

HIDDEN $SL_2(\mathbb{Z})$ -SYMMETRY IN $BC_l^{(2)}$

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ABSTRACT. A well-known Γ_θ -action on the characters of integrable highest weight modules over the affine Lie algebra of type $BC_l^{(2)}$ at a positive level is extended to an $SL_2(\mathbb{Z})$ -action at a positive even level by supplementing their twisted characters.

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INTRODUCTION

In 1972, I. G. Macdonald [9] classified the so-called (not necessarily reduced) affine root systems and obtained some identities on the Fourier expansions of products of the Dedekind η -function associated with each affine root systems. The identities he obtained are named after him. Independently, F. Bruhat and J. Tits [1] classified affine root systems, where some of their results are explained in a survey article [13].

To obtain the Macdonald identities, he started from an identity, that looks like a denominator identity up to some factor, called “missing factor”. Later, it was shown by several authors (see, e.g., our expository note [3] for the references), that the “missing factor” corresponds to the imaginary roots and the identity Macdonald used is the denominator identity of an affine Lie algebra. It would be a natural question to see what are the nature of the characters of integrable highest weight modules over an affine Lie algebra. Indeed, E. Looijenga [8] interpreted a character of an affine Lie algebra of untwisted type as a section of a line bundle over an abelian variety, which is a product of an elliptic curve. This fact inspired V. G. Kac and D. H.

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Peterson, indeed, in 1984, they [7] had studied “modular properties” of the characters of affine Lie algebras. They showed the following:

- (1) for an untwisted affine Lie algebra, the group $\mathrm{SL}_2(\mathbb{Z})$ acts on the \mathbb{C} -span of the normalized characters of each fixed central charge and
- (2) for a twisted affine Lie algebra, a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ acts on the \mathbb{C} -span of the of the normalized characters of each fixed central charge

via “modular transformation”. For $BC_l^{(2)}$ (or $A_{2l}^{(2)}$), this congruence subgroup is the so-called θ -subgroup Γ_θ . Here, the characters have been regarded as functions on a certain finite dimensional complex subdomain Y of the affine Lie algebra of type $BC_l^{(2)}$.

In Macdonald’s paper [9], he considered several possible evaluations of the denominator identities. Indeed, for $BC_l^{(2)}$, in its Appendix 1, Type BC_l , (6), he treated 4 different evaluations which concluded with 4 different η -products. The case treated by V. Kac and D. Peterson corresponds to the case (b) (cf. Remark 3.2). By computing the modular transformations of these 4 η -products, we observed that there seems to be a bigger hidden symmetry in $BC_l^{(2)}$, which inspired us to work on the subject of this article.

In this article, we show that the group $\mathrm{SL}_2(\mathbb{Z})$ acts on the vector space spanned by the characters and twisted-characters of integrable highest weight modules over the affine Lie algebra \mathfrak{g} of type $BC_l^{(2)}$, for each fixed *even* level. Here, we should consider the characters and twisted-characters as functions on a complex subdomain of the dual of the Cartan subalgebra of \mathfrak{g} and introduce *two* coordinate systems dependent on the choice of special indices. Moreover, we observe that the characters and twisted-characters of the affine Lie algebra of type $BC_l^{(2)}$ at *even* level can be identified with the characters and super-characters of the affine Lie superalgebra of type $B^{(1)}(0, l)$!

This article is organized as follows. In Section 1, we recall some basic notions and setting on $BC_l^{(2)}$, in particular, depending on the choice of a special index (see Definition 1.1), we present two coordinate systems of a complex domain in the dual of the Cartan subalgebra of the affine Lie algebra of type $BC_l^{(2)}$. In Section 2, we recall the definition of the (twisted-)character of an object of the category \mathcal{O} of Bernstein-Gelfand-Gelfand. In Sections 3, we introduce theta-series and twisted invariants of the affine Weyl group of type $BC_l^{(2)}$ and express the normalized (twisted-)characters in terms these (twisted-)invariants. In Section 4, we compute the modular transformations of (twisted-)characters. Finally, in Section 5, we interpret the computational results obtained in the previous sections. We conclude with an appendix explaining some relevant basic facts about Lie superalgebras, for the sake of reader’s convenience.

1. KAC-MOODY LIE ALGEBRAS OF TYPE $BC_l^{(2)}$

In this article, we deal with the affine root system of type $BC_l^{(2)}$ ($l \geq 1$) in the sense of K. Saito [11]. Since there are many nomenclatures for affine root systems, we give the correspondence between other nomenclatures.

Saito ([11])	Kac ([4])	Moody ([10])	Macdonald ([9])	Carter ([2])
$BC_l^{(2)}$	$A_{2l}^{(2)}$	$A_{1,2}$ $l=1$ $BC_{l,2}$ $l>1$	$BC_l = BC_l^\vee$	\tilde{C}_l'

1.1. Root datum. For a positive integer l , let $A = (a_{i,j})$ be an $(l+1) \times (l+1)$ generalized Cartan matrices (GCM for short) of type $BC_l^{(2)}$:

$$A = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix} \quad (l=1), \quad A = \begin{pmatrix} 2 & -1 & & & & \\ -2 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -2 & 2 \end{pmatrix} \quad (l \geq 2).$$

In the following, we construct a realization (in the sense of Kac [6]) of A (see (1.1.1) below).

Let F_f be an l -dimensional real vector space equipped with a positive definite symmetric bilinear form $I_f : F_f \times F_f \rightarrow \mathbb{R}$. Fix an orthonormal basis $\varepsilon_1, \dots, \varepsilon_l$ of F_f with respect to I_f . Consider a 2-dimensional extension $\hat{F} := F_f \oplus \mathbb{R}\delta \oplus \mathbb{R}\gamma$ equipped with a non-degenerate symmetric bilinear form \hat{I} on \hat{F} defined by

$$\begin{aligned} \hat{I}|_{F_f \times F_f} &= I_f, & \hat{I}(\varepsilon_i, \delta) &= \hat{I}(\varepsilon_i, \gamma) = 0 \quad (1 \leq i \leq l), \\ \hat{I}(\delta, \delta) &= \hat{I}(\gamma, \gamma) = 0, & \hat{I}(\delta, \gamma) &= 1. \end{aligned}$$

Since \hat{I} is non-degenerate, there is an induced isomorphism $\kappa : \hat{F} \xrightarrow{\sim} \hat{F}^* := \text{Hom}_{\mathbb{R}}(\hat{F}, \mathbb{R})$ defined by

$$\langle \kappa(u), v \rangle = \hat{I}(u, v) \quad \text{for } u, v \in \hat{F},$$

where $\langle \cdot, \cdot \rangle : \hat{F}^* \times \hat{F} \rightarrow \mathbb{R}$ is a canonical pairing. Set

$$(1.1.1) \quad \mathfrak{h} = \hat{F}^* \otimes \mathbb{C}.$$

Then, its complex dual space $\mathfrak{h}^* := \text{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ is naturally identified with $\hat{F} \otimes \mathbb{C}$. The complexification of the isomorphism $\kappa : \hat{F} \xrightarrow{\sim} \hat{F}^*$ is denoted by the same symbol $\kappa : \mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h}$.

Remark 1.1. In Kac's book [6], our $\kappa^{-1} : \mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$ is denoted by ν .

Set

$$\alpha_0 = \delta - 2\varepsilon_1, \quad \alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad (1 \leq i < l), \quad \alpha_l = \varepsilon_l,$$

and define a linearly independent subset Π of \mathfrak{h}^* (called the set of simple roots) by

$$\Pi = \{\alpha_0, \dots, \alpha_l\}.$$

For a non-isotropic vector $\alpha \in \widehat{F}$, denote $\alpha^\vee = 2\alpha/\widehat{I}(\alpha, \alpha)$. Set

$$\Pi^\vee = \{h_0, \dots, h_l\} \subset \mathfrak{h} \quad \text{where } h_i := \kappa(\alpha_i^\vee) \quad (0 \leq i \leq l).$$

It is called the set of simple coroots. Hence, the triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ is a realization of a GCM A . Its Dynkin diagram is given as follows.

$$\begin{array}{ccc} \textcirclearrowleft \xrightarrow{4} \textcirclearrowright & (l=1), & \textcirclearrowleft \xrightarrow{2} \textcirclearrowright \text{---} \textcirclearrowleft \text{---} \textcirclearrowleft \text{---} \textcirclearrowleft \xrightarrow{2} \textcirclearrowright \\ \alpha_0 & \alpha_1 & \alpha_0 \quad \alpha_1 \quad \quad \quad \alpha_{l-1} \quad \alpha_l \end{array} \quad (l \geq 2),$$

The labels a_i ($0 \leq i \leq l$) and co-labels a_i^\vee ($0 \leq i \leq l$) of A are by definition, relatively prime positive integers satisfying

$$\begin{aligned} A^t \mathbf{a} &= t\mathbf{0} \quad \text{where } \mathbf{a} := (a_0, \dots, a_l), \quad \mathbf{0} := (0, \dots, 0) \quad \text{and} \\ \mathbf{a}^\vee A &= \mathbf{0} \quad \text{where } \mathbf{a}^\vee := (a_0^\vee, \dots, a_l^\vee), \end{aligned}$$

respectively. Explicit forms of them are given by

$$a_i = \begin{cases} 1 & (i=0), \\ 2 & (1 \leq i \leq l), \end{cases} \quad a_i^\vee = \begin{cases} 2 & (0 \leq i \leq l-1), \\ 1 & (i=l). \end{cases}$$

It is known that the following formula holds:

$$(1.1.2) \quad \delta = \sum_{i=0}^l a_i \alpha_i.$$

1.2. Kac-Moody Lie algebras and their roots.

1.2.1. Let $\mathfrak{g} = \mathfrak{g}(A)$ be the Kac-Moody Lie algebra associated with the generalized Cartan matrix A of type $BC_l^{(2)}$. It is a Lie algebra over \mathbb{C} generated by e_i, f_i ($0 \leq i \leq l$), $h \in \mathfrak{h}$ subject to the following relations:

- (L1) $[h, h'] = 0$ for $h, h' \in \mathfrak{h}$,
- (L2) $\begin{cases} [h, e_i] = \langle h, \alpha_i \rangle e_i, \\ [h, f_i] = -\langle h, \alpha_i \rangle f_i, \end{cases}$ for $h \in \mathfrak{h}$ and $0 \leq i \leq l$,
- (L3) $[e_i, f_j] = \delta_{i,j} h_i$ for $0 \leq i, j \leq l$,
- (L4) $\begin{cases} \text{ad}(e_i)^{1-a_{i,j}}(e_j) = 0, \\ \text{ad}(f_i)^{1-a_{i,j}}(f_j) = 0, \end{cases}$ for $0 \leq i \neq j \leq l$.

The abelian subalgebra \mathfrak{h} of \mathfrak{g} is called the Cartan subalgebra of \mathfrak{g} .

In the rest of this article, we use standard notations in the theory of Kac-Moody Lie algebras following Kac's book [6]. Namely, let \mathfrak{n}_\pm be the subalgebra of \mathfrak{g} generated by $\{e_i\}_{0 \leq i \leq l}$ (resp. $\{f_i\}_{0 \leq i \leq l}$). We have the triangular decomposition of \mathfrak{g} :

$$(1.2.1) \quad \mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-.$$

Let Δ be the set of roots, Δ_{\pm} the set of positive (resp. negative) roots and Q the root lattice. The Weyl group W is the subgroup of $O(\mathfrak{h}^*, \widehat{I})$ generated by simple reflections s_{α} ($\alpha \in \Pi$).

Let $\Delta^{re} = W\Pi$ (resp. $\Delta^{im} = \Delta \setminus \Delta^{re}$) be the set of real (resp. imaginary) roots. Their explicit forms are given by

$$(1.2.2) \quad \Delta^{re} = \Delta_s^{re} \amalg \Delta_m^{re} \amalg \Delta_l^{re} \quad \text{and} \quad \Delta^{im} = \mathbb{Z}\delta$$

where

$$\begin{aligned} \Delta_s^{re} &= \{\pm\varepsilon_i + r\delta \mid 1 \leq i \leq l, r \in \mathbb{Z}\}, \\ \Delta_m^{re} &= \{\pm\varepsilon_i \pm \varepsilon_j + r\delta \mid 1 \leq i < j \leq l, r \in \mathbb{Z}\}, \\ \Delta_l^{re} &= \{\pm 2\varepsilon_i + (2r+1)\delta \mid 1 \leq i \leq l, r \in \mathbb{Z}\}. \end{aligned}$$

1.2.2.

Definition 1.1. An index $0 \leq i_0 \leq l$ of the set Π of simple roots is called *special* if there exist $p \in \mathbb{Z}_{>0}$ and $\alpha \in \Delta_+$ such that $\delta - a_{i_0}\alpha_{i_0} = p\alpha$.

There are two special indices of Π .

