The lace expansion for self-interacting RW: Aussois mini course.

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Thanks and apologies:

This mini-course includes:

- an intro. to a particular method for studying self-interacting RW
- some applications of that method

This mini-course is not:

- a survey of known results on RWRE, cookie RW or any other RW models
- going to answer all your questions about RW models

Self-interacting random walks:

- A n.n. RW path $\vec{\eta}_n$ is a sequence $\{\eta_i\}_{i=0}^{n}$ for which $\eta_i = (\eta_i^{[1]}, \ldots, \eta_i^{[d]}) \in \mathbb{Z}^d$ and $|\eta_{i+1} - \eta_i| = 1$ for each i.
- Notation: $p^{\vec{\eta}_i}(y, x)$ is conditional probability that the walk steps from $\eta_i = y$ to x, given the history $\vec{\eta}_i = (\eta_0, \ldots, \eta_i)$.
- $$Q(\vec{X}_n = (x_0, x_1, \ldots, x_n)) = \prod_{i=0}^{n-1} p^{\vec{x}_i}(x_i, x_{i+1}).$$
- Q assumed to be translation invariant w.r.t. starting point.

Self-interacting random walks include:

- simple random walk
- annealed RWRE
- reinforced random walks
- (annealed) cookie random walks

Properties of interest:

- recurrence/transience
- LLN: existence of $v := \lim_{n\to\infty} \frac{X_n}{n}$, Q-a.s.
- CLT: $\frac{X_n - nv}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Sigma)$

How do these properties change as we vary some parameter(s) of the model?

Contents of this mini-course:

- Derive the lace expansion for self-interacting random walks.
- Obtain a formula for the speed appearing in the LLN
- (Sketch)-Prove monotonicity properties for excited random walks
- Discuss other models

Primary model of interest: ERWD

- site-percolation λ-cookie environment $\omega \in \{0,1\}^{\mathbb{Z}^d}$, i.e. cookies at $\{x : \omega_x = 1\}$
- right drift (parameter β) when eat a cookie, left drift (μ) otherwise

Given ω, the ERWD $\{X_n\}_{n \geqslant 0}$ has law $\mathbb{Q}_\omega$ defined by

- $\mathbb{Q}_\omega(X_0 = o) = 1$ and

$$p^o_\omega(o, \eta_1) = \frac{1 + (\beta I_{\{\omega_o = 1\}} - \mu(1 - I_{\{\omega_o=1\}}))e_1 \cdot \eta_1}{2d}, \qquad \text{and}$$

$$p^{\vec{\eta}_i}_\omega(\eta_i, \eta_{i+1}) = \frac{1 + (\beta I_{\{\omega_{\eta_i}=1\}} I_{\{\eta_i \notin \vec{\eta}_{i-1}\}} - \mu(1 - I_{\{\omega_{\eta_i}=1\}} I_{\{\eta_i \notin \vec{\eta}_{i-1}\}}))e_1 \cdot (\eta_{i+1} - \eta_i)}{2d}.$$

Annealed ERWD

Annealed measure

$$Q(\cdot, \star) = \int_\star \mathbb{Q}_\omega(\cdot) d\mathbb{Q}.$$

Under Q, interested in $v^{[1]}(d, \beta, \mu, \lambda)$ defined by

$$v^{[1]} = \lim_{n\to\infty} \frac{X_n^{[1]}}{n}.$$

v exists Q-a.s. for $d \geqslant 6$, by a theorem of Bolthausen, Sznitman and Zeitouni (2003).

Theorem: (H, '09)

$\nu^{[1]}(d, \beta, \mu, \lambda)$ is continuous in $(\beta, \mu, \lambda) \in [0, 1]^3$ when $d \geqslant 6$ and when $d \geqslant 12$, is strictly increasing:

- in $\beta \in [0, 1]$ for each $\mu, \lambda \in (0, 1]$
- in $\lambda \in [0, 1]$ for each $\mu, \beta \in (0, 1]$

(Weaker results for monotonicity in μ).

if e.g. $\mu = 0$, we get monotonicity in $\beta, \lambda \in [0, 1]$ for $d \geqslant 9$.

The two point function

Let

$$p^o(x) = \mathbb{P}(S_1 = x)$$

be the SRW kernel (possibly with drift). Then for SRW

$$\mathbb{P}(\vec{S}_n = (x_0, x_1, \ldots, x_n)) = \prod_{i=0}^{n-1} p^o(x_{i+1} - x_i).$$

For self-interacting random walks we have

$$Q(\vec{X}_n = (x_0, x_1, \ldots, x_n)) = \prod_{i=0}^{n-1} p^{\vec{x}_i}(x_i, x_{i+1}).$$

With v.d. Hofstad we investigate the *two-point function*

$$c_n(x) = Q(X_n = x).$$

Expansion overview

- First write

$$c_{n+1}(x) = \sum_{y} p^{o}(y) c_n(x-y) + \sum_{m=2}^{n+1} \sum_{y} \pi_m(y) c_{n+1-m}(x-y).$$

  Here $\sum_x c_n(x) = 1$, which makes some of the analysis easier.
- derive bounds on the lace expansion coefficients
- analyse the recursion relation, using the bounds on the lace expansion coefficients (and induction)

Who cares?

