Enright functors for Kac-Moody superalgebras

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March 14, 2012

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1 Introduction

J. T. Enright introduced a certain completion of modules over a complex semi-simple Lie algebra in order to study fundamental series representations of real semi-simple Lie groups in [E]. Later, the idea of the completion was simplified and extended by several authors (e.g. [D], [J1], [KT], [M], [KM]). The functor taking the completion is called the Enright functor. A. Rocha-Caridi and N. R. Wallach applied the functor to the representation theory of Kac-Moody algebras and show the uniqueness of homomorphisms between Verma modules with integral support. More recently, in [KT], M. Kashiwara and T. Tanisaki generalized the Enright functor to treat modules with non-integral support. Such generalized Enright functor was also considered by O. Mathieu in [M] for different purposes.

In this paper, we first introduce a super analogue of the generalized Enright functor associated with a non-isotropic simple root of the Lie superalgebra defined by a Cartan matrix. Second, as an application, we show the uniqueness of homomorphisms between Verma modules over a Kac-Moody superalgebra (in the sense of [S]) without isotropic simple roots. Finally, we discuss explicit forms of singular vectors of Verma modules. For a Kac-Moody algebra, F. G. Malikov, B. L. Feigin and D. B. Fuks provided a singular vector formula by means of complex powers of Chevalley generators. A relation between their formula and the Enright functor was suggested by A. Joseph in [J2]. Here, we give an interpretation of complex powers of generators in Malikov-Feigin-Fuks type singular vector formulas in terms of the Enright functor.

This paper is organized as follows: In Section 2, we recall the definition of the Lie superalgebra defined by a Cartan matrix, and introduce some categories of modules. In Sections 3 and 4, we define the Enright functor and state some properties. In particular, we describe the images of Verma modules under the Enright functor. In Section 5, we show the uniqueness of homomorphism between Verma modules over a Kac-Moody superalgebra without isotropic simple roots. In Section 6, we provide the singular vector formula and state the braid relations of our Enright functors.

Throughout this paper, unless otherwise indicated, the base field of vector spaces is the complex number field $\mathbb{C}$.

2 Basic definitions

In this section, we recall the definition of the Lie superalgebra defined by a Cartan matrix, and introduce some categories of modules.

2.1 Lie superalgebra defined by Cartan matrix

Throughout this paper, for $n \in \mathbb{Z}$, we denote $n + 2\mathbb{Z} \in \mathbb{Z}/2\mathbb{Z}$ by $\bar{n}$. For a superspace (i.e., $\mathbb{Z}/2\mathbb{Z}$-graded vector space) V, we denote its even (resp. odd) part by $V_{\bar{0}}$ (resp. $V_{\bar{1}}$). For $\sigma \in \mathbb{Z}/2\mathbb{Z}$ and $v \in V_\sigma$, we set $|v| := \sigma$.

Let $I = \{1, 2, \cdots, n\}$, and let I_σ ($\sigma \in \mathbb{Z}/2\mathbb{Z}$) be subsets of I such that $I = I_{\bar{0}} \sqcup I_{\bar{1}}$ (disjoint union). We define $p : I \to \{0, 1\}$ by

$$p(i) := \begin{cases} 0 & (i \in I_{\bar{0}}) \\ 1 & (i \in I_{\bar{1}}) \end{cases}. \tag{1}$$

Let $A = (a_{i,j})_{i,j \in I}$ be a complex $n \times n$-matrix whose rank is l. Let $\mathfrak{h}$ be a $2n - l$ dimensional vector space, and let $\{\alpha_i\}_{i \in I}$ and $\{h_i\}_{i \in I}$ be subsets of $\mathfrak{h}^*$ and $\mathfrak{h}$, which satisfy (i) $\{\alpha_i\}_{i \in I}$ and $\{h_i\}_{i \in I}$ are linearly independent, (ii) $\langle h_i, \alpha_j \rangle = a_{i,j}$ for any $i, j \in I$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$. Put $Q := \sum_{i \in I} \mathbb{Z}\alpha_i$ and $\Pi := \{\alpha_i | i \in I\}$.

Let us recall the definition of the Lie superalgebra $\mathfrak{g}(A)$ associated with the Cartan matrix A. Let $\tilde{\mathfrak{g}}(A)$ be the Lie superalgebra generated by $\{e_i, f_i, h | i \in I,\ h \in \mathfrak{h}\}$ with the following commutation relations:

$$\begin{aligned} &|h| = \bar{0}\ (h \in \mathfrak{h}), \quad |e_i| = |f_i| = \sigma\ (i \in I_\sigma) \\ &[h, h'] = 0\ (h, h' \in \mathfrak{h}), \quad [e_i, f_j] = \delta_{i,j} h_i, \\ &[h, e_i] = \langle h, \alpha_i \rangle e_i, \quad [h, f_i] = -\langle h, \alpha_i \rangle f_i. \end{aligned} \tag{2}$$

Let $\mathfrak{r}$ be the maximal ideal of $\tilde{\mathfrak{g}}(A)$ which intersects $\mathfrak{h}$ trivially. We define the Lie superalgebra $\mathfrak{g}(A)$ by $\mathfrak{g}(A) := \tilde{\mathfrak{g}}(A)/\mathfrak{r}$. In the following, we often denote $\mathfrak{g}(A)$ by $\mathfrak{g}$ for simplicity. The Lie superalgebra $\mathfrak{g}$ naturally equips with Q-gradation $\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}^\alpha$ satisfying $\mathfrak{g}^0 = \mathfrak{h}$, $\mathfrak{g}^{\alpha_i} = \mathbb{C}e_i$ and $\mathfrak{g}^{-\alpha_i} = \mathbb{C}f_i$.

Since $\mathfrak{g}(A) \simeq \mathfrak{g}(DA)$ for any invertible diagonal matrix D, we may assume without loss of generality that

$$a_{k,k} \in \{0, 2\} \tag{3}$$

holds for any $k \in I$. Till the end of Section 4, the Cartan matrix A is not necessarily symmetrizable unless otherwise indicated.

Next, let us introduce some notation. Set $Q^{\pm} := \pm \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$. Let $\Delta := \{\alpha \in Q \setminus \{0\} | \mathfrak{g}^{\alpha} \neq \{0\}\}$ be the set of the root of $\mathfrak{g}$ and $\Delta^{\pm} := \Delta \cap Q^{\pm}$ the sets of the positive and the negative roots. For $\sigma \in \mathbb{Z}/2\mathbb{Z}$, we put $\Pi_{\sigma} := \{\alpha_i | i \in I_{\sigma}\}$, $\Delta_{\sigma} := \{\alpha \in \Delta | \mathfrak{g}_{\sigma} \cap \mathfrak{g}^{\alpha} \neq \{0\}\}$ and $\Delta^{\pm}_{\sigma} := \Delta_{\sigma} \cap \Delta^{\pm}$. By setting $\mathfrak{g}^{\pm} := \bigoplus_{\alpha \in \Delta^{\pm}} \mathfrak{g}^{\alpha}$, we have the triangular decomposition $\mathfrak{g} = \mathfrak{g}^{+} \oplus \mathfrak{h} \oplus \mathfrak{g}^{-}$. Set $\mathfrak{g}^{\geq} := \mathfrak{g}^{+} \oplus \mathfrak{h}$. Let $\rho \in \mathfrak{h}^*$ be an element, which satisfies

$$\langle h_k, \rho \rangle = \frac{1}{2} a_{k,k} \quad (\forall k \in I). \tag{4}$$

We fix $i \in I$ such that $a_{i,i} \neq 0$ and introduce subalgebras $\mathfrak{u}(i)$, $\mathfrak{s}(i)$ and $\mathfrak{g}(i)$ of $\mathfrak{g}$ as follows:

$$\mathfrak{u}^{\pm}(i) := \bigoplus_{n \in \mathbb{Z}_{>0}} \mathfrak{g}^{\pm n\alpha_i}, \ \mathfrak{s}(i) := \mathbb{C}h_i \oplus \bigoplus_{n \in \mathbb{Z} \setminus \{0\}} \mathfrak{g}^{n\alpha_i}, \ \mathfrak{g}^{\pm}(i) := \bigoplus_{\alpha \in \Delta^{\pm} \setminus \mathbb{Z}\alpha_i} \mathfrak{g}^{\alpha}. \tag{5}$$

Then, we have $\mathfrak{g} = \mathfrak{g}^{+}(i) \oplus (\mathfrak{s}(i) + \mathfrak{h}) \oplus \mathfrak{g}^{-}(i)$, $\mathfrak{g}^{\pm} = \mathfrak{g}^{\pm}(i) \oplus \mathfrak{u}^{\pm}(i)$ and $\mathfrak{s}(i) \simeq \mathfrak{osp}(1|2)$ if $i \in I_{\bar{1}}$ (resp. $\mathfrak{s}(i) \simeq \mathfrak{sl}(2)$ if $i \in I_{\bar{0}}$). We further introduce $s_i \in \mathrm{GL}(\mathfrak{h}^*)$ as

$$s_i(\lambda) := \lambda - \langle h_i, \lambda \rangle \alpha_i \quad (\lambda \in \mathfrak{h}^*). \tag{6}$$

2.2 Categories

Throughout this subsection, we fix $i \in I$ such that $a_{i,i} \neq 0$. By assumption (3), we have $a_{i,i} = 2$. In order to introduce the Enright functor associated with the simple root α_i, we assume that

$$\forall j \in I \setminus \{i\}; \quad a_{i,j} \in 2^{p(i)}\mathbb{Z}_{\leq 0} \wedge (a_{i,j} = 0 \Rightarrow a_{j,i} = 0). \tag{7}$$

We regard $\mathfrak{g}$ as $\mathfrak{s}(i)$-module via the adjoint action. Under the above assumption, one can easily show the following lemma:

Lemma 2.1. *Both* $\mathrm{ad}e_i$ *and* $\mathrm{ad}f_i$ *are locally nilpotent on* $\mathfrak{g}$. *Hence,* $\mathfrak{g}$ *is an integrable* $\mathfrak{s}(i)$*-module.*

By abuse of notation, let us denote by ad the adjoint action of $\mathfrak{g}$ on the enveloping algebra $U(\mathfrak{g})$ which is defined as

$$\mathrm{ad}g(X) := gX - (-1)^{|X||g|}Xg \quad (g \in \mathfrak{g}, \ X \in U(\mathfrak{g})).$$

The above lemma implies

Corollary 2.1. *Both of* $\mathrm{ad}e_i$ *and* $\mathrm{ad}f_i$ *are locally nilpotent on* $U(\mathfrak{g})$.

Next, we introduce categories $\mathbf{Mod}^{\mathbb{Z}/2\mathbb{Z}}_{\mathfrak{g}}$, $\mathcal{M}^{a}_{i}$ and $\overline{\mathcal{M}}^{a}_{i}$ of left $\mathfrak{g}$-modules.

Definition 2.1. *1. A left* $\mathfrak{g}$*-module* M *is an object of* $\mathbf{Mod}^{\mathbb{Z}/2\mathbb{Z}}_{\mathfrak{g}}$ *if and only if* M *is* $\mathbb{Z}/2\mathbb{Z}$*-graded, i.e.,* $M = M_{\bar{0}} \oplus M_{\bar{1}}$ *and* $\mathfrak{g}_{\sigma}M_{\tau} \subset M_{\sigma+\tau}$ $(\sigma, \tau \in \mathbb{Z}/2\mathbb{Z})$.

2. For $M, N \in \mathrm{Ob}(\mathbf{Mod}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}})$, we set

$$\mathrm{Hom}_{\mathbf{Mod}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}}}(M, N) := \{f \in \mathrm{Hom}_{\mathfrak{g}}(M, N) | f(M_\sigma) \subset N_\sigma \ (\forall \sigma \in \mathbb{Z}/2\mathbb{Z})\}.$$

For simplicity, we denote $\mathrm{Hom}_{\mathbf{Mod}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}}}(M, N)$ by $\mathrm{Hom}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}}(M, N)$.

For the fixed $i \in I$ and $a \in \mathbb{C}$, we introduce full subcategories $\mathcal{M}_i^a$ and $\overline{\mathcal{M}}_i^a$ of $\mathbf{Mod}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}}$ as follows.

Definition 2.2. *Let $\mathcal{M}_i^a$ be the full subcategory of $\mathbf{Mod}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}}$ whose objects M satisfies the following conditions:*

C1 M is $\mathfrak{h}^* \times (\mathbb{Z}/2\mathbb{Z})$-graded, i.e.,

$$M = \bigoplus_{(\lambda, \tau) \in \mathfrak{h}^* \times (\mathbb{Z}/2\mathbb{Z})} M_\tau^\lambda,$$

where $M^\lambda := \{m \in M | h.m = \lambda(h)m \ (\forall h \in \mathfrak{h})\}$ and $M_\tau^\lambda := M^\lambda \cap M_\tau$;

C2 $\langle h_i, \lambda \rangle \notin a + 2^{p(i)}\mathbb{Z} \implies M_\tau^\lambda = \{0\}$;

C3 e_i acts locally nilpotently on M.