- (i) Since $\delta - a_0\alpha_0 = 2\varepsilon_1$ and $\varepsilon_1 \in \Delta$, the index 0 is special.
- (ii) Since $\delta - a_l\alpha_l = \delta - 2\varepsilon_l \in \Delta$, the index l is special.

(i) We call the index $i_0 = 0$ the special index of type I. For later convenience, we denote $\varepsilon_i^{(I)} = \varepsilon_i$ ($1 \leq i \leq l$), $\alpha_i^{(I)} = \alpha_i$ ($0 \leq i \leq l$) and $F_f^{(I)} = F_f$. It follows that

$$(1.2.3) \quad F_f^{(I)} = \bigoplus_{i=1}^l \mathbb{R}\varepsilon_i^{(I)} = \bigoplus_{i=1}^l \mathbb{R}\alpha_i^{(I)} \quad \text{and} \quad F := \bigoplus_{i=0}^l \mathbb{R}\alpha_i^{(I)} = F_f^{(I)} \oplus \mathbb{R}\delta.$$

(ii) We call the index $i_0 = l$ the special index of type II. Replace the numeration of simple roots with the following:

$$\alpha_i^{(II)} := \alpha_{l-i} \quad (0 \leq i \leq l).$$

In this new numeration, the index 0 is special and the corresponding Dynkin diagram is given by

$$\begin{array}{ccc} \begin{array}{c} 4 \\ \circ \longleftarrow \circ \\ \alpha_0^{(II)} \quad \alpha_1^{(II)} \end{array} & (l=1), & \begin{array}{c} 2 \qquad \qquad \qquad 2 \\ \circ \longleftarrow \circ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \circ \longleftarrow \circ \\ \alpha_0^{(II)} \quad \alpha_1^{(II)} \qquad \qquad \qquad \alpha_{l-1}^{(II)} \quad \alpha_l^{(II)} \end{array} & (l \geq 2). \end{array}$$

Set

$$\varepsilon_i^{(II)} = -\varepsilon_{l+1-i} + \frac{1}{2}\delta \quad (1 \leq i \leq l) \quad \text{and} \quad F_f^{(II)} = \bigoplus_{i=1}^l \mathbb{R}\varepsilon_i^{(II)}.$$

Since

$$\alpha_i^{(II)} = \begin{cases} \varepsilon_i^{(II)} - \varepsilon_{i+1}^{(II)} & (1 \leq i \leq l-1), \\ 2\varepsilon_l^{(II)} & (i=l), \end{cases}$$

we have

$$(1.2.4) \quad F_f^{(\mathbb{I})} = \bigoplus_{i=1}^l \mathbb{R}\alpha_i^{(\mathbb{I})} \quad \text{and} \quad F = F_f^{(\mathbb{I})} \oplus \mathbb{R}\delta.$$

For $\sharp \in \{\mathbb{I}, \mathbb{II}\}$, let $\Lambda_0^{(\sharp)}$ be the element of \mathfrak{h}^* which satisfies

$$(1.2.5) \quad \widehat{I}((\alpha_i^{(\sharp)})^\vee, \Lambda_0^{(\sharp)}) = \delta_{i,0} \quad (0 \leq i \leq l), \quad \widehat{I}(\Lambda_0^{(\sharp)}, \Lambda_0^{(\sharp)}) = 0.$$

These conditions determine $\Lambda_0^{(\sharp)}$ uniquely, and the explicit forms of them are given by

$$(1.2.6) \quad \Lambda_0^{(\sharp)} = \begin{cases} 2\gamma & \text{if } \sharp = \mathbb{I}, \\ \gamma + \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_l) - \frac{l}{8}\delta & \text{if } \sharp = \mathbb{II}. \end{cases}$$

Remark 1.2. *By definition, one has*

$$(1.2.7) \quad \widehat{I}(\delta, \Lambda_0^{(\sharp)}) = \begin{cases} a_0^\vee (= 2) & \text{if } \sharp = \mathbb{I}, \\ a_l^\vee (= 1) & \text{if } \sharp = \mathbb{II}. \end{cases}$$

Hence, we have the following decompositions of \mathfrak{h}^* :

$$(1.2.8) \quad \mathfrak{h}^* = F_{f,\mathbb{C}}^{(\sharp)} \oplus (\mathbb{C}\delta \oplus \mathbb{C}\Lambda_0^{(\sharp)}),$$

where $F_{f,\mathbb{C}}^{(\sharp)} := F_f^{(\sharp)} \otimes \mathbb{C}$. Set

$$(1.2.9) \quad c = \kappa(\delta) \quad \text{and} \quad d^{(\sharp)} = \begin{cases} a_0\kappa(\Lambda_0^{(\mathbb{I})}) = \kappa(\Lambda_0^{(\mathbb{I})}) & \text{if } \sharp = \mathbb{I}, \\ a_l\kappa(\Lambda_0^{(\mathbb{II})}) = 2\kappa(\Lambda_0^{(\mathbb{II})}) & \text{if } \sharp = \mathbb{II}. \end{cases}$$

Then, we have decompositions of \mathfrak{h} :

$$(1.2.10) \quad \mathfrak{h} = \left(\bigoplus_{i=0}^l \mathbb{C}h_i^{(\sharp)} \right) \oplus \mathbb{C}d^{(\sharp)} = \left(\bigoplus_{i=1}^l \mathbb{C}h_i^{(\sharp)} \right) \oplus \mathbb{C}c \oplus \mathbb{C}d^{(\sharp)}.$$

1.2.3. For $\sharp \in \{\mathbb{I}, \mathbb{II}\}$, set

$$(1.2.11) \quad \Delta_f^{(\sharp)} = \Delta^{re} \cap F_{f,\mathbb{C}}^{(\sharp)}.$$

In the following, we study the structure of $\Delta_f^{(\sharp)}$ in detail.

(i) Case of $\sharp = \mathbb{I}$.

By the explicit description (1.2.2) of Δ^{re} , we have

$$\Delta_f^{(\mathbb{I})} = (\Delta_f^{(\mathbb{I})})_s \amalg (\Delta_f^{(\mathbb{I})})_m \quad \text{where} \quad \begin{cases} (\Delta_f^{(\mathbb{I})})_s := \{\pm\varepsilon_i \mid 1 \leq i \leq l\}, \\ (\Delta_f^{(\mathbb{I})})_m := \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq l\}. \end{cases}$$

This is the finite root system of type B_l with a simple system $\Pi_f^{(\mathbb{I})} := \{\alpha_1^{(\mathbb{I})}, \dots, \alpha_l^{(\mathbb{I})}\}$.

(ii) Case of $\sharp = \mathbb{II}$.

By definition, the description (1.2.2) of Δ^{re} can be rewritten as

$$\begin{aligned}\Delta_s^{re} &= \{ \pm \varepsilon_i^{(\mathbb{I})} + (r + \frac{1}{2})\delta \mid 1 \leq i \leq l, r \in \mathbb{Z} \}, \\ \Delta_m^{re} &= \{ \pm \varepsilon_i^{(\mathbb{I})} \pm \varepsilon_j^{(\mathbb{I})} + r\delta \mid 1 \leq i < j \leq l, r \in \mathbb{Z} \}, \\ \Delta_l^{re} &= \{ \pm 2\varepsilon_i^{(\mathbb{I})} + 2r\delta \mid 1 \leq i \leq l, r \in \mathbb{Z} \}.\end{aligned}$$

Therefore, we have

$$\Delta_f^{(\mathbb{I})} = (\Delta_f^{(\mathbb{I})})_m \amalg (\Delta_f^{(\mathbb{I})})_l \quad \text{where} \quad \begin{cases} (\Delta_f^{(\mathbb{I})})_m := \{ \pm \varepsilon_i^{(\mathbb{I})} \pm \varepsilon_j^{(\mathbb{I})} \mid 1 \leq i < j \leq l \}, \\ (\Delta_f^{(\mathbb{I})})_l := \{ \pm 2\varepsilon_i^{(\mathbb{I})} \mid 1 \leq i \leq l \}.\end{cases}$$

This is the finite root system of type C_l with a simple system $\Pi_f^{(\mathbb{I})} := \{\alpha_1^{(\mathbb{I})}, \dots, \alpha_l^{(\mathbb{I})}\}$.

Let $\varpi_i^{(\#)} \in F_f^{(\#)}$ ($1 \leq i \leq l$) be fundamental weights of $\Delta_f^{(\#)}$ with respect to the simple system $\Pi_f^{(\#)}$. They are characterized by $\widehat{I}(\varpi_i^{(\#)}, (\alpha_j^{(\#)})^\vee) = \delta_{i,j}$. Define

$$\rho_f^{(\#)} = \sum_{i=1}^l \varpi_i^{(\#)}.$$

Hence, one has

$$\widehat{I}(\rho_f^{(\#)}, (\alpha_j^{(\#)})^\vee) = 1 \quad \text{for every } 1 \leq j \leq l.$$

Remark 1.3. *The explicit forms of them are given as follows:*

$$\begin{aligned}\varpi_i^{(\mathbb{I})} &= \begin{cases} \sum_{j=1}^i \varepsilon_j^{(\mathbb{I})} & (1 \leq i \leq l-1), \\ \frac{1}{2} \sum_{j=1}^l \varepsilon_j^{(\mathbb{I})} & (i = l), \end{cases} & \varpi_i^{(\mathbb{II})} &= \sum_{j=1}^i \varepsilon_j^{(\mathbb{II})} \quad (1 \leq i \leq l), \\ \rho_f^{(\#)} &= \begin{cases} \sum_{i=1}^l \frac{2(l-i)+1}{2} \varepsilon_i^{(\mathbb{I})} & \text{if } \# = \mathbb{I}, \\ \sum_{i=1}^l (l-i+1) \varepsilon_i^{(\mathbb{II})} & \text{if } \# = \mathbb{II}. \end{cases}\end{aligned}$$

1.3. The affine Weyl group W . In this subsection, we study the structure of the affine Weyl group W .

1.3.1. Let $\mu \in F$ and a non-isotropic $\beta \in F$. Define linear automorphisms t_μ and s_β of \mathfrak{h}^* by

$$\begin{aligned}t_\mu(u) &= u + \widehat{I}(u, \delta)\mu - \left\{ \widehat{I}(u, \mu) + \frac{1}{2} \widehat{I}(\mu, \mu) \widehat{I}(u, \delta) \right\} \delta, \\ s_\beta(u) &= u - \widehat{I}(u, \beta^\vee)\beta,\end{aligned}$$

for $u \in \mathfrak{h}^*$, respectively. The following lemma is easily verified.

Lemma 1.4. (1) *Both t_μ and s_β belong to $O(\mathfrak{h}^*, \widehat{I})$.*

(2) *For every $\mu_1, \mu_2 \in F$, one has $t_{\mu_1} t_{\mu_2} = t_{\mu_1 + \mu_2}$.*

(3) *For every non-isotropic $\beta \in F$, one has $s_{\delta - \beta} s_\beta = t_{\beta^\vee}$.*

(4) *For $w \in W$ and $\mu \in F$, one has $w \circ t_\mu \circ w^{-1} = t_{w(\mu)}$.*

1.3.2. For $\sharp \in \{\text{I}, \text{II}\}$, let us give a semidirect product description of W attached to the choice of the special index. Recall the finite root subsystem $\Delta_f^{(\sharp)}$ of Δ^{re} introduced in (1.2.11), and let $W_f^{(\sharp)}$ be its Weyl group. Define a lattice $M^{(\sharp)}$ of rank l as follows.

(i) First, assume $\sharp = \text{I}$. Let $\theta_s^{(\text{I})} = \varepsilon_1^{(\text{I})}$ be the highest short root of $\Delta_f^{(\text{I})}$. Since $\alpha_0^{(\text{I})} = \delta - 2\theta_s^{(\text{I})}$, we have

$$s_{\alpha_0^{(\text{I})}} s_{\theta_s^{(\text{I})}} = s_{\delta - 2\theta_s^{(\text{I})}} s_{2\theta_s^{(\text{I})}} = t_{(2\theta_s^{(\text{I})})^\vee}.$$

Therefore, the group W is generated by $t_{(2\theta_s^{(\text{I})})^\vee}$ and $W_f^{(\text{I})}$. Set

$$M^{(\text{I})} = \mathbb{Z}\text{-span of } W_f^{(\text{I})}((2\theta_s^{(\text{I})})^\vee).$$

Since $(2\theta_s^{(\text{I})})^\vee = \varepsilon_1^{(\text{I})}$ and $W_f^{(\text{I})}$ is generated by $s_{\alpha_i^{(\text{I})}}$ ($1 \leq i \leq l$), we have

$$M^{(\text{I})} = \bigoplus_{i=1}^l \mathbb{Z}\varepsilon_i^{(\text{I})}.$$

(ii) Second, assume $\sharp = \text{II}$. Let $\theta_l^{(\text{II})} = 2\varepsilon_1^{(\text{II})}$ be the highest (long) root of $\Delta_f^{(\text{II})}$. As $2\alpha_0^{(\text{II})} = \delta - \theta_l^{(\text{II})}$, we have the group W is generated by $t_{(\theta_l^{(\text{II})})^\vee}$ and $W_f^{(\text{II})}$. Since $(\theta_l^{(\text{II})})^\vee = \varepsilon_1^{(\text{II})}$ and $W_f^{(\text{II})}$ is generated by $s_{\alpha_i^{(\text{II})}}$ ($1 \leq i \leq l$), we have

$$M^{(\text{II})} := \mathbb{Z}\text{-span of } W_f^{(\text{II})}((\theta_l^{(\text{II})})^\vee) = \bigoplus_{i=1}^l \mathbb{Z}\varepsilon_i^{(\text{II})}.$$

By Lemma 1.4, we have the following lemma.

