WEAK VORTICITY FORMULATION OF THE INCOMPRESSIBLE 2D EULER EQUATIONS IN BOUNDED DOMAINS

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Abstract. In this article we examine the interaction of incompressible 2D flows with material boundaries. Our focus is the dynamic behavior of the circulation of velocity around boundary components and the possible exchange between flow vorticity and boundary circulation in flows with vortex sheet initial data, in the case of bounded domains. Our point of departure is the observation that ideal flows with vortex sheet regularity have well-defined circulation around each connected component of the boundary. In addition, we show that the velocity can be uniquely reconstructed from the vorticity and boundary component circulations, which allows to recast 2D Euler evolution using vorticity and the circulations as dynamic variables. The weak form of this vortex dynamics formulation of the equations is called the *weak vorticity formulation*. Existence of a weak solution for the 2D Euler equations, in velocity form, is guaranteed by Delort's Theorem, when the initial vorticity is a bounded measure satisfying a sign condition. The main result in this article is the equivalence between the weak velocity and weak vorticity formulations, without sign assumptions. Despite their being equivalent, the qualitative information concerning weak solutions is more apparent from the weak vorticity formulation than from the velocity formulation, and the remainder of the article is devoted to several consequences which can be derived from our main result. First, we consider weak solutions obtained by mollifying initial data and passing to the limit, with the portion of vorticity singular with respect to the Lebesgue measure assumed to be nonnegative. For these solutions we prove a set of inequalities which restrict the possible generation of vorticity by the boundary. Next, we prove that, if the weak solution conserves circulation around the boundary components, then it is a *boundary coupled weak solution*, a stronger version of the weak vorticity formulation. We prove existence of a weak solution which conserves circulation around the boundary components if the initial vorticity is integrable, i.e. if the singular part vanishes. Finally, we discuss the definition of the net mechanical force which the flow exerts on each material boundary component and its relation with conservation of circulation.

1. Introduction

For two-dimensional incompressible fluid flow, a vortex sheet is a curve on which the velocity of the fluid has a tangential discontinuity. Vortex sheets are an idealized model of a thin region where the fluid is subjected to intense, strongly localized shear. Flows with vortex sheets are of critical physical interest in fluid mechanics for several reasons, specially because such flows are common in situations of practical interest, such as in the wake of an airfoil. Thus, the mathematical description of vortex sheet motion is a classical topic in fluid dynamics. The study of this problem from the point-of-view of weak solutions was pioneered by R. DiPerna and A. Majda, see [10] and references therein.

In 1990, J.-M. Delort proved global-in-time existence of weak solutions for the incompressible Euler equations having, as initial vorticity, a compactly supported, bounded Radon measure with distinguished sign in H^{-1}, plus an arbitrary, compactly supported, L^p function, with $p > 1$, see [9]. This includes a large class of examples in which the initial vorticity is actually supported on a curve (classical vortex sheets). This result was later extended to certain symmetric configurations of vorticity with sign change, see [27, 28]. Very little is known regarding Delort's weak solutions beyond their existence. Some interesting open questions are the conservation of kinetic energy, conservation of the total variation of vorticity, and the behavior of the support of vorticity.

The original work by Delort included flows in the full plane, in bounded domains and in compact manifolds without boundary. The proof was based on a compensated-compactness argument for certain quadratic expressions in the components of velocity in order to pass to the limit in the weak formulation of the momentum equations along an approximate solution sequence obtained by mollifying initial data and exactly solving the equations.

There is a large literature directly associated with Delort's Theorem. The convergence to a weak solution was extended to approximations obtained by vanishing viscosity, see [29], and by numerical approximations, see [23, 32]. The limit $\alpha \to 0$ for the Euler-α equations with vortex sheet initial data was considered in [2]. The initial data class was extended to the limiting case $p = 1$, see [11, 35], and an alternative proof using harmonic analysis was produced, see [13]. In 1995, S. Schochet presented a simplified proof of the full-plane case by introducing the *weak vorticity formulation* of the Euler equations, where the compensated compactness argument at the heart of the original result becomes an elementary algebraic trick, referred to as *Schochet symmetrization.* In addition, Schochet formulated an equivalence theorem between the weak vorticity and weak velocity formulations of the Euler equations in the full plane, see Lemma 2.1 in [31].

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Originally, this weak vorticity formulation was of interest for the simplification it provided of Delort's original argument, for enabling the extensions to some symmetric, sign-changing initial data in [27, 28] and for the general physical relevance of the vortex dynamics point of view in incompressible fluid dynamics. Recently, with the discovery and rapid development of the theory of wild solutions of the Euler equations by De Lellis and Szekelyhidi, see [4, 8] and references therein, it became clear that the weak form of the momentum formulation of the Euler equations is severely incomplete. However, the extension of the theory of wild solutions to weak vorticity formulations of the Euler equations is an important open problem, which suggests that there may be additional information encoded in the vortex dynamics which would be very interesting to uncover. Wild solutions satisfy the weak velocity formulation, and by the equivalence result in Lemma 2.1 of [31], to show that a wild solution satisfies the weak vorticity formulation it would be enough to show that the corresponding vorticity belongs to $L^\infty((0,\infty);\mathcal{BM})$. Recently, Bardos, Szekelyhidi and Wiedemann studied the construction of wild solutions in domains with boundary, showing that the presence of the boundary allows for admissible solutions, i.e. satisfying the energy inequality, which are not dissipative in the sense of P. L. Lions, see [3]. The analogous problem, in the setting of fluid domains with boundary, is whether these wild solutions satisfy the weak vorticity formulation. As a consequence of our results below, see Theorem 3.4, it would be enough to verify that the vorticity of these wild solutions belongs to $L^\infty((0,\infty);\mathcal{BM})$. In the periodic case L. Székélyhidi constructed (infinitely many) wild solutions with a vortex sheet as initial data, see [34]; he notes that these wild solutions do not, in general, have a curl which is a bounded Radon measure at any positive time.

For solutions of the weak vorticity formulation at the level of regularity of vortex sheets, uniqueness is an open problem. Kelvin-Helmholtz instability, a phenomenon of vortex sheet flows, provides a mechanism which leads to ill-posedness, see the discussion and results in [37]. Furthermore, [25] contains a numerical illustration of a way in which Kelvin-Helmholtz instability may lead to nonuniqueness.

This article concerns ideal flows in bounded domains. Let us first note that such flows, in principle, are unphysical. Indeed, ideal flows are an approximation for slightly viscous flows. It has been known since Prandtl, see [30], that, in the presence of rigid boundaries, ideal flows and slightly viscous flows behave very differently. However, the mathematical study of ideal flows in bounded domains has some physical relevance, first because the inviscid flow serves as a natural scaffolding for constructing slightly viscous flow perturbatively, and, second, because the vanishing viscosity limit for incompressible flows in a domain with boundary is an important open problem, which could, in principle, lead to some very irregular solutions of the inviscid equations. We refer the reader to [4] for a broad discussion of this problem and its relation to turbulence modeling.

The Delort theorem on vortex sheets asserts the existence of a solution on a bounded domain in the sense of distributions. This is a solution that verifies the Euler equation in the interior of the domain. There is essentially no information on the solution near the boundary, apart from it satisfying the non-penetration boundary condition. But the interaction of the flow with the boundary is a crucial qualitative problem. An important feature of the way in which slightly viscous flow interacts with a rigid boundary is vorticity production and shedding. In general, ideal flows cannot exchange vorticity with a wall, but, as we will see, at the level of regularity of vortex sheets, this becomes an interesting possibility.

Mathematically speaking, the interaction of the flow with the boundary can be brought to light in various manners. A simple approach is to take the inner product of the PDE for the velocity

$$\partial_t u + u \cdot \nabla u + \nabla p = 0$$

with divergence free test vector fields which are not necessarily compactly supported, (in contrast with Delort's weak formulation), but merely tangent to the boundary. Smooth solutions satisfy the resulting stronger weak formulation, since the pressure term vanishes when considering such test vector fields. It is not clear, however, whether vortex sheet solutions satisfy this stronger weak formulation. An analogous issue can be posed for the vorticity equation

$$\partial_t \omega + u \cdot \nabla \omega = 0.$$

In other words, can one consider test functions which are not necessarily compactly supported in the domain? A partial answer to this particular question was given by two of the authors and Zhouping Xin in [27], in the setting of the half-plane. There they introduced *boundary coupled solutions*: weak solutions for which the test functions vanish at the boundary but are not necessarily compactly supported. If we try to do the same in the case of a bounded domain, we quickly realize that we don't even know what the weak formulation of the vorticity equation should be. In fact, it is also not immediately clear what the Biot-Savart law should be in the case of a non-simply connected domain, or, in other words, how to recover the velocity from the vorticity, something which is needed in order to produce this weak formulation. Fortunately, when the vorticity is a bounded measure we can make sense of the circulations of the velocity on the various connected components of the boundary and this will allow us to write down a Biot-Savart law. Once we have the Biot-Savart law, we can try to reproduce Schochet's symmetrization trick to find the weak

Even for smooth flows, reducing ideal fluid dynamics to vortex dynamics in a domain with holes is a rather recent development. A full description, for the case of bounded domains, was developed by Flucher and Gustafsson in [14], see also [24] for a more explicit account.

vorticity formulation. It works perfectly except that the equation we find at the end is not correct (see Remark 3.5 for details)! And why is that? It is because this symmetrization trick assumes that we are dealing with smooth solutions and for smooth solutions the circulation of the velocity along the various connected components of the boundary is conserved, as a consequence of Kelvin's Circulation Theorem. The symmetrization trick fails to take into account the possible non-conservation of circulation. But for vortex sheet solutions there is not sufficient regularity to guarantee the conservation of circulation. Indeed, we don't even expect circulation to be conserved if we are to believe that the flow exchanges vorticity with the boundary.

In this article we deal with the interaction of vortex sheets with the boundary through the issues mentioned above. Let us highlight the main results in this paper. In Proposition 2.9 we give the Biot-Savart law for vortex sheet solutions, which allows to recover the velocity from a vorticity which is a bounded Radon measure, and from the circulations along connected components of the boundary. In Theorem 3.4 we give the correct weak vorticity formulation and show its equivalence to the (Delort) velocity formulation. A key result in establishing Theorem 3.4 is Proposition 2.1, where it is shown that the auxiliary symmetrized function H_φ is bounded, up to the boundary, and enjoys additional useful properties. Furthermore, the proof of Theorem 3.4 relies heavily on Proposition 2.10, which asserts the following surprising fact: if the vorticity is a bounded measure in H^{-1} and if we consider a sequence of smooth approximate vorticities obtained by cut-off and convolution, then the associated velocities converge strongly in L^2. This strong convergence does not follow from compactness, of course, and it is a strictly kinematic result which depends strongly on the precise form of the approximations. Having obtained the right weak vorticity formulation we then restrict our attention to those vortex sheet solutions which are limits of smooth, exact solutions whose initial data is a mollification of the initial vortex sheet. In Theorem 4.4 we prove that, for such solutions, if the circulations are not conserved, then the circulations on the inner parts of the boundary can only increase. This theorem also states that if the circulations are conserved then these solutions are boundary coupled and moreover, if the vorticity is integrable then their circulations are always conserved. In order to prove these results, many delicate estimates involving the boundary need to be proved. Lastly, we wish to highlight Theorem 5.5, where it is shown that the net force through boundary components is well-defined if and only if the vortex sheet solutions are boundary coupled.

The remainder of this paper is divided as follows. In Section 2 we recall the basic notation and description of smooth vortex dynamics in a smooth, connected bounded domain with a finite number of holes. Then we adapt this description to the weak solution context. The main point is that circulation of velocity around connected components of the boundary is well-defined, as long as velocity is, at least locally, a bounded measure, and vorticity is a bounded Radon measure. In Section 3, we introduce our new weak vorticity formulation and we state and prove one of the main results of this article – the equivalence between the weak velocity and weak vorticity formulations of the 2D Euler equations. In Section 4 we derive some properties of weak solutions which arise as limits of exact solutions with mollified initial data, we discuss the connection between a boundary-coupled weak solution, which is a stronger notion of weak solution, previously introduced in [27], and conservation of circulation along connected components of the boundary, and we prove existence of boundary-coupled weak solutions with integrable initial vorticity. In Section 5 we extend the equivalence between weak velocity and weak vorticity formulations to an equivalence between their boundary-coupled versions. We show that the mechanical coupling between connected components of the boundary and the fluid flow is well-defined if and only if the solution is boundary-coupled. Finally, in Section 6 we derive conclusions and we propose some open problems.

In the Appendix we prove estimates for the Green's function and the Biot-Savart kernel on bounded domains with holes which, although not new, do not appear to be easily available in the literature.

2. Incompressible 2D flows in domains with holes

In this section we introduce basic notation concerning incompressible flows in bounded domains with holes and collect a few estimates which will be needed in the remainder of the paper.

Let $\Omega \subset \mathbb{R}^2$ be a smooth, connected bounded domain in the plane with $k \geqslant 0$ disjoint holes. Let Γ be the boundary of Ω, consisting of the disjoint union of $k+1$ Jordan curves, Γ_0 the outer boundary, and $\Gamma_i, i = 1, \ldots, k$ the inner ones. We denote by $G = G(x, y)$ the Green's function of the Laplacian in Ω with Dirichlet boundary conditions. We also introduce the function $K = K(x, y) \equiv \nabla_x^\perp G(x, y)$, where $\nabla_x^\perp = (-\partial_{x_2}, \partial_{x_1})$, known as the kernel of the Biot-Savart law.

We will denote by $\mathscr{Y}^\infty$ the space of functions $f \in C^\infty(\overline{\Omega})$ such that f is constant in a neighborhood of each Γ_j (with a constant depending on j), $j = 0, 1, \ldots, k$. We also denote by $\overline{\mathscr{Y}}^\infty$ the space of functions $f \in C^\infty(\overline{\Omega})$ such that f is constant on each Γ_j (with a constant depending on j), $j = 0, 1, \ldots, k$. The space of smooth divergence free vector fields compactly supported in Ω is denoted by $C^\infty_{c,\sigma}(\Omega)$.

Next, we introduce the *harmonic measures* $\mathbf{w}_j$, $j = 1, \ldots, k$, in Ω. These are solutions of the boundary-value problem:

$$\begin{cases} \Delta \mathbf{w}_j = 0, & \text{in } \Omega, \\ \mathbf{w}_j = \delta_{j\ell}, & \text{on } \Gamma_\ell,\ \ell = 1, \ldots, k. \\ \mathbf{w}_j = 0, & \text{on } \Gamma_0. \end{cases}$$

Existence and uniqueness of the harmonic measures is well-known, and they can be expressed by means of an explicit formula in terms of the Green's function:

$$\mathbf{w}_j(y) = -\int_{\Gamma_j} \frac{\partial G(x,y)}{\partial \hat{n}_x} \,\mathrm{d}S_x$$

where $\hat{n}_x$ is the exterior unit normal vector at $\partial\Omega$ (see, for example, Section 2.2.4 of [12]).

In what follows we adopt the convention $(a,b)^\perp = (-b,a)$. We denote by $\hat{\tau}$ the unit tangent vector to $\partial\Omega$ oriented in the counterclockwise direction. Note that, in particular, $\hat{\tau} = \hat{n}^\perp$ on Γ_0 and $\hat{\tau} = -\hat{n}^\perp$ on Γ_j, $j = 1, \ldots, k$.

We call a vector field in Ω *harmonic* if it is divergence-free, irrotational and tangent to the boundary. For each $j = 1, \ldots, k$, we denote by $\mathbf{X}_j$ the unique harmonic vector field with circulation around Γ_ℓ given by $\delta_{j,\ell}$ for all $\ell = 1, \ldots, k$. By classical Hodge theory, the family of vector fields $\{\mathbf{X}_j\}$, $j = 1, \ldots, k$ are a basis for the harmonic vector fields in Ω. For each $j = 1, \ldots, k$ there exists a unique function $\Psi_j \in \overline{\mathscr{Y}}^\infty$ such that $\mathbf{X}_j = \nabla^\perp \Psi_j$ and $\Psi_j\big|_{\Gamma_0} = 0$. The restriction of each Ψ_j to Γ_ℓ, $\ell = 1, \ldots, k$, is a constant which we denote by $c_{j\ell}$. We will denote by $\mathscr{X}$ the vector space of harmonic vector fields in Ω.

Our first result is an estimate, valid up to the boundary, which is a key part of our analysis.