Taking the Fourier transform, get

$$\hat{c}_{n+1}(k) = \hat{p}^{o}(k)\hat{c}_n(k) + \sum_{m=2}^{n+1} \hat{\pi}_m(k)\hat{c}_{n+1-m}(k),$$

where

$$\hat{c}_n(k) = \sum_{x \in \mathbb{Z}^d} e^{ik\cdot x} c_n(x) = E\left[e^{ik\cdot X_n}\right].$$

Under strong* assumptions on π_m, can inductively prove

$$\hat{c}_n(kn^{-1}) = e^{ik\cdot v + e_n(k)}, \qquad \hat{c}_n(kn^{-\frac{1}{2}})e^{-ikv\sqrt{n}} = e^{-\frac{1}{2}k^t\Sigma k + \epsilon_n(k)}.$$

*The good news is that v and Σ are described in terms of the expansion coefficients π_m.

Theorem: Speed formula

If $\lim_{n\to\infty} \sum_{m=2}^{n} \sum_x x\pi_m(x)$ exists and $n^{-1}X_n \xrightarrow{Q} v$, then

$$v = \sum_x x p^o(x) + \sum_{m=2}^{\infty} \sum_x x\pi_m(x).$$

speed formula proof

- Summing recursion over x:

$$1 = 1 + \sum_{m=2}^{n+1} \sum_{x} \pi_m(x).$$

Thus $\sum_x \pi_m(x) = 0$.

- Multiply recursion by $x = y + (x - y)$ and sum over x

$$\sum_x x c_{n+1}(x) = \sum_y y p^o(y) + \sum_x x c_n(x) + \sum_{m=2}^{n+1} \sum_y y \pi_m(y).$$

i.e.

$$E[X_{n+1} - X_n] = E[X_1] + \sum_{m=2}^{n+1} \sum_y y \pi_m(y).$$

speed proof cont.

If

$$\lim_{n\to\infty} E[X_{n+1} - X_n] = \tilde{v}$$

then since $X_n = \sum_{m=1}^{n}(X_m - X_{m-1})$, we have also

$$\lim_{n\to\infty} E[n^{-1}X_n] = \tilde{v}.$$

If $n^{-1}X_n \xrightarrow{Q} v$, by bounded convergence we get

$$\lim_{n\to\infty} E[n^{-1}X_n] = v,$$

so $v = \tilde{v}$. □

Variance formula (symmetric case)

Suppose that $E[X_n] = 0$ for each n and for each $i, j \in \{1, 2, \ldots, d\}$,

$$\lim_{n\to\infty} \frac{E[X_n^{[i]} X_n^{[j]}]}{n} = \Sigma_{ij}, \quad \text{and} \quad \sum_{m=2}^{\infty} \sum_{y} y^{[i]} y^{[j]} \pi_m(y) < \infty,$$

Then

$$\Sigma_{ij} = E[X_1^{[i]} X_1^{[j]}] + \sum_{m=2}^{\infty} \sum_{y} y^{[i]} y^{[j]} \pi_m(y).$$

More notation

convolution of abs. summable functions f, g on $\mathbb{Z}^d$

$$(f * g)(x) := \sum_y f(y)g(x-y),$$

recursion becomes

$$c_{n+1}(x) = (p^o * c_n)(x) + \sum_{m=2}^{n+1} (\pi_m * c_{n+1-m})(x).$$

If $\vec{\eta}$ and $\vec{x}$ are such that $\eta_j = x_0$, then

$$(\vec{\eta}_j \circ \vec{x}_m)_i := \begin{cases} \eta_i & \text{when } 0 \leqslant i \leqslant j, \\ x_{i-j} & \text{when } j \leqslant i \leqslant m+j. \end{cases}$$

Derivation of expansion

Given $\vec{\eta}_m$, define $Q^{\vec{\eta}_m}$ on walk paths starting from η_m, by

$$Q^{\vec{\eta}_m}(\vec{X}_n = \vec{x}_n) = Q\left(\vec{X}_{m+n} = (\vec{\eta}_m, \vec{x}_n) \middle| \vec{X}_m = \vec{\eta}_m\right)$$

write

$$c_n^{\vec{\eta}_m}(\eta_m, x) := Q^{\vec{\eta}_m}(X_n = x),$$

$$c_{n+1}(x) = \sum_{\vec{x}_1^{(0)}} p^o(x_1^{(0)}) \sum_{\vec{x}_n^{(1)} : x_1^{(0)} \to x} \prod_{i=0}^{n-1} p^{\vec{x}_1^{(0)} \circ \vec{x}_i^{(1)}}(x_i^{(1)}, x_{i+1}^{(1)}).$$

expansion cont.

Write product as

$$\prod_{i=0}^{n-1}\left[p^{\vec{x}_i^{(1)}}(x_i^{(1)},x_{i+1}^{(1)})+\left(p^{\vec{x}_1^{(0)}\circ\vec{x}_i^{(1)}}(x_i^{(1)},x_{i+1}^{(1)})-p^{\vec{x}_i^{(1)}}(x_i^{(1)},x_{i+1}^{(1)})\right)\right].$$

Expand, using

$$\prod_{i=0}^{n-1}(a_i+\Delta_i)=\prod_{i=0}^{n-1}a_i+\sum_{j=0}^{n-1}\Big(\prod_{i=0}^{j-1}(a_i+\Delta_i)\Big)\Delta_j\Big(\prod_{i=j+1}^{n-1}a_i\Big),$$

(empty products are defined to be 1).

expansion cont.