Lemma 1.5. *Set $t(M^{(\sharp)}) = \{t_\mu \mid \mu \in M^{(\sharp)}\}$. This is an abelian normal subgroup of W , and we have a semidirect product description of W :*

$$W = W_f^{(\sharp)} \ltimes t(M^{(\sharp)}).$$

1.4. The complex domain \mathcal{Y} and its coordinate systems.

1.4.1. Define a complex domain $\mathcal{Y} \subset \mathfrak{h}^*$ by

$$(1.4.1) \quad \mathcal{Y} = \{v \in \mathfrak{h}^* \mid \text{Re } \widehat{I}(v, \delta) > 0\}.$$

Let \mathbb{H} be the Poincaré upper half plane. For $\sharp \in \{\text{I}, \text{II}\}$, define a map $\varphi^{(\sharp)} : \mathcal{Y} \rightarrow Y := \mathbb{H} \times \mathbb{C}^l \times \mathbb{C}$ as follows. Let φ_δ , $\varphi_i^{(\sharp)}$ ($1 \leq i \leq l$), $\varphi_{\Lambda_0^{(\sharp)}}$ be functions on \mathcal{Y} defined by

$$\varphi_\delta(v) = -\frac{\widehat{I}(v, \delta)}{2\pi\sqrt{-1}}, \quad \varphi_i^{(\sharp)}(v) = \frac{\widehat{I}(v, \varepsilon_i^{(\sharp)})}{2\pi\sqrt{-1}}, \quad \varphi_{\Lambda_0^{(\sharp)}}(v) = \frac{\widehat{I}\left(v, \frac{1}{(a_0^{(\sharp)})^\vee} \Lambda_0^{(\sharp)}\right)}{2\pi\sqrt{-1}},$$

for $v \in \mathcal{Y}$, where we set $(a_0^{(\sharp)})^\vee = a_0^\vee$ if $\sharp = \text{I}$ and $(a_0^{(\sharp)})^\vee = a_l^\vee$ if $\sharp = \text{II}$. Set

$$\varphi^{(\sharp)} : \mathcal{Y} \ni v \longmapsto (\varphi_\delta(v), (\varphi_1^{(\sharp)}(v), \dots, \varphi_l^{(\sharp)}(v)), \varphi_{\Lambda_0}^{(\sharp)}(v)) \in Y.$$

Since $\varphi^{(\sharp)}$ is an isomorphism of complex domains for each $\sharp = \{\text{I}, \text{II}\}$, we have the following commutative diagram:

$$\begin{array}{ccc} & \mathcal{Y} & \\ \varphi^{(\text{I})} \swarrow & \circlearrowleft & \searrow \varphi^{(\text{II})} \\ Y & \xleftrightarrow{\varphi^{(\text{II})} \circ (\varphi^{(\text{I})})^{-1}} & Y \\ & \xleftrightarrow{\varphi^{(\text{I})} \circ (\varphi^{(\text{II})})^{-1}} & \end{array}$$

Lemma 1.6. Denote $(\tau, \underline{z}, t) \in Y$ where $\tau \in \mathbb{H}$, $\underline{z} := (z_1, \dots, z_l) \in \mathbb{C}^l$, $t \in \mathbb{C}$. For every pair (\sharp, \flat) such that $\sharp, \flat \in \{\text{I}, \text{II}\}$ and $\sharp \neq \flat$, the explicit form of the map $\varphi^{(\sharp)} \circ (\varphi^{(\flat)})^{-1} : Y \xrightarrow{\sim} Y$ is given by

$$(1.4.2) \quad (\tau, \underline{z}, t) \longmapsto \left(\tau, \left(-z_l - \frac{1}{2}\tau, \dots, -z_1 - \frac{1}{2}\tau \right), t + \frac{l}{8}\tau + \frac{1}{2} \sum_{i=1}^l z_i \right).$$

Proof. First, we give the proof for the case that $\sharp = \text{I}$ and $\flat = \text{II}$. For $(\tau, \underline{z}, t) \in Y$, we have

$$(\varphi^{(\text{II})})^{-1}(\tau, \underline{z}, t) = 2\pi\sqrt{-1} \left(-\tau\Lambda_0^{(\text{II})} + \sum_{i=1}^l z_i \varepsilon_i^{(\text{II})} + t\delta \right).$$

By definition, we have

the right hand side

$$\begin{aligned} &= 2\pi\sqrt{-1} \left[-\tau \left(\frac{1}{2}\Lambda_0^{(\text{I})} + \frac{1}{2} \sum_{i=1}^l \varepsilon_i^{(\text{I})} - \frac{l}{8}\delta \right) + \sum_{i=1}^l z_i \left(-\varepsilon_{l+1-i}^{(\text{I})} + \frac{1}{2}\delta \right) + t\delta \right] \\ &= 2\pi\sqrt{-1} \left[-\frac{1}{2}\tau\Lambda_0^{(\text{I})} + \sum_{i=1}^l \left(-z_{l+1-i} - \frac{1}{2}\tau \right) \varepsilon_i^{(\text{I})} + \left(t + \frac{l}{8}\tau + \frac{1}{2} \sum_{i=1}^l z_i \right) \delta \right]. \end{aligned}$$

Thus, we have the formula (1.4.2) in this case.

Second, by the above result, it follows immediately that

$$(\varphi^{(\text{I})} \circ (\varphi^{(\text{II})})^{-1})^2 = \text{id}_Y.$$

Therefore, we have

$$\varphi^{(\text{II})} \circ (\varphi^{(\text{I})})^{-1} = (\varphi^{(\text{I})} \circ (\varphi^{(\text{II})})^{-1})^{-1} = \varphi^{(\text{I})} \circ (\varphi^{(\text{II})})^{-1}$$

as desired. \square

1.4.2. Introduce an isometry $\phi \in O(\mathfrak{h}^*, \hat{I})$ as follows. Define $w_0^{A_l} \in O(\mathfrak{h}^*, \hat{I})$ by $w_0^{A_l} = s_{\alpha_1^{(\text{I})}}(s_{\alpha_2^{(\text{I})}} s_{\alpha_1^{(\text{I})}}) \cdots (s_{\alpha_{l-1}^{(\text{I})}} \cdots s_{\alpha_1^{(\text{I})}})$. One has

$$w_0^{A_l}(\varepsilon_i) = \varepsilon_{l+1-i} \quad \text{for } 1 \leq i \leq l, \quad w_0^{A_l}(\delta) = \delta \quad \text{and} \quad w_0^{A_l}(\Lambda_0^{(\text{I})}) = \Lambda_0^{(\text{I})}$$

by direct computation. Introduce another isometry $\zeta_{F_f} \in \mathbf{O}(\mathfrak{h}^*, \widehat{I})$ by

$$\zeta_{F_f}(\varepsilon_i) = -\varepsilon_i \quad \text{for } 1 \leq i \leq l, \quad \zeta_{F_f}(\delta) = \delta \quad \text{and} \quad \zeta_{F_f}(\Lambda_0^{(\mathbf{I})}) = \Lambda_0^{(\mathbf{I})}.$$

Set

$$(1.4.3) \quad \phi = t_{(\varepsilon_1 + \dots + \varepsilon_l)/2} \circ w_0^A \circ \zeta_{F_f} \in \mathbf{O}(\mathfrak{h}^*, \widehat{I}).$$

The next lemma follows from direct computation.

Lemma 1.7. *One has*

- (1) $\phi(\varepsilon_i^{(\mathbf{I})}) = \varepsilon_i^{(\mathbf{II})}$ ($1 \leq i \leq l$), $\phi(\delta) = \delta$, $\phi(\Lambda_0^{(\mathbf{I})}) = 2\Lambda_0^{(\mathbf{II})}$,
- (2) ϕ is an involution.

Recall the decomposition (1.2.8) of \mathfrak{h}^* , and let $\pi^{(\sharp)} : \mathfrak{h}^* \rightarrow F_{f, \mathbb{C}}^{(\sharp)} \subset \mathfrak{h}^*$ be the canonical projection for $\sharp \in \{\mathbf{I}, \mathbf{II}\}$. The next lemma is a direct consequence of the previous lemma.

Lemma 1.8. (1) *We have*

$$\phi \circ \pi^{(\mathbf{II})} = \pi^{(\mathbf{I})} \circ \phi, \quad \phi \circ \pi^{(\mathbf{I})} = \pi^{(\mathbf{II})} \circ \phi.$$

(2) *There is the following commutative diagram.*

$$\begin{array}{ccc} & \phi & \\ \mathcal{Y} & \xleftrightarrow{\quad} & \mathcal{Y} \\ & \phi & \\ \varphi^{(\mathbf{I})} \searrow & \circlearrowleft & \swarrow \varphi^{(\mathbf{II})} \\ & Y & \end{array}$$

For later use, we set

$$(1.4.4) \quad \text{pr}^{(\sharp)} = \pi^{(\sharp)} \circ (\varphi^{(\sharp)})^{-1}.$$

1.5. Weight lattice. For $1 \leq j \leq l$ and $\sharp \in \{\mathbf{I}, \mathbf{II}\}$, define the j -th (affine) fundamental weight $\Lambda_j^{(\sharp)}$ (of type (\sharp)) by the condition that

$$\widehat{I}((\alpha_i^{(\sharp)})^\vee, \Lambda_j^{(\sharp)}) = \delta_{i,j} \quad \text{for every } 0 \leq i \leq l.$$

Note that this condition determines $\Lambda_j^{(\sharp)}$ modulo $\mathbb{C}\delta$.

Remark 1.9. *By direct computation, we have explicit forms of them as follows.*

$$\Lambda_j^{(\mathbf{I})} \equiv \begin{cases} \varpi_j^{(\mathbf{I})} + \Lambda_0^{(\mathbf{I})} \pmod{\mathbb{C}\delta} & (1 \leq j < l), \\ \varpi_l^{(\mathbf{I})} + \frac{1}{2}\Lambda_0^{(\mathbf{I})} \pmod{\mathbb{C}\delta} & (j = l), \end{cases} \quad \Lambda_j^{(\mathbf{II})} \equiv \Lambda_{l-j}^{(\mathbf{I})} \pmod{\mathbb{C}\delta}.$$

Define subsets P and P_+ of \mathfrak{h}^* by

$$P = \left(\bigoplus_{j=0}^l \mathbb{Z}\Lambda_j^{(\mathbf{I})} \right) + \mathbb{C}\delta \quad \text{and} \quad P_+ = \left(\sum_{j=0}^l \mathbb{Z}_{\geq 0}\Lambda_j^{(\mathbf{I})} \right) + \mathbb{C}\delta.$$

We call P the weight lattice, and P_+ the set of dominant weights, respectively.

Remark 1.10. As we already mentioned in Remark 1.9, we have $\Lambda_j^{(\mathbb{I})} \equiv \Lambda_{l-j}^{(\mathbb{I})} \pmod{\mathbb{C}\delta}$. Therefore, we have

$$P = \left(\bigoplus_{j=0}^l \mathbb{Z}\Lambda_j^{(\mathbb{I})} \right) + \mathbb{C}\delta \quad \text{and} \quad P_+ = \left(\sum_{j=0}^l \mathbb{Z}_{\geq 0}\Lambda_j^{(\mathbb{I})} \right) + \mathbb{C}\delta.$$

For $\lambda \in P$, the number $k := \widehat{I}(\delta, \lambda)$ is called the **level** of λ . In the case of type $BC_l^{(2)}$, the levels of fundamental weights are given in the following table.

j	0	1	\dots	$l-1$	l
level of $\Lambda_j^{(\mathbb{I})}$	2	2	\dots	2	1
level of $\Lambda_j^{(\mathbb{II})}$	1	2	\dots	2	2

For $k \in \mathbb{Z}_{\geq 0}$, set

$$P_k = \{\lambda \in P \mid \widehat{I}(\delta, \lambda) = k\} \quad \text{and} \quad P_{k,+} = P_k \cap P_+.$$

For $\sharp \in \{\mathbb{I}, \mathbb{II}\}$, define $\rho^{(\sharp)} \in P_+$ by

$$\rho^{(\sharp)} = \sum_{j=0}^l \Lambda_j^{(\sharp)}.$$

Since fundamental weights can only be determined modulo $\mathbb{C}\delta$, $\rho^{(\sharp)}$ is determined modulo $\mathbb{C}\delta$ as well. Furthermore, as $\Lambda_j^{(\mathbb{II})} \equiv \Lambda_{l-j}^{(\mathbb{I})} \pmod{\mathbb{C}\delta}$, we have

$$\rho^{(\mathbb{II})} \equiv \rho^{(\mathbb{I})} \pmod{\mathbb{C}\delta}.$$

That is, $\rho^{(\mathbb{II})}$ coincides with $\rho^{(\mathbb{I})}$ as an affine weight.