Proposition 2.1. *Let $f \in W^{1,\infty}(\overline{\Omega}; \mathbb{R}^2)$ be a vector valued function whose restriction to $\partial\Omega$ is normal to the boundary. The function $f(x) \cdot K(x,y) + f(y) \cdot K(y,x)$ is bounded, continuous on $\overline{\Omega} \times \overline{\Omega} \setminus \{(x,x)\ ;\ x \in \overline{\Omega}\}$ and vanishes when x or y are on the boundary. In particular, if $\varphi \in \overline{\mathscr{Y}}^\infty$ then we may take $f = \nabla\varphi$. It follows that:*

$$H_\varphi = H_\varphi(x,y) \equiv \frac{\nabla\varphi(x) \cdot K(x,y) + \nabla\varphi(y) \cdot K(y,x)}{2} \in L^\infty(\overline{\Omega} \times \overline{\Omega}), \tag{2.1}$$

H_φ is continuous on $\overline{\Omega} \times \overline{\Omega} \setminus \{(x,x)\ ;\ x \in \overline{\Omega}\}$, and H_φ vanishes when x or y are on the boundary.

Proof. The Green function G is known to be smooth up to the boundary, see [1, Theorem 4.17], so K is smooth on $\overline{\Omega} \times \overline{\Omega} \setminus \{(x,x)\ ;\ x \in \overline{\Omega}\}$. We infer that $f(x) \cdot K(x,y) + f(y) \cdot K(y,x)$ is continuous on $\overline{\Omega} \times \overline{\Omega} \setminus \{(x,x)\ ;\ x \in \overline{\Omega}\}$. If $y \in \partial\Omega$ we have that $G(x,y)$ and $G(y,x)$ both vanish so $K(x,y) = \nabla_x^\perp G(x,y)$ vanishes and $K(y,x) = \nabla_y^\perp G(x,y)$ is tangent to the boundary. Since f is normal to the boundary we infer that $f(x) \cdot K(x,y) + f(y) \cdot K(y,x)$ vanishes when $y \in \partial\Omega$ and the same is of course true when $x \in \partial\Omega$.

We prove now the boundedness. More precisely, we show that there exists a constant M, depending only on Ω, such that

$$|f(x) \cdot K(x,y) + f(y) \cdot K(y,x)| \leqslant M\|f\|_{W^{1,\infty}(\Omega)} \quad \forall x,y \in \Omega,\ x \neq y.$$

We use the estimates in Proposition 6.1 from Appendix A. We observe first from Proposition 6.1 that there exists a constant $M = M(\Omega)$ such that

$$|K(x,y)| \leqslant \frac{M}{|x-y|} \quad \forall\ x \neq y.$$

Define

$$H(x,y) = G(x,y) - \frac{1}{2\pi}\log|x-y|.$$

We have that the function H is harmonic in both its arguments on $\Omega \times \Omega$. Therefore

$$K(x,y) + K(y,x) = \nabla_x^\perp G(x,y) + \nabla_y^\perp G(y,x) = \nabla_x^\perp H(x,y) + \nabla_y^\perp H(y,x)$$

is also harmonic and smooth on $\overline{\Omega} \times \overline{\Omega} \setminus \{(x,x);\ x \in \partial\Omega\}$. Moreover, given that $G(x,y)$ vanishes when x or y belongs to $\partial\Omega$, we also have that

$$K(x,y) = 0 \quad \forall x \in \Omega,\ y \in \partial\Omega \qquad \text{and} \qquad K(x,y) \text{ tangent to } \partial\Omega \quad \forall x \in \partial\Omega,\ y \in \overline{\Omega} \setminus \{x\}.$$

We bound

$$\begin{aligned} |f(x) \cdot K(x,y) + f(y) \cdot K(y,x)| &\leqslant |f(x) \cdot [K(x,y) + K(y,x)]| + |[f(x) - f(y)] \cdot K(y,x)| \\ &\leqslant |f(x) \cdot [K(x,y) + K(y,x)]| + M\|\nabla f\|_{L^\infty(\Omega)}. \end{aligned}$$

Next, let us fix $x \in \Omega$. The function

$$y \mapsto f(x) \cdot [K(x,y) + K(y,x)]$$

is harmonic in Ω and smooth on $\overline{\Omega}$. By the maximum principle we therefore have that

$$\sup_{y \in \Omega} |f(x) \cdot [K(x,y) + K(y,x)]| \leqslant \max_{y \in \partial\Omega} |f(x) \cdot [K(x,y) + K(y,x)]|\big).$$

Using that f is normal to the boundary $\partial\Omega$ we have that for $y \in \partial\Omega$

$$|f(x)\cdot[K(x,y)+K(y,x)]| = |f(x)\cdot K(y,x)| = |[f(x)-f(y)]\cdot K(y,x)| \leqslant M\|\nabla f\|_{L^\infty(\Omega)}.$$

We conclude that

$$|f(x)\cdot K(x,y)+f(y)\cdot K(y,x)| \leqslant 2M\|\nabla f\|_{L^\infty(\Omega)}.$$

This completes the proof of the proposition. □

We will later use the following convergence result which is a particular case of [6, Lemma 6.3.1].

Lemma 2.2. *Let K be a compact metric space and $(\mu_n)_{n\in\mathbb{N}} \in \mathcal{BM}(K)$ a bounded sequence of bounded Radon measures on K converging weakly to a measure μ. Suppose that $(|\mu_n|)_{n\in\mathbb{N}}$ converges weakly to another measure ν. Then for any bounded borelian function f, continuous outside a closed ν-negligible set, we have that*

$$\lim_{n\to\infty}\int_K f\,d\mu_n = \int_K f\,d\mu.$$

One important dynamic variable for incompressible flow, specially in 2D, is the *vorticity*, the curl of velocity. If $u=(u_1,u_2)$ is the velocity then the vorticity is $\omega = \partial_{x_1}u_2 - \partial_{x_2}u_1 \equiv$ curl u. In bounded simply connected domains the velocity can be easily recovered from the vorticity by means of the regularizing linear operator $\nabla^\perp\Delta^{-1}$, where Δ^{-1} is the inverse Dirichlet Laplacian. The fact that Ω is not simply connected implies that we need to assign extra conditions in order to recover velocity from vorticity, for instance, the circulation of the velocity around each hole. For a given vector field u and $j=0,1,\ldots,k$, we denote by γ_j the circulation of u around Γ_j as follows

$$\gamma_j = \int_{\Gamma_j} u\cdot\hat\tau\,\mathrm{d}S.$$

Recall K is the Biot-Savart operator $\nabla^\perp\Delta^{-1}$, where Δ is the Dirichlet Laplacian in Ω, and we abuse notation, denoting also by $K=K(x,y)$ its singular kernel. We discuss how to express the velocity field u from the vorticity ω and the circulations $\gamma_1,\ldots,\gamma_k$. We start by studying the Biot-Savart operator on Ω.

2.1. **Smooth flows.** For $\omega\in C_c^\infty(\Omega)$ we denote by $K[\omega]$ the value of the operator K on ω, given by

$$K[\omega](x) = \int_\Omega K(x,y)\omega(y)\,\mathrm{d}y.$$

We will also use the analogous notation for the Green's function

$$\Delta^{-1}\omega = G[\omega](x) = \int_\Omega G(x,y)\omega(y)\,\mathrm{d}y.$$

Of course, $K[\omega]=\nabla^\perp G[\omega]$.

We have the following

Proposition 2.3. *Let $\omega\in C_c^\infty(\Omega)$. We have that $K[\omega]$ is smooth, divergence free, tangent to the boundary, such that* $\operatorname{curl} K[\omega]=\omega$. *Moreover, for each $j=1,\ldots,k$, the circulation of $K[\omega]$ on Γ_j is given by*

$$\int_{\Gamma_j} K[\omega]\cdot\hat\tau\,\mathrm{d}S = -\int_\Omega \mathbf{w}_j\omega\,\mathrm{d}x. \tag{2.2}$$

This result was explicitly stated and proved in [24], and it was implicit in the analysis contained in [14]. We include a sketch of the proof for completeness.

Proof. Recall that we defined $\hat\tau$ to be the unit tangent vector to $\partial\Omega$, oriented counterclockwise. Therefore, for each $j=1,\ldots,k$, $\hat\tau = -\hat n^\perp$ on Γ_j.

We compute:

$$\int_{\Gamma_j} K[\omega]\cdot\hat\tau\,dS = \int_{\partial\Omega}\mathbf{w}_jK[\omega]^\perp\cdot\hat n\,dS = \int_\Omega \text{div }(\mathbf{w}_jK[\omega]^\perp)\,dx = -\int_\Omega\mathbf{w}_j\omega\,dx + \int_\Omega\nabla\mathbf{w}_j\cdot K[\omega]^\perp\,dx$$

$$= -\int_\Omega\mathbf{w}_j\omega\,dx - \int_\Omega\nabla\mathbf{w}_j\cdot\nabla G[\omega]\,dx = -\int_\Omega\mathbf{w}_j\omega\,dx,$$

as $\mathbf{w}_j$ is harmonic and $G[\omega]$ vanishes on $\partial\Omega$. □

As an easy consequence of Proposition 2.3 we learn how to recover velocity from vorticity in a domain with k holes.

Proposition 2.4. *Let $\omega \in C_c^\infty(\Omega)$ and let $\gamma_j \in \mathbb{R}$, $j = 1, \ldots, k$ be given constants. Then there exists one and only one divergence free vector field $u \in C^\infty(\overline{\Omega})$, tangent to $\partial\Omega$, such that curl $u = \omega$ and*

$$\int_{\Gamma_j} u \cdot \hat{\tau} \, \mathrm{d}S = \gamma_j, \quad j = 1, \ldots, k.$$

This vector field is given by the formula:

$$u = K[\omega] + \sum_{j=1}^{k} \left(\int_\Omega \mathbf{w}_j \omega \, \mathrm{d}x + \gamma_j \right) \mathbf{X}_j. \tag{2.3}$$

Proof. The existence part is trivial since, by Proposition 2.3 the vector field defined in (2.3) has all the required properties. Now, if v is another divergence free vector field tangent to the boundary whose curl is ω and whose circulations around each Ω_j are γ_j, then consider the difference between v and u, $\bar{u} \doteq v - u$. We find that $\bar{u}$ will be harmonic and therefore, it will be a linear combination of the $\{\mathbf{X}_j\}$. In addition, it will have vanishing circulation around each hole, and the only vector field with this property is zero. This concludes the proof. □

2.2. **Vortex-sheet-type flows.** Next we concern ourselves with the technical issue of reconstructing velocity from vorticity and circulations in a less regular setting. In this paper we are interested in flows with vortex sheet regularity, and, to this end, we will consider velocities in $L^2(\Omega)$ and vorticities in $H^{-1}(\Omega) \cap \mathcal{BM}(\Omega)$. First observe that, if $\omega \in H^{-1}(\Omega)$, then we have that $G[\omega] \in H_0^1(\Omega)$, and therefore $K[\omega] \equiv \nabla^\perp G[\omega]$ is a divergence-free vector field in $L^2(\Omega)$. In addition, its normal component has vanishing trace at $\partial\Omega$, see Lemma 2.3.2 in [9], and the discussion following it. Note that, in contrast, we do not have a trace of the tangential component of $K[\omega]$ if ω is merely H^{-1}. Let us recall at this point that BV functions have well-defined traces on the boundary, see for example [38, Chapter 5]. In fact, for the velocity to have a well-defined tangential trace at the boundary it is sufficient that the vorticity be a bounded measure. In the following lemma we give a precise statement of this fact, assuming the velocity is also only a bounded measure. This parallels a result due to G.-Q. Chen and H. Frid, see [7, Theorem 3.1] with $F = u^\perp$. We give a different proof below, for the sake of completeness. We note that the tangential trace of the velocity allows to define the circulations on the various connected components of the boundary.

Lemma 2.5. *Let $v \in \mathcal{BM}(\Omega)$ be a vector field such that $\omega \equiv \operatorname{curl} v \in \mathcal{BM}(\Omega)$. Then the tangential component of v at the boundary $v \cdot \hat{n}^\perp\big|_{\partial\Omega}$ is well defined in $\mathscr{D}'(\partial\Omega)$. In particular, the circulations $\gamma_0, \gamma_1, \ldots, \gamma_k$ of v on $\Gamma_0, \Gamma_1, \ldots, \Gamma_k$ are well-defined and they are characterized by the following property:*

$$\int_\Omega \varphi \, \mathrm{d}\omega + \int_\Omega \nabla^\perp \varphi \cdot \mathrm{d}v = \gamma_0 \, \varphi\big|_{\Gamma_0} - \sum_{j=1}^{k} \gamma_j \, \varphi\big|_{\Gamma_j} \qquad \forall \varphi \in \overline{\mathscr{Y}}^\infty. \tag{2.4}$$

In addition, we have that $\gamma_0 = \displaystyle\int_\Omega \mathrm{d}\omega + \sum_{j=1}^{k} \gamma_j$.

Proof. Let us assume for a moment that v is smooth. Then the Stokes formula implies that for all $\varphi \in C^\infty(\overline{\Omega})$ we have that

$$\int_{\partial\Omega} \varphi \, v \cdot \hat{n}^\perp = \int_\Omega \operatorname{curl}(\varphi v) = \int_\Omega \varphi\omega + \int_\Omega v \cdot \nabla^\perp \varphi. \tag{2.5}$$

We observe that the right-hand side above is well-defined when v and ω are bounded Radon measures on Ω, so the left-hand side is well-defined too. This allows to define $v \cdot \hat{n}^\perp\big|_{\partial\Omega}$ in the following manner. Let $\overline{\varphi} \in C^\infty(\partial\Omega) = \mathscr{D}(\partial\Omega)$. There exists some extension $\varphi \in C^\infty(\overline{\Omega})$ such that $\|\varphi\|_{W^{1,\infty}(\overline{\Omega})} \leqslant C \|\overline{\varphi}\|_{W^{1,\infty}(\partial\Omega)}$ for some constant C independent of $\overline{\varphi}$. We define $v \cdot \hat{n}^\perp\big|_{\partial\Omega} \in \mathscr{D}'(\partial\Omega)$ by

$$\langle v \cdot \hat{n}^\perp, \overline{\varphi} \rangle_{\mathscr{D}'(\partial\Omega), \mathscr{D}(\partial\Omega)} \equiv \int_\Omega \varphi \, \mathrm{d}\omega + \int_\Omega \nabla^\perp \varphi \cdot \mathrm{d}v. \tag{2.6}$$

This definition is justified by the identity (2.5).

Because $v, \omega \in \mathcal{BM}(\Omega)$ and $\|\varphi\|_{W^{1,\infty}(\overline{\Omega})} \leqslant C \|\overline{\varphi}\|_{W^{1,\infty}(\partial\Omega)}$ we clearly have that

$$|\langle v \cdot \hat{n}^\perp, \overline{\varphi} \rangle_{\mathscr{D}'(\partial\Omega), \mathscr{D}(\partial\Omega)}| \leqslant C \|\overline{\varphi}\|_{W^{1,\infty}(\partial\Omega)}.$$

To conclude that $v \cdot \hat{n}^\perp$ is well-defined in $\mathscr{D}'(\partial\Omega)$ it suffices to prove that $\langle v \cdot \hat{n}^\perp, \overline{\varphi} \rangle_{\mathscr{D}'(\partial\Omega), \mathscr{D}(\partial\Omega)}$ does not depend on the choice of the extension φ. To do that, it suffices to show that if $\varphi \in C^\infty(\overline{\Omega})$ vanishes on the boundary then

$$\int_\Omega \varphi \, \mathrm{d}\omega + \int_\Omega \nabla^\perp \varphi \cdot \mathrm{d}v = 0. \tag{2.7}$$

Let $\varphi \in C^\infty(\overline{\Omega})$ vanishing on the boundary. We define $\varphi^n = \varphi\chi_n \in C_c^\infty(\Omega)$ where χ_n is the cutoff function at distance $\frac{1}{n}$ from the boundary, as defined on page 8. Since $\omega = \operatorname{curl} v$ in the sense of distributions and $\varphi^n \in C_c^\infty(\Omega)$ we have that

$$\int_\Omega \varphi^n \,\mathrm{d}\omega + \int_\Omega \nabla^\perp \varphi^n \cdot \mathrm{d}v = 0. \tag{2.8}$$

Because φ vanishes on the boundary, we have that φ^n and $\nabla\varphi^n$ are uniformly bounded. Moreover, $\varphi^n \to \varphi$ and $\nabla\varphi^n \to \nabla\varphi$ pointwise. Since ω and v are bounded measures, we can apply the dominated convergence theorem to deduce that $\int_\Omega \varphi^n \, d\omega \to \int_\Omega \varphi \, d\omega$ and $\int_\Omega \nabla^\perp\varphi^n \cdot \mathrm{d}v \to \int_\Omega \nabla^\perp\varphi \cdot \mathrm{d}v$ as $n \to \infty$. Passing to the limit $n \to \infty$ in (2.8) implies (2.7) and this proves that $v \cdot \hat{n}^\perp$ is well-defined in $\mathscr{D}'(\partial\Omega)$.