Let $\overline{\mathcal{M}}_i^a$ be the full subcategory of $\mathcal{M}_i^a$ consisting of the objects M on which f_i acts injectively.

Next, we recall the definition of Verma modules over $\mathfrak{g}$. For $(\lambda; \sigma) \in \mathfrak{h}^* \times (\mathbb{Z}/2\mathbb{Z})$, let $\mathbb{C}\mathbf{1}_\sigma^\lambda$ be the one-dimensional $\mathfrak{h}$-module defined by $|\mathbf{1}_\sigma^\lambda| := \sigma$ and $h.\mathbf{1}_\sigma^\lambda := \lambda(h)\mathbf{1}_\sigma^\lambda$ $(h \in \mathfrak{h})$. We regard $\mathbb{C}\mathbf{1}_\sigma^\lambda$ as $\mathfrak{g}^{\geq}$-module via $\mathfrak{g}^+|_{\mathbb{C}\mathbf{1}_\sigma^\lambda} = \{0\}$. The induced module

$$M(\lambda; \sigma) := U(\mathfrak{g}) \otimes_{\mathfrak{g}^{\geq}} \mathbb{C}\mathbf{1}_\sigma^\lambda \tag{8}$$

is called a Verma module. Notice that $M(\lambda; \sigma) \in \mathrm{Ob}(\mathcal{M}_i^a)$ for $a = \langle h_i, \lambda \rangle$.

Finally, we remark that if a satisfies

$$a \notin 2^{p(i)}\mathbb{Z} \tag{9}$$

then, Verma modules over $\mathfrak{s}(i)$ with the condition C2 in Definition 2.2 are irreducible. Hence, the action of f_i on M is injective for any $M \in \mathrm{Ob}(\mathcal{M}_i^a)$ and thus, $\overline{\mathcal{M}}_i^a = \mathcal{M}_i^a$.

3 Definition of Enright functor

In this section, we introduce the Enright functor associated with a simple root α_i satisfying $a_{i,i} \neq 0$ and (7).

3.1 Super analogue of binomial coefficients

We introduce an analogue of binomial coefficients, which is convenient to define the Enright functor. For $k \in \mathbb{Z}_{>0}$, we denote $\dfrac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} \in \mathbb{C}[\alpha]$ by $\dbinom{\alpha}{k}$, and set $\dbinom{\alpha}{0} := 1$.

For $k \in \mathbb{Z}_{\geq 0}$, we put

$$\begin{Bmatrix} \alpha \\ k \end{Bmatrix} := \begin{pmatrix} \frac{\alpha}{2} \\ [\frac{k}{2}] \end{pmatrix}, \tag{10}$$

where $[a]$ denotes the greatest integer not exceeding $a \in \mathbb{R}$. We further set

$$\begin{bmatrix} \alpha \\ k \end{bmatrix} := \begin{pmatrix} [\frac{\alpha}{2}] \\ [\frac{k}{2}] \end{pmatrix} \tag{11}$$

for $\alpha \in \mathbb{R}$. Notice that if α is an integer, then

$$\begin{bmatrix} \alpha \\ k \end{bmatrix} = \begin{Bmatrix} \alpha - \delta_{\bar{\alpha},\bar{1}} \\ k \end{Bmatrix}. \tag{12}$$

Here and after, $\delta_{\sigma,\tau}$ ($\sigma, \tau \in \mathbb{Z}/2\mathbb{Z}$) denotes the Kronecker delta. The next symbol is also useful to simplify formulas. For $n, k \in \mathbb{Z}$, $\sigma \in \mathbb{Z}/2\mathbb{Z}$, set

$$C^{\sigma}_{n,k} := \begin{cases} \delta_{\bar{k},\bar{0}} & (\bar{n} = \bar{0}) \\ (-1)^{(\bar{k}+\bar{1})\sigma} & (\bar{n} = \bar{1}) \end{cases}. \tag{13}$$

By definition,

$$C^{\sigma}_{m,k} = C^{\sigma}_{n,k} \ \text{ if } \ \bar{m} = \bar{n}. \tag{14}$$

For later use, we list some formulas related with these symbols.

Lemma 3.1. *1. For $m \in \mathbb{Z}$, $k \in \mathbb{Z}_{\geq 0}$ and $\sigma \in \mathbb{Z}/2\mathbb{Z}$,*

$$C^{\sigma}_{m,k-1} \begin{bmatrix} m \\ k-1 \end{bmatrix} + (-1)^{\sigma+\bar{k}} C^{\sigma}_{m,k} \begin{bmatrix} m \\ k \end{bmatrix} = C^{\sigma}_{m+1,k} \begin{bmatrix} m+1 \\ k \end{bmatrix}, \tag{15}$$

where we set $\begin{bmatrix} m \\ -1 \end{bmatrix} := 0$.

2. For $n \in \mathbb{Z}$, $k, l \in \mathbb{Z}_{\geq 0}$, $\sigma, \tau \in \mathbb{Z}/2\mathbb{Z}$ and $\alpha \in \mathbb{C}$,

$$\begin{aligned} &C^{\sigma}_{n,k} C^{\tau}_{n+k,l} \begin{Bmatrix} \alpha - n - \delta_{\bar{n},\bar{1}} \\ k \end{Bmatrix} \begin{Bmatrix} \alpha - n - k - \delta_{\bar{n}+\bar{k},\bar{1}} \\ l \end{Bmatrix} \\ &= C^{\sigma+\tau}_{n,k+l} C^{\sigma}_{k+l,k} \begin{Bmatrix} \alpha - n - \delta_{\bar{n},\bar{1}} \\ k+l \end{Bmatrix} \begin{bmatrix} k+l \\ k \end{bmatrix}. \end{aligned} \tag{16}$$

3. For $m \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}$ and $\sigma \in \mathbb{Z}/2\mathbb{Z}$,

$$\sum_{k=0}^{m} (-1)^{\left[\frac{1}{2}(k+\delta_{\sigma,\bar{0}})\right]} C^{\bar{k}}_{n,m} C^{\sigma}_{m,k} \begin{bmatrix} m \\ k \end{bmatrix} = \delta_{m,0}. \tag{17}$$

4. *For $m,\ n \in \mathbb{Z}$, $j \in \mathbb{Z}_{\geq 0}$, $\sigma \in \mathbb{Z}/2\mathbb{Z}$ and $\alpha,\ \beta \in \mathbb{C}$,*

$$\sum_{k=0}^{j} C^{\sigma}_{n,k} C^{\sigma+\bar{k}}_{m,j-k} \left\{ \begin{matrix} \alpha - \delta_{\bar{n},\bar{1}} \\ k \end{matrix} \right\} \left\{ \begin{matrix} \beta - \delta_{\bar{m},\bar{1}} \\ j-k \end{matrix} \right\} = C^{\sigma}_{m+n,j} \left\{ \begin{matrix} \alpha + \beta - \delta_{\bar{m}+\bar{n},\bar{1}} \\ j \end{matrix} \right\}. \tag{18}$$

Proof. All of these formulas can be shown directly by using well-known formulas for binomial coefficients. □

Using the above symbols, the action of $(\mathrm{ad} g)^n$ for $g \in \mathfrak{g}_{\bar{1}}$ can be described as follows:.

Lemma 3.2. *For $g \in \mathfrak{g}_{\bar{1}}$, $n \in \mathbb{Z}_{\geq 0}$ and $X, Y \in U(\mathfrak{g})$, we have*

$$(\mathrm{ad} g)^n (XY) = \sum_{k=0}^{n} C^{|X|}_{n,k} \begin{bmatrix} n \\ k \end{bmatrix} (\mathrm{ad} g)^k (X) (\mathrm{ad} g)^{n-k} (Y). \tag{19}$$

Proof. By using (15), one can show this lemma by induction on n. □

3.2 Definition of $U(\mathfrak{g}) f_i^{a:\mathbb{Z}}$

Let us define a $U(\mathfrak{g})$-bimodule $U(\mathfrak{g}) f_i^{a:\mathbb{Z}}$ for $i \in I$ satisfying $a_{i,i} \neq 0$ and (7). Remark that, by (3),

$$[e_i, f_i] = h_i, \quad [h_i, e_i] = 2e_i, \quad [h_i, f_i] = -2f_i,$$

and $\langle h_i, \rho \rangle = 1$ by (4).

For $a \in \mathbb{C}$ and $n \in \mathbb{Z}$, let $U(\mathfrak{g}) f_i^{a:n}$ be the $\mathbb{Z}/2\mathbb{Z}$-graded left free $U(\mathfrak{g})$-module of rank one generated from the symbol $f_i^{a:n}$, where the parity of the generator is given by

$$|f_i^{a:n}| := \begin{cases} \bar{0} & (i \in I_{\bar{0}} \vee n \in 2\mathbb{Z}) \\ \bar{1} & (i \in I_{\bar{1}} \wedge n \in 2\mathbb{Z} + 1) \end{cases}. \tag{20}$$

For $m < n$, let $\varphi_{n,m} \in \mathrm{Hom}^{\mathbb{Z}/2\mathbb{Z}}_{\mathfrak{g}}(U(\mathfrak{g}) f_i^{a:-m}, U(\mathfrak{g}) f_i^{a:-n})$ be the homomorphism of left $U(\mathfrak{g})$-modules given by $\varphi_{n,m}(X f_i^{a:-m}) = X f_i^{n-m} f_i^{a:-n}$, where $X \in U(\mathfrak{g})$. Then, $\{U(\mathfrak{g}) f_i^{a:-n}, \varphi_{n,m}\}$ is an inductive system. Let us introduce $U(\mathfrak{g}) f_i^{a:\mathbb{Z}}$ as the following inductive limit of left $U(\mathfrak{g})$-modules:

$$U(\mathfrak{g}) f_i^{a:\mathbb{Z}} := \varinjlim_{n \in \mathbb{Z}} U(\mathfrak{g}) f_i^{a:-n}.$$

Since $\varphi_{n,m}$ is injective, we have $U(\mathfrak{g}) f_i^{a:-n} \hookrightarrow U(\mathfrak{g}) f_i^{a:\mathbb{Z}}$, and thus, $U(\mathfrak{g}) f_i^{a:\mathbb{Z}} = \bigcup_{n \in \mathbb{Z}} U(\mathfrak{g}) f_i^{a:-n}$.

Next, we will introduce right $U(\mathfrak{g})$-module structure on $U(\mathfrak{g}) f_i^{a:\mathbb{Z}}$. Noticing Corollary 2.1, we define a right action of $U(\mathfrak{g})$ on $U(\mathfrak{g}) f_i^{a:\mathbb{Z}}$ as follows:

Definition 3.1. *Suppose that $X \in U(\mathfrak{g})$ is homogenous with respect to the $\mathbb{Z}/2\mathbb{Z}$-gradation. We define*

1. *Case $i \in I_{\bar 0}$:*

$$f_i^{a:n}.X := \sum_{k=0}^{\infty} \binom{a+n}{k} \{(\mathrm{ad} f_i)^k(X)\} f_i^{a:n-k}. \tag{21}$$

2. *Case $i \in I_{\bar 1}$:*

$$f_i^{a:n}.X := \sum_{k=0}^{\infty} C_{n,k}^{|X|} \left\{ \begin{matrix} a+n-\delta_{\bar n,\bar 1} \\ k \end{matrix} \right\} \{(\mathrm{ad} f_i)^k(X)\} f_i^{a:n-k}. \tag{22}$$

Indeed, to check that the above action defines $U(\mathfrak{g})$-bimodule structure on $U(\mathfrak{g})f_i^{a:\mathbb{Z}}$, it is enough to prove that $(f_i^{a:n}.X).Y = f_i^{a:n}.(XY)$ holds for any $X, Y \in U(\mathfrak{g})$. In the case where $i \in I_{\bar 1}$, this follows from (16) and (19). In the case where $i \in I_{\bar 0}$, the same argument as in [D] works. Remark that if $\mathfrak{g}$ is a Kac-Moody algebra, then $U(\mathfrak{g})f_i^{a:\mathbb{Z}}$ coincides with the $U(\mathfrak{g})$-bimodule $U(\mathfrak{g})f_i^{a+\mathbb{Z}}$ in [KT].