Get

$$c_{n+1}(x) = \sum_{\vec{x}_1^{(0)}} p^o(x_1^{(0)}) \sum_{\vec{x}_n^{(1)}:x_1^{(0)}\to x} \prod_{i=0}^{n-1} p^{\vec{x}_i^{(1)}}(x_i^{(1)}, x_{i+1}^{(1)})$$

$$+ \sum_{j=0}^{n-1} \sum_{\vec{x}_1^{(0)}} p^o(x_1^{(0)}) \sum_{\vec{x}_n^{(1)}:x_1^{(0)}\to x} \left[\prod_{i=0}^{j-1} p^{\vec{x}_1^{(0)}\circ\vec{x}_i^{(1)}}(x_i^{(1)}, x_{i+1}^{(1)}) \right]$$

$$\times \Delta_j \left[\prod_{i=j+1}^{n-1} p^{\vec{x}_i^{(1)}}(x_i^{(1)}, x_{i+1}^{(1)}) \right].$$

First term is $\sum_{\vec{x}_1^{(0)}} p^o(x_1^{(0)}) c_n(x - x_1^{(0)}) = (p^o * c_n)(x)$.

expansion cont.

second term becomes

$$\sum_{j=0}^{n-1} \sum_{\vec{x}_1^{(0)}} p^o(x_1^{(0)}) \sum_{\vec{x}_{j+1}^{(1)}: x_0^{(1)} = x_1^{(0)}} \left[\prod_{i=0}^{j-1} p^{\vec{x}_1^{(0)} \circ \vec{x}_i^{(1)}} (x_i^{(1)}, x_{i+1}^{(1)}) \right] \Delta_j^{(1)}$$

$$\times \sum_{(x_{j+2}^{(1)}, \ldots, x_n^{(1)}): x_n^{(1)} = x} \left[\prod_{i=j+1}^{n-1} p^{\vec{x}_i^{(1)}} (x_i^{(1)}, x_{i+1}^{(1)}) \right]$$

$$= \sum_{j=0}^{n-1} \sum_{\vec{x}_1^{(0)}} p^o(x_1^{(0)}) \sum_{\vec{x}_{j+1}^{(1)}} Q^{\vec{x}_1^{(0)}} (\vec{X}_j = \vec{x}_j^{(1)}) \Delta_j^{(1)}$$

$$\times \sum_{\vec{x}_{n-j-1}^{(2)}: x_{j+1}^{(1)} \to x} \left[\prod_{r=0}^{n-j-1} p^{\vec{x}_{j+1}^{(1)} \circ \vec{x}_r^{(2)}} (x_r^{(2)}, x_{r+1}^{(2)}) \right].$$

This product can be written as

$$\prod_{r=0}^{n-j-1}\left[p^{\vec{x}_r^{(2)}}(x_r^{(2)},x_{r+1}^{(2)})+p^{\vec{x}_{j+1}^{(1)}\circ\vec{x}_r^{(2)}}(x_r^{(2)},x_{r+1}^{(2)})-p^{\vec{x}_r^{(2)}}(x_r^{(2)},x_{r+1}^{(2)})\right].$$

Expand this again!

- One term involves no $\vec{x}_{j+1}^{(1)}$ history, and gives:

$$\sum_{j=0}^{n-1}\sum_{\vec{x}_1^{(0)}}p^o(x_1^{(0)})\sum_{\vec{x}_{j+1}^{(1)}}Q^{\vec{x}_1^{(0)}}(\vec{X}_j=\vec{x}_j^{(1)})\Delta_j^{(1)}\ c_{n-j-1}(x-x_{j+1}^{(1)})$$

$$=\sum_{j=0}^{n-1}\sum_{y}\left[\sum_{\vec{x}_1^{(0)}}p^o(x_1^{(0)})\sum_{\vec{x}_{j+1}^{(1)}}Q^{\vec{x}_1^{(0)}}(\vec{X}_j=\vec{x}_j^{(1)})I_{\{x_{j+1}^{(1)}=y\}}\Delta_j^{(1)}\right]c_{n-j-1}(x-$$

$$\equiv\sum_{m=2}^{n+1}\sum_{y}\pi_m^{(1)}(y)c_{n+1-m}(x-y),$$

where we have used the substitution $m=j+2$.

- Other terms all involve a $\Delta^{(2)}$.

iterate

- Recursion follows from iterating until there is nothing left to expand.

Terms appearing are:

$$\pi_m(y) = \sum_{N=1}^{\infty} \pi_m^{(N)}(y).$$

where

$$\pi_m^{(N)}(y) := \sum_{\vec{j} \in \mathcal{A}_{m,N}} \sum_{\vec{x}_1^{(0)}} p^o(x_1^{(0)}) \sum_{\vec{x}_{j_1+1}^{(1)}} \cdots \sum_{\vec{x}_{j_N+1}^{(N)}} I_{\{x_{j_N+1}^{(N)}=y\}}$$

$$\times \prod_{k=1}^{N} Q^{\vec{x}_{j_{k-1}+1}^{(k-1)}}(\vec{X}_{j_k} = \vec{x}_{j_k}) \Delta_{j_k}^{(k)},$$

and $\mathcal{A}_{m,N} = \{\vec{j} \in \mathbb{Z}_+^N : j_1 + \cdots + j_N = m - N - 1\}$ □

Excited random walk ($\lambda = 1$, $\mu = 0$)

Recall:

- $$p^o(\eta_1) = \frac{1 + \beta e_1 \cdot \eta_1}{2d}, \qquad \text{and}$$
- $$p^{\vec{\eta}_i}(\eta_i, \eta_{i+1}) = \frac{1 + \beta I_{\{\eta_i \notin \vec{\eta}_{i-1}\}} e_1 \cdot (\eta_{i+1} - \eta_i)}{2d}.$$
- $$v = E[X_1] + \sum_{m=2}^{\infty} \sum_x x \pi_m(x),$$

  if this limit exists.