Once $v \cdot \hat{n}^\perp$ is defined, the circulations $\gamma_0, \gamma_1, \ldots, \gamma_k$ are defined too. Indeed, let us recall that γ_j, the circulation of v along Γ_j, given by the formula $\gamma_j := -\int_{\Gamma_j} v \cdot \hat{n}^\perp$, $j = 1, \ldots, k$, and $\gamma_0 := \int_{\Gamma_0} v \cdot \hat{n}^\perp$. Choosing, for each $j = 0, \ldots, k$, a smooth function φ_j such that $\varphi_j = 1$ on Γ_j and $\varphi_j = 0$ on $\partial\Omega \setminus \Gamma_j$, we define

$$\gamma_j = -\langle v \cdot \hat{n}^\perp, \varphi_j \rangle_{\mathscr{D}'(\partial\Omega), \mathscr{D}(\partial\Omega)} = -\int_\Omega \varphi_j \,\mathrm{d}\omega - \int_\Omega \nabla^\perp \varphi_j \cdot \mathrm{d}v, \qquad 1 \leqslant j \leqslant k \tag{2.9}$$

and

$$\gamma_0 = \langle v \cdot \hat{n}^\perp, \varphi_0 \rangle_{\mathscr{D}'(\partial\Omega), \mathscr{D}(\partial\Omega)} = \int_\Omega \varphi_0 \,\mathrm{d}\omega + \int_\Omega \nabla^\perp \varphi_0 \cdot \mathrm{d}v \tag{2.10}$$

If we assume in (2.6) that $\varphi \in \overline{\mathscr{Y}}^\infty$ we get (2.4). In addition, the circulations γ_j are clearly uniquely determined by (2.4). Indeed, if $\gamma_0', \gamma_1', \ldots, \gamma_k'$ were to verify (2.4) as well, then we would have

$$(\gamma_0 - \gamma_0') \, \varphi\big|_{\Gamma_0} - \sum_{j=1}^k (\gamma_j - \gamma_j') \, \varphi\big|_{\Gamma_j} = 0$$

for all $\varphi \in \overline{\mathscr{Y}}^\infty$. Choosing, for each $j = 0, \ldots, k$, $\varphi = \varphi_j$ implies that $\gamma_j = \gamma_j'$.

Finally, choosing $\varphi \equiv 1$ in (2.4) implies that $\gamma_0 = \displaystyle\int_\Omega \mathrm{d}\omega + \sum_{j=1}^k \gamma_j$. This completes the proof of the lemma. □

Remark 2.6. *If we only want to define the circulations γ_j and not the full trace of $v \cdot \hat{n}$ at the boundary it is not necessary to assume that the measure v is bounded. Indeed, we can define the circulations from* (2.4) *by assuming that the test function φ belongs to $\mathscr{Y}^\infty$. For such a test function $\nabla^\perp\varphi$ is compactly supported so $\int_\Omega \nabla^\perp\varphi \cdot \mathrm{d}v$ is well-defined when v is only locally bounded.*

Our next objective is to define the Biot-Savart operator for vorticities in $H^{-1} \cap \mathcal{BM}$. First we prove that divergence free vector fields in $L^2(\Omega)$ which are tangent to the boundary are uniquely determined by their curl, together with the circulations around each boundary component.

Proposition 2.7. *Let $u \in L^2(\Omega)$ be divergence free, curl free, tangent to the boundary and with vanishing circulation on each of the connected components of the boundary. Then $u = 0$.*

Proof. Recall that the closure of $C_{c,\sigma}^\infty(\Omega)$ in $L^2(\Omega)$ is the space of square integrable, divergence free vector fields tangent to the boundary. Therefore there exists a sequence $u_j \in C_{c,\sigma}^\infty(\Omega)$ such that $u_j \to u$ in $L^2(\Omega)$ as $j \to \infty$. Since $u_j \in C_{c,\sigma}^\infty(\Omega)$ there exists some $\psi_j \in \mathscr{Y}^\infty$ such that $u_j = \nabla^\perp\psi_j$. We now use relation (2.4) together with the fact that u is curl free and has vanishing circulation on each of the connected components of the boundary, to deduce that

$$\int_\Omega u \cdot u_j = \int_\Omega u \cdot \nabla^\perp \psi_j = 0.$$

Letting $j \to \infty$, we infer that $\int_\Omega |u|^2 = 0$, so that $u = 0$. This completes the proof. □

Using Proposition 2.7, we deduce a property of the Biot-Savart operator K at the vortex-sheet level of regularity.

Corollary 2.8. *Let $\omega \in H^{-1}(\Omega) \cap \mathcal{BM}(\Omega)$. Then $K[\omega]$ is the unique vector field in $L^2(\Omega)$ which is divergence free, tangent to the boundary, with curl equal to ω and such that, for any $j = 1, \ldots, k$, its circulation on Γ_j is $-\int_\Omega \mathbf{w}_j \,\mathrm{d}\omega$.*

Proof. Since $\omega \in H^{-1}(\Omega)$ we have already observed that $K[\omega] \in L^2(\Omega)$ is divergence-free, tangent to the boundary in the trace sense, and its curl is ω. To verify the circulation information, we take $\varphi = \mathbf{w}_j$ and $v = K[\omega]$ in Lemma 2.5. We have:

$$\int_\Omega \mathbf{w}_j \,\mathrm{d}\omega + \int_\Omega K[\omega] \cdot \nabla^\perp \mathbf{w}_j = -\gamma_j.$$

Repeating the calculation done in the proof of Proposition 2.3, we see that the second integral above vanishes, which proves our contention. Finally, the uniqueness follows from Proposition 2.7. □

Given a vector field $u \in L^2(\Omega)$ which is divergence free and tangent to the boundary such that $\omega = \operatorname{curl} u \in \mathcal{BM}(\Omega)$, one can compute the circulations $\gamma_1, \ldots, \gamma_k$ on each of the connected components of the boundary $\Gamma_1, \ldots, \Gamma_k$ by using Lemma 2.5. Conversely, vorticity and circulations determine uniquely the velocity, by Proposition 2.7, and next we extend Proposition 2.4 to the weak setting to make this explicit.

Proposition 2.9. *Let $\omega \in H^{-1}(\Omega) \cap \mathcal{BM}(\Omega)$ and consider some arbitrary real numbers $\gamma_1, \ldots, \gamma_k$. There exists a unique vector field $u \in L^2(\Omega)$ such that $\operatorname{curl} u = \omega$, $\operatorname{div} u = 0$, u is tangent to the boundary and with circulations given by $\gamma_1, \ldots, \gamma_k$. More precisely, we have that*

$$u = K[\omega] + \sum_{j=1}^{k} \big(\gamma_j + \int_\Omega \mathbf{w}_j \, \mathrm{d}\omega\big) \mathbf{X}_j. \tag{2.11}$$

Proof. The uniqueness part is proved in Proposition 2.7. To prove the existence part, we observe that the vector field defined in (2.11) has all the required properties. Indeed, $K[\omega] \in L^2(\Omega)$ and it is obvious that $\operatorname{curl} u = \omega$, $\operatorname{div} u = 0$ and that u is tangent to the boundary. We conclude the proof by recalling that the circulation of $K[\omega]$ around Γ_j is given by $-\int_\Omega \mathbf{w}_j \, \mathrm{d}\omega$. $\square$

We will need, in the sequel, an approximation result. We introduce some additional notation. Let χ_n be a cutoff function at distance $\frac{1}{n}$ from the boundary. More precisely, we assume that

$$\chi_n \in C_c^\infty(\Omega; [0,1]), \quad \chi_n \equiv 1 \text{ in } \Sigma^c_{\frac{2}{n}}, \quad \chi_n \equiv 0 \text{ in } \Sigma_{\frac{1}{n}}$$
$$\|\nabla \chi_n\|_{L^\infty(\Sigma_{\frac{2}{n}} \setminus \Sigma_{\frac{1}{n}})} \leqslant Cn,$$

where

$$\Sigma_a = \{x \in \Omega \; ; \; d(x, \partial\Omega) \leqslant a\}.$$

In addition, we introduce

$$\eta \in C_c^\infty(\mathbb{R}^2; \mathbb{R}_+), \quad \operatorname{supp} \eta \subset B(0; 1/2), \quad \int \eta = 1,$$

and set

$$\eta_n(x) = n^2 \eta(nx).$$

We now prove the following approximation result.

Proposition 2.10. *Let $\omega \in H^{-1}(\Omega) \cap \mathcal{BM}(\Omega)$ and define*

$$\omega^n = (\chi_n \omega) * \eta_n. \tag{2.12}$$

Then we have that

$$\omega^n \text{ is bounded in } L^1(\Omega), \tag{2.13}$$

$$\int_\Omega \varphi \omega^n \overset{n\to\infty}{\longrightarrow} \int_\Omega \varphi \, \mathrm{d}\omega \text{ and } \quad \forall \varphi \in C^0(\overline{\Omega}), \tag{2.14}$$

$$K[\omega^n] \overset{n\to\infty}{\longrightarrow} K[\omega] \text{ strongly in } L^2(\Omega). \tag{2.15}$$

Moreover, any weak limit in the sense of measures $\mathcal{BM}(\overline{\Omega})$ of any subsequence of $|\omega^n|$ is a continuous measure on $\overline{\Omega}$.

Proof. Claim (2.13) is obvious:

$$\int_\Omega |\omega^n| = \int_\Omega |(\chi_n \omega) * \eta_n| \leqslant \int_\Omega \chi_n \, \mathrm{d}|\omega| \leqslant \int_\Omega 1 \, \mathrm{d}|\omega| = |\omega|(\Omega) < \infty.$$

Next we prove (2.14). Let $\varphi \in C^0(\overline{\Omega})$. We extend it to a compactly supported continuous function on $\mathbb{R}^2$, again denoted by φ. For f smooth, denote $\check{f} = \check{f}(z) = f(-z)$. Then we have:

$$\begin{aligned}\int_\Omega \varphi \omega^n - \int_\Omega \varphi \, \mathrm{d}\omega = \int_\Omega \varphi (\chi_n \omega) * \eta_n - \int_\Omega \varphi \, \mathrm{d}\omega = \int_\Omega \chi_n (\varphi * \check{\eta}_n) \, \mathrm{d}\omega - \int_\Omega \varphi \, \mathrm{d}\omega \\ = \int_\Omega \chi_n (\varphi * \check{\eta}_n - \varphi) \, \mathrm{d}\omega + \int_\Omega (\chi_n - 1) \varphi \, \mathrm{d}\omega \equiv I_1 + I_2.\end{aligned}$$

We first bound I_2:

$$|I_2| \leqslant \int_{\Sigma_{\frac{2}{n}}} |(\chi_n - 1)\varphi| \, \mathrm{d}|\omega| \leqslant \|\varphi\|_{L^\infty} |\omega|(\Sigma_{\frac{2}{n}}) \overset{n\to\infty}{\longrightarrow} 0.$$

Next, we know by classical results that $\varphi * \check{\eta}_n \to \varphi$ uniformly in $\mathbb{R}^2$. Therefore

$$|I_1| \leqslant \|\varphi * \check{\eta}_n - \varphi\|_{L^\infty(\mathbb{R}^2)} |\omega|(\Omega) \overset{n\to\infty}{\longrightarrow} 0.$$

This completes the proof of (2.14).

We prove now that any weak limit in the sense of measures $\mathcal{BM}(\overline{\Omega})$ of any subsequence of $|\omega^n|$ is a continuous measure. Let ω_+, respectively ω_-, be the positive part, respectively the negative part, of the measure ω. We denote by $\overline{\omega}$, $\overline{\omega}_+$ and $\overline{\omega}_-$ the extensions of ω, ω_+ and ω_- to $\mathcal{BM}(\overline{\Omega})$ with zero values on $\partial\Omega$. Since $\omega \in H^{-1}(\Omega)$ we infer that ω is a continuous measure (see [6, Lemma 6.3.2]) so $\overline{\omega}$ is continuous on $\overline{\Omega}$. Since $\omega = \omega_+ + \omega_-$ and ω_+, ω_- are orthogonal measures, we have that ω_+ and ω_- are also continuous measures. Let $\omega_+^n = (\chi_n\omega_+) * \eta_n$ and $\omega_-^n = (\chi_n\omega_-) * \eta_n$. Then ω_+^n and ω_-^n are single-signed, although they are not necessarily the positive and negative parts of ω^n. Nevertheless, we have the bound $|\omega^n| \leqslant \omega_+^n - \omega_-^n$. We can apply the result proved in relation (2.14) to the measures ω_+ and ω_-. We therefore have that $\omega_+^n \rightharpoonup \overline{\omega}_+$ and $\omega_-^n \rightharpoonup \overline{\omega}_-$ weakly in the sense of measures $\mathcal{BM}(\overline{\Omega})$. Therefore $\omega_+^n - \omega_-^n \rightharpoonup \overline{\omega}_+ - \overline{\omega}_- = |\overline{\omega}|$ weakly in the sense of measures $\mathcal{BM}(\overline{\Omega})$. From the bound $|\omega^n| \leqslant \omega_+^n - \omega_-^n$ we infer that any weak limit in the sense of measures $\mathcal{BM}(\overline{\Omega})$ of any subsequence of $|\omega^n|$ must be bounded by $|\overline{\omega}|$. Since $|\overline{\omega}|$ is a continuous measure, we deduce that any weak limit in the sense of measures of any subsequence of $|\omega^n|$ must be a continuous measure on $\overline{\Omega}$.

The proof of (2.15) is done in three steps. First we show that $K[\omega^n]$ is bounded in $L^2(\Omega)$. Second, we prove that $K[\omega^n]$ converges weakly to $K[\omega]$ in $L^2(\Omega)$. Third, we show the strong convergence of $K[\omega^n]$ to $K[\omega]$ in $L^2(\Omega)$.

We prove now that $K[\omega^n]$ is bounded in $L^2(\Omega)$. Let $F \in C^\infty(\overline{\Omega})$ be a vector field and let us define $f = G[\operatorname{curl} F]$. Clearly $f \in C^\infty(\overline{\Omega})$ is bounded and vanishes on $\partial\Omega$. We extend it to $\mathbb{R}^2$ by setting f to vanish on Ω^c. We have that

$$\int_\Omega K[\omega^n]\cdot F = \int_\Omega \nabla^\perp G[\omega^n]\cdot F = -\int_\Omega G[\omega^n]\operatorname{curl} F = -\int_\Omega G[\omega^n]\triangle f = -\int_\Omega \omega^n f. \tag{2.16}$$

Set $v = K[\omega]$ and recall that $v \in L^2(\Omega)$, $\operatorname{div} v = 0$, $\operatorname{curl} v = \omega$ and v is tangent to the boundary. Next, we observe that

$$\begin{aligned}\int_\Omega \omega^n f &= \int_\Omega (\chi_n\omega) * \eta_n\ f = \int_\Omega \chi_n(f * \check{\eta}_n)\,\mathrm{d}\omega = \langle \omega, \chi_n(f*\check{\eta}_n)\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}\\ &= \langle \operatorname{curl} v, \chi_n(f*\check{\eta}_n)\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = -\langle v, \nabla^\perp[\chi_n(f*\check{\eta}_n)]\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}\\ &= -\int_\Omega v\cdot\nabla^\perp\chi_n(f*\check{\eta}_n) - \int_\Omega \chi_n v\cdot(\nabla^\perp f * \check{\eta}_n).\end{aligned} \tag{2.17}$$

We estimate each of the two last terms above:

$$\left|\int_\Omega \chi_n v\cdot(\nabla^\perp f*\check{\eta}_n)\right| \leqslant \|v\|_{L^2(\Omega)}\|\nabla^\perp f*\check{\eta}_n\|_{L^2(\Sigma^c_{\frac{1}{n}})} \leqslant \|v\|_{L^2(\Omega)}\|\nabla^\perp f\|_{L^2(\Omega)}$$

and

$$\begin{aligned}\left|\int_\Omega v\cdot\nabla^\perp\chi_n(f*\check{\eta}_n)\right| &= \left|\int_{(\Sigma_{\frac{2}{n}}\setminus\Sigma_{\frac{1}{n}})} v\cdot\nabla^\perp\chi_n(f*\check{\eta}_n)\right|\\ &\leqslant \|v\|_{L^2(\Omega)}Cn\|f\|_{L^2(\Sigma_{\frac{5}{2n}}\setminus\Sigma_{\frac{1}{2n}})}\\ &\leqslant C\|v\|_{L^2(\Omega)}\|\nabla f\|_{L^2(\Omega)}\end{aligned}$$

where we used the fact that f vanishes on $\partial\Omega$ and the Hardy inequality (see [22, Theorem 11.3, page 65]).