Let ρ be any element of $\rho^{(\mathbb{I})} + \mathbb{C}\delta = \rho^{(\mathbb{II})} + \mathbb{C}\delta$. Note that, by definition, we have the level of ρ is $2l + 1$.

2. CHARACTERS

2.1. Algebra \mathcal{E} , formal characters. Recall the triangular decomposition (1.2.1) of \mathfrak{g} : $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$.

Let V be a \mathfrak{h} -diagonalizable module, i.e., $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ where $V_\lambda := \{v \in V \mid h.v = \langle h, \lambda \rangle v \quad h \in \mathfrak{h}\}$. Set $\mathcal{P}(V) := \{\lambda \in \mathfrak{h}^* \mid V_\lambda \neq \{0\}\}$. Let \mathcal{O} be the BGG category of \mathfrak{g} -modules, that is, it is the subcategory of \mathfrak{g} -modules whose objects are \mathfrak{h} -diagonalizable \mathfrak{g} -modules $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ satisfying

- (i) $\dim V_\lambda < \infty$ for any $\lambda \in \mathcal{P}(V)$,
- (ii) there exists $\lambda_1, \lambda_2, \dots, \lambda_r \in \mathfrak{h}^*$ such that $\mathcal{P}(V) \subset \bigcup_{i=1}^r (\lambda_i - \mathbb{Z}_{\geq 0}\Pi)$.

A typical object of this category is a so-called highest weight module defined as follows. We say that a \mathfrak{g} -module V is a highest weight module with highest weight $\Lambda \in \mathfrak{h}^*$ if

- (i) $\dim V_\Lambda = 1$

(ii) $\mathfrak{n}_+.V_\Lambda = \{0\}$ and $V = U(\mathfrak{g}).V_\Lambda$.

In particular, the last condition implies that $V = U(\mathfrak{n}_-).V_\Lambda$ as a vector space, hence

$$\mathcal{P}(V) \subset \Lambda - \mathbb{Z}_{\geq 0}\Pi.$$

Such an example is given as follows. For $\Lambda \in \mathfrak{h}^*$, let $\mathbb{C}_\Lambda = \mathbb{C}v_\Lambda$ be the one dimensional module over $\mathfrak{b}_+ := \mathfrak{h} \oplus \mathfrak{n}_+$ defined by

$$h.v_\Lambda = \langle h, \Lambda \rangle v_\Lambda \quad (h \in \mathfrak{h}), \quad \mathfrak{n}_+.v_\Lambda = 0.$$

The induced \mathfrak{g} -module $M(\Lambda) := \text{Ind}_{\mathfrak{b}_+}^{\mathfrak{g}} \mathbb{C}_\Lambda$ is called the Verma module with highest weight Λ . It can be shown that for any highest weight \mathfrak{g} -module V with highest weight $\Lambda \in \mathfrak{h}^*$, there exists a surjective \mathfrak{g} -module map $M(\Lambda) \twoheadrightarrow V$. The smallest among such V can be obtained by taking the quotient of $M(\Lambda)$ by its maximal proper submodule and the resulting \mathfrak{g} -module is the irreducible highest weight \mathfrak{g} -module with highest weight Λ , denoted by $L(\Lambda)$.

Let \mathcal{E} be the formal linear combination of e^λ ($\lambda \in \mathfrak{h}^*$) with the next condition: $\sum_\lambda c_\lambda e^\lambda \in \mathcal{E} \Rightarrow \exists \lambda_1, \lambda_2, \dots, \lambda_r \in \mathfrak{h}^*$ such that

$$\{\lambda | c_\lambda \neq 0\} \subset \bigcup_{i=1}^r (\lambda_i - \mathbb{Z}_{\geq 0}\Pi).$$

We introduce the ring structure on \mathcal{E} by $e^\lambda \cdot e^\mu := e^{\lambda+\mu}$.

The formal character of $V \in \mathcal{O}$ is, by definition, the element $\text{ch } V \in \mathcal{E}$ defined by

$$\text{ch } V = \sum_{\lambda \in \mathcal{P}(V)} (\dim V_\lambda) e^\lambda.$$

$\text{ch}(\cdot)$ can be viewed as an additive function defined on \mathcal{O} with values in \mathcal{E} . For example, The formal character of the Verma module $M(\Lambda)$ is given by

$$\text{ch } M(\Lambda) = e^\Lambda \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{-\text{mult}(\alpha)},$$

where Δ_+ is the set of positive roots of \mathfrak{g} and $\text{mult}(\alpha) = \dim \mathfrak{g}_\alpha$ is the multiplicity of the root α . Set $\varepsilon(w) = (-1)^{l(w)}$ where $l(w)$ is the length of an element $w \in W$. For $\Lambda \in P_+$, the character of $L(\Lambda)$ is known as Weyl-Kac character formula and is given by

$$(2.1.1) \quad \text{ch } L(\Lambda) = \frac{\sum_{w \in W} \varepsilon(w) e^{w(\Lambda+\rho)-\rho}}{\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\text{mult}(\alpha)}},$$

where $\rho \in \mathfrak{h}^*$ is a so-called Weyl vector, i.e., it satisfies $\langle h_i, \rho \rangle = 1$ for any $0 \leq i \leq l$. In particular, for $\Lambda = 0$, as $L(0) = \mathbb{C}$ is the trivial representation, one obtains the so-called denominator identity:

$$(2.1.2) \quad \sum_{w \in W} \varepsilon(w) e^{w(\rho)} = e^\rho \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\text{mult}(\alpha)}.$$

This implies that the Weyl-Kac character formula can be rephrased as follows:

$$(2.1.3) \quad \text{ch } L(\Lambda) = \frac{\sum_{w \in W} \varepsilon(w) e^{w(\Lambda + \rho)}}{\sum_{w \in W} \varepsilon(w) e^{w(\rho)}}.$$

2.2. Nice functions on the Weyl group W and twisted character.

Let $\Delta^{re} = \bigsqcup_{k \in S} O_k$ be the orbit space decomposition of Δ^{re} with respect to the W -action with the set S of parameters. Denote the index set of the simple system Π by $I = \{0, 1, \dots, l\}$ and set $I_k = \{i \in I \mid \alpha_i \in \Pi \cap O_k\}$. Then, we have a partition $I = \bigsqcup_{k \in S} I_k$ of the index set I of Π .

For any map $\varphi : S \rightarrow \{\pm 1\}$, there exists a morphism of groups $\psi : W \rightarrow \{\pm 1\}$ verifying $\psi(s_{\alpha_i}) = \varphi(k)$ for any $i \in I_k$. This map is well-defined as the Weyl group W is a Coxeter group. Let S_{odd} be the subset of S consisting of those k such that for any $i \in I_k$, $I(\alpha_i^\vee, \alpha_j) \in 2\mathbb{Z}$ for any $j \in I$. For a subset $S' \subset S_{\text{odd}}$, set $\tau = \bigcup_{k \in S'} I_k$. Indeed, in our case, τ can be either \emptyset or $\{l\}$. The next two maps are typical examples of the above ψ :

- (1) $\varepsilon(s_{\alpha_i}) = -1 \quad \forall i \in I$,
- (2) $\varepsilon'(s_{\alpha_i}) = 1$ for $i \in I \setminus \tau$ and $\varepsilon'(s_{\alpha_i}) = -1$ otherwise, i.e., for $i \in \tau$.

V. Kac introduced the next notion in pp. 100 of [5]: a morphism of groups $\psi : W \rightarrow \{\pm 1\}$ is said to be **nice** if there exists a morphism of groups $\bar{\psi} : \mathbb{Z}\Pi \rightarrow \mathbb{Z}/2\mathbb{Z}$ satisfying $\psi(s_\alpha) = (-1)^{\bar{\psi}(\alpha)}$. A necessary and sufficient condition for ψ to be nice is that $\psi(s_{\alpha_i}) = -1$ for an $i \in I$ only if i -th line of the GCM A , i.e., $a_{i,j}$ is even for any $j \in I$. Hence, $\psi = 1$ or $\psi = \varepsilon'$ are examples of nice morphisms.

Remark 2.1. *Regard the GCM A as the GCM of the affine Lie superalgebra of type $B^{(1)}(0, l)$. Then, for the latter case, i.e., $\psi = \varepsilon'$ with $\tau = \{l\}$, the map $\bar{\psi}$ coincides with the parity map $|\cdot| : Q(B^{(1)}(0, l)) \rightarrow \mathbb{Z}/2\mathbb{Z}$.*

In the rest of this article, we always consider the case when $A = (a_{i,j})$ is the GCM of type $BC_l^{(2)}$ and use the nice map ψ defined as follows:

$$(2.2.1) \quad \psi : W \longrightarrow \{\pm 1\}; \quad s_{\alpha_i^{(1)}} \longmapsto \begin{cases} 1 & 0 \leq i < l, \\ -1 & i = l. \end{cases}$$

Remark 2.2. *By definition, we have $\psi(s_{\alpha_i^{(\Pi)}}) = (-1)^{\delta_{i,0}}$ which implies that $W_f^{(\Pi)} \subset \text{Ker}\psi$ whereas $W_f^{(1)} \not\subset \text{Ker}\psi$. Indeed, $W_f^{(1)} \cap \text{Ker}\psi \subset W_f^{(1)}$ is a subgroup of index 2, which is isomorphic to $W(D_l)$. We denote the subgroup $W_f^{(1)} \cap \text{Ker}\psi$, which is generated by the reflections with respect to the middle length roots of $\Delta_f^{(1)}$, by $W_{f;m}^{(1)}$.*

For a highest weight $\mathfrak{g}(A)$ -module V with highest weight Λ , we define its **twisted character** $\text{ch}^\psi(V)$ as follows:

$$\text{ch}^\psi(V) = \sum_{\lambda \in \mathcal{P}(V)} (-1)^{\bar{\psi}(\Lambda - \lambda)} (\dim V_\lambda) e^\lambda \in \mathcal{E},$$

where $V = \bigoplus_{\lambda \in \mathcal{P}(V)} V_\lambda$ signifies its weight space decomposition. It has been essentially shown by V. G. Kac [5] that, for an even level $\Lambda \in P_+$, one has

$$(2.2.2) \quad \text{ch}^\psi L(\Lambda) = \frac{\sum_{w \in W} \varepsilon(w) \psi(w) e^{w(\Lambda + \rho)}}{\sum_{w \in W} \varepsilon(w) \psi(w) e^{w(\rho)}}.$$

3. THETA FUNCTIONS

For $\lambda, \mu \in \mathfrak{h}^*$, set $e^\lambda(\mu) = e^{\hat{I}(\lambda, \mu)}$. In this way, one can regard the formal (twisted-)characters as functions on a certain domain in \mathfrak{h}^* . Here, we introduce θ -series and regard them as analytic functions.

3.1. Theta functions. Let $k \in \mathbb{Z}_{>0}$ be a positive integer. For $\lambda \in P_{k,+}$ and $\sharp \in \{\text{I}, \text{II}\}$, we define the formal theta series ${}^{(\sharp)}\Theta_\lambda$ and its twisted form ${}^{(\sharp)}\Theta_\lambda^\psi$ as follows:

$$\begin{aligned} {}^{(\sharp)}\Theta_\lambda &= e^{-\frac{\hat{I}(\lambda, \lambda)}{2k}\delta} \sum_{\gamma \in M^{(\sharp)}} e^{t_\gamma(\lambda)} = \sum_{\gamma \in M^{(\sharp)}} e^{\lambda + k\gamma - \frac{1}{2k}\hat{I}(\lambda + k\gamma, \lambda + k\gamma)\delta}, \\ {}^{(\sharp)}\Theta_\lambda^\psi &= e^{-\frac{\hat{I}(\lambda, \lambda)}{2k}\delta} \sum_{\gamma \in M^{(\sharp)}} \psi(t_\gamma) e^{t_\gamma(\lambda)} = \sum_{\gamma \in M^{(\sharp)}} \psi(t_\gamma) e^{\lambda + k\gamma - \frac{1}{2k}\hat{I}(\lambda + k\gamma, \lambda + k\gamma)\delta}, \end{aligned}$$

where $\psi : W \rightarrow \{\pm 1\}$ is the nice function on W defined in (2.2.1). They can be viewed as complex analytic functions on the domain Y defined in §1.4.