Theorem: (v.d. Hofstad, H.)
For $d \geqslant 9$, $v(\beta)$ is increasing in $\beta \in [0, 1]$.

The expansion coefficients.

$$\pi_m^{(N)}(y) := \sum_{\vec{j}\in\mathcal{A}_{m,N}} \sum_{\vec{x}_1^{(0)}} p^o(x_1^{(0)}) \sum_{\vec{x}_{j_1+1}^{(1)}} \cdots \sum_{\vec{x}_{j_N+1}^{(N)}} I_{\{x_{j_N+1}^{(N)}=y\}}$$

$$\times \prod_{k=1}^{N} Q^{\vec{x}_{j_{k-1}+1}^{(k-1)}}(\vec{X}_{j_k} = \vec{x}_{j_k})\Delta_{j_k}^{(k)},$$

For ERW,

$$\Delta^{(n)} = \frac{\beta e_1 \cdot (x_{j_n+1}^{(n)} - x_{j_n}^{(n)})}{2d}\left[I_{\{x_{j_n}^{(n)} \notin \vec{x}_{j_{n-1}}^{(n-1)} \circ \vec{x}_{j_n-1}^{(n)}\}} - I_{\{x_{j_n}^{(n)} \notin \vec{x}_{j_n-1}^{(n)}\}}\right]$$

$$|\Delta^{(n)}| \leqslant \frac{\beta}{2d} I_{\{x_{j_n+1}^{(n)}=x_{j_n}^{(n)}\pm e_1\}} I_{\{x_{j_n}^{(n)} \in \vec{x}_{j_{n-1}}^{(n-1)}\}}.$$

Define

$$\pi_m^{(N)}(x,y) = \sum_{\vec{j}\in\mathcal{A}_{m,N}} \sum_{\vec{x}_1^{(0)}} \sum_{\vec{x}_{j_1+1}^{(1)}} \cdots \sum_{\vec{x}_{j_N+1}^{(N)}} I_{\{x_{j_N}^{(N)}=x,\, x_{j_N+1}^{(N)}=y\}} \cdots .$$

Since

$$\sum_{y\in\mathbb{Z}^d} \pi_m^{(N)}(x,y) = 0,$$

$$\sum_{y\in\mathbb{Z}^d} y\pi_m(y) = \sum_{x,y\in\mathbb{Z}^d} (y-x)\pi_m(x,y),$$

so that

$$v(\beta) = \frac{\beta e_1}{d} + \sum_{m=2}^{\infty}\sum_{N=1}^{\infty}\sum_{x,y\in\mathbb{Z}^d} (y-x)\pi_m^{(N)}(x,y).$$

Does speed formula converge?

- $\mathbb{P}_d$ is law of simple symmetric random walk in d dimensions,
- $D_d(x) = I_{\{|x|=1\}}/(2d)$ is SRW step distribution.
- $G_d(x) = \sum_{k=0}^{\infty} D_d^{*k}(x)$ is SRW Green's function. Then

$$G_d^{*i}(x) = \sum_{k=0}^{\infty} \frac{(k+i-1)!}{(i-1)!k!} \mathbb{P}_d(X_k = x), \quad \text{for } i \geqslant 1.$$

  Note that $G_d^{*i}(x) < \infty$ if and only if $d > 2i$.
- $G_d^{*i} := G_d^{*i}(o)$. For $i \geqslant 0$, let $q_d = (d-1)/d$

$$\mathcal{E}_i(d) = q_d^{-(i+1)} G_{d-1}^{*(i+1)} - 1.$$

bounds in terms of SRW

Lemma: For all $u \in \mathbb{Z}^d$, $\vec{\eta}_m$, and $i \in \mathbb{Z}_+$,

$$\sum_{j=0}^{\infty} \frac{(j+i)!}{j!} Q^{\vec{\eta}_m}(X_j = u) \leqslant i! q_d^{-(i+1)} G_{d-1}^{*(i+1)},$$

$$\sum_{j=1}^{\infty} \frac{(j+i)!}{j!} Q^{\vec{\eta}_m}(X_j = u) \leqslant i! \mathcal{E}_i(d).$$

sketch proof: LHS of first ineq. is

$$\sum_{j=0}^{\infty} \frac{(j+i)!}{j!} \sum_{l=0}^{j} Q^{\vec{\eta}_m}(X_j = u | \mathcal{N}_j = l) Q(\mathcal{N}_j = l)$$

$$\leqslant \sum_{l=0}^{\infty} \mathbb{P}_{d-1}(X_l = u^- - \eta_m^-) \sum_{j=l}^{\infty} \frac{(j+i)!}{j!} Q(\mathcal{N}_j = l)$$

$$\leqslant \sup_{\nu} \sum_{l=0}^{\infty} \mathbb{P}_{d-1}(X_l = \nu) \sum_{j=l}^{\infty} \frac{(j+i)!}{j!} Q(\mathcal{N}_j = l)$$

proof cont.