We infer from (2.16) and (2.17), together with the estimates performed above, that the following inequality holds true:

$$\left|\int_\Omega K[\omega^n]\cdot F\right| \leqslant C\|v\|_{L^2(\Omega)}\|\nabla f\|_{L^2(\Omega)}. \tag{2.18}$$

Next we observe that

$$\|\nabla f\|_{L^2(\Omega)} \leqslant \|F\|_{L^2(\Omega)}. \tag{2.19}$$

Indeed, we have that $\triangle f = \operatorname{curl} F$ on Ω so

$$\int_\Omega |\nabla f|^2 = -\int_\Omega \triangle f\ f = -\int_\Omega \operatorname{curl} F\ f = \int_\Omega F\cdot\nabla^\perp f \leqslant \|F\|_{L^2(\Omega)}\|\nabla f\|_{L^2(\Omega)},$$

which yields (2.19). Using (2.19) in (2.18) we obtain that

$$\left|\int_\Omega K[\omega^n]\cdot F\right| \leqslant C\|v\|_{L^2(\Omega)}\|F\|_{L^2(\Omega)},$$

for all vector fields $F \in C^\infty(\overline{\Omega})$. This implies that

$$\|K[\omega^n]\|_{L^2(\Omega)} \leqslant C\|v\|_{L^2(\Omega)}.$$

We have, hence, established that $K[\omega^n]$ is bounded in L^2. Therefore, there exists a subsequence $K[\omega^{n_k}]$ which converges weakly to some vector field $w \in L^2(\Omega)$. Given that $K[\omega^{n_k}]$ is divergence free and tangent to the boundary, the same holds true for the weak limit w. Moreover, we have that $K[\omega^{n_k}] \to w$ in $\mathscr{D}'(\Omega)$ so $\omega^{n_k} = \operatorname{curl} K[\omega^{n_k}] \to \operatorname{curl} w$ in $\mathscr{D}'(\Omega)$. Using (2.14) we have that $\omega^{n_k} \to \omega$ in $\mathscr{D}'(\Omega)$, hence $\operatorname{curl} w = \omega$. The weak convergence of $K[\omega^{n_k}]$ to w, the convergence of ω^{n_k} stated in (2.14) and the definition of the circulations given in relations (2.9)–(2.10) imply that the circulations of $K[\omega^{n_k}]$ converge towards the circulations of w. The circulations of $K[\omega^{n_k}]$ are known from Proposition 2.3 so, using again the convergence stated in (2.14), we infer that the circulation of w around Γ_j is equal to $-\int_\Omega \mathbf{w}_j \, d\omega$. Corollary 2.8 now implies that $K[\omega] = w$. We proved that the limit of any weakly convergent subsequence of $K[\omega^n]$ in L^2 must necessarily be $K[\omega]$. We conclude that the whole sequence $K[\omega^n]$ converges to $K[\omega]$ weakly in $L^2(\Omega)$.

Finally, we show the strong convergence of $K[\omega^n]$ to $K[\omega]$ in $L^2(\Omega)$. Given that we already know the weak convergence, it suffices to show the convergence of the norms. We use relations (2.16) and (2.17) with $F = F_n = K[\omega^n]$, $f = f_n = G[\operatorname{curl} F_n] = G[\omega^n]$ and $v = K[\omega]$ to obtain that

$$\int_\Omega |K[\omega^n]|^2 = \int_\Omega K[\omega^n] \cdot F_n = \int_\Omega K[\omega] \cdot \nabla^\perp \chi_n (f_n * \check{\eta}_n) + \int_\Omega \chi_n K[\omega] \cdot (\nabla^\perp f_n * \check{\eta}_n) \equiv J_1 + J_2.$$

Next, since $F_n = \nabla^\perp f_n$, we have that

$$J_2 = \int_\Omega \eta_n * (\chi_n K[\omega]) \cdot F_n.$$

By classical results, we know that $\eta_n * (\chi_n K[\omega]) \to K[\omega]$ strongly in $L^2(\Omega)$. On the other hand, we have that $F_n = K[\omega^n] \to K[\omega]$ weakly in $L^2(\Omega)$. Therefore, we can pass to the limit in the term J_2:

$$J_2 \overset{n\to\infty}{\longrightarrow} \int_\Omega |K[\omega]|^2.$$

To bound the term J_1, recall that $\operatorname{supp} \nabla \chi_n \subset \Sigma_{\frac{2}{n}}$. We have that

$$|J_1| \leqslant \|K[\omega]\|_{L^2(\Sigma_{\frac{2}{n}})} \|\nabla^\perp \chi_n (f_n * \check{\eta}_n)\|_{L^2(\Omega)}.$$

As in the estimates that follow relation (2.17), we can bound

$$\|\nabla^\perp \chi_n (f_n * \check{\eta}_n)\|_{L^2(\Omega)} \leqslant C \|\nabla f_n\|_{L^2(\Omega)} = C \|F_n\|_{L^2(\Omega)} = C \|K[\omega^n]\|_{L^2(\Omega)} \leqslant C'$$

independently of n. But clearly

$$\|K[\omega]\|_{L^2(\Sigma_{\frac{2}{n}})} \overset{n\to\infty}{\longrightarrow} 0$$

so

$$J_1 \overset{n\to\infty}{\longrightarrow} 0.$$

We conclude from the above relations $\|K[\omega^n]\|_{L^2(\Omega)} \to \|K[\omega]\|_{L^2(\Omega)}$ as $n \to \infty$. This completes the proof of Proposition 2.10. ☐

3. Weak vorticity formulation

In this section we obtain a weak formulation of vortex dynamics in domains with holes, and we establish the equivalence between this weak formulation and the standard weak formulation of the 2D Euler equations in velocity form. Let us start by recalling that the initial-boundary-value problem for the 2-D incompressible Euler equations is given by:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0, & \text{in } \mathbb{R}_+ \times \Omega, \\ \operatorname{div} u = 0, & \text{in } \overline{\mathbb{R}}_+ \times \Omega, \\ u \cdot \hat{n} = 0, & \text{on } \overline{\mathbb{R}}_+ \times \partial\Omega, \\ u(0,x) = u_0(x), & \text{on } \{t=0\} \times \Omega. \end{cases} \tag{3.1}$$

Here, $u = (u_1, u_2)$ is the velocity field and p is the scalar pressure. The evolution equation for $\omega = \operatorname{curl} u$ is known as the vorticity equation:

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, & \text{in } \mathbb{R}_+ \times \Omega \\ \operatorname{div} u = 0, \quad \operatorname{curl} u = \omega, & \text{in } \overline{\mathbb{R}}_+ \times \Omega \\ u \cdot \hat{n} = 0, & \text{on } \overline{\mathbb{R}}_+ \times \partial\Omega, \\ \omega(0,x) = \omega_0(x) = \operatorname{curl} u_0, & \text{on } \{t=0\} \times \Omega. \end{cases} \tag{3.2}$$

We begin with a precise formulation of what it means to be a weak solution of (3.1).

Definition 3.1. *Let $u_0 \in L^2(\Omega)$ and $u = u(x,t) \in L^\infty([0,\infty); L^2(\Omega)) \cap C^0([0,\infty); \mathscr{D}'(\Omega))$ be a vector field. We will say that u is a weak solution of the 2D incompressible Euler equations in Ω with initial data u_0 if the conditions below hold true:*

(a) *For any divergence free test vector field $\phi \in C_c^\infty([0,\infty)\times\Omega)$ we have the integral identity*

$$\int_0^\infty \int_\Omega \partial_t\phi \cdot u \,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty \int_\Omega \nabla\phi : u\otimes u \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega u_0\cdot\phi(0,\cdot)\,\mathrm{d}x = 0. \tag{3.3}$$

(b) *For each time $t\in[0,\infty)$, the vector field $u(\cdot,t)$ is divergence free in the sense of distributions in Ω.*
(c) *The boundary condition $u(\cdot,t)\cdot\hat{n}=0$ is satisfied in the trace sense on $\partial\Omega$, for every time.*

Remark 3.2. *The condition that $u\in C^0([0,\infty);\mathscr{D}'(\Omega))$ is a consequence of $u\in L^\infty([0,\infty);L^2(\Omega))$ and of the integral relation in (a). We chose to require it explicitly in Definition 3.1 only to make sense of parts (b) and (c). Moreover, the fact that u is $L^\infty([0,\infty);L^2)$ and $C([0,\infty);\mathscr{D}')$ implies that u is continuous into weak L^2, which, in particular, implies that it belongs to L^2 pointwise in time. This, together with the (pointwise in time) divergence free condition implies that the normal component of u has a trace at the boundary for each fixed time, so that the boundary data can be imposed pointwise in time.*

In [9], Delort proved existence of a weak solution in the sense above, for vortex-sheet initial data satisfying a sign condition, see also [35]. In both [9] and [35] the solutions were obtained by mollifying the initial data and then passing to the limit along a (sub)sequence of exact solutions with these smoothed-out initial data. We provide a precise statement below.

Theorem 3.3. *Assume that $u_0\in L^2(\Omega)$ is divergence free, tangent to the boundary and that $\omega_0 = \mathit{curl}\, u_0 \in \mathcal{BM}_+(\Omega)+L^1(\Omega)$. Then there exists a global weak solution $u\in L^\infty([0,\infty);L^2(\Omega))$ of the incompressible Euler equations with initial velocity u_0.*

Our objective is to provide a weak vorticity formulation which is equivalent to Definition 3.1. This is an extension of a work due to S. Schochet, see [31]. More precisely, we will prove the following result.

Theorem 3.4. *Let $u_0\in L^2(\Omega)$ be divergence free, tangent to the boundary and such that $\omega_0 = \mathit{curl}\, u_0\in\mathcal{BM}(\Omega)$. Let $\gamma_j(0)$ be the circulation of u_0 on Γ_j defined thanks to Lemma 2.5.*

(a) *Assume that $u\in L^\infty([0,\infty);L^2(\Omega))$ is a divergence free vector field, tangent to the boundary, and such that $\omega=\operatorname{curl} u\in L^\infty([0,\infty);\mathcal{BM}(\Omega))$. Let $\gamma_j(t)$ be the circulation of $u(t)$ on Γ_j (defined a.e. in t thanks to Lemma 2.5). Assume that u is a weak solution of the 2D incompressible Euler equations with initial data u_0. Then $\gamma_j\in L^\infty([0,\infty))$ for all j. Moreover, ω and γ_j verify the following identity:*

$$\begin{aligned}\int_0^\infty\int_\Omega \partial_t\varphi\,\mathrm{d}\omega + \sum_{j=1}^k\int_0^\infty \gamma_j(t)\partial_t\varphi(t,\cdot)\big|_{\Gamma_j}\mathrm{d}t - \int_0^\infty \gamma_0(t)\partial_t\varphi\big|_{\Gamma_0}\mathrm{d}t + \sum_{j=1}^k\int_0^\infty\Big(\gamma_j(t)+\int_\Omega \mathbf{w}_j\,\mathrm{d}\omega\Big)\int_\Omega \mathbf{X}_j\cdot\nabla\varphi\,\mathrm{d}\omega \\ + \int_0^\infty\iint_{\Omega\times\Omega}\frac{1}{2}\Big(\nabla_x\varphi(x)\cdot K(x,y)+\nabla_y\varphi(y)\cdot K(y,x)\Big)\mathrm{d}\omega(x)\,\mathrm{d}\omega(y) \\ + \int_\Omega\varphi(0,\cdot)\,\mathrm{d}\omega_0 + \sum_{j=1}^k \gamma_j(0)\varphi(0,\cdot)\big|_{\Gamma_j} - \gamma_0(0)\varphi(0,\cdot)\big|_{\Gamma_0} = 0\end{aligned} \tag{3.4}$$

for all test functions $\varphi\in C_c^\infty([0,\infty);\mathscr{Y}^\infty)$.

(b) *Conversely, assume that $\omega\in L^\infty([0,\infty);\mathcal{BM}(\Omega)\cap H^{-1}(\Omega))$ and $\gamma_j\in L^\infty([0,\infty))$ verify identity (3.4) for all test functions $\varphi\in C_c^\infty([0,\infty);\mathscr{Y}^\infty)$. Set $u=u(t)$ to be the vector field given by Proposition 2.9 in terms of $\omega=\omega(t)$ and $\gamma_j=\gamma_j(t)$. Then $u\in L^\infty([0,\infty));L^2(\Omega))$ and u is a weak solution of the 2D incompressible Euler equations with initial data $u_0=u(0)$.*

As noted in Proposition 2.1, see (2.1), the function $\nabla_x\varphi(x)\cdot K(x,y)+\nabla_y\varphi(y)\cdot K(y,x) = 2H_\varphi$ is bounded and continuous except on the diagonal. On the other hand, since $\omega=\operatorname{curl} u\in L^\infty_{\mathrm{loc}}([0,\infty);\mathcal{BM}(\Omega)\cap H^{-1}(\Omega))$ the measure $\omega\otimes\omega$ attaches no mass to the diagonal. We infer that all the terms appearing in (3.4) are well defined.

Remark 3.5. *Let us comment a bit on the relation between (3.4) and the PDE satisfied by the vorticity, see relation $(3.2)_1$. Because the test functions in (3.4) are not necessarily compactly supported in Ω, clearly relation (3.4) encodes more information than $(3.2)_1$ in the sense of distributions. We could think of (3.4) as an attempt to include the boundary behavior of $(3.2)_1$ without completely succeeding because the test function φ in (3.4) cannot be taken in $\varphi\in C_c^\infty([0,\infty);C^\infty(\overline{\Omega}))$. In fact, for such a φ the symmetrized kernel H_φ may not be bounded so we don't even know how to define the middle term in (3.4). It is also interesting to remark that if we assume the solution to be smooth and multiply $(3.2)_1$ by $\varphi\in C_c^\infty([0,\infty);\mathscr{Y}^\infty)$ and do the usual integrations by parts and symmetrize the kernel, then we obtain a relation which is not exactly (3.4). More precisely, we obtain (3.4) without the second and the third term on the first line and without the last two terms on the third line. Indeed the sum of these four terms vanishes, for every test function $\varphi\in C_c^\infty([0,\infty);\mathscr{Y}^\infty)$, if and only if Kelvin's circulation holds true along boundary components. This is of course the case for smooth solutions. The point we want to make in this remark is the following: the correct weak* *vorticity formulation cannot be obtained by formally multiplying* $(3.2)_1$ *by a test function and do the usual integrations by parts and symmetrize the kernel. At this level of regularity, one has to start from the velocity equation and prove an identity relating the bilinear term in velocity form and in vorticity form. This will be detailed below.*

We remark now that (3.4) can be written as an equation in $\mathscr{D}'(\mathbb{R}_+)$:

$$(3.5)\quad \partial_t \int_\Omega \theta \,\mathrm{d}\omega + \sum_{j=1}^k \gamma_j' \theta\big|_{\Gamma_j} - \gamma_0'\theta\big|_{\Gamma_0} = \sum_{j=1}^k \big(\gamma_j + \int_\Omega \mathbf{w}_j \,\mathrm{d}\omega\big) \int_\Omega \mathbf{X}_j \cdot \nabla\theta \,\mathrm{d}\omega \\ + \iint_{\Omega\times\Omega} \frac{1}{2}\Big(\nabla_x\theta(x)\cdot K(x,y) + \nabla_y\theta(y)\cdot K(y,x)\Big)\,\mathrm{d}\omega(x)\,\mathrm{d}\omega(y)$$

for all test functions $\theta \in \mathscr{Y}^\infty$. In fact, (3.4) is equivalent to (3.5) plus the initial conditions ω_0 and $\gamma_j(0)$ given.

Proof. To show Theorem 3.4, we observe first that $\nabla^\perp \mathscr{Y}^\infty = C^\infty_{c,\sigma}(\Omega)$ as sets. Therefore, the velocity formulation in the sense of distributions (see relation (3.3)) can be written as follows

$$(3.6)\qquad \int_\Omega u_0 \cdot \nabla^\perp\varphi(0,\cdot) + \int_0^\infty \int_\Omega u\cdot\partial_t\nabla^\perp\varphi + \int_0^\infty \int_\Omega (u\otimes u)\cdot\nabla\nabla^\perp\varphi = 0$$

for all test functions $\varphi \in C_c^\infty([0,\infty); \mathscr{Y}^\infty)$. Using Lemma 2.5, we have that

$$(3.7)\qquad \int_\Omega u_0\cdot\nabla^\perp\varphi(0,\cdot) = -\int_\Omega \varphi(0,\cdot)\,\mathrm{d}\omega_0 + \gamma_0(0)\varphi(0,\cdot)\big|_{\Gamma_0} - \sum_{j=1}^k \gamma_j(0)\varphi(0,\cdot)\big|_{\Gamma_j}$$

and

$$(3.8)\qquad \int_0^\infty \int_\Omega u\cdot\partial_t\nabla^\perp\varphi = -\int_0^\infty\int_\Omega \partial_t\varphi\,\mathrm{d}\omega + \int_0^\infty \gamma_0(t)\partial_t\varphi(t,\cdot)\big|_{\Gamma_0} - \sum_{j=1}^k \int_0^\infty \gamma_j(t)\partial_t\varphi(t,\cdot)\big|_{\Gamma_j}.$$

In order to show Theorem 3.4 it suffices to prove the following proposition.

