 0$ represent the horizontal and vertical viscosities.

Concerning the physical significance of these equations, we refer to [2] and to the references therein. We will simply say that systems of this type can be found in the theory of rotating fluids and also in the study of the Ekman layers for rotating fluids.

As specified above, the vertical viscosity $\nu_{_V}$ may vanish (or converge to 0). For this reason, the classical theory of the Navier–Stokes equations does not apply. Some L^2 energy estimates still hold for (NS_h), but these are not enough to pass to the limit and obtain a weak solution. The strong solution theory doesn't apply either, unless we work in the framework of hyperbolic symmetric systems by ignoring completely the viscosity terms and requiring a lot of regularity for the initial data. Actually, the only result concerning the situation described to be found in the literature is given by [2, Theorems 2, 3].

THEOREM 0.1 (see Chemin et al.). *Let $s > 1/2$ be a real number, and let $v_0\in\mathrm{H}^{0,s}$ be a divergence-free vector field. Then a positive time T and a solution v of (NS_h) defined on $[0,T]\times\mathbb{R}^3$ exist such that*

$$v\in\mathrm{L}^\infty\big(0,T;\mathrm{H}^{0,s}\big)\cap\mathrm{L}^2\big(0,T;\mathrm{H}^{1,s}\big).$$

Furthermore, there exists a constant c such that if $\|v_0\|_{0,s}$ is less than $c\nu$, then we can choose $T=+\infty$. Finally, this solution is unique, provided that $s > 3/2$.

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†IRMAR, Université de Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France (iftimie@maths.univ-rennes1.fr).

The space $\mathrm{H}^{s,s'}$ is a space with Sobolev regularity H^s in (x_1, x_2) and $\mathrm{H}^{s'}$ in x_3, whose precise definition will be given in section 1.

Let us first make some observations regarding the isotropic Navier–Stokes equations. Critical spaces for the three-dimensional (3D) Navier–Stokes equations are the spaces whose homogeneous norm is invariant under the scaling $f(\cdot) \leftrightarrow \lambda f(\lambda\cdot)$. There is no result on existence and uniqueness of solutions for general initial data in a subcritical space (i.e., a space whose homogeneous norm is invariant under the scaling $f(\cdot) \leftrightarrow \lambda^\alpha f(\lambda\cdot)$ for some $\alpha > 1$). For critical spaces, there are many existence and uniqueness results, starting with the classical result for $\mathrm{H}^{\frac12}$ of Fujita and Kato [3] and continuing with Besov spaces, Triebel–Lizorkin spaces, etc. We refer to Cannone [1] for details. Let us just note that a borderline case, that of initial data in BMO^{-1} (divergences of BMO vector fields), was recently proved by Koch and Tataru [6]. Initial data in anisotropic critical spaces were considered by the author in [5]. That work contains an existence and uniqueness result [5, Theorem 3.1] for initial data in a critical anisotropic Besov-type space that contains $\mathrm{H}^{0,s}$ for all $s > 1/2$ (but is contained in $\mathrm{H}^{0,\frac12}$). The problem of well-posedness for initial data in $\mathrm{H}^{0,\frac12}$ seems to be very difficult for the following two reasons. First, the homogeneous version of $\mathrm{H}^{0,\frac12}$ is not well defined since it would require defining $\mathrm{H}^{\frac12}$ homogeneous regularity in x_3 and it is well known that the homogeneous space $\mathrm{H}^{\frac12}$ is not well defined in dimension 1 (or rather it is not a Banach space). Second, the space $\mathrm{H}^{0,\frac12}$ is not included in C^{-1} (see [5, Proposition 4.1]) and it seems to be very difficult to prove existence and uniqueness for initial data which is not C^{-1}. (All the spaces of initial data for which existence and uniqueness of solutions are known are embedded in C^{-1}.)

In accordance with what is observed above, the existence part of Theorem 0.1 is very similar to results known for the isotropic Navier–Stokes equations. The key observation is that, although there is not enough regularity in the vertical direction, the partial derivative ∂_3 is always multiplied by u_3 in the nonlinear term, and the divergence-free condition implies that u_3 has enough vertical regularity. Nevertheless, some technical difficulties persist.

In the result of Chemin et al. there is a gap between the existence result and the uniqueness result. This gap is unexpected, especially since, for the full Navier–Stokes equations, $s > 1/2$ is sufficient to get uniqueness within the framework of anisotropic spaces (see [5, Theorem 3.1]). The aim of this work is to close this gap, proving that uniqueness holds when existence does, i.e., $s > 1/2$.

The gap in the proof of uniqueness given by [2] is due to the term $w_3\partial_3 v$ (w is the difference of two solutions and v is one of the two solutions). Roughly speaking, to estimate this term in $\mathrm{H}^{0,\frac12}$ one needs at least $\mathrm{H}^{\frac12}$ regularity for $\partial_3 v$ in the vertical direction, that is, $\mathrm{H}^{\frac32}$ regularity for v in the vertical direction. This demands, of course, that $s > 3/2$. To overcome this difficulty, we propose to estimate the $\mathrm{H}^{0,-\frac12}$ norm of w instead of the $\mathrm{H}^{0,\frac12}$ norm. This will require only $\mathrm{H}^{\frac12}$ regularity for v in the vertical direction, so the hypothesis $s > 1/2$ will suffice.

This point of view implies some difficulties. First, we will require a product theorem for the anisotropic Sobolev spaces where the regularities in the vertical direction are supercritical for one of the terms and subcritical for the other; see Theorem 1.4. Some product theorems are available but only when both regularities are subcritical (see [5, Theorem 1.1] and [4, Theorem 1.1]) or supercritical (see [2, Lemma 1]).