By direct computations, one has, for $\sharp \in \{\text{I}, \text{II}\}$ and $\underline{y} = (\tau, \underline{z}, t) \in Y$,

$$\begin{aligned} & ({}^{(\sharp)}\Theta_\lambda \circ (\varphi^{(\sharp)})^{-1})(\tau, \underline{z}, t) \\ &= e^{2\hat{I}(\lambda, \delta)\pi\sqrt{-1}t} \sum_{\gamma \in M^{(\sharp)}} e^{\hat{I}(\lambda, \delta)\pi\sqrt{-1}\tau \|\gamma + \frac{1}{k}\pi^{(\sharp)}(\lambda)\|^2 + 2\hat{I}(\lambda, \delta)\pi\sqrt{-1}\hat{I}(\gamma + \frac{1}{k}\pi^{(\sharp)}(\lambda), \text{pr}^{(\sharp)}(\underline{y}))}, \\ & ({}^{(\sharp)}\Theta_\lambda^\psi \circ (\varphi^{(\sharp)})^{-1})(\tau, \underline{z}, t) \\ &= e^{2\hat{I}(\lambda, \delta)\pi\sqrt{-1}t} \sum_{\gamma \in M^{(\sharp)}} \psi(t_\gamma) e^{\hat{I}(\lambda, \delta)\pi\sqrt{-1}\tau \|\gamma + \frac{1}{k}\pi^{(\sharp)}(\lambda)\|^2 + 2\hat{I}(\lambda, \delta)\pi\sqrt{-1}\hat{I}(\gamma + \frac{1}{k}\pi^{(\sharp)}(\lambda), \text{pr}^{(\sharp)}(\underline{y}))}, \end{aligned}$$

where $\text{pr}^{(\sharp)}$ is defined in (1.4.4). Fix $k \in \mathbb{Z}_{>0}$. Now, we introduce W -anti invariants, or equivalently, ε -twisted W -invariants: for $\lambda \in P_{k,+}$,

$$A_{\lambda+\rho} := e^{-\frac{\hat{I}(\lambda+\rho, \lambda+\rho)}{2(k+2l+1)}\delta} \sum_{w \in W} \varepsilon(w) e^{w(\lambda+\rho)}.$$

By Lemma 1.5 and the fact that $\varepsilon(t_\gamma) = 1$ for any $\gamma \in M^{(\sharp)}$, it follows that

$$A_{\lambda+\rho} = \sum_{u \in W_f^{(\sharp)}} \varepsilon(u) {}^{(\sharp)}\Theta_{u(\lambda+\rho)},$$

for any $\sharp \in \{I, II\}$. Similarly, we introduce $\varepsilon\psi$ -twisted W -invariants: for $\lambda \in P_{k,+}$,

$$A_{\lambda+\rho}^\psi := e^{-\frac{\hat{I}(\lambda+\rho, \lambda+\rho)}{2(k+2l+1)}\delta} \sum_{w \in W} \varepsilon(w)\psi(w)e^{w(\lambda+\rho)}.$$

By Remark 2.2, we can rewrite it as follows:

$$\begin{aligned} A_{\lambda+\rho}^\psi &= \sum_{u \in W_f^{(\sharp)}} \varepsilon(u)\psi(u)^{(\sharp)} \Theta_{u(\lambda+\rho)}^\psi \\ &= \begin{cases} \sum_{u \in W_{f;m}^{(I)}} \varepsilon(u) \left({}^{(I)}\Theta_{u(\lambda+\rho)}^\psi + {}^{(I)}\Theta_{u s_{\alpha_l^{(I)}}(\lambda+\rho)}^\psi \right) & \sharp = I, \\ \sum_{u \in W_f^{(II)}} \varepsilon(u) {}^{(II)}\Theta_{u(\lambda+\rho)}^\psi & \sharp = II, \end{cases} \end{aligned}$$

where the group $W_{f;m}^{(I)}$ is defined in Remark 2.2.

Remark 3.1. For $\lambda = 0$, one has the following (*twisted-*)denominator:

$$\begin{aligned} A_\rho &= e^{\rho - \frac{\hat{I}(\rho, \rho)}{2(2l+1)}\delta} \prod_{n=1}^{\infty} (1 - e^{-n\delta})^l \prod_{\alpha \in (\Delta_f^{(I)})_s^+} (1 - e^{-\alpha}) \prod_{\alpha \in (\Delta_f^{(I)})_m^+} (1 - e^{-\alpha}) \\ &\quad \times \prod_{n=1}^{\infty} \left(\prod_{\alpha \in (\Delta_f^{(I)})_s} (1 - e^{-\alpha - n\delta})(1 - e^{-2\alpha - (2n-1)\delta}) \prod_{\alpha \in (\Delta_f^{(I)})_m} (1 - e^{-\alpha - n\delta}) \right), \\ A_\rho^\psi &= e^{\rho - \frac{\hat{I}(\rho, \rho)}{2(2l+1)}\delta} \prod_{n=1}^{\infty} (1 - e^{-n\delta})^l \prod_{\alpha \in (\Delta_f^{(I)})_s^+} (1 + e^{-\alpha}) \prod_{\alpha \in (\Delta_f^{(I)})_m^+} (1 - e^{-\alpha}) \\ &\quad \times \prod_{n=1}^{\infty} \left(\prod_{\alpha \in (\Delta_f^{(I)})_s} (1 + e^{-\alpha - n\delta})(1 - e^{-2\alpha - (2n-1)\delta}) \prod_{\alpha \in (\Delta_f^{(I)})_m} (1 - e^{-\alpha - n\delta}) \right). \end{aligned}$$

When we regard the twisted invariants $A_{\lambda+\rho}$ and $A_{\lambda+\rho}^\psi$ as functions on Y , for $\underline{y} = (\tau, \underline{z}, t) \in Y$, we have

$$\begin{aligned} (A_{\lambda+\rho} \circ (\varphi^{(\sharp)})^{-1})(\underline{y}) &= \sum_{u \in W_f^{(\sharp)}} \varepsilon(u)^{(\sharp)} \Theta_{u(\lambda+\rho)}(\underline{y}), \\ (A_{\lambda+\rho}^\psi \circ (\varphi^{(\sharp)})^{-1})(\underline{y}) &= \sum_{u \in W_f^{(\sharp)}} \varepsilon(u)\psi(u)^{(\sharp)} \Theta_{u(\lambda+\rho)}^\psi(\underline{y}). \end{aligned}$$

Remark 3.2. An appropriate specialization of $A_\rho \circ (\varphi^{(\sharp)})^{-1}$ and $A_\rho^\psi \circ (\varphi^{(\sharp)})^{-1}$ ($\sharp \in \{I, II\}$) gives η -product identities appeared in the Appendix 1, Type BC_l , (6) in [9]:

Twisted-invariant	$A_\rho \circ (\varphi^{(I)})^{-1}$	$A_\rho \circ (\varphi^{(II)})^{-1}$	$A_\rho^\psi \circ (\varphi^{(I)})^{-1}$	$A_\rho^\psi \circ (\varphi^{(II)})^{-1}$
Table in [9]	(a)	(b)	(c)	(d)

3.2. Normalized (twisted-)characters. For $\Lambda \in P_+$, the rational number

$$c_\Lambda = \frac{\widehat{I}(\Lambda + \rho, \Lambda + \rho)}{2\widehat{I}(\Lambda + \rho, \delta)} - \frac{\widehat{I}(\rho, \rho)}{2\widehat{I}(\rho, \delta)}$$

is called the **conformal anomaly**. We define the normalized (twisted-)character χ_Λ (resp. χ_Λ^ψ) as follows:

$$\chi_\Lambda := e^{-c_\Lambda \delta} \text{ch } L(\Lambda), \quad \chi_\Lambda^\psi := e^{-c_\Lambda \delta} \text{ch}^\psi L(\Lambda).$$

It follows from the Weyl-Kac character formula (2.1.3) and its twisted version (2.2.2) that for an even level $\Lambda \in P_+$, one has

$$(3.2.1) \quad \chi_\Lambda = \frac{A_{\Lambda+\rho}}{A_\rho}, \quad \chi_\Lambda^\psi = \frac{A_{\Lambda+\rho}^\psi}{A_\rho^\psi}.$$

4. ACTIONS OF $\text{SL}_2(\mathbb{Z})$

Recall the action of the group $\text{SL}_2(\mathbb{Z})$ on the complex domain Y : for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ and $\underline{y} = (\tau, z, t) \in Y$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\tau, z, t) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}, t - \frac{c \|\text{pr}^{(\#)}(\underline{y})\|^2}{2(c\tau + d)} \right).$$

In particular, in this section, we study how the (twisted) W -invariant θ -functions behave under the action of $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

4.1. Poisson's resummation formula. Let $E = \mathbb{R}^l$ be an affine space and I be a positive definite symmetric bilinear form on E . For a full sublattice M of E , denote by $\text{Vol}(M)$ the discriminant of M , i.e., the volume of E/M . For a rapidly decreasing function $f : E \rightarrow \mathbb{C}$ and $\underline{y} \in \mathbb{R}^l$, set

$$\widehat{f}(\underline{y}) := \int_E f(\underline{x}) e^{-2\pi\sqrt{-1}I(\underline{x}, \underline{y})} d\underline{x}.$$

The so-called Poisson's resummation formula is described as follows:

$$(4.1.1) \quad \sum_{\underline{m} \in M} f(\underline{m}) = \frac{1}{\text{Vol}(M)^{\frac{1}{2}}} \sum_{\underline{m} \in M^\vee} \widehat{f}(\underline{m}).$$

Here,

$$M^\vee := \{ \underline{n} \in E \mid I(\underline{m}, \underline{n}) \in \mathbb{Z}, \forall \underline{m} \in M \}$$

is the dual lattice of M . In particular, for a self-dual lattice M , $\underline{a} \in \mathbb{C}^l$, $\tau \in \mathbb{H}$ and $f(\underline{x}) := e^{\pi\sqrt{-1}(-\frac{1}{\tau})\|\underline{x}+\underline{a}\|^2}$, one obtains

$$(4.1.2) \quad \sum_{\underline{m} \in M} e^{\pi\sqrt{-1}(-\frac{1}{\tau})\|\underline{m}+\underline{a}\|^2} = \left(\frac{\tau}{\sqrt{-1}} \right)^{\frac{1}{2}l} \sum_{\underline{m} \in M} e^{\pi\sqrt{-1}\tau\|\underline{m}\|^2 + 2\pi\sqrt{-1}I(\underline{a}, \underline{m})}.$$

See, e.g., [12] for more information.

4.2. Modular transformations on twisted-invariants.

As applications of (4.1.2), we compute the modular transformation of the anti-invariants $A \circ (\varphi^{(\sharp)})^{-1}$ and $A^\psi \circ (\varphi^{(\sharp)})^{-1}$. Since the concrete computations are entirely similar to the cases studied by V. G. Kac and D. H. Peterson [7], we omit the detail and write only the results.

For $\lambda, \mu \in P_{k,+}$, set

$$\begin{aligned} a^{(\text{I})}(\lambda, \mu) &= \sum_{u \in W_f^{(\text{I})}} \varepsilon(u) \psi(u) e^{-\frac{2\pi\sqrt{-1}}{k+2l+1} \cdot \hat{I}(u(\pi^{(\text{I})}(\lambda) + \rho_f^{(\text{I})}), \pi^{(\text{I})}(\mu) + \rho_f^{(\text{I})})}, \\ a^{(\text{I}),(\text{II})}(\lambda, \mu) &= \sum_{u \in W_f^{(\text{II})}} \varepsilon(u) e^{-\frac{2\pi\sqrt{-1}}{k+2l+1} \cdot \hat{I}(u(\pi^{(\text{II})}(\lambda) + \phi(\rho_f^{(\text{I})})), \pi^{(\text{II})}(\mu) + \rho_f^{(\text{II})})}, \\ a^{(\text{II}),(\text{I})}(\lambda, \mu) &= \sum_{u \in W_f^{(\text{I})}} \varepsilon(u) e^{-\frac{2\pi\sqrt{-1}}{k+2l+1} \cdot \hat{I}(u(\pi^{(\text{I})}(\lambda) + \phi(\rho_f^{(\text{II})})), \pi^{(\text{I})}(\mu) + \rho_f^{(\text{I})})}, \\ a^{(\text{II})}(\lambda, \mu) &= \sum_{u \in W_f^{(\text{II})}} \varepsilon(u) e^{-\frac{2\pi\sqrt{-1}}{k+2l+1} \cdot \hat{I}(u(\pi^{(\text{II})}(\lambda) + \rho_f^{(\text{II})}), \pi^{(\text{II})}(\mu) + \rho_f^{(\text{II})})}. \end{aligned}$$

Remark 4.1. Let $k \in 2\mathbb{Z}_{\geq 0}$.