Proposition 3.6. *Let* $u \in L^2(\Omega)$ *be a divergence free vector field tangent to the boundary such that* $\omega = \operatorname{curl} u \in \mathcal{BM}(\Omega)\cap H^{-1}(\Omega)$. *Let* $\gamma_1,\dots,\gamma_k$ *be the circulations of* u *on each of the connected components of the boundary* $\Gamma_1,\dots,\Gamma_k$. *Let* $\varphi \in \overline{\mathscr{Y}}^\infty$. *The following identity holds true:*

$$(3.9)\qquad \int_\Omega (u\otimes u):\nabla\nabla^\perp\varphi = -\iint_{\Omega\times\Omega} H_\varphi(x,y)\,\mathrm{d}\omega(x)\,\mathrm{d}\omega(y) - \sum_{j=1}^k \alpha_j \int_\Omega \mathbf{X}_j\cdot\nabla\varphi\,\mathrm{d}\omega$$

where

$$H_\varphi(x,y) = \frac{1}{2}\Big(\nabla_x\varphi(x)\cdot K(x,y) + \nabla_y\varphi(y)\cdot K(y,x)\Big) \quad \textit{and} \quad \alpha_j = \gamma_j + \int_\Omega \mathbf{w}_j\,\mathrm{d}\omega.$$

Indeed, assume that this proposition is proved and let us prove part (a). Since $u\in L^\infty([0,\infty);L^2(\Omega))$ and $\omega = \operatorname{curl} u \in L^\infty([0,\infty);\mathcal{BM}(\Omega))$ we immediately deduce from the definition of the circulations, see relations (2.9)–(2.10), that $\gamma_j \in L^\infty([0,\infty))$. Moreover, we have that $u(t)\in L^2(\Omega)$ and $\omega(t) = \operatorname{curl} u(t) \in \mathcal{BM}(\Omega)\cap H^{-1}(\Omega)$ for almost all times t. For those times t we can apply Proposition 3.6 to obtain relation (3.9) for $u(t)$ and $\omega(t)$. Integrating this relation in time implies that

$$(3.10)\quad \int_0^\infty\int_\Omega (u\otimes u)\cdot\nabla\nabla^\perp\varphi = -\int_0^\infty \iint_{\Omega\times\Omega}\frac{1}{2}\Big(\nabla_x\varphi(x)\cdot K(x,y) + \nabla_y\varphi(y)\cdot K(y,x)\Big)\,\mathrm{d}\omega(x)\,\mathrm{d}\omega(y) \\ - \sum_{j=1}^k \int_0^\infty \big(\gamma_j(t) + \int_\Omega \mathbf{w}_j\,\mathrm{d}\omega\big)\int_\Omega \mathbf{X}_j\cdot\nabla\varphi\,\mathrm{d}\omega\,\mathrm{d}t.$$

Combining relations (3.7), (3.8) and (3.10) shows that the left-hand side of (3.4) is equal up to a sign to the left-hand side of (3.6).

Conversely, let us prove part (b). We begin by noting that, since $\omega\in L^\infty([0,\infty);H^{-1}(\Omega))$, it follows that $K[\omega] \in L^\infty([0,\infty);L^2(\Omega))$. Furthermore, since $\omega\in L^\infty([0,\infty);\mathcal{BM}(\Omega))$ and $\gamma_j\in L^\infty([0,\infty))$, we find that u given by Proposition 2.9,

$$u = K[\omega] + \sum_{i=1}^k (\gamma_j + \int_\Omega \mathbf{w}_j\,\mathrm{d}\omega)\mathbf{X}_j,$$

belongs to $L^\infty([0,\infty);L^2(\Omega))$. Choosing φ constant in the x variable in relation (3.4) implies that $\gamma_0=\displaystyle\int_\Omega \mathrm{d}\omega+\sum_{j=1}^k\gamma_j$. Lemma 2.5 therefore implies that the circulation of u on Γ_0 is γ_0. Then identities (3.7), (3.8) hold true, and Proposition 3.6 implies, as before, that (3.10) is valid as well. This implies (3.6), so u is a weak solution of the 2D incompressible Euler equations with initial data $u_0=u(0)$.

This completes the proof of Theorem 3.4, once we establish Proposition 3.6. □

We prove now Proposition 3.6.

Proof of Proposition 3.6. We show first (3.9) when u and ω are smooth. More precisely, we assume that $\omega\in C_c^\infty(\Omega)$. Then, by Proposition 2.4 we have that

$$u=K[\omega]+\sum_{j=1}^k\alpha_j\mathbf{X}_j,\qquad \alpha_j=\gamma_j+\int_\Omega\mathbf{w}_j\omega.$$

We integrate by parts the left-hand side of (3.9):

$$\int_\Omega(u\otimes u):\nabla\nabla^\perp\varphi=-\int_\Omega\operatorname{div}(u\otimes u)\cdot\nabla^\perp\varphi=-\int_\Omega u\cdot\nabla u\cdot\nabla^\perp\varphi=\int_\Omega\operatorname{curl}(u\cdot\nabla u)\varphi-\int_{\partial\Omega}u\cdot\nabla u\cdot\hat n^\perp\varphi.$$

We claim that the boundary integral above vanishes. Indeed, let $C_j=\varphi\big|_{\Gamma_j}$ and write $u\big|_{\Gamma_j}=\beta\hat n^\perp$ with β a scalar function defined on Γ_j. Then

$$\int_{\Gamma_j}u\cdot\nabla u\cdot\hat n^\perp\varphi=C_j\int_{\Gamma_j}\beta\hat n^\perp\cdot\nabla u\cdot\hat n^\perp=C_j\int_{\Gamma_j}\hat n^\perp\cdot\nabla u\cdot u=\frac{C_j}{2}\int_{\Gamma_j}\hat n^\perp\cdot\nabla(|u|^2)=0$$

since Γ_j is a closed curve. Since $\operatorname{curl}(u\cdot\nabla u)=u\cdot\nabla\omega$ we infer that

$$\begin{aligned}\int_\Omega(u\otimes u):\nabla\nabla^\perp\varphi=\int_\Omega u\cdot\nabla\omega\varphi=-\int_\Omega u\cdot\nabla\varphi\omega=-\int_\Omega K[\omega]\cdot\nabla\varphi\omega-\sum_{j=1}^k\alpha_j\int_\Omega\mathbf{X}_j\cdot\nabla\varphi\omega\\ =-\iint_{\Omega\times\Omega}K(x,y)\cdot\nabla\varphi(x)\omega(x)\omega(y)\,\mathrm{d}x\,\mathrm{d}y-\sum_{j=1}^k\alpha_j\int_\Omega\mathbf{X}_j\cdot\nabla\varphi\omega.\end{aligned}$$

Symmetrizing the term involving the kernel of the Biot-Savart law $K(x,y)$ implies (3.9).

We prove now the general case. Let ω^n be defined as in Proposition 2.10 and let us introduce

$$u^n=K[\omega^n]+\sum_{j=1}^k\big(\gamma_j+\int_\Omega\mathbf{w}_j\omega^n\big)\mathbf{X}_j.$$

From Proposition 2.9, we know that

$$u=K[\omega]+\sum_{j=1}^k\big(\gamma_j+\int_\Omega\mathbf{w}_j\,\mathrm{d}\omega\big)\mathbf{X}_j.$$

Since ω^n and u^n are smooth, we have that

$$\int_\Omega(u^n\otimes u^n):\nabla\nabla^\perp\varphi=-\iint_{\Omega\times\Omega}H_\varphi(x,y)\omega^n(x)\omega^n(y)\,\mathrm{d}x\,\mathrm{d}y-\sum_{j=1}^k\alpha_j^n\int_\Omega\mathbf{X}_j\cdot\nabla\varphi\,\omega^n$$

where

$$\alpha_j^n=\gamma_j+\int_\Omega\mathbf{w}_j\omega^n.$$

Given Proposition 2.10 it not difficult to pass to the limit in the above relation and obtain (3.9). First, from (2.14) we infer that $\alpha_j^n\to\alpha_j$ as $n\to\infty$ for all $j\in\{1,\dots,k\}$. Clearly, we also have

$$\int_\Omega\mathbf{X}_j\cdot\nabla\varphi\,\omega^n\to\int_\Omega\mathbf{X}_j\cdot\nabla\varphi\,\mathrm{d}\omega.$$

Next, from (2.15) we infer that

$$u^n\to u\quad\text{strongly in }L^2(\Omega).$$

so that

$$\int_\Omega(u^n\otimes u^n):\nabla\nabla^\perp\varphi\overset{n\to\infty}{\longrightarrow}\int_\Omega(u\otimes u):\nabla\nabla^\perp\varphi.$$

It remains to show that

$$\iint_{\Omega\times\Omega} H_\varphi(x,y)\omega^n(x)\omega^n(y)\,\mathrm{d}x\,\mathrm{d}y \overset{n\to\infty}{\longrightarrow} \iint_{\Omega\times\Omega} H_\varphi(x,y)\,\mathrm{d}\omega(x)\,\mathrm{d}\omega(y).$$

This follows immediately from Proposition 2.1, Lemma 2.2 and Proposition 2.10. Indeed, let $\overline{\omega}$ be the extension of ω to a measure in $\mathcal{BM}(\overline{\Omega})$ with zero values on $\partial\Omega$. By Proposition 2.1 we have that the function $H_\varphi(x,y)$ is bounded and continuous on $\overline{\Omega}\times\overline{\Omega}\setminus\{(x,x)\ ;\ x\in\overline{\Omega}\}$. From Proposition 2.10 we know that the sequence ω^n is bounded in L^1, that it converges in the sense of measures $\mathcal{BM}(\overline{\Omega})$ towards $\overline{\omega}$ and that any weak limit μ in the sense of measures $\mathcal{BM}(\overline{\Omega})$ of $|\omega^n|$ is a continuous measure on $\overline{\Omega}$. Because μ is continuous we obviously have that the measure $\mu\otimes\mu$ of the diagonal $\{(x,x)\ ;\ x\in\overline{\Omega}\}$ vanishes. Then Lemma 2.2 applied on $\overline{\Omega}\times\overline{\Omega}$ implies that

$$\iint_{\Omega\times\Omega} H_\varphi(x,y)\omega^n(x)\omega^n(y)\,\mathrm{d}x\,\mathrm{d}y \overset{n\to\infty}{\longrightarrow} \iint_{\overline{\Omega}\times\overline{\Omega}} H_\varphi(x,y)\,\mathrm{d}\overline{\omega}(x)\,\mathrm{d}\overline{\omega}(y) = \iint_{\Omega\times\Omega} H_\varphi(x,y)\,\mathrm{d}\omega(x)\,\mathrm{d}\omega(y).$$

This completes the proof of Proposition 3.6. □

Definition 3.7. *Let $\omega_0\in\mathcal{BM}(\Omega)\cap H^{-1}(\Omega)$ and consider real numbers $\gamma_j(0)$, $j=1,\dots,k$. We say that the $(k+1)$-tuple $(\omega,\gamma_1,\dots,\gamma_k)$, is a solution of the weak vorticity formulation of the incompressible 2D Euler equations in Ω, with initial vorticity ω_0 and initial circulations $\gamma_j(0)$, if $\omega\in L^\infty([0,\infty);\mathcal{BM}(\Omega)\cap H^{-1}(\Omega))$, if $\gamma_j\in L^\infty([0,\infty))$, $j=1,\dots,k$, and if the identity* (3.4) *with $\gamma_0=\int_\Omega \mathrm{d}\omega+\sum_{j=1}^k\gamma_j$ holds true for every test function $\varphi\in C_c^\infty([0,\infty);\mathscr{Y}^\infty)$.*

Remark 3.8. *We have shown, in Theorem 3.4, that there is a one-to-one correspondence between weak solutions of the incompressible 2D Euler equations in Ω and solutions of the weak vorticity formulation. Notice that the weak vorticity formulation allows for weak solutions for which the Kelvin circulation theorem is no longer valid along boundary components.*

Remark 3.9. *In Definition 3.7, let us introduce a continuous measure $\widetilde{\omega}\in L^\infty(\mathbb{R}_+;\mathcal{BM}(\overline{\Omega}))$ such that $\widetilde{\omega}\big|_\Omega=\omega$ and, for all $t\geqslant 0$, $\widetilde{\omega}(t,\Gamma_j)=\gamma_j(t)$ for $j=1,\dots,k$ and $\widetilde{\omega}(t,\Gamma_0)=-\gamma_0(t)$. Then, with this notation, identity* (3.4) *becomes*

$$(3.11)\quad \int_\Omega \varphi(0,\cdot)\,\mathrm{d}\widetilde{\omega}_0+\int_0^\infty\int_\Omega\partial_t\varphi\,\mathrm{d}\widetilde{\omega}+\sum_{j=1}^k\int_0^\infty\Big(\int_{\overline{\Omega}}\mathbf{w}_j\,\mathrm{d}\widetilde{\omega}\Big)\Big(\int_{\overline{\Omega}}\mathbf{X}_j\cdot\nabla\varphi\,\mathrm{d}\widetilde{\omega}\Big)+\int_0^\infty\iint_{\overline{\Omega}\times\overline{\Omega}}H_\varphi(x,y)\,\mathrm{d}\widetilde{\omega}(x)\,\mathrm{d}\widetilde{\omega}(y)=0.$$

Indeed, we observe that for all test functions $\varphi\in C_c^\infty([0,\infty);\mathscr{Y}^\infty)$ the functions $\mathbf{X}_j\cdot\nabla\varphi$ and H_φ vanish on the boundary and, in addition,

$$\gamma_j(t)+\int_\Omega\mathbf{w}_j\,\mathrm{d}\omega=\int_{\overline{\Omega}}\mathbf{w}_j\,\mathrm{d}\widetilde{\omega}.$$

Note that this does not give rise to a weak formulation for a completely determined evolution problem since nothing is said about the density of $\widetilde{\omega}$ on the boundary beyond its mass. One could postulate a complete problem by setting $\widetilde{\omega}$ to be a uniform measure on each Γ_i, however this is an arbitrary choice.

Next, our aim is to collect as much information as possible on the circulations γ_j of a weak solution. Suppose that $(\omega,\gamma_1,\dots,\gamma_k)$ is a solution of the weak vorticity formulation as in Definition 3.7. Then the circulations γ_j, $j=1,\dots,k$ can be expressed uniquely in terms of ω. Indeed, take in (3.5) successive test functions $\theta_\ell\in\mathscr{Y}^\infty$, vanishing in a neighborhood of the boundary except in the neighborhood of Γ_ℓ, where it equals 1. We then obtain the following linear system of ODEs for the circulations:

$$\begin{aligned}\partial_t\int_\Omega\theta_\ell\,\mathrm{d}\omega+\gamma_\ell'=\sum_{j=1}^k\big(\gamma_j+\int_\Omega\mathbf{w}_j\,\mathrm{d}\omega\big)\int_\Omega\mathbf{X}_j\cdot\nabla\theta_\ell\,\mathrm{d}\omega\\ +\iint_{\Omega\times\Omega}\frac{1}{2}\Big(\nabla_x\theta_\ell(x)\cdot K(x,y)+\nabla_y\theta_\ell(y)\cdot K(y,x)\Big)\,\mathrm{d}\omega(x)\,\mathrm{d}\omega(y),\end{aligned}$$

$\ell=1,\dots,k$.

Clearly, this system of ODEs for the circulations has a unique solution. So we can express the circulations γ_j in terms of ω. Plugging the formula for the circulations in γ_j in (3.5) and taking a test function $\theta\in C_c^\infty(\Omega)$, it is possible to obtain an equation in the sense of distributions for ω only.

4. Weak solutions obtained by mollifying initial data

In this section we will study some properties enjoyed by weak solutions which are weak limits of smooth solutions, obtained by smoothing out initial data. Motivated by a a stronger notion of weak solution, introduced by two of the authors in [27] and [28] in the setting of half-plane and exterior-domain flows, called *boundary-coupled weak solution*, we begin with a natural extension of these weak solutions to bounded domain flows. Roughly speaking, a boundary-coupled weak solution is a solution satisfying the weak vorticity formulation (3.4) for a wider class of test functions, whose derivatives do not necessarily vanish in a neighborhood of the boundary.