It is easy to see that, for any $a, b \in \mathbb{C}$ satisfying $a - b \in 2\mathbb{Z}$ if $i \in I_{\bar 1}$ (resp. $a - b \in \mathbb{Z}$ if $i \in I_{\bar 0}$), there exists an isomorphism of $U(\mathfrak{g})$-bimodule

$$U(\mathfrak{g})f_i^{a:\mathbb{Z}} \simeq U(\mathfrak{g})f_i^{b:\mathbb{Z}}. \tag{23}$$

The left and the right multiplications are related as follows:

Lemma 3.3. *Suppose that $n \in \mathbb{Z}_{\geq 0}$ and $X \in U(\mathfrak{g})$. Then, we have*

1. *Case $i \in I_{\bar 0}$:*

$$X.f_i^{a:-n} = \sum_{k=0}^{\infty} (-1)^k \binom{a-n}{k} f_i^{a:-n-k} \{(\mathrm{ad} f_i)^k(X)\}.$$

2. *Case $i \in I_{\bar 1}$:*

$$X.f_i^{a:-n} = \sum_{k=0}^{\infty} (-1)^{\left[\frac{1}{2}(k+\delta_{|X|,\bar 0})\right]} C_{n,k}^{|X|} \left\{ \begin{matrix} a-n-\delta_{\bar n,\bar 1} \\ k \end{matrix} \right\} f_i^{a:-n-k} \{(\mathrm{ad} f_i)^k(X)\}. \tag{24}$$

Proof. In the first case, we can check this formula in a way similar to the Lie algebra case. In the second case, the formula follows from (22), (16) and (17). □

This lemma implies

$$U(\mathfrak{g})f_i^{a:\mathbb{Z}} = \varinjlim_{n \in \mathbb{Z}} f_i^{a:-n} U(\mathfrak{g}) = \bigcup_{n \in \mathbb{Z}} f_i^{a:-n} U(\mathfrak{g}). \tag{25}$$

By using (19) and (25), the following isomorphism of $U(\mathfrak{g})$-bimodule can be easily shown:

$$U(\mathfrak{g})f_i^{a+b:\mathbb{Z}} \simeq U(\mathfrak{g})f_i^{a:\mathbb{Z}} \otimes_{\mathfrak{g}} U(\mathfrak{g})f_i^{b:\mathbb{Z}}. \tag{26}$$

For $M \in \mathrm{Ob}(\mathbf{Mod}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}})$, we naturally regard $U(\mathfrak{g})f_i^{a:\mathbb{Z}} \otimes_{\mathfrak{g}} M$ as left $\mathfrak{g}$-module. Then, $U(\mathfrak{g})f_i^{a:\mathbb{Z}} \otimes_{\mathfrak{g}} (\cdot)$ defines a functor on the category $\mathbf{Mod}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}}$. In a way similar to [KT], one can show the next proposition:

Proposition 3.1. *$U(\mathfrak{g})f_i^{a:\mathbb{Z}} \otimes_{\mathfrak{g}} (\cdot) : \mathbf{Mod}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}} \to \mathbf{Mod}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}}$ is exact.*

3.3 Definition of $T_i(a)$

Using the functor $U(\mathfrak{g})f_i^{a:\mathbb{Z}} \otimes_{\mathfrak{g}} (\cdot)$, we define the Enright functor $T_i(a)$.

We first state a general fact. Let $\mathfrak{a}$ be a Lie superalgebra, and let $\mathfrak{b}$ be a subalgebra of $\mathfrak{a}$. Suppose that the adjoint action of $\mathfrak{b}$ on $\mathfrak{a}$ is locally finite.

Lemma 3.4. *Suppose that V is a left $\mathfrak{a}$-module. Then, $\{v \in V \,|\, \dim U(\mathfrak{b})v < \infty\}$ is an $\mathfrak{a}$-submodule of V.*

By this lemma, we see that $V \mapsto \{v \in V \,|\, \dim U(\mathfrak{b})v < \infty\}$ defines a functor on $\mathbf{Mod}_{\mathfrak{a}}^{\mathbb{Z}/2\mathbb{Z}}$. We denote this functor by $F_{\mathfrak{b}}$. By definition,

Lemma 3.5. *The functor $F_{\mathfrak{b}}$ is a left exact functor.*

Let $\mathfrak{u}^+(i)$ be the subalgebra of $\mathfrak{g}$ given in (5). Recall that $\mathfrak{g}$ is $\mathfrak{u}^+(i)$-locally finite. We define the functor $T_i(a) : \mathbf{Mod}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}} \to \mathbf{Mod}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}}$ as follows:

$$T_i(a)(M) := F_{\mathfrak{u}^+(i)}(U(\mathfrak{g})f_i^{a:\mathbb{Z}} \otimes_{\mathfrak{g}} M) \quad (M \in \mathrm{Ob}(\mathbf{Mod}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}}))$$

Combining Proposition 3.1 with Lemma 3.5, we see that $T_i(a)$ is a left exact functor.

Since $h_i.(f_i^{a:n} \otimes m) = (\mu(h_i) - 2(a+n))f_i^{a:n} \otimes m$ for $m \in M^\mu$ $(\mu \in \mathfrak{h}^*)$, $T_i(a)$ defines a functor from $\mathcal{M}_i^a$ to $\mathcal{M}_i^{-a}$, which is denoted by the same symbol.

4 Properties of Enright functor

In this section, we study the properties of the functor $T_i(a)$ for $i \in I$ such that $a_{i,i} \neq 0$. In the case where $i \in I_{\bar{0}}$, the same arguments as in [KT] work. Hence, for simplicity, we concentrate on the case where $i \in I_{\bar{1}}$, unless otherwise is indicated.

4.1 Preliminaries

We collect some formulas which are necessary in the following subsections.

Lemma 4.1. *Suppose that $g \in \mathfrak{g}_{\bar{1}}$ and $X \in U(\mathfrak{g})$. For any $m \in \mathbb{Z}_{\geq 0}$,*

$$g^m X = \sum_{k=0}^{m} C_{m,m-k}^{|X|} \begin{bmatrix} m \\ k \end{bmatrix} \{(\mathrm{ad} g)^{m-k} X\} g^k.$$

Proof. Using (15), one can show the formula by induction on m. □

The next lemma is a super analogue of the formula (2.4.4) in [KT].

Lemma 4.2. *For any* $a \in \mathbb{C}$, $m \in \mathbb{Z}_{\geq 0}$ *and* $n \in \mathbb{Z}$,

$$\begin{aligned} e_i^m f_i^{a:n} =& \sum_{j=0}^{m} \tilde{C}_{m,n}^j f_i^{a:n-j} e_i^{m-j} 2^j [\frac{j}{2}]![\frac{j+1}{2}]! \\ & \times \left\{ \begin{matrix} m - \delta_{\bar{m},\bar{1}} \\ j+1-\delta_{\bar{m},\bar{1}} \end{matrix} \right\} \left\{ \begin{matrix} a+n-\delta_{\bar{n},\bar{1}} \\ j+1-\delta_{\bar{n},\bar{1}} \end{matrix} \right\} \left\{ \begin{matrix} h_i - a - n + m - \delta_{\bar{m}-\bar{n},\bar{1}} \\ j+1-\delta_{\bar{m}-\bar{n},\bar{1}} \end{matrix} \right\}, \end{aligned}$$

where we put

$$\tilde{C}_{m,n}^j := (-1)^{[\frac{1}{2}(j+1)]} \times \begin{cases} \delta_{\bar{j},\bar{0}} & (\bar{m} = \bar{n} = \bar{0}) \\ -1 & (\bar{m} = \bar{n} = \bar{1}) \\ 1 & (\bar{m} \neq \bar{n}) \end{cases} .$$

Proof. This lemma can be proved by induction on m. □

4.2 Image of Verma modules

Let $\mathfrak{g}^{\pm}(i)$, $\mathfrak{s}(i)$ and $\mathfrak{u}^{\pm}(i)$ be the subalgebras of $\mathfrak{g}$ defined in (5). Since $\mathfrak{g}^- = \mathfrak{g}^-(i) \oplus \mathfrak{u}^-(i)$, we have $U(\mathfrak{g}^-) \simeq U(\mathfrak{g}^-(i)) \otimes_{\mathbb{C}} \mathbb{C}[f_i]$ as vector space. Hence, by Definition 3.1, we have

Lemma 4.3. *For any* $a \in \mathbb{C}$, *there exists the following isomorphism of vector spaces:*

$$\begin{aligned} U(\mathfrak{g}^-(i)) \otimes_{\mathbb{C}} \mathbb{C}[f_i, f_i^{-1}] f_i^{a:0} \otimes_{\mathbb{C}} U(\mathfrak{g}^{\geq}) &\xrightarrow{\sim} U(\mathfrak{g}) f_i^{a:\mathbb{Z}}. \\ X^- \otimes P f_i^{a:0} \otimes X^+ &\longmapsto X^- P f_i^{a:0} X^+ \end{aligned} \tag{27}$$

Let $M(\lambda;\sigma)$ be the Verma module over $\mathfrak{g}$ defined in (8). The above lemma implies

$$U(\mathfrak{g}) f_i^{a:\mathbb{Z}} \otimes_{\mathfrak{g}} M(\lambda;\sigma) \simeq U(\mathfrak{g}^-(i)) \otimes_{\mathbb{C}} \mathbb{C}[f_i, f_i{}^{-1}] f_i^{a:0} \otimes_{\mathbb{C}} \mathbb{C}\mathbf{1}_{\sigma}^{\lambda}. \tag{28}$$

The next theorem is one of the main results of this paper.

Theorem 4.1 (cf. [KT] Proposition 2.4.6)**.** *Let* $(\lambda;\sigma) \in \mathfrak{h}^* \times (\mathbb{Z}/2\mathbb{Z})$, *and set* $a := \langle h_i, \lambda \rangle$. *Then, we have*

$$T_i(a)(M(\lambda;\sigma)) \simeq \begin{cases} M(s_i \circ \lambda; \sigma + \bar{1}) & (\ a \notin 2\mathbb{Z}_{\geq 0}\ ) \\ M(\lambda;\sigma) & (\ a \in 2\mathbb{Z}_{\geq 0}\ ) \end{cases},$$

where

$$s_i \circ \lambda := s_i(\lambda + \rho) - \rho. \tag{29}$$

Proof. We first consider the case where $a \notin 2\mathbb{Z}_{\geq 0}$. Lemma 4.2 implies

$$f_i^{a:1} \otimes \mathbf{1}_\sigma^\lambda \in \left\{ U(\mathfrak{g}) f_i^{a:\mathbb{Z}} \otimes_{\mathfrak{g}} M(\lambda;\sigma) \right\}^{\mathfrak{g}^+}. \tag{30}$$

Hence, by (28), we have

$$U(\mathfrak{g}).f_i^{a:1} \otimes \mathbf{1}_\sigma^\lambda \simeq M(s_i \circ \lambda; \sigma + \bar{1}),$$

and thus, it suffices to show that $T_i(a)(M(\lambda;\sigma))$ is generated by $f_i^{a:1} \otimes \mathbf{1}_\sigma^\lambda$. Since any vector $v \in U(\mathfrak{g}) f_i^{a:\mathbb{Z}} \otimes_{\mathfrak{g}} M(\lambda;\sigma)$ can be uniquely written as

$$v = \sum_{n \in \mathbb{Z}} P_n f_i^{a:n} \otimes \mathbf{1}_\sigma^\lambda \quad (P_n \in U(\mathfrak{g}^-(i))),$$

we may show that if $e_i^m.v = 0$ ($\exists m \in \mathbb{Z}_{>0}$) then $P_n = 0$ ($\forall n \in \mathbb{Z}_{\leq 0}$).

Suppose that $e_i^m.v = 0$. In a way similar to [KT], Lemma 4.1 and 4.2 imply

$$\sum_{k=0}^{m} C_{m,m-k}^{|P_{n+k}|} \tilde{C}_{k,n+k}^{k} 2^k [\frac{k}{2}]! [\frac{k+1}{2}]! \\ \times \begin{bmatrix} m \\ k \end{bmatrix} \left\{ \begin{matrix} a+n+k-\delta_{\bar{n}+\bar{k},\bar{1}} \\ k+1-\delta_{\bar{n}+\bar{k},\bar{1}} \end{matrix} \right\} \left\{ \begin{matrix} -n-\delta_{\bar{n},\bar{1}} \\ k+1-\delta_{\bar{n},\bar{1}} \end{matrix} \right\} \{(\mathrm{ad} e_i)^{m-k} P_{n+k}\} = 0.$$

Let us assume that $P_{-n} \neq 0$ for some $n \in \mathbb{Z}_{\geq 0}$ and lead to the contradiction. Suppose that c is the maximal non-negative integer such that $P_{-c} \neq 0$. Then, we have

$$C_{m,0}^{|P_{-c}|} \tilde{C}_{m,-c}^{m} 2^m [\frac{m}{2}]! [\frac{m+1}{2}]! \left\{ \begin{matrix} a-c-\delta_{\bar{c},\bar{1}} \\ m+1-\delta_{\bar{c},\bar{1}} \end{matrix} \right\} \left\{ \begin{matrix} c+m-\delta_{\bar{c}+\bar{m},\bar{1}} \\ m+1-\delta_{\bar{c}+\bar{m},\bar{1}} \end{matrix} \right\} = 0,$$

and thus,

$$\left\{ \begin{matrix} a-c-\delta_{\bar{c},\bar{1}} \\ m+1-\delta_{\bar{c},\bar{1}} \end{matrix} \right\} = 0.$$

Since $c + \delta_{\bar{c},\bar{1}} \in 2\mathbb{Z}_{\geq 0}$, this contradicts to $a \notin 2\mathbb{Z}_{\geq 0}$. Hence, $P_n = 0$ ($\forall n \in \mathbb{Z}_{\leq 0}$).