- (1) For any $\lambda, \mu \in P_{k,+}$, one has $a^{(\text{I}),(\text{II})}(\phi(\lambda), \mu) = a^{(\text{II}),(\text{I})}(\phi(\mu), \lambda)$.
- (2) The definition of $a^{(\text{I})}(\lambda, \mu)$ can be rewritten as follows:

$$\begin{aligned} a^{(\text{I})}(\lambda, \mu) &= \sum_{u \in W_{f;m}^{(\text{I})}} \varepsilon(u) \left(e^{-\frac{2\pi\sqrt{-1}}{k+2l+1} \cdot \hat{I}(u(\pi^{(\text{I})}(\lambda) + \rho_f^{(\text{I})}), \pi^{(\text{I})}(\mu) + \rho_f^{(\text{I})})} \right. \\ &\quad \left. + e^{-\frac{2\pi\sqrt{-1}}{k+2l+1} \cdot \hat{I}(u s_{\alpha_l^{(\text{I})}}(\pi^{(\text{I})}(\lambda) + \rho_f^{(\text{I})}), \pi^{(\text{I})}(\mu) + \rho_f^{(\text{I})})} \right), \end{aligned}$$

where the group $W_{f;m}^{(\text{I})}$ is defined in Remark 2.2.

The first lemma concerns the W -anti invariants and the coordinate system of type (I) :

Lemma 4.2. Let $k \in 2\mathbb{Z}_{>0}$. For $\lambda \in P_{k,+}$ and $\underline{y} = (\tau, \underline{z}, t) \in Y$,

$$\begin{aligned} &(A_{\lambda+\rho} \circ (\varphi^{(\text{I})})^{-1}) \left(-\frac{1}{\tau}, \frac{\underline{z}}{\tau}, t - \frac{\|\text{pr}^{(\text{I})}(\underline{y})\|^2}{2\tau} \right) \\ &= (k+2l+1)^{-\frac{1}{2}l} \left(\frac{\tau}{\sqrt{-1}} \right)^{\frac{1}{2}l} \sum_{\mu \in P_{k,+} \bmod \mathbb{C}\delta} a^{(\text{I}),(\text{II})}(\phi(\lambda), \mu) (A_{\mu+\rho}^\psi \circ (\varphi^{(\text{II})})^{-1})(\underline{y}), \\ &(A_{\lambda+\rho} \circ (\varphi^{(\text{I})})^{-1})(\tau+1, \underline{z}, t) = e^{\frac{\pi\sqrt{-1}}{k+2l+1} \|\pi^{(\text{I})}(\lambda+\rho)\|^2} (A_{\lambda+\rho} \circ (\varphi^{(\text{I})})^{-1})(\underline{y}). \end{aligned}$$

In particular,

$$(A_{\rho} \circ (\varphi^{(\text{I})})^{-1}) \left(-\frac{1}{\tau}, \frac{\underline{z}}{\tau}, t - \frac{\|\text{pr}^{(\text{I})}(\underline{y})\|^2}{2\tau} \right) = \left(\frac{\tau}{\sqrt{-1}} \right)^{\frac{1}{2}l} (\sqrt{-1})^{-l^2} (A_{\rho}^{\psi} \circ (\varphi^{(\text{II})})^{-1})(\underline{y}).$$

The second lemma concerns the $\varepsilon\psi$ -twisted W -invariants and the coordinate of type (I) :

Lemma 4.3. *Let $k \in 2\mathbb{Z}_{>0}$. For $\lambda \in P_{k,+}$ and $\underline{y} = (\tau, \underline{z}, t) \in Y$,*

$$\begin{aligned} & (A_{\lambda+\rho}^{\psi} \circ (\varphi^{(\text{I})})^{-1}) \left(-\frac{1}{\tau}, \frac{\underline{z}}{\tau}, t - \frac{\|\text{pr}^{(\text{I})}(\underline{y})\|^2}{2\tau} \right) \\ &= (k+2l+1)^{-\frac{1}{2}l} \left(\frac{\tau}{\sqrt{-1}} \right)^{\frac{1}{2}l} \sum_{\mu \in P_{k,+} \bmod \mathbb{C}\delta} a^{(\text{I})}(\lambda, \mu) (A_{\mu+\rho}^{\psi} \circ (\varphi^{(\text{I})})^{-1})(\underline{y}), \\ & (A_{\lambda+\rho}^{\psi} \circ (\varphi^{(\text{I})})^{-1})(\tau+1, \underline{z}, t) = e^{\frac{\pi\sqrt{-1}}{k+2l+1} \|\pi^{(\text{I})}(\lambda+\rho)\|^2} (A_{\lambda+\rho}^{\psi} \circ (\varphi^{(\text{I})})^{-1})(\underline{y}). \end{aligned}$$

In particular,

$$(A_{\rho}^{\psi} \circ (\varphi^{(\text{I})})^{-1}) \left(-\frac{1}{\tau}, \frac{\underline{z}}{\tau}, t - \frac{\|\text{pr}^{(\text{I})}(\underline{y})\|^2}{2\tau} \right) = \left(\frac{\tau}{\sqrt{-1}} \right)^{\frac{1}{2}l} (-1)^{\frac{1}{2}l(l-1)} (A_{\rho}^{\psi} \circ (\varphi^{(\text{I})})^{-1})(\underline{y}).$$

The third lemma concerns the W -anti invariants and the coordinate of type (II) :

Lemma 4.4. *Let $k \in 2\mathbb{Z}_{>0}$. For $\lambda \in P_{k,+}$ and $\underline{y} = (\tau, \underline{z}, t) \in Y$,*

$$\begin{aligned} & (A_{\lambda+\rho} \circ (\varphi^{(\text{II})})^{-1}) \left(-\frac{1}{\tau}, \frac{\underline{z}}{\tau}, t - \frac{\|\text{pr}^{(\text{II})}(\underline{y})\|^2}{2\tau} \right) \\ &= (k+2l+1)^{-\frac{1}{2}l} \left(\frac{\tau}{\sqrt{-1}} \right)^{\frac{1}{2}l} \sum_{\mu \in P_{k,+} \bmod \mathbb{C}\delta} a^{(\text{II})}(\lambda, \mu) (A_{\mu+\rho} \circ (\varphi^{(\text{II})})^{-1})(\underline{y}), \\ & (A_{\lambda+\rho} \circ (\varphi^{(\text{II})})^{-1})(\tau+1, \underline{z}, t) = e^{\frac{\pi\sqrt{-1}}{k+2l+1} \|\pi^{(\text{II})}(\lambda+\rho)\|^2} (A_{\lambda+\rho}^{\psi} \circ (\varphi^{(\text{II})})^{-1})(\underline{y}). \end{aligned}$$

In particular,

$$(A_{\rho} \circ (\varphi^{(\text{II})})^{-1}) \left(-\frac{1}{\tau}, \frac{\underline{z}}{\tau}, t - \frac{\|\text{pr}^{(\text{II})}(\underline{y})\|^2}{2\tau} \right) = \left(\frac{\tau}{\sqrt{-1}} \right)^{\frac{1}{2}l} (\sqrt{-1})^{-l^2} (A_{\rho} \circ (\varphi^{(\text{II})})^{-1})(\underline{y}).$$

The fourth lemma concerns the $\varepsilon\psi$ -twisted W -invariants and the coordinate of type (II) :

Lemma 4.5. *Let $k \in 2\mathbb{Z}_{>0}$. For $\lambda \in P_{k,+}$ and $\underline{y} = (\tau, z, t) \in Y$,*

$$\begin{aligned} & (A_{\lambda+\rho}^\psi \circ (\varphi^{(\mathbb{I})})^{-1}) \left(-\frac{1}{\tau}, \frac{z}{\tau}, t - \frac{\|\text{pr}^{(\mathbb{I})}(\underline{y})\|^2}{2\tau} \right) \\ = & (k+2l+1)^{-\frac{1}{2}l} \left(\frac{\tau}{\sqrt{-1}} \right)^{\frac{1}{2}l} \sum_{\mu \in P_{k,+} \bmod \mathbb{C}\delta} a^{(\mathbb{I}),(\mathbb{I})}(\phi(\lambda), \mu) (A_{\mu+\rho} \circ (\varphi^{(\mathbb{I})})^{-1})(\underline{y}), \\ & (A_{\lambda+\rho}^\psi \circ (\varphi^{(\mathbb{I})})^{-1})(\tau+1, z, t) = e^{\frac{\pi\sqrt{-1}}{k+2l+1} \|\pi^{(\mathbb{I})}(\lambda+\rho)\|^2} (A_{\lambda+\rho} \circ (\varphi^{(\mathbb{I})})^{-1})(\underline{y}). \end{aligned}$$

In particular,

$$(A_{\rho}^\psi \circ (\varphi^{(\mathbb{I})})^{-1}) \left(-\frac{1}{\tau}, \frac{z}{\tau}, t - \frac{\|\text{pr}^{(\mathbb{I})}(\underline{y})\|^2}{2\tau} \right) = \left(\frac{\tau}{\sqrt{-1}} \right)^{\frac{1}{2}l} (\sqrt{-1})^{-l^2} (A_{\rho} \circ (\varphi^{(\mathbb{I})})^{-1})(\underline{y}).$$

We remark the one of the key formulas in the proof of the above 4 lemmas for $\lambda = 0$ are the next well-known formula: for $N \in \mathbb{Z}_{>1}$,

$$\prod_{k=1}^{N-1} \sin\left(\frac{k\pi}{N}\right) = \frac{N}{2^{N-1}}.$$

4.3. Modular transformations on (twisted-)characters.

Let $k \in 2\mathbb{Z}_{\geq 0}$ and $\lambda \in P_{k,+}$. Thanks to (3.2.1), the results stated in the previous section imply the following results:

By Lemma 4.2, we have

Proposition 4.6. *Let $k \in 2\mathbb{Z}_{>0}$. For $\lambda \in P_{k,+}$ and $\underline{y} = (\tau, z, t) \in Y$,*

$$\begin{aligned} & (\chi_\lambda \circ (\varphi^{(\mathbb{I})})^{-1}) \left(-\frac{1}{\tau}, \frac{z}{\tau}, t - \frac{\|\text{pr}^{(\mathbb{I})}(\underline{y})\|^2}{2\tau} \right) \\ = & (k+2l+1)^{-\frac{1}{2}l} (\sqrt{-1})^{l^2} \sum_{\mu \in P_{k,+} \bmod \mathbb{C}\delta} a^{(\mathbb{I}),(\mathbb{I})}(\phi(\lambda), \mu) (\chi_\mu^\psi \circ (\varphi^{(\mathbb{I})})^{-1})(\underline{y}), \\ & (\chi_\lambda \circ (\varphi^{(\mathbb{I})})^{-1})(\tau+1, z, t) \\ = & e^{\pi\sqrt{-1} \left(\frac{\|\pi^{(\mathbb{I})}(\lambda+\rho)\|^2}{k+2l+1} - \frac{\|\pi^{(\mathbb{I})}(\rho)\|^2}{2l+1} \right)} (\chi_\lambda \circ (\varphi^{(\mathbb{I})})^{-1})(\underline{y}). \end{aligned}$$

Lemma 4.3 implies

Proposition 4.7. *Let $k \in 2\mathbb{Z}_{>0}$. For $\lambda \in P_{k,+}$ and $\underline{y} = (\tau, \underline{z}, t) \in Y$,*

$$\begin{aligned} & (\chi_\lambda^\psi \circ (\varphi^{(\mathbb{I})})^{-1}) \left(-\frac{1}{\tau}, \frac{\underline{z}}{\tau}, t - \frac{\|\mathrm{pr}^{(\mathbb{I})}(\underline{y})\|^2}{2\tau} \right) \\ &= (k+2l+1)^{-\frac{1}{2}l} (-1)^{\frac{1}{2}l(l-1)} \sum_{\mu \in P_{k,+} \bmod \mathbb{C}\delta} a^{(\mathbb{I})}(\lambda, \mu) (\chi_\mu^\psi \circ (\varphi^{(\mathbb{I})})^{-1})(\underline{y}), \\ & (\chi_\lambda^\psi \circ (\varphi^{(\mathbb{I})})^{-1})(\tau+1, \underline{z}, t) \\ &= e^{\pi\sqrt{-1} \left(\frac{\|\pi^{(\mathbb{I})}(\lambda+\rho)\|^2}{k+2l+1} - \frac{\|\pi^{(\mathbb{I})}(\rho)\|^2}{2l+1} \right)} (\chi_\lambda^\psi \circ (\varphi^{(\mathbb{I})})^{-1})(\underline{y}). \end{aligned}$$