Definition 4.1. *Let $\omega_0 \in \mathcal{BM}(\Omega) \cap H^{-1}(\Omega)$ and consider real numbers $\gamma_j(0)$, $j = 1, \ldots, k$. We say that the $(k+1)$-tuple $(\omega, \gamma_1, \ldots, \gamma_k)$, is a boundary-coupled weak solution of the incompressible 2D Euler equations in Ω, with initial vorticity ω_0 and initial circulations $\gamma_j(0)$, if $\omega \in L^\infty_{\text{loc}}([0,\infty); \mathcal{BM}(\Omega) \cap H^{-1}(\Omega))$, if $\gamma_j \in L^\infty_{\text{loc}}([0,\infty))$, $j = 1, \ldots, k$, and if the identity* (3.4) *with $\gamma_0 = \int_\Omega \mathrm{d}\omega + \sum_{j=1}^k \gamma_j$ holds true for every test function $\varphi \in C_c^\infty([0,\infty); \overline{\mathscr{Y}}^\infty)$.*

Remark 4.2. *We note that in the special case of a fluid domain which is bounded, connected and simply-connected, the test functions in the definition above can be assumed, without loss of generality, to vanish on the boundary. Indeed, in this special case, we will have that $\gamma_0 = \displaystyle\int_\Omega \mathrm{d}\omega$ so there is no advantage in incorporating γ_0 in the identity* (3.4)*.*

Remark 4.3. *Note that the observation in Remark 3.9 goes through for boundary coupled weak solutions as well. Indeed, for all test functions $\varphi \in C_c^\infty([0,\infty); \overline{\mathscr{Y}}^\infty)$ the functions $\mathbf{X}_j \cdot \nabla\varphi$ and H_φ vanish on the boundary, see Proposition 2.1 for the latter.*

Our next result contains three statements about weak solutions obtained by smoothing out initial data, when the singular part of the initial vorticity has a distinguished sign. First we show that the circulations obey a one-sided inequality. Then we establish a sufficient condition for the existence of boundary-coupled weak solutions, namely, that the Kelvin circulation theorem hold true. Our third result is that, if $\omega_0 \in L^1(\Omega) \cap H^{-1}(\Omega)$, then any solution of the weak vorticity formulation constructed as in Theorem 3.3, *i.e.* by smoothing out the initial vorticity as in (2.12) and passing to the limit, has no circulation defect and is boundary coupled. Thus, for those weak solutions obtained by Delort in Theorem 3.3, the phenomena of circulation defect may happen only at the level of regularity of vortex sheets.

Theorem 4.4. *Let $\omega_0 \in (\mathcal{BM}_+(\Omega) + L^1(\Omega)) \cap H^{-1}(\Omega)$, and consider $\gamma^j(0) \in \mathbb{R}$, $j = 1, \ldots, k$. Let $(\omega, \gamma_1, \ldots, \gamma_k)$ be a solution of the weak vorticity formulation as constructed in Theorem 3.3, i.e. by smoothing out the initial vorticity as in* (2.12) *and passing to the limit. Recall that $\gamma_0 = \int_\Omega \mathrm{d}\omega + \sum_{j=1}^k \gamma_j$. Then we have that*

(a) *$\gamma_j(t) \geqslant \gamma_j(0)$, $j = 1, \ldots, k$, and $\gamma_0(t) \leqslant \gamma_0(0)$, for almost every $t \geqslant 0$.*
(b) *If $\gamma_j(t) \equiv \gamma_j(0)$ for almost all $t > 0$, $j = 0, 1, \ldots, k$, then the solution is a boundary-coupled weak solution.*
(c) *If $\omega_0 \in L^1(\Omega) \cap H^{-1}(\Omega)$ then $\gamma_j(t) \equiv \gamma_j(0)$ for almost all $t > 0$, $j = 0, 1, \ldots, k$, and the solution is boundary-coupled.*

Proof. The proof of this theorem consists of 3 steps. First, we smooth-out the initial data, we construct the approximate smooth solutions and extract a sub-sequence with the required properties. Second, we write down that the circulations of the approximate smooth solutions are conserved and we pass to the limit in these circulations. This will imply part (a). Third, we pass to the limit in the PDE and prove parts (b) and (c).

Construction of the approximate smooth solutions. Let us introduce some notation and recall Delort's original construction. Recall first that

$$u_0 = K[\omega_0] + \sum_{j=1}^{k} \left(\gamma_j(0) + \int_\Omega \mathbf{w}_j \, \mathrm{d}\omega_0 \right) \mathbf{X}_j.$$

Let ω_0^n be the sequence of smooth functions constructed from ω_0 as in (2.12). The corresponding approximate initial velocities are chosen as follows:

$$u_0^n = K[\omega_0^n] + \sum_{j=1}^{k} \left(\gamma_j(0) + \int_\Omega \mathbf{w}_j \, \mathrm{d}\omega_0 \right) \mathbf{X}_j. \tag{4.1}$$

Let us denote by $\overline{\omega}_0$ the extension of ω_0 to a measure in $\mathcal{BM}(\overline{\Omega})$ by setting zero values on the boundary $\partial\Omega$. According to Proposition 2.10 we have the following convergence results at the level of the initial data:

$$\omega_0^n \rightharpoonup \overline{\omega}_0 \text{ weak}* \text{ in } \mathcal{BM}(\overline{\Omega}) \text{ and } u_0^n \to u_0 \text{ strongly in } L^2(\Omega), \text{ as } n \to \infty. \tag{4.2}$$

Delort's argument proceeds as follows. Let u^n be an exact, smooth, solution of the 2D incompressible Euler equations (3.1) with initial velocity u_0^n. Set $\omega^n = \operatorname{curl} u^n$. We know from Proposition 2.10 that ω_0^n is bounded in L^1, so ω^n is also bounded in $L^\infty_{\text{loc}}(\mathbb{R}_+; L^1(\Omega))$. Therefore there exists some $\overline{\omega} \in L^\infty_{\text{loc}}(\mathbb{R}_+; \mathcal{BM}(\overline{\Omega}))$ and a subsequence ω^{n_p} such that

$$\omega^{n_p} \rightharpoonup \overline{\omega} \quad \text{ in } L^\infty_{\text{loc}}(\mathbb{R}_+; \mathcal{BM}(\overline{\Omega})) \text{ weak}* . \tag{4.3}$$

Let $\omega = \overline{\omega}\big|_\Omega$. Let us split ω_0 into its positive and negative parts, so that $\omega_0^+ \in \mathcal{BM}_+(\Omega)$ and $\omega_0^- \in L^1(\Omega)$. Then, $\omega_0^{n,-} = \eta_n * (\chi_n \omega_0^-)$ converges weakly (even strongly) in $L^1(\Omega)$ to ω_0^-. Let $\Phi^n = \Phi^n(t,x)$ denote the flow map associated to u^n. Write the corresponding smooth ω^n as $\omega^n = \omega^{n,+} + \omega^{n,-}$, where $\omega^{n,+} = \omega^{n,+}(t,x) = \omega_0^{n,+}((\Phi^n)^{-1}(t,x))$ and $\omega^{n,-} = \omega^{n,-}(t,x) = \omega_0^{n,-}((\Phi^n)^{-1}(t,x))$. We have that $\omega^{n,+} \geqslant 0$, while $\omega^{n,-} \leqslant 0$. Since $\omega^{n,-}(t,\cdot)$ is a rearrangement of $\omega_0^{n,-}(\cdot)$ and since, by the Dunford-Pettis theorem, $\omega_0^{n,-}$ is uniformly integrable, it follows that $\omega^{n,-}$ is also uniformly integrable, uniformly in time. We deduce, once using again the Dunford-Pettis theorem, that $\omega^{n_p,-}$ converges weak$*$ in $L^\infty_{\text{loc}}(\mathbb{R}_+; L^1(\Omega))$ to some $\omega^- \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^1(\Omega))$, $\omega^- \leqslant 0$. Let us denote by $\overline{\omega}^-$ the extension of ω^- to $\overline{\Omega}$ with zero values on the boundary $\partial\Omega$. We conclude that

$$\omega^{n_p,-} \rightharpoonup \overline{\omega}^- \quad \text{in } L^\infty(\mathbb{R}_+; \mathcal{BM}(\overline{\Omega})) \text{ weak} * \qquad \text{and} \quad \overline{\omega}^-\big|_{\partial\Omega} = 0. \tag{4.4}$$

Passing to further subsequences as necessary, we have that there exists $\overline{\omega}^+ \in L^\infty_{\text{loc}}(\mathbb{R}_+; \mathcal{BM}_+(\overline{\Omega}))$ such that $\omega^{n_p,+} \rightharpoonup \overline{\omega}^+$ weak$*$ in $L^\infty_{\text{loc}}(\mathbb{R}_+; \mathcal{BM}_+(\overline{\Omega}))$. Recalling that $\omega^n = \omega^{n,+} + \omega^{n,-}$, that ω^{n_p} converges weakly to $\overline{\omega}$ and that $\omega^{n_p,-}$ converges weakly to $\overline{\omega}^-$ we infer that $\overline{\omega} - \overline{\omega}^- = \overline{\omega}^+ \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathcal{BM}_+(\overline{\Omega}))$. But $\overline{\omega}^-$ vanishes on the boundary, so

$$\overline{\omega}\big|_{\partial\Omega} = \overline{\omega}^+\big|_{\partial\Omega} \tag{4.5}$$

where the trace is taken in the sense of the restriction of measures on measurable subsets.

Delort proved, in [9], see also [35], that, passing to subsequences as needed without relabeling,

$$u^{n_p} \rightharpoonup u \text{ weak} * \text{ in } L^\infty_{\text{loc}}(\mathbb{R}_+; L^2(\Omega)) \tag{4.6}$$

and, furthermore, u is a weak solution of the 2D incompressible Euler equations with initial velocity u_0. This is the content of Theorem 3.3.

Passing to the limit in the circulations. Since the solution u^n is smooth, the Kelvin circulation theorem holds true so the circulation of u^n on Γ_j is constant in time, denoted by γ_j^n. We write down the fact that $\gamma_j^{n_p}$ is the circulation of u^{n_p} by using (2.4):

$$\int_0^\infty \int_\Omega \varphi \omega^{n_p} + \int_0^\infty \int_\Omega u^{n_p} \cdot \nabla^\perp \varphi = \int_0^\infty \gamma_0^{n_p} \varphi\big|_{\Gamma_0} - \sum_{j=1}^k \int_0^\infty \gamma_j^{n_p} \varphi\big|_{\Gamma_j} \tag{4.7}$$

for all $\varphi \in C_c^\infty([0,\infty); \overline{\mathscr{Y}}^\infty)$. We shall pass to the limit $p \to \infty$ above. Thanks to (4.3) and (4.6) we can immediately pass to the limit on the left-hand side:

$$\int_0^\infty \int_\Omega \varphi \omega^{n_p} + \int_0^\infty \int_\Omega u^{n_p} \cdot \nabla^\perp \varphi \to \int_0^\infty \int_{\overline{\Omega}} \varphi \, \mathrm{d}\overline{\omega} + \int_0^\infty \int_\Omega u \cdot \nabla^\perp \varphi \quad \text{as } p \to \infty.$$

To pass to the limit on the right-hand side we need to determine the limit of γ_j^n. Recalling that γ_j^n is the circulation of the initial smoothed-out velocity u_0^n we get, using relations (2.2) and (4.1), that

$$\gamma_j^n = \gamma_j(0) + \int_\Omega \mathbf{w}_j \, \mathrm{d}\omega_0 - \int_\Omega \mathbf{w}_j \omega_0^n$$

for all $1 \leqslant j \leqslant k$. We deduce from (4.2) that

$$\gamma_j^n \to \gamma_j(0), \quad j = 1, \dots, k, \quad \text{as } n \to \infty.$$

In addition,

$$\gamma_0^n = \int_\Omega \mathrm{d}\omega_0^n + \sum_{j=1}^k \gamma_j^n \to \int_\Omega \mathrm{d}\omega_0 + \sum_{j=1}^k \gamma_j(0) = \gamma_0(0) \quad \text{as } n \to \infty.$$

We can now pass to the limit $p \to \infty$ in (4.7) to obtain

$$\int_0^\infty \int_{\overline{\Omega}} \varphi \, \mathrm{d}\overline{\omega} + \int_0^\infty \int_\Omega u \cdot \nabla^\perp \varphi = \gamma_0(0) \int_0^\infty \varphi\big|_{\Gamma_0} - \sum_{j=1}^k \gamma_j(0) \int_0^\infty \varphi\big|_{\Gamma_j}.$$

Using again Lemma 2.5 we have that

$$\int_0^\infty \int_\Omega \varphi \, \mathrm{d}\omega + \int_0^\infty \int_\Omega u \cdot \nabla^\perp \varphi = \int_0^\infty \gamma_0(t) \varphi\big|_{\Gamma_0} - \sum_{j=1}^k \int_0^\infty \gamma_j(t) \varphi\big|_{\Gamma_j}.$$

Subtracting the two previous relations we get that

$$\int_0^\infty \big[\gamma_0(0) - \gamma_0(t) - \overline{\omega}(\Gamma_0)\big] \varphi\big|_{\Gamma_0} - \sum_{j=1}^k \int_0^\infty \big[\gamma_j(0) - \gamma_j(t) + \overline{\omega}(\Gamma_j)\big] \varphi\big|_{\Gamma_j} = 0.$$

Recalling (4.5) we conclude that

$$\begin{gathered} \forall 1 \leqslant j \leqslant k \quad \gamma_j(t) = \gamma_j(0) + \overline{\omega}^+(\Gamma_j) \quad \text{a.e. in } t \text{ and} \\ \gamma_0(0) = \gamma_0(t) + \overline{\omega}^+(\Gamma_0) \quad \text{a.e. in } t. \end{gathered} \tag{4.8}$$

Since $\overline{\omega}^+ \geqslant 0$, part (a) follows.

Passing to the limit in the PDE. To show part (b) we need to pass the limit in the PDE. Recall that $\omega^n = \operatorname{curl} u^n$, where u^n is the approximate solution considered above. We use the Kelvin circulation theorem together with Theorem 3.4 applied to the smooth solutions ω^n to obtain that ω^n verifies the formulation

$$(4.9)\quad \int_0^\infty \int_\Omega \partial_t\varphi\omega^n + \sum_{j=1}^k \int_0^\infty \gamma_j^n \partial_t\varphi(t,\cdot)\big|_{\Gamma_j}\,\mathrm{d}t - \int_0^\infty \gamma_0^n\partial_t\varphi\big|_{\Gamma_0}\,\mathrm{d}t + \sum_{j=1}^k \int_0^\infty \Big(\gamma_j^n + \int_\Omega \mathbf{w}_j\omega^n\Big)\int_\Omega \mathbf{X}_j\cdot\nabla\varphi\omega^n$$
$$+ \int_0^\infty \iint_{\Omega\times\Omega} \frac{1}{2}\Big(\nabla_x\varphi(x)\cdot K(x,y) + \nabla_y\varphi(y)\cdot K(y,x)\Big)\omega^n(x)\omega^n(y)$$
$$+ \int_\Omega \varphi(0,\cdot)\omega_0^n + \sum_{j=1}^k \gamma_j^n\varphi(0,\cdot)\big|_{\Gamma_j} - \gamma_0^n\varphi(0,\cdot)\big|_{\Gamma_0} = 0$$

for all test functions $\varphi \in C_c^\infty([0,\infty);\mathscr{Y}^\infty)$. Given that ω^n is compactly supported in Ω it is easy to see that (4.9) actually holds true with $\varphi \in C_c^\infty([0,\infty);\overline{\mathscr{Y}}^\infty)$; that is the smooth solutions are boundary-coupled. Indeed, let $\overline{\varphi} \in C_c^\infty([0,\infty);\overline{\mathscr{Y}}^\infty)$ and define $\varphi^m = \overline{\varphi}\chi_m$ where χ_m was defined on page 8. Then it is trivial to see that after we put φ^m as test function in (4.9) and send $m\to\infty$ we obtain the desired relation. More precisely, if m is sufficiently large then, due to the compact support of ω^n, all terms in (4.9) except for

$$(4.10)\qquad \sum_{j=1}^k \int_0^\infty \gamma_j^n\partial_t\varphi(t,\cdot)\big|_{\Gamma_j}\,\mathrm{d}t - \int_0^\infty \gamma_0^n\partial_t\varphi\big|_{\Gamma_0}\,\mathrm{d}t + \sum_{j=1}^k \gamma_j^n\varphi(0,\cdot)\big|_{\Gamma_j} - \gamma_0^n\varphi(0,\cdot)\big|_{\Gamma_0}$$

do not change if we replace φ with φ^m. However, recalling that γ_j^n does not depend on the time we immediately see that (4.10) vanishes for all $\varphi \in C_c^\infty([0,\infty);\overline{\mathscr{Y}}^\infty)$. This completes the proof that (4.9) holds true for all $\varphi \in C_c^\infty([0,\infty);\overline{\mathscr{Y}}^\infty)$.