- $\mathcal{N}_j$ is # steps that $\vec{X}_j$ takes in coordinates $2, 3, \dots, d$
- $\{\mathcal{N}_j\}_{j \geqslant 0}$ is a RW on $\mathbb{Z}_+$, steps +1,0 w.p. q_d, $1 - q_d$
- $\mathcal{N}_j \sim$Bin(j, q_d), thus

$$\frac{(j+i)!}{j!} Q(\mathcal{N}_j = l) = q_d^{-i} \frac{(l+i)!}{l!} Q(\mathcal{N}_{j+i} = l+i).$$

- $\mathcal{N}_j$-local time of level $l \sim$Geom(q_d), thus for $m \leqslant l$,

$$\sum_{j=m}^{\infty} \frac{(j+i)!}{j!} Q(\mathcal{N}_j = l) = q_d^{-(i+1)} \frac{(l+i)!}{l!}.$$

Finally

$$\sum_{l=0}^{\infty} \mathbb{P}_{d-1}(X_l = v) \frac{(l+i)!}{l!} = i! G_{d-1}^{*(i+1)}(v)$$

□

Proposition $\approx$ formula converges

Define
$$a_d = \frac{d}{(d-1)^2} G^{*2}_{d-1}.$$

$a_d < 1$ when $d \geqslant 6$.

Proposition:

- $\sum_{x,y\in\mathbb{Z}^d} \sum_m |\pi^{(1)}_m(x,y)| \leqslant \beta d^{-1} \mathcal{E}_0(d)$,
- $N \geqslant 2$,

$$\sum_{x,y\in\mathbb{Z}^d} \sum_m |\pi^{(N)}_m(x,y)| \leqslant \beta^N d^{-1} (d-1)^{-1} G_{d-1} \mathcal{E}_1(d) a_d^{N-2}.$$

Piecewise bounds

Given $\vec{\eta}_m$ and $\vec{z}_{j+1}$, define

$$\Delta(\vec{z}_{j+1}) = \left(p^{\vec{\eta}_m \circ \vec{z}_j}\left(z_j, z_{j+1}\right) - p^{\vec{z}_j}\left(z_j, z_{j+1}\right)\right) I_{\{z_0 = \eta_m\}}.$$

Lemma: For any $\vec{\eta}_m$,

$$\sum_{j=0}^{\infty} \sum_{\vec{z}_{j+1}} |\Delta(\vec{z}_{j+1})| Q^{\vec{\eta}_m}(\vec{X}_j = \vec{z}_j) \leqslant m\beta \frac{G_{d-1}}{d-1},$$

$$\sum_{j=0}^{\infty} (j+1) \sum_{\vec{z}_{j+1}} |\Delta(\vec{z}_{j+1})| Q^{\vec{\eta}_m}(\vec{X}_j = \vec{z}_j) \leqslant m\beta a_d,$$

$$\sum_{j=1}^{\infty} \sum_{\vec{z}_{j+1}} |\Delta(\vec{z}_{j+1})| Q^{\vec{\eta}_m}(\vec{X}_j = \vec{z}_j) \leqslant m\beta \frac{\mathcal{E}_0(d)}{d},$$

$$\sum_{j=1}^{\infty} (j+1) \sum_{\vec{z}_{j+1}} |\Delta(\vec{z}_{j+1})| Q^{\vec{\eta}_m}(\vec{X}_j = \vec{z}_j) \leqslant m\beta \frac{\mathcal{E}_1(d)}{d}.$$

Lemma sketch proof: (first one)

LHS bounded by

$$\sum_{j=0}^{\infty}\sum_{\vec{z}_j} Q^{\vec{\eta}_m}(\vec{X}_j = \vec{z}_j) I_{\{z_j \in \vec{\eta}_{m-1}\}} \frac{\beta}{2d} \sum_{z_{j+1}} I_{\{z_{j+1} = z_j \pm e_1\}}$$

$$= \frac{\beta}{d} \sum_{j=0}^{\infty}\sum_{\vec{z}_j} Q^{\vec{\eta}_m}(\vec{X}_j = \vec{z}_j) I_{\{z_j \in \vec{\eta}_{m-1}\}},$$

This is equal to

$$\frac{\beta}{d}\sum_{j=0}^{\infty} Q^{\vec{\eta}_m}(X_j \in \vec{\eta}_{m-1}) \leqslant \frac{\beta}{d} \sum_{l=0}^{m-1}\sum_{j=0}^{\infty} Q^{\vec{\eta}_m}(X_j = \eta_l).$$

□

sketch proof of Proposition

From definition of $\pi_m^{(N)}(x,y)$, $\sum_{x,y\in\mathbb{Z}^d}\sum_m |\pi_m^{(N)}(x,y)|$ is bounded by

$$\sum_{x_1^{(0)}} p^o(x_1^{(0)}) \sum_{j_1=1}^{\infty} \sum_{\vec{x}_{j_1+1}^{(1)}} |\Delta^{(1)}| Q^{\vec{x}_1^{(0)}}(\vec{X}_{j_1} = \vec{x}_{j_1}^{(1)}) \dots$$

$$\dots \sum_{j_N=0}^{\infty} \sum_{\vec{x}_{j_N+1}^{(N)}} |\Delta^{(N)}| Q^{\vec{x}_{j_{N-1}}^{(N-1)}}(\vec{X}_{j_N} = \vec{x}_{j_N}^{(N)}).$$

- $\Delta^{(1)}$ is non-zero only if j_1 is odd.
- Use Lemma repeatedly to get desired bounds.