A second difficulty is to estimate a symmetric term of the type $\int v_3\partial_3 w \cdot \Lambda_3^{-1} w \,\mathrm{d}x$ (Λ_3 is roughly ∂_3; see the next section for the precise definition). Such a term does not appear when making L^2 estimates instead of $\mathrm{H}^{0,-\frac12}$. The estimates for symmetric terms are usually complicated when the indices of regularity are not integers. In the previously cited works, the estimates of this type are long and require dyadic decompositions. We will be able to obtain such an estimate through elementary techniques.

The following theorem completes Theorem 0.1 in the sense that the hypothesis $s > 3/2$ is no longer required to get uniqueness of solutions.

THEOREM 0.2. *Let v and $\widetilde{v}$ be two solutions of (NS_h) on $(0,T)$ belonging to $\mathrm{L}^\infty\big(0,T;\mathrm{H}^{0,s}\big)\cap \mathrm{L}^2\big(0,T;\mathrm{H}^{1,s}\big)$, where $s>1/2$. If v and $\widetilde{v}$ have the same initial data, then $v\equiv\widetilde{v}$.*

Although the regularity invoked in the hypothesis of this theorem is not sufficient by itself to define a trace of the velocity v at time $t=0$, it is a classical observation that v satisfying (NS_h) implies some continuity in time of v, namely $v\in \mathrm{C}^0([0,T];\mathrm{H}^{0,r})$ for all $r<s$ (for a proof, see the remarks before (4)). It therefore makes sense to say that v and $\widetilde{v}$ have the same initial data.

The author is able to prove neither uniqueness nor existence (in the regularity class of Theorem 0.1) in the case $s=1/2$. To this respect, we have nothing to add to the comments made for the isotropic Navier–Stokes equations.

In the following section we introduce notation and prove a new product theorem for anisotropic Sobolev spaces. The last section contains the proof of Theorem 0.2.

1. Notation and preliminary results. In the following, C will denote a constant which may change from one relation to another and which may depend on the different parameters $s, s', \dots$ introduced. The constant K is a universal constant which can also change from one relation to another. Two quantities A and B are said to verify the relation $A\simeq B$ if and only if the ratio A/B stays between two positive constants. We denote by $\langle x\rangle$ the quantity $\langle x\rangle=(1+|x|^2)^{\frac12}$.

DEFINITION 1.1. *For $s,s'\in\mathbb{R}$ we define the anisotropic Sobolev space $\mathrm{H}^{s,s'}$ to be the space of those tempered distributions f which satisfy*

$$\|f\|_{s,s'}\overset{def}{=}\big\|\langle\xi'\rangle^s\langle\xi_3\rangle^{s'}\widehat{f}(\xi)\big\|_{\mathrm{L}^2}<\infty,$$

where $\xi'=(\xi_1,\xi_2)$.

The space $\mathrm{H}^{s,s'}$ endowed with the norm $\|\cdot\|_{s,s'}$ is a Hilbert space.

The partial derivative $\partial/\partial x_j$ is denoted by ∂_j. We denote by Λ_3 the operator $\Lambda_3=\big(1-\partial_3^2\big)^{\frac12}$, that is, the operator of multiplication by $\langle\xi_3\rangle$ in the frequency space. Clearly, Λ_3 is an isometry from $\mathrm{H}^{s,s'}$ to $\mathrm{H}^{s,s'-1}$ for all real numbers s and s'.

When we apply an operator to a vector field, we mean that we apply it to each component of the vector field. The $\mathrm{H}^{s,s'}$ norm of a vector field is the Euclidean norm of the $\mathrm{H}^{s,s'}$ norms of the components. If u, v, and w are three vector fields, then $u\cdot\nabla v$ denotes the vector field $\sum_i u_i\partial_i v$, and $u\cdot\nabla v\cdot w$ denotes the scalar $\sum_{i,j}u_i\,\partial_i v_j\,w_j$.

We will need in the proof of Theorem 0.2 certain interpolation properties of the spaces $\mathrm{H}^{s,s'}$. The following proposition is very easy to prove (see [4, Proposition 1.1]).

PROPOSITION 1.2 (interpolation). *Let $s,t,s',t'\in\mathbb{R}$ and $\alpha\in[0,1]$. If $f\in \mathrm{H}^{s,s'}\cap\mathrm{H}^{t,t'}$, then we have that $f\in\mathrm{H}^{\alpha s+(1-\alpha)t,\alpha s'+(1-\alpha)t'}$ and*

$$\|f\|_{\alpha s+(1-\alpha)t,\alpha s'+(1-\alpha)t'}\le\|f\|_{s,s'}^{\alpha}\|f\|_{t,t'}^{1-\alpha}.$$

The multiplicative properties of the anisotropic Sobolev spaces have been studied in several papers [2, 4, 5, 8, 9]. The following result is proved in [4, Theorem 1.1], valid in the periodic case.

Theorem 1.3. *Let $s,t<1$, $s+t>0$, and $s',t'<1/2$, $s'+t'>0$. If $f\in \mathrm{H}^{s,s'}$ and $g\in \mathrm{H}^{t,t'}$, then $fg\in \mathrm{H}^{s+t-1,s'+t'-1/2}$ and there exists a constant C such that*

$$\|fg\|_{s+t-1,s'+t'-\frac12}\le C\|f\|_{s,s'}\|g\|_{t,t'}.$$

The proof in [4], which uses dyadic decompositions, carries over to the case of the full space. Nevertheless, a more elementary proof can be given, as in Theorem 1.4. For further details, see Remark 3.

Theorem 1.3 is not enough for our purposes. Indeed, the regularity we need is "supercritical" in the vertical direction, i.e., greater than 1/2, a situation which is not covered by Theorem 1.3. The purpose of the following theorem is to deal with this difficulty.