Next, let us suppose that $a \in 2\mathbb{Z}_{\geq 0}$. Since $T_i(a) = T_i(0)$ by (23), it suffices to show that $T_i(0)(M(\lambda;\sigma))$ is generated by $1 \otimes \mathbf{1}_\sigma^\lambda$. For $w \in U(\mathfrak{g}) f_i^{0:\mathbb{Z}} \otimes_{\mathfrak{g}} M(\lambda;\sigma)$ such that

$$w = \sum_{n \in \mathbb{Z}} Q_n f_i^{0:n} \otimes \mathbf{1}_\sigma^\lambda \quad (Q_n \in U(\mathfrak{g}^-(i))),$$

we show that if $e_i^m.w = 0$ ($\exists m > 0$) then $Q_n = 0$ ($\forall n \in \mathbb{Z}_{<0}$). Let d be the maximal positive integer such that $Q_{-d} \neq 0$. Then, by looking at the coefficient of $f_i^{0:-d-m} \otimes \mathbf{1}_\sigma^\lambda$ in $e_i^m.w$, we have

$$C_{m,0}^{|Q_{-d}|} \tilde{C}_{m,-d}^{m} 2^m [\frac{m}{2}]! [\frac{m+1}{2}]! \left\{ \begin{matrix} -d-\delta_{\bar{d},\bar{1}} \\ m+1-\delta_{\bar{d},\bar{1}} \end{matrix} \right\} \left\{ \begin{matrix} a+d+m-\delta_{\bar{d}+\bar{m},\bar{1}} \\ m+1-\delta_{\bar{d}+\bar{m},\bar{1}} \end{matrix} \right\} = 0.$$

But, this is impossible, since $d, a+d \in \mathbb{Z}_{>0}$. Hence, $Q_n = 0$ ($\forall n \in \mathbb{Z}_{<0}$).

□

One can similarly show

Theorem 4.2. *In the case where $i \in I_{\bar{0}}$, we have*

$$T_i(a)(M(\lambda;\sigma)) \simeq \begin{cases} M(s_i \circ \lambda;\sigma) & (\ a \notin \mathbb{Z}_{\geq 0}\ ) \\ M(\lambda;\sigma) & (\ a \in \mathbb{Z}_{\geq 0}\ ) \end{cases}.$$

Remark that for a Kac-Moody algebra $\mathfrak{g}$, the theorem was proved in [RW] for $a \in \mathbb{Z}$ and in [KT] for $a \notin \mathbb{Z}_{\geq 0}$.

4.3 Homomorphisms between Verma modules

We compare two spaces

$$\mathrm{Hom}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}}(M(\mu;\tau), M(\lambda;\sigma)), \quad \mathrm{Hom}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}}(T_i(a)(M(\mu;\tau)), T_i(a)(M(\lambda;\sigma)))$$

where $a := \langle h_i, \lambda\rangle$. Let us divide this subsection into two parts, Case $a \notin 2\mathbb{Z}$ and Case $a \in 2\mathbb{Z}$.

Case $a \notin 2\mathbb{Z}$ We show a preliminary lemma.

Lemma 4.4. *For any $a \in \mathbb{C}$, $M \in \mathrm{Ob}(\mathcal{M}_i^a)$ and $u \in M$, we have $f_i^{a:n} \otimes u \in T_i(a)(M)$ $(n \gg 0)$.*

Proof. It is enough to show that for any $u \in M$, there exist n, $m \in 2\mathbb{Z}_{>0}$ such that $e_i^m.(f_i^{a:n} \otimes u) = 0$. We may also assume that $h_i.u = au$. By Lemma 4.2, we have

$$e_i^m.(f_i^{a:n} \otimes u) = \sum_{k=0}^{\frac{m}{2}} 2^{2k}(k!)^2 \binom{\frac{m}{2}}{k}\binom{\frac{a+n}{2}}{k}\binom{\frac{m-n}{2}}{k} f_i^{a:n-2k} \otimes e_i^{m-2k}u.$$

Hence, by an argument similar to Lemma 2.4.3 in [KT], one can show this lemma. $\square$

Hence, there exists a morphism of functor

$$\mathrm{id}_{\mathcal{M}_i^a} \longrightarrow T_i(-a) \circ T_i(a). \tag{31}$$

By using (31) and Theorem 4.1, we obtain

Proposition 4.1. *Suppose that $\lambda \in \mathfrak{h}^*$ satisfies $\langle h_i, \lambda\rangle \notin 2\mathbb{Z}$. For any $\mu \in \mathfrak{h}^*$,*

$$\mathrm{Hom}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}}(M(\mu;\tau), M(\lambda;\sigma)) \simeq \mathrm{Hom}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}}(M(s_i \circ \mu;\tau'), M(s_i \circ \lambda;\sigma')), \tag{32}$$

where $\sigma' := \sigma + \bar{1}$ and $\tau' := \tau + \bar{1}$.

Case $a \in 2\mathbb{Z}$ In this case, (23) implies that $T_i(a) = T_i(0)$. Let $\overline{\mathcal{M}}_i^0$ be the category defined in Definition 2.2.

Lemma 4.5. *$T_i(0)(M) \in \mathrm{Ob}(\overline{\mathcal{M}}_i^0)$ for any $M \in \mathrm{Ob}(\overline{\mathcal{M}}_i^0)$.*

Proof. By definition, f_i acts injectively on $U(\mathfrak{g})f_i^{0:\mathbb{Z}} \otimes M$ if it does on M. Hence, the action of f_i on $T_i(0)(M)$ is injective, and thus, the lemma holds. $\square$

Hence, $T_i(0)$ defines a functor on $\overline{\mathcal{M}}_i^0$. Moreover, we have

Lemma 4.6. *For any $M \in \mathrm{Ob}(\overline{\mathcal{M}}_i^0)$, we have $M \hookrightarrow T_i(0)(M)$ $(m \mapsto 1 \otimes m)$.*

Proof. Since f_i acts injectively on $M \in \mathrm{Ob}(\overline{\mathcal{M}}_i^0)$, we have $M \hookrightarrow U(\mathfrak{g})f_i^{0:\mathbb{Z}} \otimes_{\mathfrak{g}} M$. Moreover, the action of e_i on M is locally finite, $M \hookrightarrow T_i(0)(M)$ holds. $\square$

Using this lemma, one can show the next proposition.

Proposition 4.2. *The functor $T_i(0) : \overline{\mathcal{M}}_i^0 \to: \overline{\mathcal{M}}_i^0$ is faithful.*

Proof. For $M, N \in \mathrm{Ob}(\overline{\mathcal{M}}_i^0)$ and $f \in \mathrm{Hom}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}}(M, N)$, we have $T_i(0)(f)|_M = f$, since $M \hookrightarrow T_i(0)(M)$. Hence,

$$T_i(0) : \mathrm{Hom}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}}(M, N) \to \mathrm{Hom}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}}(T_i(0)(M), T_i(0)(N))$$

is injective, and thus, the proposition holds. $\square$

Corollary 4.1. *Suppose that $\lambda, \mu \in \mathfrak{h}^*$ satisfy $\langle h_i, \lambda\rangle, \langle h_i, \mu\rangle \in 2\mathbb{Z}$. Then, the following inequality holds:*

$$\begin{aligned} &\dim \mathrm{Hom}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}}(M(\mu;\tau), M(\lambda;\sigma)) \\ &\leq \dim \mathrm{Hom}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}}\left(T_i(0)(M(\mu;\tau)), T_i(0)(M(\lambda;\sigma))\right). \end{aligned} \tag{33}$$

5 Verma modules over Kac-Moody superalgebra without isotropic simple roots

As an application of the Enright functors introduced in the previous sections, we show the uniqueness of homomorphisms between Verma modules over a symmetrizable Kac-Moody superalgebra without isotropic simple roots. For representations of such Lie superalgebras, the reader may consult [K2].

5.1 Definitions and main result

Throughout this section, let us always assume that $\mathfrak{g} = \mathfrak{g}(A)$ is a Kac-Moody superalgebra whose simple roots are non-isotropic. Hence, we assume that the Cartan matrix $A = (a_{i,j})_{i,j\in I}$ satisfies (i) $a_{i,j} \in \mathbb{Z}$ $(i, j \in I)$, (ii) $a_{i,i} = 2$, (iii) $a_{i,j} \leq 0$ $(i \neq j)$, (iv) if $a_{i,j} = 0$, then $a_{j,i} = 0$, (v) $a_{i,j} \in 2\mathbb{Z}$ $(i \in I_{\bar{1}})$. Here, we also assume that A is symmetrizable. Hence, in particular, we identify $\mathfrak{h}$ and $\mathfrak{h}^*$ so that the duality pairing induces a bilinear form on $\mathfrak{h}^*$ which will be denoted by $(\cdot, \cdot)$.

Next, let us introduce some notation for the root system. Set

$$\overline{\Delta}_{\bar{0}} := \{\alpha \in \Delta_{\bar{0}} | \frac{1}{2}\alpha \notin \Delta_{\bar{1}}\}, \quad \Delta'_{\bar{0}} := \{\beta \in \Delta_{\bar{0}} \setminus \overline{\Delta}_{\bar{0}} | \dim \mathfrak{g}^{\beta} \neq \dim \mathfrak{g}^{\frac{1}{2}\beta}\}, \tag{34}$$

$\overline{\Delta}_{\bar{0}}^{\pm} := \overline{\Delta}_{\bar{0}} \cap Q^{\pm}$ and $\Delta'^{\pm}_{\bar{0}} := \Delta'_{\bar{0}} \cap Q^{\pm}$.

Let $s_i \in \mathrm{GL}(\mathfrak{h}^*)$ be the simple reflection defined in (6) and $W := \langle s_i | i \in I \rangle \subset \mathrm{GL}(\mathfrak{h}^*)$ the Weyl group of $\mathfrak{g}$. An element in

$$\Delta_{\mathrm{re}} := W(\Pi_{\bar{0}} \sqcup \Pi_{\bar{1}} \sqcup 2\Pi_{\bar{1}}) \quad (\text{resp. } \Delta_{\mathrm{im}} := \Delta \setminus \Delta_{\mathrm{re}})$$

is called a real (resp. an imaginary) root. We set $\Delta^+_{\mathrm{re}} := \Delta_{\mathrm{re}} \cap Q^+$, $\Delta^+_{\mathrm{im}} := \Delta_{\mathrm{im}} \cap Q^+$.

For $\alpha \in \Delta_{\mathrm{re}}$, $s_\alpha \in \mathrm{GL}(\mathfrak{h}^*)$ denotes the reflection given by $s_\alpha(\lambda) := \lambda - (\alpha^\vee, \lambda)\alpha$, where $\alpha^\vee := 2\alpha/(\alpha, \alpha)$. Recall that the shift action $\circ$ of W on $\mathfrak{h}^*$ is defined by

$$w \circ \lambda := w(\lambda + \rho) - \rho \quad (\lambda \in \mathfrak{h}^*) \tag{35}$$

where $\rho \in \mathfrak{h}^*$ is given in Section 2.1. Remark that $(\rho, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i)$ holds for any $i \in I$.

The next theorem is a consequence of the Shapovalov determinant formula ([K1], see also [IK1]) and the injectivity of non-trivial homomorphisms between Verma modules over $\mathfrak{g}$ ([IK1]).

