

Dimensions and unions

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1 Geometric rank computations

Let \mathcal{M} be a first-order structure with a *definable* and *additive* finite rank function rk on definable sets. We work with interpretable sets throughout, that we still call definable.

Definability means: for any definable function f from A to B , the set of elements of B with fibers of a given rank is definable.

Additivity means: for any definable function f from A to B , with fibers of constant rank r , we have $\text{rk}(A) = \text{rk}(B) + r$.

Examples of such structures: Groups of finite Morley rank, o-minimal structures, ...

Let $X(a)$ be a uniformly definable family of sets, where the parameters a vary in a definable set A . By passing to a further equivalence relation, saying that the parameter a is equivalent to the parameter a' if and only if $X(a) = X(a')$, we may always assume that A exactly parametrizes the family of definable sets $X(a)$, i.e., $X(a) = X(a')$ if and only if $a = a'$.

The following geometric rank computation, which gives rank of unions, follows exactly as in [Jal06] and [Jal08].

Lemma 1.1 *Let \mathcal{M} and $X(a)$ as above, where A parametrizes exactly the family of sets $X(a)$. For r between 0 and $\text{rk}(A)$, let $X(a)_r$ denote the set of elements x of $X(a)$ such that the set of sets of the form $X(a')$ containing x has rank r . If $X(a)_r$ is nonempty for some r and has constant rank when a varies in A , then*

$$\text{rk}\left(\bigcup_{a \in A} X(a)_r\right) + r = \text{rk}(A) + \text{rk}(X(a)_r)$$

Proof. One considers the geometry where points are the elements of $\bigcup_{a \in A} X(a)$ and lines are the sets of the form $X(a)$, with the natural incidence relation.

Then one considers the set of flags, that is the set of couples (point, line) where the point belongs to the line.

One considers then the two projections of the set of flags, on the set of points on the one hand and on the set of lines on the other.

Notice that the *definability* of the rank, applied to the first projection on the set of points, gives the definability of all subsets of the form $X(a)_r$.

One projects then the natural subflag associated to the set of points belonging to a set of lines of rank exactly r . By projecting on the set of points and on the set of lines respectively, our assumption and the *additivity* of the rank gives immediately the left and the right term of the desired equality for the rank of this subflag. This proves the equality. \square

Remarks:

1- When $X(a)_r$ does not have constant rank, one can always make some computations “generically”.

2- The definability of sets of the form $X(a)_0$ is always given by *elimination of \exists^∞* , if such an elimination is valid in \mathcal{M} . Notice that the set $X(a)_0$ is exactly the set of elements of $X(a)$ contained in only finitely many sets of the form $X(a')$. With elimination of \exists^∞ , we have uniform bounds: there exists an m such that any elements of $X(a)$ contained in only finitely many sets of the form $X(a')$ is contained in at most m such. In this case this is, a priori, stronger than the definability of the rank. (Groups of finite Morley rank do have a definable rank *and* elimination of \exists^∞).

3- The additivity of the rank is true in supersimple structures of finite SU-rank, which in this case equals the D -rank and the S_1 -rank. It is not clear whether it is definable. Is it definable when the whole structure is a group?

Corollary 1.2 *Let (G, Ω) be a permutation group with a definable and additive rank function, and H a definable subset of Ω . Assume $\text{rk}(G) = \text{rk}(\Omega)$ and $\text{rk}(H) \leq \text{rk}(N(H))$ (where $N(H)$ denotes the setwise stabilizer of H in G). Then $\text{rk}(H^G) = \text{rk}(\Omega)$ if and only if $\text{rk}(H_0) = \text{rk}(N(H))$.*

In this case $\text{rk}(H_0) = \text{rk}(H) = \text{rk}(N(H))$, Ω has a definable generic subset whose elements are all contained in only finitely many G -conjugates of H , and the same applies to H .

Proof. Exactly as in [Jal08, Corollary 5]. \square

Cases where the preceding corollary applies:

1- G is a group with a definable and additive rank function, with the action on itself by conjugation, and H is a definable subset of G such that $\text{rk}(H) \leq \text{rk}(N(H))$.

This latter condition is guaranteed for all H of the form $H = x\Gamma$, where Γ is a definable subgroup of G and x is an element normalizing Γ : in this case one has $x\Gamma \subseteq N(x\Gamma)$ (in fact $N(x\Gamma)$ is exactly the preimage in $N(\Gamma)$ of $C_{N(\Gamma)/\Gamma}(x \bmod \Gamma)$) and one gets $\text{rk}(x\Gamma) \leq \text{rk}(N(x\Gamma))$ (provided the rank is monotone!).

References

[Jal06] E. Jaligot. Generix never gives up. *J. Symbolic Logic*, 71(2):599–610, 2006.

[Jal08] E. Jaligot. Cosets, genericity, and the weyl group. Submitted:
arXiv:0801.2243, 2008.