Theorem 1.4. *Let $s,t<1$, $s+t>0$, and $s'>1/2$. If $f\in \mathrm{H}^{s,s'}$ and $g\in \mathrm{H}^{t,-\frac12}$, then $fg\in \mathrm{H}^{s+t-1,-\frac12}$ and there exists a constant C such that*

$$\|fg\|_{s+t-1,-\frac12}\le C\|f\|_{s,s'}\|g\|_{t,-\frac12}.$$

The proof will use the following easy lemma.

Lemma 1.5. *Let $s\in\mathbb{R}$ and $n\in\mathbb{N}^*$. A constant C exists such that*

$$\int_{|x|\le R}\langle x\rangle^s\,\mathrm{d}x\le\begin{cases}C\max(1,\langle R\rangle^{s+n}) & \text{if } s+n\neq 0,\\ \sigma_{n-1}\sqrt2(1+\log\langle R\rangle) & \text{if } s+n=0,\end{cases}$$

where the variable of integration x belongs to $\mathbb{R}^n$ and σ_{n-1} denotes the area of the unit sphere in $\mathbb{R}^n$. Moreover, if s is not large (for instance, if $|s|\le 100$), then the constant C can be chosen of the form $C=\frac{K(n)}{|s+n|}$.

Proof of the lemma. Clearly

$$\int_{|x|\le R}\langle x\rangle^s\,\mathrm{d}x=\sigma_{n-1}\int_0^R\langle r\rangle^s r^{n-1}\,\mathrm{d}r\le\sigma_{n-1}\int_0^R\langle r\rangle^{s+n-1}\,\mathrm{d}r.$$

Since $\frac{1+r}{\sqrt2}\le\langle r\rangle\le 1+r$, we deduce that $\langle r\rangle^{s+n-1}\le(1+r)^{s+n-1}$ if $s+n\ge1$, and $\langle r\rangle^{s+n-1}\le\frac{(1+r)^{s+n-1}}{(\sqrt2)^{s+n-1}}$ if $s+n\le1$. It follows that

$$\begin{aligned}\int_{|x|\le R}\langle x\rangle^s\,\mathrm{d}x&\le\sigma_{n-1}\max(1,2^{\frac{1-n-s}{2}})\int_0^R(1+r)^{s+n-1}\,\mathrm{d}r\\&=\begin{cases}\sigma_{n-1}\max(1,2^{\frac{1-n-s}{2}})\frac{(1+R)^{s+n}-1}{s+n} & \text{if } s+n\neq0,\\ \sigma_{n-1}\sqrt2\log(1+R) & \text{if } s+n=0.\end{cases}\end{aligned}$$

The conclusion follows by using that $(1+R)^{s+n}\simeq\langle R\rangle^{s+n}$ and $\log(1+R)\le 1+\log\langle R\rangle$. ∎

Remark 1. In the following we don't need to know the behavior of the constant C of Lemma 1.5 as $|s|\to\infty$. Nevertheless, for the sake of completeness we indicate that the constant C can be chosen of the form $\frac{K(n)}{|s+n|}$ as $|s|\to\infty$, too. The proof of this fact is very easy in the case $n\ge2$, and so we include it here. As in the proof of the lemma, we have for $s+n\neq0$ that

$$\begin{aligned}\int_{|x|\le R}\langle x\rangle^s\,\mathrm{d}x&=\sigma_{n-1}\int_0^R\langle r\rangle^s r^{n-1}\,\mathrm{d}r\le\sigma_{n-1}\int_0^R\langle r\rangle^{s+n-2}r\,\mathrm{d}r\\&=\frac{\sigma_{n-1}}{s+n}\int_0^R\frac{\mathrm{d}}{\mathrm{d}r}\big(\langle r\rangle^{s+n}\big)\,\mathrm{d}r=\frac{\sigma_{n-1}}{s+n}\big(\langle R\rangle^{s+n}-1\big)\le\frac{\sigma_{n-1}}{|s+n|}\max\big(1,\langle R\rangle^{s+n}\big).\end{aligned}$$

Proof of Theorem 1.4. Let $f \in \mathrm{H}^{s,s'}$ and $g \in \mathrm{H}^{t,-\frac{1}{2}}$. We have to estimate the norm

$$\|fg\|_{s+t-1,-\frac{1}{2}} = (2\pi)^{-3}\|\langle\xi'\rangle^{s+t-1}\langle\xi_3\rangle^{-\frac{1}{2}}\widehat{f}*\widehat{g}(\xi)\|_{\mathrm{L}^2}.$$

By duality,

$$\begin{aligned}(2\pi)^3\|fg\|_{s+t-1,-\frac{1}{2}} &= \sup_{\|h\|_{\mathrm{L}^2}\le 1}\int\langle\xi'\rangle^{s+t-1}\langle\xi_3\rangle^{-\frac{1}{2}}\widehat{f}*\widehat{g}(\xi)h(\xi)\,\mathrm{d}\xi\\ &= \sup_{\|h\|_{\mathrm{L}^2}\le 1}\iint\langle\xi'+\eta'\rangle^{s+t-1}\langle\xi_3+\eta_3\rangle^{-\frac{1}{2}}\widehat{f}(\xi)\widehat{g}(\eta)h(\xi+\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta.\end{aligned}$$

We can further write

(1)
$$\begin{aligned}(2\pi)^3\|fg\|_{s+t-1,-\frac{1}{2}} \le \sup_{\|h\|_{\mathrm{L}^2}\le 1}&\underbrace{\iint_{2|\xi'|\ge|\eta'|}\langle\xi'+\eta'\rangle^{s+t-1}\langle\xi_3+\eta_3\rangle^{-\frac{1}{2}}\widehat{f}(\xi)\widehat{g}(\eta)h(\xi+\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta}_{I_1}\\ &+\sup_{\|h\|_{\mathrm{L}^2}\le 1}\underbrace{\iint_{2|\xi'|\le|\eta'|}\langle\xi'+\eta'\rangle^{s+t-1}\langle\xi_3+\eta_3\rangle^{-\frac{1}{2}}\widehat{f}(\xi)\widehat{g}(\eta)h(\xi+\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta}_{I_2}.\end{aligned}$$

In what follows, whenever we study the dependence of constants on s', we will assume that s' stays bounded (for instance, $s' \le 100$ or any other universal constant).