Theorem 5.1. *For $(\lambda; \sigma)$ and $(\mu; \tau) \in \mathfrak{h}^* \times (\mathbb{Z}/2\mathbb{Z})$, the following conditions are equivalent:*

1. *there exists an sequence $\{(\lambda_k; \sigma_k)\}_{k=0}^{l} \subset \mathfrak{h}^* \times (\mathbb{Z}/2\mathbb{Z})$ such that*
   - *(a) $(\lambda_0; \sigma_0) = (\lambda; \sigma)$, $(\lambda_l; \sigma_l) = (\mu; \tau)$ and*
   - *(b) there exists $\{(\beta_k, n_k)\}_{k=1}^{l}$ which satisfies*

$$\begin{aligned} &(\beta_k, n_k) \in \Big\{(\overline{\Delta}_{\bar{0}}^{+} \sqcup \Delta'^{+}_{\bar{0}}) \times \mathbb{Z}_{>0}\Big\} \cup \big\{\Delta^+_{\bar{1}} \times (2\mathbb{Z}_{\geq 0} + 1)\big\}, \\ &2(\lambda_{k-1} + \rho, \beta_k) = n_k(\beta_k, \beta_k), \\ &\lambda_k = \lambda_{k-1} - n_k\beta_k, \quad \sigma_k = \begin{cases} \sigma_{k-1} & (\beta_k \in \overline{\Delta}_{\bar{0}}^{+} \sqcup \Delta'^{+}_{\bar{0}}) \\ \sigma_{k-1} + \bar{1} & (\beta_k \in \Delta^+_{\bar{1}}) \end{cases}; \end{aligned}$$

2. *$\mathrm{Hom}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}}(M(\mu; \tau), M(\lambda; \sigma)) \neq \{0\}$;*
3. *there exists an injective homomorphism $M(\mu; \tau) \hookrightarrow M(\lambda; \sigma)$.*

We describe the first condition of the theorem in terms of the integral Weyl group. The integral root system $\Delta(\lambda)$ and the integral Weyl group $W(\lambda)$ associated with $\lambda \in \mathfrak{h}^*$ are defined as follows:

$$\begin{aligned}
&\overline{\Delta}_{\bar{0}}(\lambda) := \{\alpha \in \Delta_{\mathrm{re}} \cap \overline{\Delta}_{\bar{0}} | (\lambda+\rho, \alpha^\vee) \in \mathbb{Z}\}, \\
&\Delta_{\bar{1}}(\lambda) := \{\alpha \in \Delta_{\mathrm{re}} \cap \Delta_{\bar{1}} | (\lambda+\rho, \alpha^\vee) \in 2\mathbb{Z}+1\}, \\
&\Delta_{\bar{0}}(\lambda) := \overline{\Delta}_{\bar{0}}(\lambda) \cup 2\Delta_{\bar{1}}(\lambda),\ \Delta(\lambda) := \Delta_{\bar{0}}(\lambda) \cup \Delta_{\bar{1}}(\lambda),\ \Delta^{\pm}(\lambda) := \Delta(\lambda) \cap Q^{\pm}, \\
&\Pi_{\bar{0}}(\lambda) := \{\alpha \in \overline{\Delta}_{\bar{0}}(\lambda) \cap Q^+ | s_\alpha(\Delta^+(\lambda) \setminus \{\alpha\}) \subset \Delta^+(\lambda)\}, \\
&\Pi_{\bar{1}}(\lambda) := \{\alpha \in \Delta_{\bar{1}}(\lambda) \cap Q^+ | s_\alpha(\Delta^+(\lambda) \setminus \{\alpha, 2\alpha\}) \subset \Delta^+(\lambda)\}, \\
&\Pi(\lambda) := \Pi_{\bar{0}}(\lambda) \cup \Pi_{\bar{1}}(\lambda), \quad W(\lambda) := \langle s_\alpha | \alpha \in \Pi(\lambda)\rangle.
\end{aligned} \tag{36}$$

As we will see in the next subsection, $W(\lambda)$ is an integral Weyl group of a certain Kac-Moody algebra contained in $\mathfrak{g}$, and hence, $(\{s_\alpha | \alpha \in \Pi(\lambda)\}, W(\lambda))$ is a Coxeter system.

To state the main result of this section, let us introduce a set $\mathcal{K}$ of highest weights. For $\lambda \in \mathfrak{h}^*$, set

$$\begin{aligned}
&\overline{R}^{\lambda}_{\bar{0}} := \{\beta \in \Delta^+_{\mathrm{re}} \cap \overline{\Delta}_{\bar{0}} | (\lambda+\rho, \beta^\vee) \in \mathbb{Z}_{<0}\}, \\
&R^{\lambda}_{\bar{1}} := \{\beta \in \Delta^+_{\mathrm{re}} \cap \Delta_{\bar{1}} | (\lambda+\rho, \beta^\vee) \in 2\mathbb{Z}_{\leq 0} - 1\}, \\
&R^{\lambda}_{\bar{0}} := \overline{R}^{\lambda}_{\bar{0}} \sqcup 2R^{\lambda}_{\bar{1}}, \quad R^{\lambda} := R^{\lambda}_{\bar{0}} \sqcup R^{\lambda}_{\bar{1}},
\end{aligned} \tag{37}$$

and define $\mathcal{K} \subset \mathfrak{h}^*$ as follows:

$$\mathcal{K} := \{\lambda \in \mathfrak{h}^* | \sharp R^\lambda < \infty \wedge 2(\lambda+\rho, \beta) \neq (\beta, \beta)\ (\forall \beta \in \Delta^+_{\mathrm{im}})\}. \tag{38}$$

Then, as an immediate corollary of Theorem 5.1, we obtain

Corollary 5.1. *Suppose that $\lambda \in \mathcal{K}$. If $\mathrm{Hom}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}}(M(\mu;\tau), M(\lambda;\sigma)) \neq \{0\}$, then there exists $w \in W(\lambda)$ such that $\mu = w \circ \lambda$.*

The following is the main theorem of this section:

Theorem 5.2. *Suppose that $(\lambda;\sigma) \in \mathcal{K} \times (\mathbb{Z}/2\mathbb{Z})$. Then,*

$$\dim \mathrm{Hom}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}}(M(\mu;\tau), M(\lambda;\sigma)) \leq 1 \tag{39}$$

holds for any $(\mu;\tau) \in \mathfrak{h}^* \times (\mathbb{Z}/2\mathbb{Z})$.

Remark 5.1. *Set $\mathcal{K}^- := \{2\rho - \lambda \in \mathfrak{h}^* | \lambda \in \mathcal{K}\}$. By the tilting equivalence for Lie superalgebras ([Br], [IK2]), the inequality (39) holds for $\lambda \in \mathcal{K}^-$.*

5.2 Kac-Moody algebra contained in $\mathfrak{g}$

We introduce a certain subalgebra of $\mathfrak{g}_{\bar{0}}$, which is isomorphic to a Kac-Moody algebra and shares the same Weyl group with $\mathfrak{g}$. Let $\{e_i, f_i, h_i | i \in I\}$ be the Chevalley generators of $\mathfrak{g}$. We set $\widetilde{h}_i := 2^{-p(i)} h_i$,

$$\widetilde{e}_i := \begin{cases} e_i & (i \in I_{\bar{0}}) \\ \frac{1}{4}[e_i, e_i] & (i \in I_{\bar{1}}) \end{cases}, \quad \widetilde{f}_i := \begin{cases} f_i & (i \in I_{\bar{0}}) \\ -\frac{1}{4}[f_i, f_i] & (i \in I_{\bar{1}}) \end{cases},$$

and $\widetilde{\alpha}_i := 2^{p(i)}\alpha_i$. Then, we have

$$[\widetilde{e}_i, \widetilde{f}_j] = \delta_{i,j}\widetilde{h}_i, \quad [h, \widetilde{e}_i] = \widetilde{\alpha}_i(h)\widetilde{e}_i, \quad [h, \widetilde{f}_i] = -\widetilde{\alpha}_i(h)\widetilde{f}_i \quad (h \in \mathfrak{h})$$

by direct computation. Let us introduce the matrix $\widetilde{A} := (\widetilde{a}_{i,j})_{i,j \in I}$ as follows:

$$\widetilde{a}_{i,j} := \begin{cases} 2a_{i,j} & (i \in I_{\bar{0}} \wedge j \in I_{\bar{1}}) \\ \frac{1}{2}a_{i,j} & (i \in I_{\bar{1}} \wedge j \in I_{\bar{0}}) \\ a_{i,j} & (\text{others}) \end{cases}, \tag{40}$$

where $A = (a_{i,j})_{i,j \in I}$ is the Cartan matrix of $\mathfrak{g}$. One can easily check that $\widetilde{a}_{i,j} \in \mathbb{Z}$ for any $i, j \in I$ and $\widetilde{A}$ is symmetrizable. Let $\widetilde{\mathfrak{g}}$ be the subalgebra of $\mathfrak{g}$ generated by $\{\widetilde{e}_i, \widetilde{f}_i, h | i \in I,\ h \in \mathfrak{h}\}$. Remark that V. Serganova introduced subalgebras similar to $\tilde{\mathfrak{g}}$ for more general Kac-Moody superalgebras in [S].

Proposition 5.1 (cf. [K2], [S])**.** *The subalgebra $\widetilde{\mathfrak{g}}$ is isomorphic to the Kac-Moody algebra $\mathfrak{g}(\tilde{A})$ associated with the Cartan matrix $\tilde{A}$.*

Proof. We may assume that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}(\tilde{A})$, since $(\mathfrak{h}, \{\tilde{\alpha}_i\}_{i \in I}, \{\tilde{h}_i\}_{i \in I})$ is a realization of the Cartan matrix $\tilde{A}$ in the sense of Chapter one in [K3].

The commutation relation $(\mathrm{ad}e_i)^{1-a_{i,j}}(e_j) = 0 = (\mathrm{ad}f_i)^{1-a_{i,j}}(f_j)$ in $\mathfrak{g}$ implies $(\mathrm{ad}\widetilde{e}_i)^{1-\widetilde{a}_{i,j}}(\widetilde{e}_j) = 0 = (\mathrm{ad}\widetilde{f}_i)^{1-\widetilde{a}_{i,j}}(\widetilde{f}_j)$. Hence, there exists a surjective homomorphism $\mathfrak{g}(\tilde{A}) \to \tilde{\mathfrak{g}}$ whose restriction to $\mathfrak{h}$ is the identity. The kernel of the homomorphism intersects $\mathfrak{h}$ trivially, and thus, it is an isomorphism. □

Next, let us introduce some notation for the root system of $\widetilde{\mathfrak{g}}$. We put $\widetilde{\Pi} := \{\widetilde{\alpha}_i | i \in I\}$, $\widetilde{Q} := \sum_{i \in I} \mathbb{Z}\widetilde{\alpha}_i$ and $\widetilde{Q}^{\pm} := \pm\sum_{i \in I} \mathbb{Z}_{\geq 0}\widetilde{\alpha}_i$. Let $\widetilde{\Delta} := \{\alpha \in \Delta | \widetilde{\mathfrak{g}} \cap \mathfrak{g}^{\alpha} \neq \{0\}\}$ be the set of the roots of $\widetilde{\mathfrak{g}}$ and $\widetilde{W} := \langle s_\alpha | \alpha \in \widetilde{\Pi}\rangle \subset \mathrm{GL}(\mathfrak{h}^*)$ the Weyl group of $\widetilde{\mathfrak{g}}$. Let $\widetilde{\Delta}_{\mathrm{re}} := \tilde{W}\widetilde{\Pi}$ (resp. $\widetilde{\Delta}_{\mathrm{im}} := \widetilde{\Delta} \setminus \widetilde{\Delta}_{\mathrm{re}}$) be the sets of the real (resp. the imaginary) roots of $\tilde{\mathfrak{g}}$. By definition, $\widetilde{W}$ coincides with the Weyl group W of $\mathfrak{g}$, and thus,

$$\widetilde{\Delta}_{\mathrm{re}} = \Delta_{\mathrm{re}} \cap \widetilde{Q}, \quad \widetilde{\Delta}_{\mathrm{im}} \subset \Delta_{\mathrm{im}} \cap \widetilde{Q}$$

hold. Put $\widetilde{\Delta}^{\pm} := \widetilde{\Delta} \cap \widetilde{Q}^{\pm}$, $\widetilde{\Delta}^{\pm}_{\mathrm{re}} := \widetilde{\Delta}_{\mathrm{re}} \cap \widetilde{Q}^{\pm}$ and $\widetilde{\Delta}^{\pm}_{\mathrm{im}} := \widetilde{\Delta}_{\mathrm{im}} \cap \widetilde{Q}^{\pm}$. Let $\widetilde{\rho}$ be an element of $\mathfrak{h}^*$ such that $(\widetilde{\rho}, \widetilde{\alpha}_i) = \frac{1}{2}(\widetilde{\alpha}_i, \widetilde{\alpha}_i)$ holds for any $i \in I$. We denote the shift action of W on $\mathfrak{h}^*$ with respect to $\widetilde{\rho}$ by $\star$, namely,

$$w \star \lambda := w(\lambda + \widetilde{\rho}) - \widetilde{\rho}.$$

We define the integral root system $\widetilde{\Delta}(\lambda)$ and the integral Weyl group $\widetilde{W}(\lambda)$ of $\widetilde{\mathfrak{g}}$ associated with $\lambda \in \mathfrak{h}^*$ as follows:

$$\begin{aligned} &\widetilde{\Delta}(\lambda) := \{\alpha \in \widetilde{\Delta}_{\mathrm{re}} | (\lambda + \widetilde{\rho}, \alpha^{\vee}) \in \mathbb{Z}\}, \quad \widetilde{\Delta}^{\pm}(\lambda) := \widetilde{\Delta}(\lambda) \cap \widetilde{\Delta}^{\pm}, \\ &\widetilde{\Pi}(\lambda) := \{\alpha \in \widetilde{\Delta}^{+}(\lambda) | s_\alpha(\widetilde{\Delta}^{+}(\lambda) \setminus \{\alpha\}) \subset \widetilde{\Delta}^{+}(\lambda)\}, \\ &\widetilde{W}(\lambda) := \langle s_\alpha | \alpha \in \widetilde{\Pi}(\lambda)\rangle. \end{aligned}$$

Proposition 5.2. *For* $\lambda \in \mathfrak{h}^*$, $\widetilde{\Delta}(\lambda) = \Delta(\lambda) \cap \widetilde{Q} = \overline{\Delta}_{\bar{0}}(\lambda) \sqcup 2\Delta_{\bar{1}}(\lambda)$ *and* $\widetilde{\Pi}(\lambda) = \Pi_{\bar{0}}(\lambda) \cup 2\Pi_{\bar{1}}(\lambda)$.