By Lemma 4.4, we have (cf. the first identity had been obtained in [7])

Proposition 4.8. *Let $k \in 2\mathbb{Z}_{>0}$. For $\lambda \in P_{k,+}$ and $\underline{y} = (\tau, \underline{z}, t) \in Y$,*

$$\begin{aligned} & (\chi_\lambda \circ (\varphi^{(\mathbb{II})})^{-1}) \left(-\frac{1}{\tau}, \frac{\underline{z}}{\tau}, t - \frac{\|\mathrm{pr}^{(\mathbb{II})}(\underline{y})\|^2}{2\tau} \right) \\ &= (k+2l+1)^{-\frac{1}{2}l} (\sqrt{-1})^{l^2} \sum_{\mu \in P_{k,+} \bmod \mathbb{C}\delta} a^{(\mathbb{II})}(\lambda, \mu) (\chi_\mu \circ (\varphi^{(\mathbb{II})})^{-1})(\underline{y}), \\ & (\chi_\lambda \circ (\varphi^{(\mathbb{II})})^{-1})(\tau+1, \underline{z}, t) \\ &= e^{\pi\sqrt{-1} \left(\frac{\|\pi^{(\mathbb{II})}(\lambda+\rho^{(\mathbb{II})})\|^2}{k+2l+1} - \frac{\|\pi^{(\mathbb{II})}(\rho^{(\mathbb{II})})\|^2}{2l+1} \right)} (\chi_\lambda \circ (\varphi^{(\mathbb{II})})^{-1})(\underline{y}). \end{aligned}$$

Lemma 4.5 implies

Proposition 4.9. *Let $k \in 2\mathbb{Z}_{>0}$. For $\lambda \in P_{k,+}$ and $\underline{y} = (\tau, \underline{z}, t) \in Y$,*

$$\begin{aligned} & (\chi_\lambda^\psi \circ (\varphi^{(\mathbb{II})})^{-1}) \left(-\frac{1}{\tau}, \frac{\underline{z}}{\tau}, t - \frac{\|\mathrm{pr}^{(\mathbb{II})}(\underline{y})\|^2}{2\tau} \right) \\ &= (k+2l+1)^{-\frac{1}{2}l} (\sqrt{-1})^{l^2} \sum_{\mu \in P_{k,+} \bmod \mathbb{C}\delta} a^{(\mathbb{II}),(\mathbb{I})}(\phi(\lambda), \mu) (\chi_\mu \circ (\varphi^{(\mathbb{I})})^{-1})(\underline{y}), \\ & (\chi_\lambda^\psi \circ (\varphi^{(\mathbb{II})})^{-1})(\tau+1, \underline{z}, t) \\ &= e^{\pi\sqrt{-1} \left(\frac{\|\pi^{(\mathbb{II})}(\lambda+\rho^{(\mathbb{II})})\|^2}{k+2l+1} - \frac{\|\pi^{(\mathbb{II})}(\rho^{(\mathbb{II})})\|^2}{2l+1} \right)} (\chi_\lambda \circ (\varphi^{(\mathbb{II})})^{-1})(\underline{y}). \end{aligned}$$

5. CONCLUSION AND OBSERVATION

In 1984, V. Kac and D. Peterson [7] showed that there exists an action of the subgroup Γ_θ of $\mathrm{SL}_2(\mathbb{Z})$, generated by S and T^2 , on the vector space

$$V_k^{(\mathbb{II})} := \bigoplus_{\lambda \in P_{k,+} \bmod \mathbb{C}\delta} \mathbb{C}(\chi_\lambda \circ (\varphi^{(\mathbb{II})})^{-1}).$$

As Γ_θ is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of index 3, the dimension of the $\mathrm{SL}_2(\mathbb{Z})$ -module

$$V_k := \mathrm{Ind}_{\Gamma_\theta}^{\mathrm{SL}_2(\mathbb{Z})} V_k^{(\mathbb{II})}$$

is $3 \dim V_k^{(\mathbb{I})}$. Let $V_k^{(\mathbb{I})}$ and $V_k^{\psi,(\mathbb{II})}$ be the vector spaces generated by $\chi_\lambda \circ (\varphi^{(\mathbb{I})})^{-1}$ ($\lambda \in P_{k,+} \bmod \mathbb{C}\delta$) and $\chi_\lambda^\psi \circ (\varphi^{(\mathbb{II})})^{-1}$ ($\lambda \in P_{k,+} \bmod \mathbb{C}\delta$), respectively. From the previous section, we see that two vector spaces $V_k^{(\mathbb{I})}$ and $V_k^{\psi,(\mathbb{II})}$ are subspaces of V_k and that

$$V_k = V_k^{(\mathbb{I})} \oplus V_k^{(\mathbb{II})} \oplus V_k^{\psi,(\mathbb{II})}.$$

Indeed, by Propositions 4.6, 4.8 and 4.9, the $\mathrm{SL}_2(\mathbb{Z})$ -action on V_k , more precisely, the elements S and T map

$$\begin{aligned} S : V_k^{(\mathbb{II})} &\hookrightarrow V_k^{(\mathbb{II})}, & V_k^{(\mathbb{I})} &\longrightarrow V_k^{\psi,(\mathbb{II})}, & V_k^{\psi,(\mathbb{II})} &\longrightarrow V_k^{(\mathbb{I})} \\ T : V_k^{(\mathbb{I})} &\hookrightarrow V_k^{(\mathbb{I})}, & V_k^{(\mathbb{II})} &\longrightarrow V_k^{\psi,(\mathbb{II})}, & V_k^{\psi,(\mathbb{II})} &\longrightarrow V_k^{(\mathbb{II})}. \end{aligned}$$

Therefore, we have

Theorem 5.1. *Fix $k \in 2\mathbb{Z}_{\geq 0}$. Then, $\mathrm{SL}_2(\mathbb{Z})$ acts on the \mathbb{C} -vector space spanned by $\{ \chi_\lambda \circ (\varphi^{(\mathbb{I})})^{-1}, \chi_\lambda \circ (\varphi^{(\mathbb{II})})^{-1}, \chi_\lambda^\psi \circ (\varphi^{(\mathbb{II})})^{-1} \}_{\lambda \in P_{k,+} \bmod \mathbb{C}\delta}$.*

Set

$$V_k^{\psi,(\mathbb{I})} = \bigoplus_{\lambda \in P_{k,+} \bmod \mathbb{C}\delta} \mathbb{C} \chi_\lambda^\psi \circ (\varphi^{(\mathbb{I})})^{-1}.$$

Surprisingly, by Proposition 4.7, we have the following theorem:

Theorem 5.2. *Fix $k \in 2\mathbb{Z}_{\geq 0}$. Then, $\mathrm{SL}_2(\mathbb{Z})$ acts on the \mathbb{C} -vector space spanned by $\{ \chi_\lambda^\psi \circ (\varphi^{(\mathbb{I})})^{-1} \mid \lambda \in P_{k,+} \bmod \mathbb{C}\delta \}$.*

For an *odd* k , the authors are not aware of any such interpretation.

There is an interesting connection between this theorem and the affine Lie superalgebra \mathfrak{g} of type $B^{(1)}(0, l)$. This is a Lie superalgebra whose set of real roots, forgetting the $\mathbb{Z}/2\mathbb{Z}$ -structure, is the non-reduced affine root system of type BCC_l . By V. Kac [5], it appears that both $\chi_\lambda \circ (\varphi^{(\mathbb{I})})^{-1}$ ($\lambda \in P_{k,+}$) and $\chi_\lambda \circ (\varphi^{(\mathbb{II})})^{-1}$ ($\lambda \in P_{k,+}$) can be also viewed as the character of integrable highest weight modules of level k over \mathfrak{g} , since the non-negative integer k is supposed to be even. With this identification, the twisted characters $\chi_\lambda^\psi \circ (\varphi^{(\mathbb{I})})^{-1}$ and $\chi_\lambda^\psi \circ (\varphi^{(\mathbb{II})})^{-1}$ can be viewed as the **super-character** of integrable highest weight modules of level k over \mathfrak{g} (cf. §A). There is 4 type of affine Lie superalgebras whose universal enveloping algebras are integral domain. In addition, the set of real roots of such affine Lie superalgebras, forgetting their $\mathbb{Z}/2\mathbb{Z}$ -graded structure, are non-reduced affine root systems:

Non-reduced type	BCC_l	$C^\vee BC_l$	BB_l^\vee	$C^\vee C_l$
Affine super	$B^{(1)}(0, l)$	$A^{(4)}(0, 2l)$	$A^{(2)}(0, 2l - 1)$	$C^{(2)}(l + 1)$

This article deals with $BC_l^{(2)}$ which is connected with $B^{(1)}(0, l)$. For 3 other cases, there might be something similar, as can be seen from the appendix of

[9]. The $\mathrm{SL}_2(\mathbb{Z})$ -symmetry, we have shown in this article, is quite mysterious and its « raison d'être » are to be revealed.

APPENDIX A. LIE SUPERALGEBRA

Here, we recall some relevant basic facts about Lie superalgebras. The basic reference here is [5].

A.1. Lie superalgebra $\mathfrak{osp}(1|2)$ and its irreducible representations.

The Lie superalgebra $\mathfrak{osp}(1|2)$ is the superspace

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1, \quad \mathfrak{g}_0 := \mathbb{C}E \oplus \mathbb{C}H \oplus \mathbb{C}F, \quad \mathfrak{g}_1 := \mathbb{C}e \oplus \mathbb{C}f$$

equipped with the Lie super-bracket $[\cdot, \cdot]$ satisfying

$$\begin{aligned} [H, E] &= 4E, & [H, F] &= -4F, & [E, F] &= 2H, \\ [H, e] &= 2e, & [H, f] &= -2f, \\ [e, f] &= H, & [e, e] &= 2E, & [f, f] &= -2F. \end{aligned}$$

Let $\alpha \in (\mathbb{C}H)^*$ such that $\alpha(H) = 2$. Then, it is clear that $\{\pm\alpha\} \cup \{\pm 2\alpha\}$ is the set of roots of \mathfrak{g} with respect to $\mathfrak{h} = \mathbb{C}H$. Set $\mathfrak{b} = \mathbb{C}H \oplus \mathbb{C}e \oplus \mathbb{C}F$. For $\lambda \in \mathfrak{h}^*$, let $\mathbb{C}v_\lambda$ be the even \mathfrak{b} -module defined by

$$H.v_\lambda = \lambda(H)v_\lambda, \quad e.v_\lambda = 0.$$

It is clear that this implies $E.v_\lambda = 0$. The \mathfrak{g} -module $M(\lambda) := \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}v_\lambda$ is the Verma module with highest weight λ . For $i \in \mathbb{Z}_{\geq 0}$, set $w_i = \frac{1}{i!} f^i \otimes v_\lambda$. By direct computation, one sees

$$\begin{aligned} H.w_i &= (\lambda(H) - 2i)w_i, & e.w_i &= \begin{cases} -w_{i-1} & i \equiv 0 \pmod{2}, \\ i(\lambda(H) - i + 1)w_{i-1} & i \equiv 1 \pmod{2}, \end{cases} \\ f.w_i &= (i + 1)w_{i+1}. \end{aligned}$$

Hence, the \mathfrak{g} -module $M(\lambda)$ is reducible if and only if $\lambda(H) \in 2\mathbb{Z}_{\geq 0}$, in which case the subspace $N(\lambda) := \bigoplus_{i > \lambda(H)} \mathbb{C}w_i$ gives the maximal proper \mathfrak{g} -submodule. Denote by $L(\lambda)$ its irreducible quotient. The next lemma is important:

Lemma A.1. *Let V be an irreducible finite dimensional \mathfrak{g} -module. Then, there exists $N \in \mathbb{Z}_{\geq 0}$ such that $V \cong L(N\alpha)$. In particular, V is of odd dimension.*

A.2. Lie superalgebra of type $B^{(1)}(0, l)$. Fix a finite set I . Recall that a Kac Moody Lie algebra is a Lie algebra associated with a GCM $A = (a_{i,j})_{i,j \in I}$, i.e., a square matrix with $a_{i,j} \in \mathbb{Z}$ such that i) $a_{i,i} = 2$ for any $i \in I$, ii) $a_{i,j} \leq 0$ for any $i \neq j$ and iii) $a_{i,j} = 0$ implies $a_{j,i} = 0$. Its building blocks are isomorphic to $\mathfrak{sl}(2)$. Similarly, a class of Lie superalgebras had been considered in the article [5] associated with a pair (A, τ) of a GCM A and the subset $\tau \subset I$ satisfying the additional condition iv) $a_{i,j} \in 2\mathbb{Z}$ for

any $i \in \tau$ and $j \in I$. Its building blocks are isomorphic either to $\mathfrak{sl}(2)$ or to $\mathfrak{osp}(1|2)$.