We now write I_1 under the form

$$I_1 = \iint_{2|\xi'|\ge|\eta'|}\langle\xi_3+\eta_3\rangle^{-\frac{1}{2}}\frac{\langle\xi'+\eta'\rangle^{s+t-1}}{\langle\eta'\rangle^t}\widehat{f}(\xi)\langle\eta'\rangle^t\widehat{g}(\eta)h(\xi+\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta,$$

and we apply Hölder's inequality in the variables ξ' and η' to obtain that

$$\begin{aligned}I_1 \le \iint\langle\xi_3+\eta_3\rangle^{-\frac{1}{2}}\Bigg(&\underbrace{\iint_{2|\xi'|\ge|\eta'|}\frac{\langle\xi'+\eta'\rangle^{2(s+t-1)}}{\langle\eta'\rangle^{2t}}|\widehat{f}(\xi)|^2\,\mathrm{d}\xi'\,\mathrm{d}\eta'}_{I_3}\\ &\times\iint\langle\eta'\rangle^{2t}|\widehat{g}(\eta)|^2|h(\xi+\eta)|^2\,\mathrm{d}\xi'\,\mathrm{d}\eta'\Bigg)^{\frac{1}{2}}\mathrm{d}\xi_3\,\mathrm{d}\eta_3.\end{aligned}$$

To estimate I_3, we first integrate with respect to η' and then decompose

$$\begin{aligned}\int_{2|\xi'|\ge|\eta'|}\frac{\langle\xi'+\eta'\rangle^{2(s+t-1)}}{\langle\eta'\rangle^{2t}}\,\mathrm{d}\eta' = &\int_{|\eta'|\le|\xi'|/2}\frac{\langle\xi'+\eta'\rangle^{2(s+t-1)}}{\langle\eta'\rangle^{2t}}\,\mathrm{d}\eta'\\ &+\int_{|\xi'|/2\le|\eta'|\le 2|\xi'|}\frac{\langle\xi'+\eta'\rangle^{2(s+t-1)}}{\langle\eta'\rangle^{2t}}\,\mathrm{d}\eta'.\end{aligned}$$

If $|\eta'| \le |\xi'|/2$, then $\langle\xi'+\eta'\rangle \simeq \langle\xi'\rangle$. If $|\xi'|/2 \le |\eta'| \le 2|\xi'|$, then $\langle\eta'\rangle \simeq \langle\xi'\rangle$. We deduce that

$$\int_{|\eta'|\le|\xi'|/2}\frac{\langle\xi'+\eta'\rangle^{2(s+t-1)}}{\langle\eta'\rangle^{2t}}\,\mathrm{d}\eta' \simeq \langle\xi'\rangle^{2(s+t-1)}\int_{|\eta'|\le|\xi'|/2}\frac{1}{\langle\eta'\rangle^{2t}}\,\mathrm{d}\eta' \le \frac{K}{1-t}\langle\xi'\rangle^{2s}$$

and that

$$\int_{|\xi'|/2\le|\eta'|\le 2|\xi'|}\frac{\langle\xi'+\eta'\rangle^{2(s+t-1)}}{\langle\eta'\rangle^{2t}}\,\mathrm{d}\eta' \simeq \frac{1}{\langle\xi'\rangle^{2t}}\int_{|\xi'|/2\le|\eta'|\le 2|\xi'|}\langle\xi'+\eta'\rangle^{2(s+t-1)}\,\mathrm{d}\eta'$$
$$\le \frac{K}{\langle\xi'\rangle^{2t}}\int_{|\zeta|\le 3|\xi'|}\langle\zeta\rangle^{2(s+t-1)}\,\mathrm{d}\zeta \le \frac{K}{s+t}\langle\xi'\rangle^{2s},$$

where we have used Lemma 1.5 and the change of variables $\zeta=\xi'+\eta'$.

According to the definition of I_3, we obtain from the previous relations that

$$I_3 \le C\int\langle\xi'\rangle^{2s}|\widehat{f}(\xi)|^2\,\mathrm{d}\xi',$$

which yields the following estimate for I_1:

$$I_1 \le C\iint\left(\int\langle\xi'\rangle^{2s}\langle\xi_3\rangle^{2s'}|\widehat{f}(\xi)|^2\,\mathrm{d}\xi'\int|h(\zeta,\xi_3+\eta_3)|^2\,\mathrm{d}\zeta\right)^{\frac12}$$
$$\times\left(\langle\xi_3+\eta_3\rangle^{-1}\langle\xi_3\rangle^{-2s'}\int\langle\eta'\rangle^{2t}|\widehat{g}(\eta)|^2\,\mathrm{d}\eta'\right)^{\frac12}\mathrm{d}\xi_3\,\mathrm{d}\eta_3.$$

Hölder's inequality applied in the variable (ξ_3,η_3) now gives that

$$I_1 \le C\|f\|_{s,s'}\|h\|_{\mathrm{L}^2}\left(\int\langle\eta'\rangle^{2t}\varphi(\eta_3)|\widehat{g}(\eta)|^2\,\mathrm{d}\eta\right)^{\frac12}, \tag{2}$$

where

$$\varphi(\eta_3)=\int\frac{1}{\langle\xi_3+\eta_3\rangle\langle\xi_3\rangle^{2s'}}\,\mathrm{d}\xi_3.$$