Proof. We first show that

$$(\widetilde{\rho} - \rho, \beta^\vee) \in \mathbb{Z} + \frac{1}{2}\delta_{\sigma,\bar{1}} \tag{41}$$

holds for $\beta \in \widetilde{\Delta}_{\mathrm{re}}$, where $\sigma \in \mathbb{Z}/2\mathbb{Z}$ such that $\beta \in \Delta_\sigma$.

There exist $i \in I_\sigma$ and $w \in W$ such that $\beta = w(\widetilde{\alpha}_i)$. If $\beta = \alpha_i$ then (41) holds, since $(\widetilde{\rho} - \rho, \widetilde{\alpha}_i^\vee) = \frac{1}{2}\delta_{\sigma,\bar{1}}$. Hence, we show that if (41) holds for β, then it does for $s_k(\beta)$. Suppose that $\alpha_k \in \Pi_\tau$ ($\tau \in \mathbb{Z}/2\mathbb{Z}$). Since $s_k(\beta)^\vee = s_k(\beta^\vee) = \beta^\vee - (\beta^\vee, \widetilde{\alpha}_k)\widetilde{\alpha}_k^\vee$, we have

$$(\widetilde{\rho} - \rho, s_k(\beta)^\vee) = (\widetilde{\rho} - \rho, \beta^\vee) - \frac{1}{2}\delta_{\tau,\bar{1}}(\beta^\vee, \widetilde{\alpha}_k).$$

If α_k is an odd root, then $(\beta^\vee, \alpha_k) \in 2\mathbb{Z}$, since $\langle h_j, \alpha_k \rangle \in 2\mathbb{Z}$ ($\forall j \in I$). Hence, (41) holds for any $\beta \in \widetilde{\Delta}_{\mathrm{re}}$.

Using (41), we prove the proposition. Suppose that $\beta \in \widetilde{\Delta}_{\mathrm{re}} \cap \Delta_\sigma$. In the case where $\sigma = \bar{1}$, we have $\beta \in 2\Delta_{\bar{1}}$. Setting $\gamma := \frac{1}{2}\beta$, we have $(\lambda + \widetilde{\rho}, \beta^\vee) \in \mathbb{Z} \Leftrightarrow (\lambda + \rho, \gamma^\vee) \in 2\mathbb{Z} + 1$ by (41). Hence, $\widetilde{\Delta}(\lambda) \cap 2\Delta_{\bar{1}} = 2\Delta_{\bar{1}}(\lambda)$. In the case where $\sigma = \bar{0}$, we have $\beta \in \overline{\Delta}_{\bar{0}}$. Hence, $(\lambda + \widetilde{\rho}, \beta^\vee) \in \mathbb{Z} \Leftrightarrow (\lambda + \rho, \beta^\vee) \in \mathbb{Z}$ by (41). and thus, $\widetilde{\Delta}(\lambda) \cap \Delta_{\bar{0}} = \overline{\Delta}_{\bar{0}}(\lambda)$. Now, we obtain $\widetilde{\Delta}(\lambda) = \overline{\Delta}_{\bar{0}}(\lambda) \sqcup 2\Delta_{\bar{1}}(\lambda)$. Since $\Delta(\lambda) = \overline{\Delta}_{\bar{0}}(\lambda) \sqcup \Delta_{\bar{1}}(\lambda) \sqcup 2\Delta_{\bar{1}}(\lambda)$ and $\Delta_{\bar{1}}(\lambda) \cap \widetilde{Q} = \emptyset$ by (36), we have $\widetilde{\Delta}(\lambda) = \Delta(\lambda) \cap \widetilde{Q}$. Moreover, $\widetilde{\Pi}(\lambda) = \Pi_{\bar{0}}(\lambda) \sqcup 2\Pi_{\bar{1}}(\lambda)$ holds by the definition of $\Pi(\lambda)$ and $\widetilde{\Pi}(\lambda)$. □

By Proposition 5.2, for $\lambda \in \mathfrak{h}^*$, we have $W(\lambda) = \widetilde{W}(\lambda)$. Hence, Proposition 2.2.9 in [KT] implies

Corollary 5.2. $W(\lambda)$ *is the Coxeter group with the Coxeter system* $\{s_\alpha | \alpha \in \widetilde{\Pi}(\lambda)\}$, *whose length function* $\ell_\lambda : W(\lambda) \to \mathbb{Z}_{\geq 0}$ *is given by*

$$\ell_\lambda(w) = \sharp(\widetilde{\Delta}^-(\lambda) \cap w(\widetilde{\Delta}^+(\lambda))) \quad (w \in W(\lambda)). \tag{42}$$

We need the following lemma to prove Theorem 5.2:

Lemma 5.1. *Suppose that* $\lambda \in \mathfrak{h}^*$.

1. *For* $\widetilde{\alpha}_i \in \widetilde{\Pi}$, *if* $\widetilde{\alpha}_i \in \widetilde{\Delta}(\lambda)$, *then* $\widetilde{\alpha}_i \in \widetilde{\Pi}(\lambda)$.
2. *If* $\widetilde{\alpha}_i \in \widetilde{\Pi}$ *and* $\alpha \in \widetilde{\Pi}(\lambda)$ *satisfy* $(\widetilde{\alpha}_i, \alpha) > 0$, *then* $\alpha = \widetilde{\alpha}_i$ *or* $\widetilde{\alpha}_i \notin \widetilde{\Delta}(\lambda)$.
3. *For* $\widetilde{\alpha}_i \notin \widetilde{\Delta}(\lambda)$, *set* $\lambda' := s_i \circ \lambda$. *Then,* $\widetilde{\Delta}(\lambda') = s_i\widetilde{\Delta}(\lambda)$, $\widetilde{\Delta}^\pm(\lambda') = s_i\widetilde{\Delta}^\pm(\lambda)$, $\widetilde{\Pi}(\lambda') = s_i\widetilde{\Pi}(\lambda)$ *and* $W(\lambda') = s_iW(\lambda)s_i$ *hold. Moreover, for* $w \in W(\lambda)$, *set* $w' := s_iws_i$. *Then,* $\ell_{\lambda'}(w') = \ell_\lambda(w)$.

Proof. The first and the second statements follow from Lemma 2.2.1 and 2.2.3 of [KT]. Hence, we show the third one. We have

$$\lambda' - s_i \star \lambda = s_i(\rho - \widetilde{\rho}) - (\rho - \widetilde{\rho}) = \delta_{\sigma,\bar{1}}\alpha_i,$$

where $\sigma \in \mathbb{Z}/2\mathbb{Z}$ such that $i \in I_\sigma$. Since $(\alpha_i, \beta^\vee) \in \mathbb{Z}$ for any $\beta \in \widetilde{\Delta}_{\mathrm{re}}$, we have $\widetilde{\Delta}(\lambda') = \widetilde{\Delta}(s_i \star \lambda)$. On the other hand, $\widetilde{\Delta}(s_i \star \lambda) = s_i\widetilde{\Delta}(\lambda)$ by the definition of $\widetilde{\Delta}(\lambda)$, and thus, $\widetilde{\Delta}(\lambda') = s_i\widetilde{\Delta}(\lambda)$.

Lemma 2.2.2 of [KT] implies $\widetilde{\Pi}(\lambda') = s_i\widetilde{\Pi}(\lambda)$ and $W(\lambda') = s_iW(\lambda)s_i$. Hence,

$$\begin{aligned}\ell_{\lambda'}(w') =& \sharp(\widetilde{\Delta}^-(\lambda') \cap w'\widetilde{\Delta}^+(\lambda')) = \sharp(s_i\widetilde{\Delta}^-(\lambda) \cap s_iws_i^2\widetilde{\Delta}^+(\lambda)) \\ &= \sharp s_i(\widetilde{\Delta}^-(\lambda) \cap w\widetilde{\Delta}^+(\lambda)) = \ell_\lambda(w).\end{aligned}$$

□

Next, we state some properties of the set $\mathcal{K}$. To do it, we show a preliminary lemma.

Lemma 5.2.

$$\Delta^+(\lambda) \subset \sum_{\alpha\in\Pi(\lambda)} \mathbb{Z}_{\geq 0}\alpha. \tag{43}$$

Proof. Define the hight of $\gamma = \sum_{i\in I} k_i\alpha_i \in Q$ by $\mathrm{ht}\gamma := \sum_{i\in I} k_i$. For $\gamma \in \Delta^+(\lambda)$, set $\widetilde{\gamma} := 2^{\delta_{\sigma,\bar{1}}}\gamma$, where $\sigma \in \mathbb{Z}/2\mathbb{Z}$ such that $\gamma \in \Delta_\sigma(\lambda)$. By Proposition 5.2, $\widetilde{\gamma} \in \widetilde{\Delta}^+(\lambda)$ holds.

For $\beta \in \Delta^+(\lambda)$, let us prove $\beta \in \sum_{\alpha\in\Pi(\lambda)} \mathbb{Z}_{\geq 0}\alpha$ by induction of $\mathrm{ht}\beta$. Lemma 2.2.6 of [KT] implies that there exists $\alpha \in \widetilde{\Pi}(\lambda)$ such that $(\widetilde{\beta}, \alpha) > 0$. Hence, $(\beta, \alpha) > 0$. If one of $\beta = \alpha$ and $\beta = \frac{1}{2}\alpha$ holds, then $\beta \in \sum_{\alpha\in\Pi(\lambda)} \mathbb{Z}_{\geq 0}\alpha$. If it does not holds, then for $\gamma := s_\alpha\beta$ we have $\gamma \in \Delta^+(\lambda)$ and $\mathrm{ht}\gamma < \mathrm{ht}\beta$. By induction hypothesis, $\gamma \in \sum_{\alpha\in\Pi(\lambda)} \mathbb{Z}_{\geq 0}\alpha$. Hence,

$$\beta = s_\alpha\gamma = \gamma + (\beta, \alpha^\vee)\alpha \in \sum_{\alpha\in\Pi(\lambda)} \mathbb{Z}_{\geq 0}\alpha.$$

and thus, (43) holds. □

Lemma 5.3. *1. $W \circ \mathcal{K} \subset \mathcal{K}$.*

2. Set $\mathcal{C} := \left\{\lambda \in \mathfrak{h}^* \,\middle|\, R^\lambda = \emptyset \wedge 2(\lambda + \rho, \beta) \neq (\beta, \beta)\ (\forall\beta \in \Delta^+_{\mathrm{im}})\right\}$. Then, for any $\lambda \in \mathcal{K}$, we have $\{W(\lambda) \circ \lambda\} \cap \mathcal{C} \neq \emptyset$.