□

PART 2:

Monotonicity for excited random walk

Excited random walk ($\lambda = 1$, $\mu = 0$)

Recall:

- $$p^{o}(\eta_1) = \frac{1 + \beta e_1 \cdot \eta_1}{2d}, \qquad \text{and}$$
- $$p^{\vec{\eta}_i}(\eta_i, \eta_{i+1}) = \frac{1 + \beta I_{\{\eta_i \notin \vec{\eta}_{i-1}\}} e_1 \cdot (\eta_{i+1} - \eta_i)}{2d}.$$
- $$v^{[1]} = \frac{\beta}{d} + \sum_{m=2}^{\infty} \sum_{x,y} (y^{[1]} - x^{[1]}) \pi_m(x, y).$$

Theorem: (v.d. Hofstad, H.)
For $d \geqslant 9$, $v(\beta)$ is increasing in $\beta \in [0, 1]$.

Differentiate!

Let $\varphi_m^{(N)}(x,y) = \frac{\partial}{\partial\beta}\pi_m^{(N)}(x,y)$ we have

$$\frac{\partial v^{[1]}}{\partial\beta} = \frac{1}{d} + \sum_{N=1}^{\infty}\sum_{m=2}^{\infty}\sum_{x,y\in\mathbb{Z}^d}(y^{[1]} - x^{[1]})\varphi_m^{(N)}(x,y).$$

$$\left|\frac{\partial v^{[1]}}{\partial\beta} - \frac{1}{d}\right| \leqslant \sum_{N=1}^{\infty}\sum_{m=2}^{\infty}\sum_{x,y\in\mathbb{Z}^d}|\varphi_m^{(N)}(x,y)|.$$

Write

$$\varphi_m^{(N)}(x,y) = \varphi_m^{(N,1)}(x,y) + \varphi_m^{(N,2)}(x,y) + \varphi_m^{(N,3)}(x,y)$$

where these terms arise from differentiating

- $p^o(x_1^{(0)})$,
- $\prod_{n=1}^{N}\prod_{i_n=0}^{j_n-1} p^{\vec{x}_{j_{n-1}+1}^{(n-1)}\circ\vec{x}_{i_n}^{(n)}}\left(x_{i_n}^{(n)}, x_{i_n+1}^{(n)}\right)$
- $\prod_{n=1}^{N}\Delta^{(n)}$,

with respect to β.

piecewise derivatives

$$\frac{\partial}{\partial\beta} p^{\vec{\eta}_m}(\eta_m, x) = \frac{I_{\{\eta_m \notin \vec{\eta}_{m-1}\}}}{2d} \left(I_{\{x-\eta_m = e_1\}} - I_{\{x-\eta_m=-e_1\}} \right),$$

$$\frac{\partial}{\partial\beta} \left(p^{\vec{\eta}_m}(\eta_m, x) - p^{\vec{x}_n \circ \vec{\eta}_m}(\eta_m, x) \right)$$

$$= \frac{1}{2d} I_{\{\eta_m \notin \vec{\eta}_{m-1}, \eta_m \in \vec{x}_{n-1}\}} \left(I_{\{x-\eta_m = e_1\}} - I_{\{x-\eta_m=-e_1\}} \right).$$

- define $\rho^{(N)}$ as $\sum_{x,y,m} \pi^{(N)}(x,y)$ with p^o replaced with bound on its derivative, and $\Delta^{(n)}$ by $|\Delta^{(n)}|$
- define $\chi_k^{(N)}$ ... replacing $\Delta^{(k)}$ with its derivative ...
- $\gamma_k^{(N)}$ ... replacing kth product of the trans. probabilities with bound on derivative and $\Delta^{(n)}$ by $|\Delta^{(n)}|$

derivative bounds

Letting $\gamma^{(N)} = \sum_{k=1}^{N} \gamma_k^{(N)}$ and $\chi^{(N)} = \sum_{k=1}^{N} \chi_k^{(N)}$, we obtain

$$\sum_{m} \sum_{x,y \in \mathbb{Z}^d} |\varphi_m^{(N,1)}(x,y)| \leqslant \rho^{(N)}$$

$$\sum_{m} \sum_{x,y \in \mathbb{Z}^d} |\varphi_m^{(N,2)}(x,y)| \leqslant \gamma^{(N)},$$

$$\sum_{m} \sum_{x,y \in \mathbb{Z}^d} |\varphi_m^{(N,3)}(x,y)| \leqslant \chi^{(N)}.$$

Bound all of these terms separately, as done for $\sum_{x,y,m} \pi_m^{(N)}(x,y)$

Summary of bounds

$$\sum_N \rho^{(N)} + \gamma^{(N)} + \chi^{(N)} \leqslant \text{ stuff } (d)$$

where

- we need $a_d < 1$ for "stuff" to converge
- "stuff" involves G_{d-1}^{*i} for $i = 1, 2, 3$, so need $d \geqslant 8$
- "stuff" is $O(d^{-2})$ and stuff$(d) \leqslant d^{-1}$ when $d \geqslant 9$ using

$$G_8 \leqslant 1.07865, \quad G_8^{*2} \leqslant 1.2891, \quad G_8^{*3} \leqslant 1.8316.$$

Monotonicity result follows!