To estimate φ, we proceed as for I_3 by investigating several pieces and using Lemma 1.5:

$$\int_{|\xi_3|\ge 2|\eta_3|}\frac{1}{\langle\xi_3+\eta_3\rangle\langle\xi_3\rangle^{2s'}}\,\mathrm{d}\xi_3 \simeq \int_{2|\eta_3|}^{\infty}\frac{1}{\langle\xi_3\rangle^{2s'+1}}\,\mathrm{d}\xi_3$$
$$\simeq \int_{2|\eta_3|}^{\infty}\frac{1}{(1+\xi_3)^{2s'+1}}\,\mathrm{d}\xi_3 \le \frac{K}{\langle\eta_3\rangle^{2s'}} \le \frac{K}{\langle\eta_3\rangle},$$
$$\int_{|\xi_3|\le|\eta_3|/2}\frac{1}{\langle\xi_3+\eta_3\rangle\langle\xi_3\rangle^{2s'}}\,\mathrm{d}\xi_3 \simeq \frac{1}{\langle\eta_3\rangle}\int_{|\xi_3|\le|\eta_3|/2}\frac{1}{\langle\xi_3\rangle^{2s'}}\,\mathrm{d}\xi_3 \le \frac{K}{(s'-\frac12)\langle\eta_3\rangle},$$
$$\int_{|\eta_3|/2\le|\xi_3|\le 2|\eta_3|}\frac{1}{\langle\xi_3+\eta_3\rangle\langle\xi_3\rangle^{2s'}}\,\mathrm{d}\xi_3 \simeq \frac{1}{\langle\eta_3\rangle^{2s'}}\int_{|\eta_3|/2\le|\xi_3|\le 2|\eta_3|}\frac{1}{\langle\xi_3+\eta_3\rangle}\,\mathrm{d}\xi_3$$
$$\le \frac{K}{\langle\eta_3\rangle^{2s'}}\int_{|\zeta|\le 3|\eta_3|}\frac{1}{\langle\zeta\rangle}\,\mathrm{d}\zeta$$
$$\le \frac{K}{\langle\eta_3\rangle^{2s'}}(1+\log\langle\eta_3\rangle) \le \frac{K}{(s'-\frac12)\langle\eta_3\rangle},$$

where we have used in the last relation that $\log\alpha\le\frac{\alpha^\varepsilon}{e\varepsilon}$ for all $\alpha\ge 1$ and $\varepsilon>0$. We deduce from the previous relations that

$$\varphi(\eta_3)\le C\langle\eta_3\rangle^{-1},$$

which, plugged into (2), yields the estimate

$$I_1 \leq C\|f\|_{s,s'}\|g\|_{t,-\frac{1}{2}}\|h\|_{\mathrm{L}^2}. \tag{3}$$

To complete the proof, it remains to estimate I_2 (defined in relation (1)). By Hölder's inequality,

$$I_2 \leq (J_1 J_2)^{\frac{1}{2}},$$

where

$$J_1 = \iint \langle \xi' \rangle^{2s} \langle \xi_3 \rangle^{2s'} |\widehat{f}(\xi)|^2 |h(\xi+\eta)|^2 \, \mathrm{d}\xi \, \mathrm{d}\eta$$

and

$$J_2 = \iint_{2|\xi'|\leq|\eta'|} \frac{\langle \xi'+\eta' \rangle^{2(s+t-1)}}{\langle \xi' \rangle^{2s}} \frac{\langle \xi_3+\eta_3 \rangle^{-1}}{\langle \xi_3 \rangle^{2s'}} |\widehat{g}(\eta)|^2 \, \mathrm{d}\xi \, \mathrm{d}\eta.$$

Clearly, $J_1 = \|h\|^2_{\mathrm{L}^2}\|f\|^2_{s,s'}$. To estimate J_2, we first integrate in ξ' and ξ_3. As in the estimate for I_3,

$$\int_{2|\xi'|\leq|\eta'|} \frac{\langle \xi'+\eta' \rangle^{2(s+t-1)}}{\langle \xi' \rangle^{2s}} \, \mathrm{d}\xi' \simeq \langle \eta' \rangle^{2(s+t-1)} \int_{2|\xi'|\leq|\eta'|} \frac{1}{\langle \xi' \rangle^{2s}} \, \mathrm{d}\xi' \leq \frac{K}{1-s} \langle \eta' \rangle^{2t}.$$

Therefore, again using the bound for φ, one deduces that

$$J_2 \leq C\|g\|^2_{t,-\frac{1}{2}}.$$

We conclude that

$$I_2 \leq C\|f\|_{s,s'}\|g\|_{t,-\frac{1}{2}}\|h\|_{\mathrm{L}^2},$$

which, combined with relations (1) and (3), completes the proof of Theorem 1.4. □

Remark 2. It might be useful to know how the constant C in the statement of Theorem 1.4 depends on s, s', and t. Actually, tracking the constants in the proof, the constant C is of the form

$$C = K\left(\frac{1}{\sqrt{1-s}} + \frac{1}{\sqrt{1-t}} + \frac{1}{\sqrt{s+t}}\right)\frac{1}{\sqrt{s'-1/2}},$$

where we have assumed that s' stays bounded (say, $s' \leq 100$).

Remark 3. Theorem 1.4 is a special case of a more general theorem. More precisely, instead of considering $g \in \mathrm{H}^{t,-\frac{1}{2}}$, one may consider $g \in \mathrm{H}^{t,t'}$, where t' verifies $t' \leq s'$ and $s'+t' > 0$. The conclusion is then that $fg \in \mathrm{H}^{s+t-1,t'}$. We chose to prove the special case $t' = -1/2$, sufficient for our purposes, because the proof is considerably simpler. The proof in the general case does not involve any new ideas. In fact, the complication in the proof comes from the fact that the decomposition in ξ' given in relation (1) has to be done in the variable ξ_3 also; therefore one has to examine four pieces instead of just two, but the techniques are identical. Finally, let us note that if we add the hypothesis $t' < s' - 1/2$, then the additional decomposition in ξ_3 is not necessary and the proof given here carries over with no modification other than the replacement of $-1/2$ by t'.