Recall the definition of the Lie superalgebra $\mathfrak{g}(A, \tau)$. Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of the GCM A so that $\Pi = \{\alpha_i\}_{i \in I} \subset \mathfrak{h}^*$ and $\Pi^\vee = \{h_i\}_{i \in I}$ are sets of linearly independent vectors satisfying $\langle \alpha_i, h_j \rangle = a_{j,i}$ for any $i, j \in I$. $\mathfrak{g}(A, \tau)$ is the Lie superalgebra generated by e_i, f_i ($i \in I$) and \mathfrak{h} with the parity

$$|e_i| = |f_i| := \begin{cases} \bar{0} & i \in I \setminus \tau, \\ \bar{1} & i \in \tau, \end{cases} \quad |h| := \bar{0} \quad h \in \mathfrak{h},$$

equipped with the Lie superbracket $[\cdot, \cdot]$ satisfying

$$\begin{aligned} \text{(S1)} \quad & [h, h'] = 0 \text{ for } h, h' \in \mathfrak{h}, \\ \text{(S2)} \quad & \begin{cases} [h, e_i] = \langle h, \alpha_i \rangle e_i, \\ [h, f_i] = -\langle h, \alpha_i \rangle f_i, \end{cases} \text{ for } h \in \mathfrak{h} \text{ and } 0 \leq i \leq l, \\ \text{(S3)} \quad & [e_i, f_j] = \delta_{i,j} h_i \text{ for } 0 \leq i, j \leq l, \\ \text{(S4)} \quad & \begin{cases} \text{ad}(e_i)^{1-a_{i,j}}(e_j) = 0, \\ \text{ad}(f_i)^{1-a_{i,j}}(f_j) = 0, \end{cases} \text{ for } 0 \leq i \neq j \leq l. \end{aligned}$$

A non-zero element $\alpha \in \mathfrak{h}^*$ is called a root if

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g}(A, \tau) \mid [h, x] = \alpha(h)x \ \forall h \in \mathfrak{h}\} \neq \{0\}.$$

We denote the set of all roots by Δ . We have the so-called root space decomposition with respect to \mathfrak{h} : $\mathfrak{g}(A, \tau) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$. A root $\alpha \in \Delta$ is called *even* (resp. *odd*) if any non-zero element of \mathfrak{g}_α is even (resp. *odd*). It is clear that a root $\alpha = \sum_{i \in I} m_i \alpha_i \in \Delta$ is even iff so is $\sum_{i \in \tau} m_i$. Denote by $\Delta_{\bar{0}}$ and $\Delta_{\bar{1}}$ the set of even and odd roots, respectively. By definition, one has the following parity decomposition:

$$\mathfrak{g}(A, \tau) = \mathfrak{g}(A, \tau)_{\bar{0}} \oplus \mathfrak{g}(A, \tau)_{\bar{1}},$$

where $\mathfrak{g}(A, \tau)_{\bar{0}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_{\bar{0}}} \mathfrak{g}_\alpha$ and $\mathfrak{g}(A, \tau)_{\bar{1}} = \bigoplus_{\alpha \in \Delta_{\bar{1}}} \mathfrak{g}_\alpha$. For $i \in I$, Let s_{α_i} be the reflection in \mathfrak{h}^* defined by $s_{\alpha_i}(x) = x - x(h_i)\alpha_i$ and $W(A)$ be the subgroup of $GL(\mathfrak{h}^*)$ generated by s_{α_i} ($i \in I$). An element of $\Delta^{re} := W \cdot \Pi$ (resp. $\Delta^{im} := \Delta \setminus \Delta^{re}$) is called a *real* (resp. an *imaginary*) root.

The finite Lie superalgebra \mathfrak{g}_f of type $B(0, l)$ is, by definition, the Lie superalgebra $\mathfrak{g}(A_f, I_{f, odd})$ associated with the Cartan matrix $A_f = (a_{i,j})_{1 \leq i, j \leq l}$ of type B_l , i.e.,

$$A_f = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -2 & 2 \end{pmatrix}, \quad I_{f, odd} = \{l\} \subset I_f = \{1, 2, \dots, l\}.$$

This Lie superalgebra is also called $\mathfrak{osp}(1|2l)$.

Remark A.2. The root system Δ_f of the Lie superalgebra \mathfrak{g}_f is of type BC_l , i.e., $\Delta_f = (\Delta_f)_s \cup (\Delta_f)_m \cup (\Delta_f)_l$ where

$$\begin{aligned} (\Delta_f)_s &:= \{\pm\varepsilon_i \mid 1 \leq i \leq l\}, & (\Delta_f)_l &:= \{\pm 2\varepsilon_i \mid 1 \leq i \leq l\}, \\ (\Delta_f)_m &:= \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq l\}. \end{aligned}$$

Its parity decomposition is given as follows: $\Delta_f = (\Delta_f)_{\bar{0}} \amalg (\Delta_f)_{\bar{1}}$ where

$$(\Delta_f)_{\bar{0}} = (\Delta_f)_m \amalg (\Delta_f)_l, \quad (\Delta_f)_{\bar{1}} = (\Delta_f)_s.$$

The affine Lie superalgebra of type $B^{(1)}(0, l)$ is, by definition, the Lie superalgebra $\mathfrak{g}(A, I_{\text{odd}})$ associated with the same GCM $A = (a_{i,j})_{0 \leq i, j \leq l}$ as for type $BC_l^{(2)}$ introduced in §1.1 and the subset $I_{\text{odd}} := \{l\}$ of the index set $I := \{0, 1, \dots, l\}$. Thus, its Dynkin diagram is

$$\begin{array}{ccc} \begin{array}{c} 4 \\ \circ \rightarrow \bullet \\ \alpha_0 \quad \alpha_1 \end{array} & (l = 1), & \begin{array}{c} 2 \qquad \qquad \qquad 2 \\ \circ \rightarrow \circ \text{---} \circ \text{---} \circ \text{---} \circ \rightarrow \bullet \\ \alpha_0 \quad \alpha_1 \qquad \qquad \alpha_{l-1} \quad \alpha_l \end{array} \quad (l \geq 2). \end{array}$$

Here, the black nodes signify that the corresponding simple root α_l is non-isotropic and odd. The Weyl group W is isomorphic to the Weyl group of type $BC_l^{(2)}$, and the set of real roots Δ^{re} is of type BCC_l , i.e., $\Delta^{re} = \Delta_f + \mathbb{Z}\delta$, where $\delta := \alpha_0 + 2(\alpha_1 + \alpha_2 + \dots + \alpha_l)$ is the *even* positive isotropic root such that $\Delta^{im} = \mathbb{Z}\delta$. The parity decomposition $\Delta = \Delta_{\bar{0}} \amalg \Delta_{\bar{1}}$ is given by

$$\Delta_{\bar{0}} = ((\Delta_f)_m + \mathbb{Z}\delta) \amalg ((\Delta_f)_l + \mathbb{Z}\delta) \amalg \Delta^{im}, \quad \Delta_{\bar{1}} = (\Delta_f)_s + \mathbb{Z}\delta.$$

We remark that the node α_l is the only node verifying $a_{l,j} \in 2\mathbb{Z}$ for any $0 \leq j \leq l$. In particular, the map ψ defined in (2.2.1) is nice.

Remark A.3. It is known that the derived subalgebra of $\mathfrak{g}(A, I_{\text{odd}})$ can be realized as the universal central extension of the loop algebra over \mathfrak{g}_f . In particular, it follows that $\text{mult}(n\delta) = l$ for any $n \in \mathbb{Z} \setminus \{0\}$.

For later use, let \mathfrak{n}_{\pm} be the Lie sub superalgebras of $\mathfrak{g}(A, I_{\text{odd}})$ generated by e_i ($i \in I$) and f_i ($i \in I$), respectively. One can define the category \mathcal{O} of $\mathfrak{g}(A, I_{\text{odd}})$ -modules as in the Kac-Moody case. Set $\Delta_+ := \Delta \cap \mathbb{Z}_{\geq 0}\Pi$.

A.3. Integrability highest weight modules. A highest weight $\mathfrak{g}(A, I_{\text{odd}})$ -module is defined similarly. For $\Lambda \in \mathfrak{h}^*$, let $\mathbb{C}_{\Lambda} = \mathbb{C}v_{\Lambda}$ be the one dimensional *even* module over $\mathfrak{b}_+ := \mathfrak{h} \oplus \mathfrak{n}_+$ defined by

$$h.v_{\Lambda} = \langle h, \Lambda \rangle v_{\Lambda} \quad (h \in \mathfrak{h}), \quad \mathfrak{n}_+.v_{\Lambda} = 0.$$

The induced $\mathfrak{g}(A, I_{\text{odd}})$ -module $M(\Lambda) := \text{Ind}_{\mathfrak{b}_+}^{\mathfrak{g}(A, I_{\text{odd}})} \mathbb{C}_{\Lambda}$ is called the Verma module with highest weight Λ . The Verma module $M(\Lambda)$ possesses the maximal proper submodule, say $N(\Lambda)$. Denote its irreducible quotient $M(\Lambda)/N(\Lambda)$ by $L(\Lambda)$. The next lemma can be shown, with the aide of Lemma A.1:

Lemma A.4. The highest weight $\mathfrak{g}(A, I_{\text{odd}})$ -module $L(\Lambda)$ is integrable if and only if $\Lambda(h_i) \in \mathbb{Z}_{\geq 0}$ for $0 \leq i < l$ and $\Lambda(h_l) \in 2\mathbb{Z}_{\geq 0}$, i.e., $\Lambda \in P_+$ and the level of Λ is even.

Note that the level of i -th fundamental weight $\Lambda_i \in \mathfrak{h}^*$ is the co-label a_i^\vee .

As any highest weight module is \mathfrak{h} -diagonalisable, we can speak of its formal (super-)character which is an element of \mathcal{E} . For an object V of the category \mathcal{O} , let $V = \bigoplus_{\lambda \in \mathcal{P}(V)} V_\lambda$ be its weight space decomposition. Its character $\text{ch}(V)$ and super-character $\text{sch}(V)$ are the elements of \mathcal{E} defined by

$$\text{ch}(V) = \sum_{\lambda \in \mathcal{P}(V)} (\dim V_\lambda) e^\lambda, \quad \text{sch}(V) = \sum_{\lambda \in \mathcal{P}(V)} (\text{sdim } V_\lambda) e^\lambda,$$

where for a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space $E = E_{\bar{0}} \oplus E_{\bar{1}}$, we set

$$\text{sdim}(E) = \dim E_{\bar{0}} - \dim E_{\bar{1}}.$$

Let $\rho := \sum_{i=0}^l \in \Lambda_i$ be a Weyl vector of $\mathfrak{g}(A, I_{\text{odd}})$. For a $\Lambda \in P_+$ with even level, the formal character of the irreducible highest weight module $L(\Lambda)$ is given by the Weyl-Kac character formula:

$$\text{ch } L(\Lambda) = \sum_{w \in W} \varepsilon(w) \text{ch } M(w(\Lambda + \rho) - \rho) = \frac{\sum_{w \in W} \varepsilon(w) e^{w(\Lambda + \rho)}}{\sum_{w \in W} \varepsilon(w) e^{w(\rho)}}.$$

This is exactly the same as for the affine Lie algebra $\mathfrak{g}(A)$ (cf. (2.1.3)). Hence, the super-character of $L(\Lambda)$ can be given by

$$\text{sch } L(\Lambda) = \sum_{w \in W} \varepsilon(w) \psi(w) \text{sch } M(w(\Lambda + \rho) - \rho),$$

where the map $\psi : W \rightarrow \{\pm 1\}$ is defined in (2.2.1). By the Poincaré-Birkhoff-Witt theorem, it follows that

$$\text{sch } M(\lambda) = e^\lambda \frac{\prod_{\alpha \in \Delta_+ \cap \Delta_{\bar{1}}} (1 - e^{-\alpha})^{\text{mult}(\alpha)}}{\prod_{\alpha \in \Delta_+ \cap \Delta_{\bar{0}}} (1 - e^{-\alpha})^{\text{mult}(\alpha)}}$$

which implies the super-denominator identity:

$$(A.3.1) \quad \sum_{w \in W} \varepsilon(w) \psi(w) e^{w\rho} = e^\rho \frac{\prod_{\alpha \in \Delta_+ \cap \Delta_{\bar{0}}} (1 - e^{-\alpha})^{\text{mult}(\alpha)}}{\prod_{\alpha \in \Delta_+ \cap \Delta_{\bar{1}}} (1 - e^{-\alpha})^{\text{mult}(\alpha)}}.$$

As a corollary, one sees that

$$\text{sch } L(\Lambda) = \frac{\sum_{w \in W} \varepsilon(w) \psi(w) e^{w(\Lambda + \rho)}}{\sum_{w \in W} \varepsilon(w) \psi(w) e^{w(\rho)}}.$$

Thus, it is exactly the twisted-character (2.2.2) !

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