Proof. We show the first assertion. Since Δ^+_{im} is W-invariant, for any $w \in W$, the following holds:

$$2(\lambda + \rho, \beta) \neq (\beta, \beta)\ (\forall\beta \in \Delta^+_{\mathrm{im}}) \Rightarrow 2(w \circ \lambda + \rho, \beta) \neq (\beta, \beta)\ (\forall\beta \in \Delta^+_{\mathrm{im}}). \tag{44}$$

Hence, we may show that $\sharp R^\lambda < \infty \Rightarrow \sharp R^{s_i\circ\lambda} < \infty$ for any $i \in I$. For $i \in I_{\bar{1}}$, we have $\sharp R^{s_i\circ\lambda} \leq \sharp R^\lambda + 2$, since

$$s_i(\Delta^+(\lambda) \setminus \{\alpha_i, 2\alpha_i\}) = \Delta^+(\lambda) \setminus \{\alpha_i, 2\alpha_i\}, \quad s_i(\{\alpha_i, 2\alpha_i\}) = \{-\alpha_i, -2\alpha_i\}.$$

For $i \in I_{\bar{0}}$, one can show $\sharp R^{s_i \circ \lambda} \leq \sharp R^{\lambda} + 1$ in a similar way. Hence, the first assertion holds.

Let us prove the second assertion. By (44), we may show that

$$\exists w \in W(\lambda), \quad R^{w \circ \lambda} = \emptyset \tag{45}$$

holds for any $\lambda \in \mathcal{K}$. We prove (45) by induction on $\sharp R^{\lambda}$. In the case where $\sharp R^{\lambda} = 0$, by definition, $\lambda \in \mathcal{C}$. Hence, suppose that $\sharp R^{\lambda} \neq 0$. If $\Pi(\lambda) \cap R^{\lambda} = \emptyset$, then $(\lambda+\rho, \beta) \geq 0$ holds for $\beta \in \Delta^{+}(\lambda)$ by (43). This means that $R^{\lambda} = \emptyset$. Hence, we may assume that there exists $\alpha \in \Pi(\lambda) \cap R^{\lambda}$. In the case where $\alpha \in \Pi_{\bar{1}}(\lambda)$, we have $\sharp R^{s_\alpha \circ \lambda} = \sharp R^{\lambda} - 2$, since $s_\alpha(\Delta^{+}(\lambda) \setminus \{\alpha, 2\alpha\}) = \Delta^{+}(\lambda) \setminus \{\alpha, 2\alpha\}$ and $s_\alpha(\{\alpha, 2\alpha\}) = \{-\alpha, -2\alpha\}$. In the case where $\alpha \in \Pi_{\bar{0}}(\lambda)$, we have $\sharp R^{s_\alpha \circ \lambda} = \sharp R^{\lambda} - 1$. By induction hypothesis, (45) holds for $s_\alpha \circ \lambda$, and thus, it does for λ. We have completed the proof. □

5.3 Proof of Theorem 5.2

In general, the root system Δ of $\mathfrak{g}$ is not reduced. Here, we modify the arguments in the proof of Theorem 2.5.3 in [KT] to deal with non-reduced root systems.

Proof of Theorem 5.2. Since $\{W(\lambda) \circ \lambda\} \cap \mathcal{C} \neq \emptyset$ by Lemma 5.3, Theorem 5.1 ensures that $M(\lambda; \sigma) \hookrightarrow M(\lambda'; \sigma')$ for some $\lambda' \in \mathcal{C}$ and $\sigma' \in \mathbb{Z}/2\mathbb{Z}$. Hence, we may assume that $\lambda \in \mathcal{C}$.

Suppose that $\mathrm{Hom}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}}(M(\mu; \tau), M(\lambda; \sigma)) \neq \{0\}$. By Corollary 5.1, there exists $w \in W(\lambda)$ such that $\mu = w \circ \lambda$. We show the theorem by induction on $\ell_\lambda(w)$.

If $\ell_\lambda(w) = 0$, then the theorem clearly holds. Suppose that $\ell_\lambda(w) \geq 1$. There exists $\widetilde{\alpha} \in \widetilde{\Pi}(\lambda)$ such that $\ell_\lambda(s_{\widetilde{\alpha}} w) < \ell_\lambda(w)$, and hence, $w^{-1}\widetilde{\alpha} \in \widetilde{\Delta}^{-}(\lambda) \subset \widetilde{Q}^{-}$. We express $\widetilde{\alpha}$ as $\widetilde{\alpha} = \sum_{i \in I} k_i \widetilde{\alpha}_i \in \widetilde{Q}$ and set $\widetilde{\mathrm{ht}}\widetilde{\alpha} := \sum_{i \in I} k_i$. We further use induction on $\widetilde{\mathrm{ht}}\widetilde{\alpha}$.

<u>Case $\widetilde{\mathrm{ht}}\widetilde{\alpha} = 1$</u> There exists i such that $\widetilde{\alpha} = \widetilde{\alpha}_i$.

Case $i \in I_{\bar{1}}$: By Proposition 5.2, $\alpha_i = \frac{1}{2}\widetilde{\alpha}_i \in \Pi_{\bar{1}}(\lambda)$. Hence, $(\lambda+\rho, \alpha_i^{\vee}) \in 2\mathbb{Z}+1$. Since $R^{\lambda} = \emptyset$, we have $(\lambda+\rho, \alpha_i^{\vee}) \notin 2\mathbb{Z}_{<0}+1$. Hence, $(\lambda+\rho, \alpha_i^{\vee}) \in 2\mathbb{Z}_{\geq 0}+1$. On the other hand, by $(\mu+\rho, \alpha_i^{\vee}) = (\lambda+\rho, w^{-1}(\alpha_i^{\vee}))$ and $w^{-1}(\widetilde{\alpha}_i) \in \widetilde{\Delta}(\lambda)$, we have $(\mu+\rho, \alpha_i^{\vee}) \in 2\mathbb{Z}+1$. Since $w^{-1}(\widetilde{\alpha}_i) \in \widetilde{Q}^{-}$, we have $(\mu+\rho, \alpha_i^{\vee}) < 0$, and hence, $(\mu + \rho, \alpha_i^{\vee}) \in 2\mathbb{Z}_{<0} + 1$. Noticing that $(\rho, \alpha_i^{\vee}) = 1$, we obtain

$$\langle h_i, \lambda \rangle \in 2\mathbb{Z}_{\geq 0}, \quad \langle h_i, \mu \rangle \in 2\mathbb{Z}_{<0}. \tag{46}$$

Case $i \in I_{\bar{0}}$: In a way similar to the above case, we have $(\lambda + \rho, \alpha_i^{\vee}) \in \mathbb{Z}_{\geq 0}$ and $(\mu + \rho, \alpha_i^{\vee}) \in \mathbb{Z}_{\leq 0}$. In the case where $(\mu + \rho, \alpha_i^{\vee}) = 0$, we have $s_{\widetilde{\alpha}} \circ \mu = \mu$, and thus, the proof can be reduced to the case where the length $\ell_\lambda(w)$ is smaller. Hence, we may assume that $(\mu + \rho, \alpha_i^{\vee}) \neq 0$ and

$$\langle h_i, \lambda \rangle \in \mathbb{Z}_{\geq -1}, \quad \langle h_i, \mu \rangle \in \mathbb{Z}_{\leq -2}. \tag{47}$$

It follows from (46), (47) and Theorem 4.1 that

$$T_i(0)(M(\lambda;\sigma)) \simeq M(\lambda;\sigma), \quad T_i(0)(M(\mu;\tau)) \simeq M(s_i \circ \mu; \tau').$$

Here and after, we set $\tau' := \tau + \overline{p(i)}$ for $\tau \in \mathbb{Z}/2\mathbb{Z}$. By Proposition 4.2,

$$\dim \mathrm{Hom}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}}(M(\mu;\tau), M(\lambda;\sigma)) \leq \dim \mathrm{Hom}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}}(M(s_i \circ \mu;\tau'), M(\lambda;\sigma)).$$

Since $\ell_\lambda(s_i \circ w) < \ell_\lambda(w)$, the dimension in the right hand side is less than or equal to one by induction hypothesis. Hence, in the case where $\widetilde{\mathrm{ht}}\widetilde{\alpha} = 1$, (39) holds.

<u>Case $\widetilde{\mathrm{ht}}\widetilde{\alpha} > 1$</u> There exists $i \in I$ such that $(\widetilde{\alpha}_i^\vee, \widetilde{\alpha}) > 0$. Lemma 5.1.2 implies $\widetilde{\alpha}_i \notin \widetilde{\Delta}(\lambda)$. Putting $\lambda' := s_i \circ \lambda$ and $\widetilde{\alpha}' := s_i \widetilde{\alpha}$, we have $\widetilde{\alpha}' \in \widetilde{\Pi}(\lambda')$ by Lemma 5.1.3. Moreover, putting $\mu' := s_i \circ \mu$, $w' := s_i w s_i$, we have $\mu' = w' \circ \lambda'$. Hence, Lemma 5.1.3 implies $w' \in W(\lambda')$ and $\ell_{\lambda'}(w') = \ell_\lambda(w)$. Since $\widetilde{\alpha}_i \notin \widetilde{\Delta}(\lambda)$, Proposition 5.2 implies $\alpha_i \notin \Delta(\lambda)$. Since $M(\lambda;\sigma)$ and $M(\mu;\tau)$ are objects in $\mathcal{M}_i^a$ for $a := \langle h_i, \lambda \rangle$, Theorem 4.1 implies that

$$T_i(a)(M(\lambda;\sigma)) \simeq M(\lambda';\sigma'), \quad T_i(a)(M(\mu;\tau)) \simeq M(\mu';\tau').$$

Applying Proposition 4.1, we have

$$\mathrm{Hom}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}}(M(\mu;\tau), M(\lambda;\sigma)) \simeq \mathrm{Hom}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}}(M(\mu';\tau'), M(\lambda';\sigma')).$$

It follows from $(\widetilde{\alpha}_i, \widetilde{\alpha}) > 0$ that $\widetilde{\mathrm{ht}}\widetilde{\alpha}' < \widetilde{\mathrm{ht}}\widetilde{\alpha}$. Hence, by induction hypothesis,

$$\dim \mathrm{Hom}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}}(M(\mu';\tau'), M(\lambda';\sigma')) \leq 1.$$

Now, we obtain $\dim \mathrm{Hom}_{\mathfrak{g}}^{\mathbb{Z}/2\mathbb{Z}}(M(\mu;\tau), M(\lambda;\sigma)) \leq 1$. Hence, (39) holds in the case where $\widetilde{\mathrm{ht}}\widetilde{\alpha} > 1$, and we have completed the proof. $\square$

6 Singular vectors of Verma modules

In the previous section, we have shown the uniqueness of homomorphism between Verma modules. This means that for $\lambda \in \mathcal{K}$, a singular vector in $M(\lambda;\sigma)^\mu$ is unique up to scalar. In this section, we discuss explicit forms of singular vectors.

6.1 Malikov-Feigin-Fuks singular vector formula

We first recall the singular vector formula given in [MFF]. For a while, let $\mathfrak{g}$ be a Kac-Moody algebra with a Cartan subalgebra $\mathfrak{h}$ and $\lambda \in \mathfrak{h}^*$ satisfying $(\lambda + \rho, \alpha^\vee) \in \mathbb{Z}_{>0}$ for $\alpha \in \Delta_{\mathrm{re}}^+$. Suppose that the reflection s_α is written as $s_\alpha = s_{k_P} \cdots s_{k_1}$ by simple reflections. Define $\lambda_j \in \mathfrak{h}^*$ and $a_j \in \mathbb{C}$ $(j = 1, \cdots, P)$ by $\lambda_1 := \lambda$, $\lambda_{j+1} := s_{k_j} \circ \lambda_j$, $a_j := \langle h_{k_j}, \lambda_j \rangle$. F. G. Malikov, B. L. Feigin and D. B. Fuks show the following formula:

$$f_{k_P}^{a_P+1} \cdots f_{k_1}^{a_1+1} \mathbf{1}^\lambda \in M(\lambda)^{\mathfrak{g}^+} \setminus \{0\},$$

where $M(\lambda)$ and $\mathbf{1}^\lambda$ denote the Verma module over $\mathfrak{g}$ and its highest weight vector.

In [MFF], they first show that $f_{k_P}^{a_P+1}\cdots f_{k_1}^{a_1+1}$ makes sense for regular dominant integral weights and their descendants, i.e., $\lambda \in W\circ\mu$ (μ is a regular dominant integral weight), and they conclude that the formula holds for any λ such that $(\lambda+\rho,\alpha^\vee)\in\mathbb{Z}_{>0}$ by 'analytic continuation', since the set of the above λ is Zariski dense in $\mathfrak{h}^*$.