2. Proof of the main theorem. We can assume without loss of generality that $s<1$. We will prove that the $\mathrm{H}^{0,-\frac{1}{2}}$ norm of $w=v-\widetilde{v}$ vanishes. In order to estimate $\|w\|_{0,-\frac{1}{2}}$, let us prove that the regularity available is enough to allow us to multiply the equation for $v-\widetilde{v}$ by $\Lambda_3^{-1}w$. First, note that we can write

$$v\cdot\nabla v=\sum_i\partial_i(v_iv).$$

By interpolation and by hypothesis, one has that $v\in\mathrm{L}^4(0,T;\mathrm{H}^{\frac{1}{2},s})$ (see relation (16)). The product theorem 1.3 easily implies that $v_iv\in\mathrm{L}^2(0,T;\mathrm{H}^{0,1-s})$ so $v\cdot\nabla v\in\mathrm{L}^2(0,T;\mathrm{H}^{-1,-s})\subset\mathrm{L}^2(0,T;\mathrm{H}^{-1,-\frac{3}{2}})$. Clearly, $\nu(\partial_1^2+\partial_2^2)v+\nu_{\scriptscriptstyle V}\partial_3^2v\in\mathrm{L}^2(0,T;\mathrm{H}^{-1,s})+\mathrm{L}^2(0,T;\mathrm{H}^{1,s-2})\subset\mathrm{L}^2(0,T;\mathrm{H}^{-1,-\frac{3}{2}})$. From the equation (NS_h) it follows that $\partial_tv\in\mathrm{L}^2(0,T;\mathrm{H}^{-1,-\frac{3}{2}})$. We deduce that every term in the equations for v and $\widetilde{v}$ belongs to $\mathrm{L}^2(0,T;\mathrm{H}^{-1,-\frac{3}{2}})$ and can therefore be multiplied by $\Lambda_3^{-1}w$, which belongs to $\mathrm{L}^2(0,T;\mathrm{H}^{1,1+s})\subset\mathrm{L}^2(0,T;\mathrm{H}^{1,\frac{3}{2}})$.

Note that the fact that $\partial_tv\in\mathrm{L}^2(0,T;\mathrm{H}^{-1,-\frac{3}{2}})$ and $v\in\mathrm{L}^2(0,T;\mathrm{H}^{1,s})$ implies, by the interpolation theory developed by Lions and Magenes [7, Chapter 1], that $v\in\mathrm{C}^0([0,T];\mathrm{H}^{0,\frac{2s-3}{4}})$. The interpolation property stated in Proposition 1.2 along with the fact that $v\in\mathrm{L}^\infty(0,T;\mathrm{H}^{0,s})$ imply in a classical manner that $v\in\mathrm{C}^0([0,T];\mathrm{H}^{0,r})$ for all $r<s$.

Multiplying the equation for $v-\widetilde{v}$ by $\Lambda_3^{-1}w$, integrating on $(\varepsilon,t)\times\mathbb{R}^3$, letting $\varepsilon\to0$, and using the continuity in time of $\|w\|_{0,-\frac{1}{2}}$ yields

(4)
$$\begin{aligned}\|w(t)\|_{0,-\frac{1}{2}}^2&+2\nu\int_0^t\big(\|\partial_1w(\tau)\|_{0,-\frac{1}{2}}^2+\|\partial_2w(\tau)\|_{0,-\frac{1}{2}}^2\big)\,\mathrm{d}\tau+2\nu_{\scriptscriptstyle V}\int_0^t\|\partial_3w(\tau)\|_{0,-\frac{1}{2}}^2\,\mathrm{d}\tau\\&=-2\int_0^t\int v(\tau,x)\cdot\nabla w(\tau,x)\cdot\Lambda_3^{-1}w(\tau,x)\,\mathrm{d}\tau\,\mathrm{d}x\\&\qquad-2\int_0^t\int w(\tau,x)\cdot\nabla\widetilde{v}(\tau,x)\cdot\Lambda_3^{-1}w(\tau,x)\,\mathrm{d}\tau\,\mathrm{d}x.\end{aligned}$$

To simplify the notation, we will write v instead of $v(\tau,x)$ and so on. We consider τ fixed, and we evaluate

$$\int v\cdot\nabla w\cdot\Lambda_3^{-1}w\,\mathrm{d}x=\underbrace{\int(v_1\partial_1w+v_2\partial_2w)\cdot\Lambda_3^{-1}w\,\mathrm{d}x}_{L_1}+\underbrace{\int v_3\partial_3w\cdot\Lambda_3^{-1}w\,\mathrm{d}x}_{L_2}\tag{5}$$

and

$$\int w\cdot\nabla\widetilde{v}\cdot\Lambda_3^{-1}w\,\mathrm{d}x=\underbrace{\int(w_1\partial_1\widetilde{v}+w_2\partial_2\widetilde{v})\cdot\Lambda_3^{-1}w\,\mathrm{d}x}_{L_3}+\underbrace{\int w_3\partial_3\widetilde{v}\cdot\Lambda_3^{-1}w\,\mathrm{d}x}_{L_4}.\tag{6}$$

We will now estimate each of these integrals.