General case

Recall, in site-perc λ-cookie environment ω,

$$p^o_\omega(o,\eta_1) = \frac{1 + (\beta I_{\{\omega_o=1\}} - \mu(1 - I_{\{\omega_o=1\}}))e_1 \cdot \eta_1}{2d}, \qquad \text{and}$$

$$p^{\vec{\eta}_i}_\omega(\eta_i,\eta_{i+1})$$

$$= \frac{1 + (\beta I_{\{\omega_{\eta_i}=1\}} I_{\{\eta_i \notin \vec{\eta}_{i-1}\}} - \mu(1 - I_{\{\omega_{\eta_i}=1\}} I_{\{\eta_i \notin \vec{\eta}_{i-1}\}}))e_1 \cdot (\eta_{i+1} - \eta_i)}{2d}.$$

Under Q:

$$p^o(o,\eta_1) = \frac{1 + ((\beta+\mu)\lambda - \mu)e_1 \cdot \eta_1}{2d} \quad \text{and similarly,}$$

$$p^{\vec{\eta}_i}(\eta_i,\eta_{i+1}) = \frac{1 + ((\beta+\mu)\lambda I_{\{\eta_i \notin \vec{\eta}_{i-1}\}} - \mu)e_1 \cdot (\eta_{i+1} - \eta_i)}{2d}.$$

Theorem: (H.)

$\nu^{[1]}(d, \beta, \mu, \lambda)$ is continuous in $(\beta, \mu, \lambda) \in [0,1]^3$ when $d \geqslant 6$ and when $d \geqslant 12$, is strictly increasing:

- in $\beta \in [0,1]$ for each $\mu, \lambda \in (0,1]$
- in $\lambda \in [0,1]$ for each $\mu, \beta \in (0,1]$

(Weaker results for monotonicity in μ).

if e.g. $\mu = 0$, we get monotonicity in $\beta, \lambda \in [0,1]$ for $d \geqslant 9$.

Strategy

- Speed exists ($d \geqslant 6$) by a theorem of Bolthausen, Sznitman and Zeitouni
- Show that speed formula converges

$$v^{[1]} = \frac{(\beta + \mu)\lambda - \mu}{d} + \sum_{m=2}^{\infty} \sum_{x,y} (y^{[1]} - x^{[1]}) \pi_m(x, y)$$

- differentiate speed formula, show that "leading" term dominates

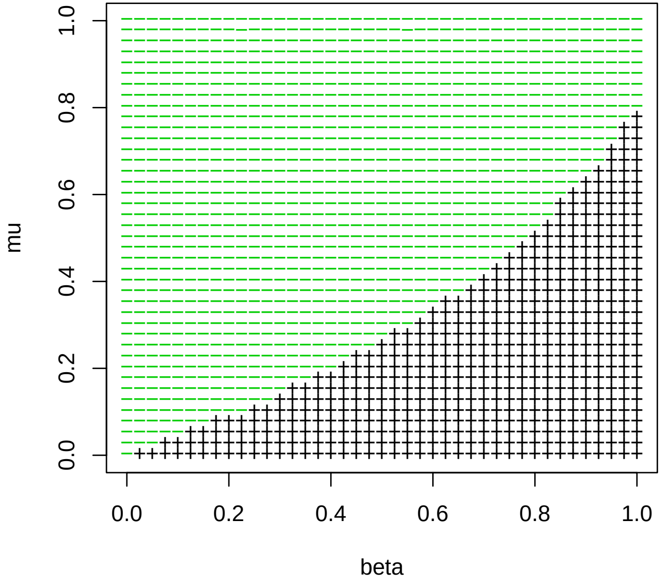
More interesting

Conjecture:

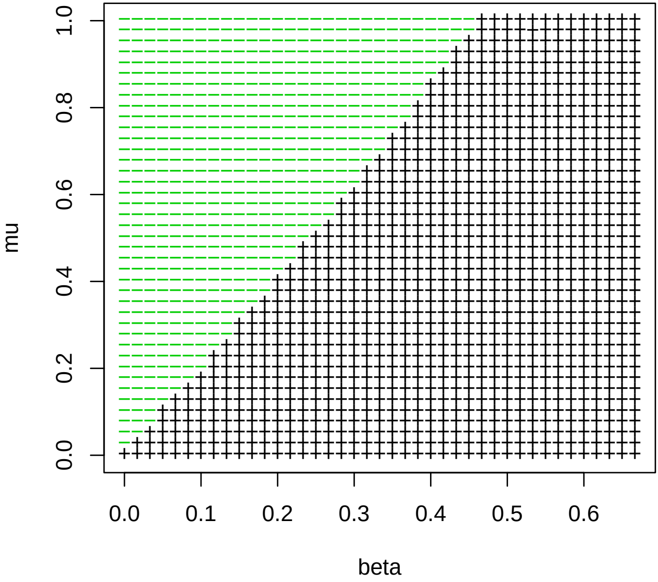
- For all $d \geqslant 2$, $(\mu, \beta, \lambda) \in [0,1]^3$, $\nu^{[1]}$ exists and is monotone increasing in β for fixed μ, λ and decreasing in μ for fixed β, λ respectively.
- For each $d \geqslant 3$ and $\mu \in [0,1]$ and all λ sufficiently large, $\exists!$ $\beta_0(\mu, d, \lambda) \in [0,1]$ such that $\nu(d, \mu, \beta_0, \lambda) = 0$. The same is true if the roles of λ and β are reversed.