Assume now that $\gamma_j(t) = \gamma_j(0)$, $j = 0,1,\dots,k$. Then (4.8) implies that $\overline{\omega}^+$ attaches no mass to the boundary. Relation (4.5) now implies that $\overline{\omega}$ vanishes on the boundary (the restriction to the boundary is taken in the sense of measures). Let us fix some $\varphi \in C_c^\infty([0,\infty);\overline{\mathscr{Y}}^\infty)$ and use it as test function in (4.9). We let $n = n_p \to \infty$. Recalling that

- $\omega^{n_p} \rightharpoonup \overline{\omega}$ weak$*$ in $L^\infty(\mathbb{R}_+;\mathcal{BM}(\overline{\Omega}))$,
- $\overline{\omega}\big|_{\partial\Omega} = 0$ and $\overline{\omega}\big|_{\Omega} = \omega$,
- $\gamma_j^n \to \gamma_j(0)$ for all j

we easily observe that all terms in (4.9) except for the term on the middle line converge to the expected limit, that is the same relation where ω^n is replaced by ω and γ_j^n is replaced by $\gamma_j(0)$. Indeed, we have for example that

$$\int_0^\infty \int_\Omega \partial_t\varphi\omega^n \to \int_0^\infty \int_{\overline{\Omega}} \partial_t\varphi\,\mathrm{d}\overline{\omega} = \int_0^\infty \int_\Omega \partial_t\varphi\,\mathrm{d}\omega.$$

To pass to the limit in the term on the middle line of (4.9) we need to show that the weak limit of $|\omega^{n_p}|$ is a continuous measure on $\overline{\Omega}$. Recall that $\omega^n = \omega^{n,+} + \omega^{n,-}$, with $\omega^{n,+} \geqslant 0$ and $\omega^{n,-} \leqslant 0$ so that $|\omega^n| \leqslant \omega^{n,+} - \omega^{n,-}$. Let ν be a weak limit of $|\omega^{n_p}|$. Recalling that $\overline{\omega}^+$ and $\overline{\omega}^-$ are the weak limits of $\omega^{n,+}$ and $\omega^{n,-}$, we get that $\nu \leqslant \overline{\omega}^+ - \overline{\omega}^-$. By construction of $\overline{\omega}^-$ we know that it vanishes on the boundary of Ω, see relation (4.4). We observed above that $\overline{\omega}^+$ also vanishes on the boundary. We conclude that ν vanishes on the boundary. Next, ω^- and ω are continuous measures (on Ω) because ω^- belongs to $L^\infty_{loc}(\mathbb{R}_+;L^1(\Omega))$ and ω belongs to $L^\infty_{loc}(\mathbb{R}_+;H^{-1}(\Omega))$. Thus $\overline{\omega}^+\big|_\Omega = \omega - \omega^-$ is a continuous measure on Ω as well. Then $\nu \leqslant \overline{\omega}^+ - \overline{\omega}^-$ is also continuous on Ω. We proved that ν has no discrete part both in Ω and on $\partial\Omega$, so ν is a continuous measure on $\overline{\Omega}$.

Now that we proved that ν is a continuous measure on $\overline{\Omega}$, the term on the middle line of (4.9) converges to the required limit as a consequence of a time dependent version of Lemma 2.2 applied on $\overline{\Omega}\times\overline{\Omega}$ and of Proposition 2.1. Indeed, Proposition 2.1 implies that the kernel $\nabla_x\varphi(x)\cdot K(x,y) + \nabla_y\varphi(y)\cdot K(y,x)$ is bounded and continuous on $\overline{\Omega}\times\overline{\Omega}\setminus\{(x,x)\ ;\ x\in\overline{\Omega}\}$ while the fact that ν is continuous on $\overline{\Omega}$ implies that the measure $\nu\otimes\nu$ of the diagonal $\{(x,x)\ ;\ x\in\overline{\Omega}\}$ vanishes. This completes the proof of part (b).

Finally, let us prove part (c). We assume now that $\omega_0 \in L^1(\Omega)\cap H^{-1}(\Omega)$. Then $\omega_0^+ \in L^1$ and we can argue as for ω^- to deduce that $\overline{\omega}^+$ vanishes on the boundary. So, in view of (4.8), the circulations are conserved. By part (b), the solution is boundary-coupled. This completes the proof of the theorem. ☐

Remark 4.5. *The problem in passing to the limit in* (4.9) *with test functions* $\varphi \in C_c^\infty([0,\infty);\overline{\mathscr{Y}}^\infty)$ *is the passing to the limit in the nonlinear term, i.e. the term on the middle line. We were able to do that because, by conservation of circulations, the limit measure* $\overline{\omega}$ *attaches no mass to the boundary. However, less is required to pass to the limit. Indeed, the proof only uses that* $\overline{\omega}$ *attaches no mass to the points of the boundary. If we can ensure, in some other* *way, that $\overline{\omega}$ is continuous on the boundary, then we can pass to the limit in* (4.9)*. However, the limit solution is not necessarily boundary-coupled because when $\overline{\omega}$ is continuous on the boundary but not vanishing on the boundary, the limit weak formulation has some additional boundary terms.*

Remark 4.6. *With respect to the notation in Remarks 3.9 and 4.3, we observe that $\overline{\omega}$ verifies relation* (3.11)*. However, there is no reason for $\overline{\omega}$ to be a uniform measure on each Γ_i so that* (3.11) *provides an incomplete PDE for $\overline{\omega}$.*

5. Net force on boundary components

For smooth fluid flow, the net force exerted by the fluid on an immersed solid object with boundary Γ is given by

$$\int_\Gamma p\hat{n}\,dS,$$

where p is the scalar pressure and $\hat{n}$ is the unit exterior normal to the fluid at Γ. It is natural to ask whether it is possible to make sense of this net force at the level of regularity of vortex sheets. This is the subject of the present section.

Clearly, vortex sheet flows are not regular enough to allow for traces of pressure at the boundary. To circumvent this, we will use the PDE itself. First, let us assume for a moment that the solution is smooth and let us try to find a weak formulation for the net force.

Let $C^\infty_\sigma(\overline{\Omega})$ denote the space of smooth divergence free vector fields on $\overline{\Omega}$.

Lemma 5.1. *Let u be a smooth solution of the incompressible Euler equations in Ω. Let $\Phi^1, \Phi^2 \in C^\infty_\sigma(\overline{\Omega})$ be two smooth divergence free vector fields such that $\Phi^j \cdot \hat{n} = \hat{n}^j$ on $\partial\Omega$. Then the net force exerted by the fluid through $\partial\Omega$ is given by*

$$\int_{\partial\Omega} p\hat{n}\,dS = -(F_u(\Phi^1), F_u(\Phi^2))$$

where F_u is the functional defined on $C^\infty_\sigma(\overline{\Omega})$ by

$$F_u(\Phi) = \partial_t \int_\Omega u\cdot\Phi\,dx - \int_\Omega [(u\cdot\nabla)\Phi]\cdot u\,dx.$$

Remark 5.2. *Before proving this lemma, let us observe that such vector fields Φ^1 and Φ^2 do indeed exist. One could take, for instance, $\Phi^1 = (1,0)$ and $\Phi^2 = (0,1)$.*

Proof. To compute the horizontal component of the net force exerted by the fluid through $\partial\Omega$ we take the inner product of the Euler equations with Φ^1 and integrate on Ω:

$$\int_\Omega [\partial_t u + (u\cdot\nabla)u]\cdot\Phi^1\,dx = -\int_\Omega \nabla p\cdot\Phi^1\,dx.$$

We then observe that, since Φ^1 is divergence free, we have

$$-\int_\Omega \nabla p\cdot\Phi^1\,dx = -\int_{\partial\Omega} p\Phi^1\cdot\hat{n}\,dS = -\int_{\partial\Omega} p\hat{n}^1\,dS.$$

A simple integration by parts shows immediately that the horizontal component of the net force exerted by the fluid through $\partial\Omega$ can be written under the following form

$$\int_{\partial\Omega} p\hat{n}^1\,dS = -\int_\Omega [\partial_t u + (u\cdot\nabla)u]\cdot\Phi^1\,dx = -\partial_t\int_\Omega u\cdot\Phi^1 + \int_\Omega [(u\cdot\nabla)\Phi^1]\cdot u\,dx = -F_u(\Phi^1).$$

Similarly, we have that

$$\int_{\partial\Omega} p\hat{n}^2\,dS = -F_u(\Phi^2).$$

This completes the proof of the lemma. □

Now, let us go back to the problem of defining the net force for vortex sheet solutions. Let us remark that for such solutions, the functional F_u is well-defined and, for each test vector field $\Phi \in C^\infty_\sigma(\overline{\Omega})$, we have that $F_u(\Phi) \in W^{-1,\infty}(0,+\infty)$. So, in the case of vortex sheets, a natural candidate for the net force exerted by the fluid through $\partial\Omega$ is given by

$$-(F_u(\Phi^1), F_u(\Phi^2)), \tag{5.1}$$

where $\Phi^1, \Phi^2 \in C^\infty_\sigma(\overline{\Omega})$ are such that $\Phi^j\cdot\hat{n} = \hat{n}^j$ on $\partial\Omega$. In order for such a definition to make sense, we need to make sure that it does not depend on the choice of Φ^1 and Φ^2, that is, we wish to have $F_u(\Phi)$ depending on Φ only through its normal component at $\partial\Omega$. Since $F_u(\Phi)$ is linear with respect to Φ, this means we would like to have $F_u(\Phi)$ vanishing whenever $\Phi \in C^\infty_{\sigma,\tan}(\overline{\Omega})$, i.e. whenever $\Phi \in C^\infty_\sigma(\overline{\Omega})$ is tangent to the boundary.

Definition 5.3. *Let u be a solution in the sense of Definition 3.1. Assume that $F_u(\Phi) = 0$ for all $\Phi \in C^\infty_{\sigma,\tan}(\overline{\Omega})$. Then, the net force of the fluid through $\partial\Omega$, denoted* $\mathbf{f}$, *is*

$$\mathbf{f} = -(F_u(\Phi^1), F_u(\Phi^2),$$

where Φ^1, $\Phi^2 \in C^\infty_\sigma(\overline{\Omega})$ are such that $\Phi^j \cdot \hat{n} = \hat{n}^j$ on $\partial\Omega$.

Note that $\mathbf{f}$ is well-defined if and only if $F_u(\Phi) = 0$ for all $\Phi \in C^\infty_{\sigma,\tan}(\overline{\Omega})$.

As we will see, this brings us naturally to the notion of boundary-coupled weak solution. We have defined boundary-coupled weak solution using the vorticity formulation of the Euler equations. However we require now a definition of boundary-coupled weak solution arising from the velocity formulation.

We allow test vector fields in Definition 3.1 which are merely tangent to the boundary instead of compactly supported in Ω, i.e., substitute divergence free $\phi \in C^\infty_c([0,\infty) \times \Omega)$ for $\phi \in C^\infty_c([0,\infty); C^\infty_{\sigma,\tan}(\overline{\Omega}))$.

Theorem 5.4. *Let $\omega_0 \in \mathcal{BM}(\Omega) \cap H^{-1}(\Omega)$ and fix real numbers $\gamma_j(0)$, $j = 1, \dots, k$. Let $\omega \in L^\infty_{\text{loc}}([0,\infty); \mathcal{BM}(\Omega) \cap H^{-1}(\Omega))$, together with $\gamma_j \in L^\infty_{\text{loc}}([0,\infty))$, $j = 1, \dots, k$, be a boundary-coupled weak solution of the Euler equations with initial vorticity ω_0 and initial circulations $\gamma_j(0)$. Then $u = u(t)$ given by Proposition 2.9 in terms of ω and γ_j is a weak solution of the incompressible 2D Euler equations for which the integral identity in part (a) of Definition 3.1 holds for all test vector fields $\phi \in C^\infty_c([0,\infty); C^\infty_{\sigma,\tan}(\overline{\Omega}))$.*

Conversely, assume that $u \in L^\infty_{\text{loc}}([0,\infty); L^2(\Omega)) \cap C^0([0,\infty); \mathscr{D}'(\Omega))$ is a weak solution which satisfies the identity in Definition 3.1, item (a), for all $\phi \in C^\infty_c([0,\infty); C^\infty_{\sigma,\tan}(\overline{\Omega}))$. Suppose, additionally, that $\omega = \operatorname{curl} u \in L^\infty_{\text{loc}}([0,\infty); \mathcal{BM}(\Omega))$. Let γ_j denote the circulation of u around Γ_j. Then $(\omega, \gamma_1, \dots, \gamma_k)$ is a boundary-coupled weak solution of the 2D incompressible Euler equations.

This is the analogue, for boundary-coupled weak solutions, of Theorem 3.4. The proof, which we omit, is essentially identical to that of Theorem 3.4. Indeed, the three main ingredients of that proof go through in this case. One main ingredient is that $\nabla^\perp \overline{\mathscr{Y}}^\infty = C^\infty_{\sigma,\tan}(\overline{\Omega})$ as sets. The second ingredient is Lemma 2.5. And the third main ingredient is the identity proved in Proposition 3.6, already stated for test functions in $\overline{\mathscr{Y}}^\infty$.

We henceforth refer to u as being a boundary-coupled weak solution (of the velocity formulation of the incompressible 2D Euler equations) if the set of test vector fields allowed in identity (3.3) includes all $\phi \in C^\infty_c([0,\infty); C^\infty_{\sigma,\tan}(\overline{\Omega}))$.

It should be noted that boundary coupled weak solutions for the velocity formulation have appeared in different contexts. A version of the velocity formulation of boundary coupled weak solutions was used in [33] in order to deal with the motion of a non-stationary rigid body under the net force of a vortex sheet flow. Moreover, in [36], see page 20, it was observed that boundary coupled weak solutions which satisfy in addition the following energy inequality

$$\int_\Omega |u(t,x)|^2 \, dx \leqslant \int_\Omega |u_0(x)|^2 \, dx \quad \text{a.e. } t \geqslant 0$$

are unique in the class of strong solutions (they satisfy the weak-strong uniqueness property).

Now, given a boundary-coupled weak solution u it can be readily verified that the functional we defined above, $F_u(\Phi)$, vanishes when $\Phi \in C^\infty_{\sigma,\tan}(\overline{\Omega})$. Indeed, taking $\eta = \eta(t) \in C^\infty_c([0,\infty))$ and setting $\phi = \eta\Phi$, with $\Phi \in C^\infty_{\sigma,\tan}(\overline{\Omega})$, in the integral identity in Definition 3.1, (a), we get

$$\int \eta'(t) \int_\Omega u \cdot \Phi \, dxdt + \int \eta(t) \int_\Omega [(u \cdot \nabla)\Phi] \cdot u \, dxdt + \int_\Omega \eta(0)\Phi \cdot u_0 \, dx = 0.$$

Now,

$$\int \eta'(t) \int_\Omega u \cdot \Phi \, dxdt + \int_\Omega \eta(0)\Phi \cdot u_0 \, dx = -\int \eta \partial_t \int_\Omega u \cdot \Phi \, dxdt,$$

where the right-hand-side derivative is interpreted in the sense of distributions. It follows that $F_u(\Phi) = 0$ in $W^{-1,\infty}((0,\infty))$. Conversely, if we assume that $F_u(\Phi) = 0$ in $W^{-1,\infty}((0,\infty))$ we can reverse the argument above and prove that relation (3.3) holds true for all test functions of the form $\phi = \eta\Phi$. By density of linear combinations of such test functions, we infer that (3.3) holds true for all test functions $\phi \in C^\infty_c([0,\infty); C^\infty_{\sigma,\tan}(\overline{\Omega}))$.

We just proved that u is a boundary coupled weak solution if and only if $F_u(\Phi) = 0$ in $W^{-1,\infty}((0,\infty))$ for all $\Phi \in C^\infty_{\sigma,\tan}(\overline{\Omega})$. In view of Definition 5.3 we conclude that the net force is well-defined if and only if the solution is boundary coupled. More precisely, in view of Theorems 4.4 and 5.4, we have the following result.

Theorem 5.5. *Let $\omega_0 \in \mathcal{BM}(\Omega) \cap H^{-1}(\Omega)$ and fix real numbers $\gamma_i(0)$, $i = 1, \dots, k$. Let $(\omega, \gamma_1, \dots, \gamma_k)$ be a solution of the weak vorticity formulation with initial data $(\omega_0, \gamma_1(0), \dots, \gamma_k(0))$. Let u be given by Proposition 2.9 in terms of ω and γ_i. Then the the net force exerted by the fluid through $\partial\Omega$ is well-defined if and only if $(\omega, \gamma_1, \dots, \gamma_k)$ is a boundary-coupled weak solution.*

If $(\omega, \gamma_1, \dots, \gamma_k)$ is a solution constructed as in Theorem 3.3 for which $\gamma_i(t) = \gamma_i(0)$, $i = 0, \dots, k$, $t > 0$, then the the net force exerted by the fluid through $\partial\Omega$ is well-defined. In particular, when $\omega_0 \in L^1(\Omega) \cap H^{-1}(\Omega)$ there exists a weak solution with well-defined net forces.

Let us observe that we can define in the same manner the *net force which the i-th hole exerts on the fluid.* It suffices to use the same formula (5.1) but with Φ^1 and Φ^2 vanishing on $\partial\Omega \setminus \Gamma_i$ and $\Phi^j \cdot \hat{n} = \hat{n}^j$ on Γ_i. For instance, we can take $\Phi^1 = \nabla^\perp(-x_2\chi_i(x))$ and $\Phi^2 = \nabla^\perp(x_1\chi_i(x))$, where χ_i is a smooth cut-off function of a neighborhood of Γ_i: $\chi_i \equiv 1$ in the neighborhood of Γ_i and χ_i vanishes in a neighborhood of all the remaining Γ_j, for $j \neq i$.