Estimate of L_1. According to the product theorem 1.4, one can bound L_1 as follows:

$$|L_1|\le\|v_1\partial_1w+v_2\partial_2w\|_{-\frac{1}{2},-\frac{1}{2}}\|\Lambda_3^{-1}w\|_{\frac{1}{2},\frac{1}{2}}\le C\|v\|_{\frac{1}{2},s}\|w\|_{1,-\frac{1}{2}}\|w\|_{\frac{1}{2},-\frac{1}{2}}.$$

By the interpolation property given in Proposition 1.2, one has that

$$\|w\|_{\frac{1}{2},-\frac{1}{2}} \leq \|w\|_{0,-\frac{1}{2}}^{\frac{1}{2}} \|w\|_{1,-\frac{1}{2}}^{\frac{1}{2}}, \tag{7}$$

which leads to

$$|L_1| \leq C\|v\|_{\frac{1}{2},s} \|w\|_{0,-\frac{1}{2}}^{\frac{1}{2}} \|w\|_{1,-\frac{1}{2}}^{\frac{3}{2}}. \tag{8}$$

Estimate of L_3. Again by the product theorem 1.4, we have that

$$|L_3| \leq \|w_1\partial_1\widetilde{v} + w_2\partial_2\widetilde{v}\|_{-\frac{3}{4},-\frac{1}{2}} \|\Lambda_3^{-1}w\|_{\frac{3}{4},\frac{1}{2}} \leq C\|\widetilde{v}\|_{\frac{1}{2},s}\|w\|_{\frac{3}{4},-\frac{1}{2}}^2.$$

By interpolation,

$$\|w\|_{\frac{3}{4},-\frac{1}{2}} \leq \|w\|_{0,-\frac{1}{2}}^{\frac{1}{4}} \|w\|_{1,-\frac{1}{2}}^{\frac{3}{4}},$$

so that

$$|L_3| \leq C\|\widetilde{v}\|_{\frac{1}{2},s} \|w\|_{0,-\frac{1}{2}}^{\frac{1}{2}} \|w\|_{1,-\frac{1}{2}}^{\frac{3}{2}}. \tag{9}$$

Estimate of L_4. We proceed by using Theorem 1.3:

$$\begin{aligned}|L_4| \leq \|w_3\partial_3\widetilde{v}\|_{-\frac{1}{2},\frac{2s-3}{4}} \|\Lambda_3^{-1}w\|_{\frac{1}{2},\frac{3-2s}{4}} &\leq C\|w_3\|_{0,\frac{3-2s}{4}} \|\partial_3\widetilde{v}\|_{\frac{1}{2},s-1} \|w\|_{\frac{1}{2},\frac{-2s-1}{4}}\\ &\leq C\|\widetilde{v}\|_{\frac{1}{2},s}\|w_3\|_{0,\frac{1}{2}}\|w\|_{\frac{1}{2},-\frac{1}{2}}.\end{aligned}$$

But it is trivial to see that

$$\|f\|_{s,s'} = \big(\|f\|_{s,s'-1}^2 + \|\partial_3 f\|_{s,s'-1}^2\big)^{\frac{1}{2}} \leq \|f\|_{s,s'-1} + \|\partial_3 f\|_{s,s'-1}.$$

Therefore, because w is divergence free,

$$\|w_3\|_{0,\frac{1}{2}} \leq \|w\|_{0,-\frac{1}{2}} + \|\partial_3 w_3\|_{0,-\frac{1}{2}} = \|w\|_{0,-\frac{1}{2}} + \|\partial_1 w_1 + \partial_2 w_2\|_{0,-\frac{1}{2}} \leq \|w\|_{0,-\frac{1}{2}} + 2\|w\|_{1,-\frac{1}{2}}.$$

Also using relation (7), we infer that

$$|L_4| \leq C\|\widetilde{v}\|_{\frac{1}{2},s}\big(\|w\|_{0,-\frac{1}{2}}^{\frac{3}{2}}\|w\|_{1,-\frac{1}{2}}^{\frac{1}{2}} + \|w\|_{0,-\frac{1}{2}}^{\frac{1}{2}}\|w\|_{1,-\frac{1}{2}}^{\frac{3}{2}}\big). \tag{10}$$

Estimate of L_2. The proof for L_2 is more delicate. It is a commutator-type estimate and requires an integration by parts. Applying Parseval's formula gives

$$\begin{aligned} L_2 &= \int v_3\partial_3 w\cdot\Lambda_3^{-1}w\,\mathrm{d}x\\ &= (2\pi)^{-3}\int \widehat{v_3\partial_3 w}(\xi)\cdot\widehat{\Lambda_3^{-1}w}(-\xi)\,\mathrm{d}\xi\\ &= (2\pi)^{-6}\int \frac{1}{\langle\xi_3\rangle}\widehat{v}_3 * \widehat{\partial_3 w}(\xi)\cdot\widehat{w}(-\xi)\,\mathrm{d}\xi\\ &= i(2\pi)^{-6}\iint \frac{\eta_3}{\langle\xi_3\rangle}\widehat{v}_3(\xi-\eta)\widehat{w}(\eta)\cdot\widehat{w}(-\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta. \end{aligned} \tag{11}$$

Using the change of variables $(\xi,\eta)\leftrightarrow(-\eta,-\xi)$, one can write

$$L_2 = \frac{i}{2}(2\pi)^{-6}\iint\left(\frac{\eta_3}{\langle\xi_3\rangle} - \frac{\xi_3}{\langle\eta_3\rangle}\right)\widehat{v}_3(\xi-\eta)\widehat{w}(\eta)\cdot\widehat{w}(-\xi)\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{12}$$

Now, for $x, y \in \mathbb{R}$, one can check the following identity:

$$\frac{x}{\langle y\rangle} - \frac{y}{\langle x\rangle} = \frac{x-y}{\langle y\rangle} + \frac{(x-y)y(x+y)}{\langle x\rangle\langle y\rangle(\langle x\rangle + \langle y\rangle)}.$$

As $|y| < \langle y\rangle$ and $|x+y| < \langle x\rangle + \langle y\rangle$, we infer that

$$\left|\frac{x}{\langle y\rangle} - \frac{y}{\langle x\rangle}\right| \leq |x-y|\left(\frac{1}{\langle x\rangle} + \frac{1}{\langle y\rangle}\right).$$

Therefore, we obtain from (12) that

$$|L_2| \leq \frac{1}{2}(2\pi)^{-6}\sum_j \iint |\xi_3 - \eta_3|\left(\frac{1}{\langle \xi_3\rangle} + \frac{1}{\langle \eta_3\rangle}\right)|\widehat{v}_3(\xi-\eta)|\,|\widehat{w_j}(\eta)|\,|\widehat{w}_j(-\xi)|\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

