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1 Introduction

My research is in model theory, specifically o-minimality. O-minimal structures are linearly ordered structures in which every definable subset is a finite union of points and intervals. Examples include the reals, considered as a field, and the real field expanded by the exponential function ([Wil96]). However, general o-minimal structures need not have any connection with the reals, and many challenges lie in both extending known results on the reals to o-minimal structures, and investigating the model theory of these structures in their own right.
My research has focused on classification of types in o-minimal theories, as first proposed by [Mar86] and used extensively by [MS94] and [Dol04]. There is a simple dichotomy of Dedekind cuts in a linear order: when both sides of the cut are open sets, denoted *non-principal*, and when at least one side is closed, denoted *principal*. Since the order type induces the type in o-minimal structures, this classification proves quite useful. For instance, principal cuts are precisely the definable 1-types. This classification also allows for very tight descriptions of definable functions, because there is no definable function that takes an element realizing a non-principal cut to one realizing a principal cut.
While the classification in [Mar86] is extremely descriptive for 1-types, and [Dol04] applied it somewhat to n-types, much remains to be done both in the analysis of n-types, and in the analysis of types over a pair of sets – for instance, we may want a type over a larger set (perhaps an elementary extension, or superstructure) that possesses some properties with respect to a smaller one. Such issues have connections to further research that I wish to pursue – see §4.

2 Extending Functions on Curves

The bulk of my dissertation ([Ram08]) stemmed from a question of Speissegger's, about extending a bounded function to a closed set containing a (not necessarily definable) curve. It arose for him in the study of differential equations, specifically trying to prove an analogue to a theorem of Malgrange ([Mal74]) in which the existence of a formal solution to an ordinary differential equation in a space of generalized series implies that there is actually a C^∞ function that is a solution, and that is asymptotic to this formal solution. While the proof has not been completed, the question it prompted is of independent interest. It can be stated in its general form as follows:

Question 1. *Let γ be a curve in M^n, where M is an o-minimal structure, and a curve is a continuous map from $(0,1)$ to M^n, and let $\lim_{t\to 0^+}\gamma(t) = a$, with $a \in M^n$. Let $f : M^n \to M$ be an M-definable function, and suppose that $\{f(\gamma(t)) \mid t \in (0,1)\}$ is bounded. Does there exist an M-definable C such that $\gamma \restriction (0,s) \subseteq C$ for some $s > 0$, f is continuous on C, and $f \restriction C$ can be extended continuously to* $\mathrm{cl}(C)$*?*

It is clear that some basic restrictions must be placed on γ to prevent oscillation that would defeat any effort to put γ in an appropriate M-definable set. [LMS03] calls curves with the appropriate properties "non-oscillatory." Speissegger's hope was that this question had a positive answer for any such non-oscillatory γ. However, I discovered the following counterexample.

Example 2. *Let $M = (\mathbb{R}, +, \cdot, 0, 1, <)$, let $\gamma(t) = \langle t, -t/\ln t\rangle$, and let $f(x,y) = y/x$. Then there is no M-definable set containing γ on which f is continuous, and onto whose closure f extends continuously. In particular, f cannot be extended continuously to the origin.*

This example, and one other, closely related one, turn out to be the prototypes for all instances of failure. However, isolating the precise property that causes failure is non-trivial. The first step is to note the easy fact that to each curve is associated a complete n-type – intuitively, the type of $\gamma(\epsilon)$ for ϵ an infinitesimal above 0 (although this can be formalized). This allows us to translate Question 1 into the language of types.

Question 3. *Given an n-type, p, and a definable function, f, bounded in a neighborhood of p, is there a definable set, C, containing p on which f is continuous, and $f \restriction C$ extends continuously to* $\mathrm{cl}(C)$*?*

Example 2 then becomes

Example 4. *Let $M = (\mathbb{R}, +, \cdot, 0, 1, <)$, let $\bar{c} = \langle c_1, c_2\rangle$ be a pair satisfying the formulas $\{x < r \mid r > 0, r \in \mathbb{R}\} \cup \{y < rx \mid r > 0, r \in \mathbb{R}\} \cup \{y > x^d \mid d \in \mathbb{Q}, d > 1\}$. Then there is no M-definable set containing $\bar{c}$ on which f is continuous, and onto whose closure f extends continuously. In particular, f cannot be extended continuously to the origin.*

Answering Question 3 led me to define the following two concepts, Definitions 5 and 6. The first is derived from [MS94], although in a somewhat modified form.

Definition 5. *Let $A \subseteq B$ be sets. Let $c \notin B$ be non-principal over B. We say that* $\mathrm{tp}(c/B)$ *is* in scale on A *if, for some A-definable function, $f(x,y)$, with x a tuple and y a singleton, and some tuple $b \in B$, $f(b, \mathrm{dcl}(A))$ is cofinal and coinitial at c in* $\mathrm{dcl}(B)$*. Say* $\mathrm{tp}(c/B)$ *is* near scale on A *if there is a function and tuple, as before, such that $f(b, \mathrm{dcl}(A))$ is cofinal (or coinitial) at c in* $\mathrm{dcl}(B)$*. Say* $\mathrm{tp}(c/B)$ *is* out of scale on A *otherwise.*

[MS94] needed a definition like this because they were examining definable n-tuples. To see why we need it, consider Example 4 and $f(c_1, -)$. As the argument of $f(c_1, -)$ approaches c_2 from above, $f(c_1, -)$ takes on all positive real values, and there is no

Mc-definable point below which $f(c_1, -)$ stops taking positive real values. Equivalently, the inverse of $f(c_1, -)$ witnesses that $\mathrm{tp}(c_2/Mc)$ is near scale on M, and that equivalence is why scale is useful. However, while scale is a crucial component to the solution, it is not a particularly stable property – for example, it can change under reordering of coordinates. For that reason, I introduced the following notion.