Here, we remark that their singular vector formula (including its super analogue) can be directly interpreted in terms of the Enright functor. From now on, let $\mathfrak{g}$ be a Kac-Moody superalgebra without isotropic simple roots. Suppose that $\lambda\in\mathfrak{h}^*$ and $\alpha\in\Delta_{\mathrm{re}}^+\cap(\overline{\Delta}_{\bar 0}\sqcup\Delta_{\bar 1})$ satisfy

$$(\lambda+\rho,\alpha^\vee)\in\begin{cases}\mathbb{Z}_{>0} & (\text{if } \alpha\in\overline{\Delta}_{\bar 0})\\ 2\mathbb{Z}_{\geq 0}+1 & (\text{if } \alpha\in\Delta_{\bar 1})\end{cases}.$$

There exist $w\in W$ and $\alpha_k\in\overline{\Pi}_{\bar 0}\sqcup\Pi_{\bar 1}$ such that $\alpha=w(\alpha_k)$. Let $w=s_{i_1}\cdots s_{i_N}$ be a reduced expression. Then, $s_\alpha = ws_kw^{-1}=s_{i_1}\cdots s_{i_N}s_ks_{i_N}\cdots s_{i_1}$. We choose the sequence $(k_1,\cdots,k_P):=(i_1,\cdots,i_N,k,i_N,\cdots,i_1)$ with $P:=2N+1$ and set $\lambda_1:=\lambda$, $\lambda_{j+1}:=s_{k_j}\circ\lambda_j$, $a_j:=\langle h_{k_j},\lambda_j\rangle$ and

$$F(s_\alpha;\lambda):=f_{k_P}^{a_P:1}\otimes\cdots\otimes f_{k_1}^{a_1:1}\in U(\mathfrak{g})f_{k_P}^{a_P:\mathbb{Z}}\otimes_{\mathfrak{g}}\cdots\otimes_{\mathfrak{g}}U(\mathfrak{g})f_{k_1}^{a_1:\mathbb{Z}}.$$

One can show that $a_{N+1}\in 2^{p(k)}\mathbb{Z}_{\geq 0}$ and $a_{2N+2-j}+a_j\in 2^{p(i_j)}\mathbb{Z}$. Hence,

$$F(s_\alpha;\lambda)\in U(\mathfrak{g})f_{i_1}^{-a_1:\mathbb{Z}}\otimes_{\mathfrak{g}}\cdots\otimes_{\mathfrak{g}}U(\mathfrak{g})f_{i_N}^{-a_N:\mathbb{Z}}\otimes_{\mathfrak{g}}U(\mathfrak{g})f_{i_N}^{a_N:\mathbb{Z}}\otimes_{\mathfrak{g}}\cdots\otimes_{\mathfrak{g}}U(\mathfrak{g})f_{i_1}^{a_1:\mathbb{Z}}.$$

Proposition 6.1. *Suppose that the following condition holds:*

$$a_j\notin 2^{p(i_j)}\mathbb{Z}\quad(\forall j=1,\cdots,N). \tag{48}$$

Then, $F(s_\alpha;\lambda)\otimes\mathbf{1}_\sigma^\lambda$ gives a non-trivial singular vector in

$$T_{i_1}(-a_1)\cdots T_{i_N}(-a_N)T_{i_N}(a_N)\cdots T_{i_1}(a_1)(M(\lambda;\sigma))\simeq M(\lambda;\sigma).$$

Successive use of the following lemma shows this proposition.

Lemma 6.1. *Suppose that there exists an inclusion $M(\nu;\tau)\hookrightarrow M(\mu;\sigma)$ which maps $\mathbf{1}_\tau^\nu\mapsto S\mathbf{1}_\sigma^\mu$ ($S\in U(\mathfrak{g}^-)$). For $i\in I$ such that $\langle h_i,\mu\rangle\notin 2^{p(i)}\mathbb{Z}$, we set*

$$F:=f_i^{\langle h_i,\nu\rangle:1}\otimes Sf_i^{\langle h_i,s_i\circ\mu\rangle:1}\in U(\mathfrak{g})f_i^{-a:\mathbb{Z}}\otimes_{\mathfrak{g}}U(\mathfrak{g})f_i^{a:\mathbb{Z}},$$

where $a:=\langle h_i,s_i\circ\mu\rangle$. Then, we have

1. *$F\otimes\mathbf{1}_{\sigma'}^{s_i\circ\mu}\in T_i(-a)T_i(a)(M(s_i\circ\mu;\sigma'))\simeq M(s_i\circ\mu;\sigma')$,*
2. *$M(s_i\circ\nu;\tau')\hookrightarrow M(s_i\circ\mu;\sigma')$ ($\mathbf{1}_{\tau'}^{s_i\circ\nu}\mapsto F\otimes\mathbf{1}_{\sigma'}^{s_i\circ\mu}$),*

where $\sigma':=\sigma+\overline{p(i)}$, $\tau':=\tau+\overline{p(i)}$.

Proof. Under the identification

$$M(s_i \circ \nu; \tau') \simeq T_i(-a)(M(\nu;\tau)), \quad M(\mu;\sigma) \simeq T_i(a)(M(s_i \circ \mu; \sigma')),$$

we have

$$\mathbf{1}_{\tau'}^{s_i \circ \nu} = f_i^{\langle h_i, \nu \rangle : 1} \otimes \mathbf{1}_{\tau}^{\nu}, \quad \mathbf{1}_{\sigma}^{\mu} = f_i^{\langle h_i, s_i \circ \mu \rangle : 1} \otimes \mathbf{1}_{\sigma'}^{s_i \circ \mu}.$$

Hence, the image of a highest weight vector $\mathbf{1}_{\tau'}^{s_i \circ \nu}$ under the composition

$$M(s_i \circ \nu; \tau') \simeq T_i(-a)(M(\nu;\tau)) \hookrightarrow T_i(-a)(M(\mu;\sigma)) \simeq T_i(-a)T_i(a)(M(s_i \circ \mu; \sigma'))$$

is given by

$$\mathbf{1}_{\tau'}^{s_i \circ \nu} = f_i^{\langle h_i, \nu \rangle : 1} \otimes \mathbf{1}_{\tau}^{\nu} \longmapsto f_i^{\langle h_i, \nu \rangle : 1} \otimes S\mathbf{1}_{\sigma}^{\mu} = F \otimes \mathbf{1}_{\sigma'}^{s_i \circ \mu},$$

and thus, the lemma follows. □

Remark that by the above proposition, we conclude that

$$F(s_\alpha; \lambda) \otimes \mathbf{1}_{\sigma}^{\lambda} \in \{M(\lambda;\sigma)\}^{\mathfrak{g}^+} \setminus \{0\}$$

holds without the assumption (48).

6.2 Braid relations

In the case where $\mathfrak{g}$ is one of the simple Lie algebras of type A_2, B_2 and G_2, the uniqueness of singular vectors of Verma modules with regular dominant integral highest weight implies the so-called Verma relations ([V]). By means of the Verma relations, V. V. Deodhar and A. Bouaziz independently proved the braid relations of the Enright functors. More conceptual and less computational approaches were proposed by A. Joseph in [J1] and S. König and V. Mazorchuk in [KM].

In this subsection, we make brief comments on the Verma relation for the simple Lie superalgebra of type $B(0,2)$ and the braid relation of the Enright functors $T_i(0)$. For simplicity, we denote $T_i(0)$ by T_i. Let $\mathfrak{g}$ be a Kac-Moody superalgebra without isotropic simple roots, and W the Weyl group of $\mathfrak{g}$. Suppose that

$$w = s_{i_1} s_{i_2} \cdots s_{i_k} = s_{j_1} s_{j_2} \cdots s_{j_k} \quad (i_1, \cdots, i_k, j_1, \cdots, j_k \in I)$$

are two reduced expressions of $w \in W$. One can prove the next theorem by arguments similar to [D].

Theorem 6.1. *For any $M \in \bigcap_{i \in I} \mathrm{Ob}(\overline{\mathcal{M}}_i^0)$, there exists an isomorphism of $\mathfrak{g}$-module*

$$T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_k}(M) \xrightarrow{\sim} T_{j_1} \circ T_{j_2} \circ \cdots \circ T_{j_k}(M)$$

whose restriction to M is the identity on M.

As stated in [D], we can reduce the proof to the cases where $\mathfrak{g}$ is one of finite dimensional simple Lie superalgebras of type A_2, B_2, G_2 and $B(0,2)$. The cases of A_2, B_2 and G_2 have been done in [D]. In the remaining case, $I_{\bar{0}} := \{1\}$, $I_{\bar{1}} = \{2\}$ and the Cartan matrix is given by

$$A_{B(0,2)} := \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}.$$

Proposition 6.2. *Suppose that* $\mathfrak{g} = \mathfrak{g}(A_{B(0,2)})$. *For* $M \in \mathrm{Ob}(\overline{\mathcal{M}}_1^0) \cap \mathrm{Ob}(\overline{\mathcal{M}}_2^0)$, *there exists an isomorphism of* $\mathfrak{g}$*-module*

$$\phi : T_1 \circ T_2 \circ T_1 \circ T_2(M) \to T_2 \circ T_1 \circ T_2 \circ T_1(M)$$

such that $\phi|_M = \mathrm{id}_M$.

This proposition can be proved by using arguments similar to [D] and the following lemma:

Lemma 6.2 (Verma relation for $B(0,2)$). *For any* $p \in \mathbb{Z}_{\geq 0}$ *and* $q \in 2\mathbb{Z}_{\geq 0} + 1$, *we have*

$$f_1^p f_2^{2p+q} f_1^{p+q} f_2^q = f_2^q f_1^{p+q} f_2^{2p+q} f_1^p.$$

Acknowledgement Y.K was partly supported by the Japan Society for the Promotion of Science Grant-in-Aid for Scientific Research.

References

[Bo] A. Bouaziz, *Sur les représentations des algèbres de Lie semi-simples construites par T. Enright,* Lect. Notes in Math. **880** (1981), 57–68.

[Br] J. Brundan, *Tilting modules for Lie superalgebras*, Comm. Alg. **32** (2004), 2251–2268.

[D] V. V. Deodhar, *On a construction of representations and a problem of Enright*, Invent. Math. **57** (1980), 101–118.

[E] T. J. Enright, *On the fundamental series of a real semi-simple Lie algebra: their irreducibility, resolutions and multiplicity formulae*, Ann. Math. **110** (1979), 1–82.

[J1] A. Joseph, *The Enright functor on the Bernstein-Gelfand-Gelfand category $\mathcal{O}$*, Invent. Math. **67** (1982), no. 3, 423–445.

[J2] A. Joseph, Mathematical Review (MR1289323) on the article: by K. Iohara and F. Malikov, *Rings of skew polynomials and Gelfand-Kirillov conjecture for quantum groups*, Comm. Math. Phys. **164** (1994), no. 2, 217–237.

[IK1] K. Iohara and Y. Koga, *Fusion algebras for N=1 superconformal field theories through coinvariants I:*$\hat{\mathfrak{osp}}(1|2)$*-symmetry*, J. Reine Angew. Math. **531** (2001), 1–34.

[IK2] K. Iohara and Y. Koga, *Tilting equivalence for superconformal algebra*, Math. Scand. **99** (2006), 17–52.

[K1] V. G. Kac, *Contravariant form for infinite-dimensional Lie algebras and superalgebras*, Lect. Notes in Phys. **94** (1979), 441–445.

[K2] V. G. Kac, *Infinite-dimensional algebras, Dedekind's* η*-function, classical Möbius function and the very strange formula*, Adv. in Math. **30** (1978), no. 2, 85–136.

[K3] V. G. Kac, *Infinite-dimensional Lie algebras*, Third edition, Cambridge University Press, Cambridge (1990).

[KM] S. König and V. Mazorchuk, *Enright's completions and injectively copresented modules*, Trans. Amer. Math. Soc. **354** (2002), no. 7, 2725-2743.

[KT] M. Kashiwara and T. Tanisaki, *Kazhdan-Lusztig conjecture for symmetrizable Kac-Moody Lie algebra. III – Positive rational case*, Asian J. Math. **2**, no. 4, (1998), 799–832.

[M] O. Mathieu, *Classification of irreducible weight modules*, Ann. Inst. Fourier **50**, no.2, (2000), 537–592.

[MFF] F. G. Malikov, B. L. Feigin and D. B. Fuks, *Singular vectors in Verma modules over Kac-Moody algebras*, Funkts. Anal. Prilozhen., **20**, No. 2, (1988), 25–37.

[RW] A. Rocha-Caridi and N.R. Wallach, *Projective modules over graded Lie algebras I*, Math. Z. **180** (1982), 151–177.

[S] V. Serganova, *Kac-Moody superalgebras and integrability*, Prog. Math. **288** (2010), 169–218.

[V] D.-N. Verma, *Structure of certain induced representations of complex semisimple Lie algebras* Bull. Amer. Math. Soc. **74** (1968), 160–166.