Using again the change of variables $(\xi,\eta) \leftrightarrow (-\eta,-\xi)$, we deduce

$$|L_2| \leq (2\pi)^{-6}\sum_j \iint \frac{|\xi_3-\eta_3|}{\langle\xi_3\rangle}|\widehat{v}_3(\xi-\eta)|\,|\widehat{w_j}(\eta)|\,|\widehat{w}_j(-\xi)|\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

As v is divergence free, one has that $\xi_3\widehat{v}_3(\xi) = -\xi_1\widehat{v}_1(\xi) - \xi_2\widehat{v}_2(\xi)$, so $|\xi_3|\,|\widehat{v}_3(\xi)| \leq |\xi_1|\,|\widehat{v}_1(\xi)| + |\xi_2|\,|\widehat{v}_2(\xi)|$. It follows that

$$|L_2| \leq (2\pi)^{-6}\sum_j \iint \frac{|\xi_1-\eta_1|\,|\widehat{v}_1(\xi-\eta)| + |\xi_2-\eta_2|\,|\widehat{v}_2(\xi-\eta)|}{\langle\xi_3\rangle}|\widehat{w_j}(\eta)|\,|\widehat{w}_j(-\xi)|\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{13}$$

Let V be the vector field whose components verify

$$\widehat{V}_j = |\widehat{v}_j|.$$

Obviously, $\|V_j\|_{r,r'} = \|v_j\|_{r,r'}$ for all r, r', and j. We define in the same manner the vector field W. Using the reversed argument of (11), we observe that relation (13) is equivalent to

$$|L_2| \leq \int (|D_1|V_1 + |D_2|V_2)W \cdot \Lambda_3^{-1}W\,\mathrm{d}x,$$

where $|D_j|$ denotes the operator of multiplication in the frequency space by $|\xi_j|$. As $|D_j|V_j$ and $\partial_j V_j$ have the same $\mathrm{H}^{r,r'}$ norm for all r, r', and j, the same argument as in the estimate of L_3 shows that

$$\begin{aligned}|L_2| &\leq \int (|D_1|V_1 + |D_2|V_2)W\cdot\Lambda_3^{-1}W\,\mathrm{d}x \leq C\|V\|_{\frac12,s}\|W\|_{0,-\frac12}^{\frac12}\|W\|_{1,-\frac12}^{\frac32} \\ &= C\|v\|_{\frac12,s}\|w\|_{0,-\frac12}^{\frac12}\|w\|_{1,-\frac12}^{\frac32}.\end{aligned} \tag{14}$$

Collecting relations (4), (5), (6), (8), (9), (10), and (14), we get

$$\begin{aligned}&\|w(t)\|_{0,-\frac12}^2 + 2\nu\int_0^t \left(\|\partial_1 w\|_{0,-\frac12}^2 + \|\partial_2 w\|_{0,-\frac12}^2\right)\mathrm{d}\tau \\ &\leq C\int_0^t \|w\|_{1,-\frac12}^{\frac32}\|w\|_{0,-\frac12}^{\frac12}(\|v\|_{\frac12,s} + \|\widetilde{v}\|_{\frac12,s})\,\mathrm{d}\tau \\ &+ C\int_0^t \|w\|_{1,-\frac12}^{\frac12}\|w\|_{0,-\frac12}^{\frac32}\|\widetilde{v}\|_{\frac12,s}\,\mathrm{d}\tau.\end{aligned}$$

Using that $ab \leq \frac{a^4}{4} + \frac{3b^{\frac{4}{3}}}{4}$ for suitable choices of a and b, we infer that

$$\|w(t)\|_{0,-\frac{1}{2}}^2 + 2\nu \int_0^t \left(\|\partial_1 w\|_{0,-\frac{1}{2}}^2 + \|\partial_2 w\|_{0,-\frac{1}{2}}^2\right) \mathrm{d}\tau \leq \nu \int_0^t \|w\|_{1,-\frac{1}{2}}^2 \,\mathrm{d}\tau + C \int_0^t \|w\|_{0,-\frac{1}{2}}^2 \left(\|v\|_{\frac{1}{2},s}^4 + \|\widetilde{v}\|_{\frac{1}{2},s}^4 + \|\widetilde{v}\|_{\frac{1}{2},s}^{\frac{4}{3}}\right) \mathrm{d}\tau.$$

As

$$\|w\|_{1,-\frac{1}{2}}^2 = \|\partial_1 w\|_{0,-\frac{1}{2}}^2 + \|\partial_2 w\|_{0,-\frac{1}{2}}^2 + \|w\|_{0,-\frac{1}{2}}^2,$$

we further deduce that

$$\|w(t)\|_{0,-\frac{1}{2}}^2 \leq \int_0^t \|w(\tau)\|_{0,-\frac{1}{2}}^2 h(\tau) \,\mathrm{d}\tau, \tag{15}$$

where

$$h(t) = \nu + C(\|v\|_{\frac{1}{2},s}^4 + \|\widetilde{v}\|_{\frac{1}{2},s}^4 + \|\widetilde{v}\|_{\frac{1}{2},s}^{\frac{4}{3}}).$$

By interpolation,

$$\|v\|_{\frac{1}{2},s} \leq \|v\|_{0,s}^{\frac{1}{2}} \|v\|_{1,s}^{\frac{1}{2}}. \tag{16}$$

The hypothesis made on v implies that $\|v\|_{\frac{1}{2},s} \in \mathrm{L}^4(0,T)$. The same holds for $\widetilde{v}$, so $h \in \mathrm{L}^1(0,T)$. Gronwall's lemma applied in (15) now implies that $w \equiv 0$. The proof is completed.

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