Definition 6. *Let M be a model, and $\bar{a} = \langle a_1, \ldots, a_k \rangle$ be a sequence, with $a_i \notin M$ for $i \leq k$. Say that $\bar{a}$ is* decreasing *if, for each $i < j \leq k$, $(0, b) \cap \mathrm{dcl}(M\bar{a}_{<i}a_j)$ is non-empty for all $b \in \mathrm{dcl}(M\bar{a}_{\leq i})$.*

This notion of decreasing captures precisely that earlier elements are never infinitesimal over later ones. It is not hard to show, as well, that every n-type can be reordered to be decreasing. With this in hand, we can finally state the theorem.

Theorem 7. *Let M be an o-minimal real closed field, possibly with some additional structure. Let $p \in S_n(M)$ be a decreasing type. Then the following two conditions are equivalent:*

1. *For $\bar{c} = \langle c_1, \ldots, c_n \rangle$, some (any) realization of p, $\mathrm{tp}(c_i/\bar{c}_{<i}M)$ is principal, or out of scale on M, for $i = 1, \ldots, n$.*
2. *For every M-definable function, f, bounded on some M-definable set in p, there is an M-definable set, C, in p, such that f is continuous on C and extends continuously to $\mathrm{cl}(C)$.*

Note that the first condition can be equivalently phrased as $\mathrm{tp}(c_i/\bar{c}_{<i}M)$ is *not* near scale or in scale on M. Example 4 was near scale, and shows the essential obstacle there. Since in scale is basically near scale on both sides, it is easy to see why failure occurs there as well.
The theorem's proof, while technically delicate due to the multiple dimensions, follows the spirit of the above discussion. A key point is that a decreasing type can be contained in a definable set in which any curve with one coordinate going to a limit point must have all later coordinates going to limit points as well, and going to those limit points at least as "fast." This allows us to choose boundary functions for our cell that come together quickly enough to ensure that the function f continuously extends to all boundary points.

3 Pure model theory

The above result led to several new avenues of research in pure model theory. First, while the above concept of "scale" is due to [MS94], their definition is actually somewhat looser, which makes their characterization of definable types in o-minimal theories not as tight as it could be. The optimal statement is

Theorem 8. *Let p be an n-type over M, and let $\bar{c} = \langle c_1, \ldots, c_n \rangle \models p$. Then p is definable iff for $i \leq n$, $\mathrm{tp}(c_i/M\bar{c}_{<i})$ is principal over $M\bar{c}_{<i}$, or near scale or out of scale on M.*

Scale permits a very fine-grained distinction among types in o-minimal theories. [Mar86] divided types into three non-interdefinable categories, meaning that no definable function could take a realization of one to a realization of another. With scale, we can further subdivide so that we have five categories of types, all non-interdefinable.

3.1 Decreasing types

As well, while decreasing types were useful in the above result as a way to control later variables of a type in terms of earlier ones, they can also be studied in their own right. That every type of finite length can be reordered so as to be decreasing is not difficult, but as well, every type of finite length is interdefinable with a decreasing type with the property that it contains no in scale or near scale elements in the sequence. Moreover, every sequence, not just finite, is interdefinable with a decreasing sequence with the above property – that it contains no in scale or near scale elements in the sequence. This is a refinement of a result of [BP98], which states

Lemma 9. *If $M \prec N$, then there is some N' with $M \prec N' \prec N$ such that the type of every element of $N' \setminus M$ is definable, and the type of every element of $N \setminus N'$ is not definable, and similarly there is some N'' with $M \prec N'' \prec N$ such that the type of every element of $N'' \setminus M$ is not definable, and the type of every element of $N \setminus N''$ is definable.*

My result gives a more detailed description of these extensions – in particular, a basis (under the operation of dcl) and a partition of the basis such that adding elements in the first part gives an extension with no realizations of definable types, and adding elements in the second gives an extension with only realizations of definable types. Moreover, the second part of this basis is a linearly ordered sequence whose order type gives an invariant of the extension.

4 Further work

I think that the tools developed in the solution of Question 1 have two applications. First, they may well aid in solving other problems coming from o-minimal analysis. Second, and more interestingly to me, they can help to find more invariants of types that are preserved/modified in a predictable way under definable maps. Having such invariants allows us to better understand both the behavior of definable functions in elementary extensions, and the behavior of the types themselves.

Decreasing types are very closely bound up with T-convex subrings, as defined in [vdDL95], in that a decreasing sequence generates an increasing sequence of T-convex subrings. Scale is connected generally to convex subrings, about which there is current work, especially in the realm of determining the behavior of the quotient field that results from a convex subring and its maximal ideal of infinitesimals (for instance, [MvdD09]). I believe that more precise information on types will be very useful here. For instance, if we have $M \prec N$, an out of scale type on M

corresponds to a non-principal cut in the value group $N^\times/M^\times$, while near scale types are principal. When the value group is o-minimal, or amenable to o-minimal methods, we obtain even finer classifications of o-minimal types, by considering their images in this value group.

Another question I hope to answer is suggested by Tressl – whether any o-minimal elementary extension can be written using a basis of orthogonal types. Invariants that prevent or guarantee interdefinability of types may hold the key to this question, and decreasing types will allow me to build the elementary extension in a controlled manner.

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