We conclude this section by observing that, as with the net forces on Γ_i, we can discuss the torque $\tau_{\mathbf{i}}$ which the fluid exerts on each Γ_i. In the smooth setting this corresponds to

$$\int_{\Gamma_i} p(x - \overline{x}_i)^\perp \cdot \hat{n}\, dS,$$

where

$$\overline{x}_i = \frac{\int_{\Omega_i} x\, dx}{\int_{\Omega_i} dx}$$

is the centroid of the i-th hole Ω_i.

For each $i = 1, \ldots, k$ set

$$\Psi_i = \Psi_i(x) = \nabla^\perp\left(\frac{1}{2}|x - \overline{x}_i|^2\chi_i(x)\right)$$

where χ_i is the same as before.

At the level of regularity of vortex sheets the torque $\tau_{\mathbf{i}}$ across each boundary component Γ_i corresponds to $-F_u(\Psi_i)$. The conclusions for the torques, analogous to those in Theorem 5.5 for the net forces, hold true.

6. Final remarks and conclusion

The main point of the present article was to develop the vortex dynamics formulation of the incompressible 2D Euler equations in domains with boundaries, with vortex sheet initial data. We chose to formulate our results for flow in a smooth, connected, bounded domain, as it is the simplest context in which relevant issues arise. However, it is possible to extend the analysis carried out in the present work to flows in a smooth domain, *exterior* to a finite number of holes. This introduces a few technical complications, see [19].

We remark that, recently, there has been considerable progress on the theme of incompressible inviscid planar fluid flow in non-simply connected domains. In particular, the papers [15, 16, 20] concern weak solutions; we note, however, that in all these works the vorticity belongs to L^p, with, at best, $p > 1$.

All results in this article have been about the interaction of 2D vortex sheet flows with *compact* boundary components. It is natural to investigate the relation between boundary circulation and vorticity in the presence of non-compact boundaries. The simplest such situation is flow in the half-plane. In [27], the authors introduced a weak vorticity formulation for vortex sheet flows in the half-plane $\mathbb{H}$ and, also, the notion of boundary-coupled weak solution. In that same paper, existence of a boundary-coupled weak solution was established assuming the initial vorticity was a nonnegative measure in $H^{-1}(\mathbb{H})$. The corresponding existence result remains open for flows in domains with compact boundaries, basically due to not being able to exclude concentration of vorticity at each boundary component. In the present work, we proved that weak solutions obtained by mollifying initial data, for which the circulation along boundary components is conserved, are, in fact, boundary-coupled. For half-plane flows the circulation along the boundary is naturally defined as the integral of vorticity in the bulk of the fluid. Conservation of circulation in half-plane flows is, therefore, equivalent to conservation of mass of vorticity. We do not know whether mass of vorticity is conserved for vortex sheet flows in the half-plane, but we do not expect this to be the case. This raises the question of whether the converse to Theorem 4.4 holds in general, i.e., if there exist boundary-coupled weak solutions which do not conserve circulation along boundary components.

The physically relevant solutions of the incompressible Euler equations are those obtained from the vanishing viscosity limit of solutions of the Navier-Stokes system. It is an important open problem whether solutions of the Navier-Stokes equations with a fixed initial condition, on a domain with boundary, satisfying the no-slip boundary conditions, converge to a weak solution of the incompressible Euler equations in the vanishing viscosity limit. See [4] for the state of the art concerning this problem, and its connection with turbulence modelling. Weak solutions which are limits of vanishing viscosity, if such solutions exist, are expected to exchange vorticity with the boundary, see the discussion in [4], and have, at best, vortex sheet regularity, see [26]. This means that the weak vorticity formulation in bounded domains, as developed here, provides an appropriate context to seek vanishing viscosity limits. Weak solutions obtained by mollifying initial data and solutions obtained as vanishing viscosity limits ought to behave differently with respect to their interaction with boundaries. To illustrate this, we observe that, according to the proof of Theorem 4.4, weak solutions must satisfy identity (4.8). In contrast, if we consider the approximate solution sequence $\{\omega^\nu\}$ given by (9.1)-(9.4) in [26], the limit velocity is time-independent, given by u_0, but, by (9.49) and (9.54) in the same reference, $\overline{\omega}(\{|x| = 1\}) = \alpha(t-)$, which, if it does not vanish, implies that (4.8) is not satisfied in this limit. For initial vorticity in L^1, we have obtained weak solutions which conserve circulation around boundary components and for the half-plane and similar domains, existence of a weak solution in [27, 28] follows from the construction of an approximate solution sequence which does not concentrate vorticity at the boundary. In both cases, we obtain boundary-coupled weak solutions. Vanishing viscosity limits, on the other hand might not, maybe should not be boundary-coupled, which, in light of the discussion in Section 5, makes the discussion of solid-fluid interaction rather delicate.

The system formed by the incompressible 2D Euler equations in the exterior of a compact rigid body together with the equations for the motion of the rigid body under the fluid force was studied in [17, 33]. The existence of a weak solution for the coupled system was proved in two cases: initial vorticity in L^p, $p > 1$ and symmetric body with symmetric vortex sheet data with a sign condition. These are two situations where boundary coupled weak solutions are known to exist, by the present work in the first case and by [28] in the second. Our work suggests two natural extensions of the results in [17, 33]: to the limit case $p = 1$ and to motion with more bodies. The case of motion of a rigid body coupled with a general Delort solution is physically very interesting, but our analysis in Section 5 highlights the difficulty in defining the coupling, and makes this case a more challenging open problem.

Another natural avenue for investigation related to the present work is to adapt those results, proved in this article for approximate solution sequences obtained by mollifying initial data, to other approximation schemes, such as numerical approximations, approximation by Euler-α solutions, and vanishing viscosity on Navier-Stokes solutions with Lions' free boundary or other, more general, Navier boundary conditions.

Appendix: Uniform estimates of the Green's function and of the Biot-Savart kernel

In this appendix we establish uniform estimates for the Green's function and for the Biot-Savart kernel of a general, smooth, bounded domain in the plane. Such estimates can be found in classical textbooks, see, for instance, Theorem 4.17 in [1], but the constants depend on the distance to the boundary. Here we show they are uniformly bounded in the whole domain.

Proposition 6.1. *Let $U \subset \mathbb{R}^2$ be a smooth, bounded domain. Let $G_U = G_U(x, y)$ be the Green's function for the Dirichlet Laplacian in U and set $K_U = K_U(x, y) = \nabla_x^\perp G_U(x, y)$. Then there exists $M = M(U) > 0$ such that*

$$|G_U(x, y)| \leqslant M(1 + |\log |x - y||), \tag{6.1}$$

and

$$|K_U(x, y)| \leqslant \frac{M}{|x - y|}, \tag{6.2}$$

for all $(x, y) \in U \times U$, $x \neq y$.

Remark 6.2. *Estimate* (6.2) *is due to L. Lichtenstein, see* [21]. *For convenience sake, we include a proof.*

Proof. The proof proceeds in several steps.

First we establish the result when the domain U is the unit disk $D = B(0; 1)$. We have, for the disk,

$$G_D(x, y) = \frac{1}{2\pi} \log \frac{|x - y|}{|x - y^*||y|},$$

where $y^* = \dfrac{y}{|y|^2}$. Now, it is easy to verify that

$$\lim_{y \to 0} |x - y^*||y| = 1,$$

and this limit is uniform with respect to $x \in D$. Hence, there exists $0 < r_0 \leqslant 1$ such that $1/2 < |x - y^*||y| < 2$ for all $y \in D$ such that $|y| < r_0$ and all $x \in D$. Therefore, if $(x, y) \in D \times D$, $|y| < r_0$, then (6.1) follows immediately. If, on the other hand, $(x, y) \in D \times D$ and $|y| \geqslant r_0$ then $|x - y^*| \leqslant 1 + 1/r_0$, so that

$$\left| \log \frac{|x - y|}{|x - y^*||y|} \right| \leqslant \log \frac{1 + 1/r_0}{|x - y|} - \log r_0 \leqslant M(1 + |\log |x - y||).$$

This establishes (6.1) if U is the unit disk.

Next we analyze K_D: we have

$$K_D(x, y) = \frac{1}{2\pi} \frac{(x - y)^\perp}{|x - y|^2} - \frac{1}{2\pi} \frac{(x - y^*)^\perp}{|x - y^*|^2}.$$

Together with the fact that $|x - y| \leqslant |x - y^*|$, this trivially yields (6.2).

The second step consists of extending these estimates to a general smooth, bounded, simply connected domain D_0. This can be done through a conformal map $T : D_0 \to D$. The existence of the biholomorphism T follows from the Riemann mapping theorem; that it can be extended smoothly up to the boundary was established in [5]. Furthermore, since both T and T^{-1} are smooth diffeomorphisms and $\overline{D_0}$ is a compact set with smooth boundary, it follows that T and T^{-1} are globally Lipschitz and, therefore, there exist $C_1 > 0$ and $C_2 > 0$ such that

$$C_1|x - y| \leqslant |T(x) - T(y)| \leqslant C_2|x - y|,$$

for all $(x,y) \in D_0 \times D_0$. With this map in place we note that

$$G_{D_0}(x,y) = G_D(T(x),T(y)),$$

whence we obtain (6.1). Also,

$$K_{D_0}(x,y) = K_D(T(x),T(y))DT(x).$$

This yields, analogously, (6.2), since DT is uniformly bounded, which is a consequence of T extending smoothly up to the boundary.

The third step consists of examining the case of a(n unbounded) domain which is exterior to a single, simply connected, bounded, smooth domain. We begin with the domain exterior to the unit disk, $\Pi = \mathbb{R}^2 \setminus D$. In this case the Green's function is given by

$$G_\Pi(x,y) = G_D(x^*,y^*),$$

with $x^* = x/|x|^2$ and $y^* = y/|y|^2$, as before. A straightforward calculation yields

$$|x^* - y^*| = \frac{|x-y|}{|x||y|}$$

so that, since $|x|$, $|y| \geqslant 1$, it follows from (6.1) for the unit disk that, for any $R > 0$,

$$|G_\Pi(x,y)| \leqslant M(1 + |\log|x-y||),$$

for all x, $y \in \Pi$, $|x|$, $|y| \leqslant R$, which is a local version of (6.1). Furthermore, the Biot-Savart kernel K_Π can be computed from G_Π and we find

$$|K_\Pi(x,y)| \leqslant \frac{C}{|x|^2}\frac{1}{|x^*-y^*|} = \frac{C|y|}{|x||x-y|}.$$

Hence we can also deduce a version of (6.2), valid in bounded subsets of Π.

Let, now, Π_0 denote a general unbounded exterior domain, which is exterior to a single, simply connected, bounded, smooth obstacle. Using a biholomorphism $T : \Pi_0 \to \mathbb{R}^2 \setminus \overline{D}$, with smooth extension to the boundary, as was done in [18], we obtain local versions of (6.1) and (6.2) for this kind of domain from the estimates in the domain exterior to the unit disk.

The fourth step consists of analyzing the case of a smooth, bounded, connected domain D_1 in the plane with one obstacle, i.e., let D_0 be a bounded, simply connected smooth domain and let $\mathcal{V}_1 \subset D_0$ be another simply connected smooth domain. Set $D_1 = D_0 \setminus \mathcal{V}_1$ and denote $\Gamma_1 \equiv \partial\mathcal{V}_1$ and $\Gamma_0 \equiv \partial D_0$, so that $\partial D_1 = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$. Let us begin by establishing (6.1) in D_1. For each $y \in D_1$ fixed, consider the function $\psi = \psi(x) = G_{D_1}(x,y) - G_{D_0}(x,y)$. Recall that, by the maximum principle, both G_{D_1} and G_{D_0} are nonpositive everywhere. We have:

$$\begin{cases} \Delta_x \psi = 0, x \in D_1, \\ \psi = 0, \quad x \in \Gamma_0, \\ \psi = -G_{D_0} \geqslant 0, \quad x \in \Gamma_1. \end{cases}$$

Therefore, by the maximum principle we find $0 \leqslant -G_{D_1}(x,y) \leqslant -G_{D_0}(x,y)$, for all $(x,y) \in D_1 \times D_1$. Together with the estimate for G_{D_0}, this yields the desired result for G_{D_1}.

We cannot use the maximum principle in such a simple manner for the components $K^j_{D_1}$, $j = 1, 2$, because we have no boundary information for either of K_{D_0} or K_{D_1} on any of the two boundaries. So, to establish (6.2) for K_{D_1} we split the problem in three cases: (i) y in the interior of D_1, far from either boundary, (ii) y near the outer boundary Γ_0 and (iii) y near the inner boundary Γ_1. In the first case, (i), we obtain (6.2) from Theorem 4.17 of [1], where this estimate is deduced with a constant depending on the distance to the boundary. Next we assume that y is close to the outer boundary Γ_0, as in (ii). Let $r > 0$ and suppose that $y \in B(z_0; r/2) \cap D_1$ for some $z_0 \in \Gamma_0$. Assume that r is sufficiently small so that $\partial B(z_0; 2r) \cap D_1$ is a connected set. Let $\phi = \phi(x)$ be a smooth cut-off function for $B(z_0; r) \cap D_1$, i.e., $\phi \geqslant 0$, $\phi \equiv 1$ inside $B(z_0; r) \cap D_1$, $\phi \equiv 0$ outside $B(z_0; 2r) \cap D_1$, $\phi \in C^\infty(D_1)$. Set $\varphi = \phi(x)G_{D_1}(x,y)$ and extend φ to D_0 by setting it to vanish inside the domain bounded by Γ_1. We then have:

$$\begin{aligned} \Delta_x \varphi &= \phi(x)\Delta G_{D_1}(x,y) + 2\nabla\phi(x)\nabla G_{D_1}(x,y) + \Delta\phi(x)G_{D_1}(x,y) \\ &= \phi(y)\delta_y(x) + 2\nabla\phi(x)\nabla G_{D_1}(x,y) + \Delta\phi(x)G_{D_1}(x,y). \end{aligned}$$

Observe that $\phi(y) = 1$ and notice that the last two terms above are supported in $B(z_0; 2r) \setminus B(z_0; r)$ and, hence, are smooth functions. Let $\psi = \varphi(x) - G_{D_0}(x,y)$. We have:

$$\begin{cases} \Delta_x \psi = 2\nabla\phi(x)\nabla G_{D_1}(x,y) + \Delta\phi(x)G_{D_1}(x,y) \equiv f_1, \quad x \in D_0, \\ \psi = 0, \quad x \in \Gamma_0. \end{cases}$$

We now use the representation formula for ψ in terms of f_1 to show that ψ and its derivatives are bounded, uniformly with respect to x. We have:

$$\psi(x) = \int_{D_0} f_1(z) G_{D_0}(x,z)\, dz,$$

so that

$$\nabla_x^\perp \psi(x) = \int_{D_0} f_1(z) K_{D_0}(x,z)\, dz.$$

Now, f_1 is a bounded function and K_{D_0} satisfies (6.2), so it is easy to show that $\nabla_x^\perp \psi$ is bounded in D_0. Since $\psi = \phi(x) G_{D_1}(x,y) - G_{D_0}(x,y)$ and $\phi \equiv 1$ in $B(z_0;r) \cap D_1$, we can compute K_{D_1} in terms of $\nabla_x^\perp \psi$ for x near the boundary Γ_0:

$$K_{D_1}(x,y) = \nabla_x^\perp \psi(x) + K_{D_0}(x,y), \quad \text{for } x \in B(z_0;r) \cap D_1.$$

If $y \in B(z_0;r/2) \cap D_1$ and $x \in D_1$, $x \notin B(z_0;r)$ then $K_{D_1}(x,y)$ is trivially bounded, since $G_{D_1}(x,y)$ is smooth away from the diagonal $\{x=y\}$. Hence, we conclude that $|K_{D_1}(x,y)| \leqslant C(1 + 1/|x-y|) \leqslant C/|x-y|$ for all $x \in D_1$ and $y \in B(z_0;r/2) \cap D_1$, $z_0 \in \Gamma_0$. The analysis of case (iii) proceeds in a similar fashion, except that we must use the exterior domain $\Pi_0 \equiv \mathbb{R}^2 \setminus \mathcal{V}_1$ in place of D_0 everywhere in the argument above. We note that, despite the fact that estimates (6.1) and (6.2) were only shown to hold in bounded subsets of an exterior domain, this is enough to estimate K_{D_1} for $x \in D_1$ and y near the inner boundary Γ_1, as the new auxiliary function f_1 will be compactly supported and as $x \in D_1$, y near Γ_1, remain bounded. To conclude the proof of (6.2) in D_1 we argue by compactness of the boundaries Γ_0 and Γ_1 to obtain a constant $M = M(D_1) > 0$ which is uniform near the boundary, i.e., $y \in B(z_0;r/2) \cap D_1$ and $x \in B(z_0;r) \cap D_1$. This, together with Theorem 4.17 in [1], yields the desired estimate in D_1.

The fifth and final step in the proof is to analyze the case of a general bounded, smooth, domain with N holes. The proof proceeds by induction with respect to the number of holes by repeating the argument presented in the fourth step, when adding one hole at a time.

This concludes the proof. □

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