Theorem: (H.) True in high dimensions.

Sign of velocity of ERW in 2 dimensions with competing drifts beta and mu



Sign of velocity of ERW in 3 dimensions with competing drifts beta and mu



Speed formula converges, $d \geqslant 6$

For ERWD,

$$\Delta^{(n)} = \frac{(\beta+\mu)\lambda e_1 \cdot (x^{(n)}_{j_n+1} - x^{(n)}_{j_n})}{2d} \left[I_{\{x^{(n)}_{j_n} \notin \vec{x}^{(n-1)}_{j_{n-1}} \circ \vec{x}^{(n)}_{j_n - 1}\}} - I_{\{x^{(n)}_{j_n} \notin \vec{x}^{(n)}_{j_n - 1}\}} \right]$$

$$|\Delta^{(n)}| \leqslant \frac{(\beta+\mu)\lambda}{2d} I_{\{x^{(n)}_{j_n+1} = x^{(n)}_{j_n} \pm e_1\}} I_{\{x^{(n)}_{j_n} \in \vec{x}^{(n-1)}_{j_{n-1}}\}}.$$

Same as before except for $(\beta+\mu)\lambda$ instead of β in all bounds.

- Repeat procedure to get convergence when $2a_d < 1$ $(d \geqslant 6)$.
- Get continuity of speed as a function of (λ, β, μ) for free. $(d \geqslant 6)$
- $d \geqslant 9$, for any μ, speed is positive for $\lambda\beta$ large enough.

Partial derivatives of speed formula

As before,

$$\left|\frac{\partial v^{[1]}}{\partial \beta} - \frac{\lambda}{d}\right| \leqslant \sum_{N=1}^{\infty} \sum_{m=2}^{\infty} \sum_{x,y} |\varphi_{\beta,m}^{(N)}(x,y)|$$

$$\left|\frac{\partial v^{[1]}}{\partial \lambda} - \frac{\beta+\mu}{d}\right| \leqslant \sum_{N=1}^{\infty} \sum_{m=2}^{\infty} \sum_{x,y} |\varphi_{\lambda,m}^{(N)}(x,y)|$$

$$\left|\frac{\partial v^{[1]}}{\partial \mu} - \frac{\lambda-1}{d}\right| \leqslant \sum_{N=1}^{\infty} \sum_{m=2}^{\infty} \sum_{x,y} |\varphi_{\mu,m}^{(N)}(x,y)|,$$

Derivatives of pieces

$$\frac{\partial}{\partial\beta} p^{\vec{\eta}_m}(\eta_m, x) = \frac{\lambda I_{\{\eta_m \notin \vec{\eta}_{m-1}\}}}{2d} \left(I_{\{x-\eta_m = e_1\}} - I_{\{x-\eta_m=-e_1\}} \right),$$

$$\frac{\partial}{\partial\lambda} p^{\vec{\eta}_m}(\eta_m, x) = \frac{(\beta+\mu) I_{\{\eta_m \notin \vec{\eta}_{m-1}\}}}{2d} \dots$$

$$\frac{\partial}{\partial\mu} p^{\vec{\eta}_m}(\eta_m, x) = \frac{\lambda I_{\{\eta_m \notin \vec{\eta}_{m-1}\}} - 1}{2d} \dots$$

and

$$\frac{\partial}{\partial\beta} \left(p^{\vec{\eta}_m}(\eta_m, x) - p^{\vec{x}_n \circ \vec{\eta}_m}(\eta_m, x) \right)$$

$$= \frac{\lambda}{2d} I_{\{\eta_m \notin \vec{\eta}_{m-1}, \eta_m \in \vec{x}_{n-1}\}} \left(I_{\{x-\eta_m=e_1\}} - I_{\{x-\eta_m=-e_1\}} \right).$$

The other terms are similar.

- Proceed as before using these slightly different bounds. Get

$$\left|\frac{\partial v^{[1]}}{\partial \beta} - \frac{\lambda}{d}\right| \leqslant \lambda \cdot \mathsf{stuff}(d)$$

$$\left|\frac{\partial v^{[1]}}{\partial \lambda} - \frac{\beta + \mu}{d}\right| \leqslant (\beta + \mu) \cdot \mathsf{stuff}(d)$$

$$\left|\frac{\partial v^{[1]}}{\partial \mu} - \frac{\lambda - 1}{d}\right| \leqslant \mathsf{stuff}(d),$$

- stuff(d) is order d^{-2}
- Need to take d higher to beat $(\beta + \mu)\lambda \leqslant 2$
- Doesn't quite work for μ derivative for large λ

Negative speeds:

Lemma: For each $d \geqslant 2$ and $\mu > 0$, the speed** is negative for $\lambda\beta$ sufficiently small.

Corollary: Fix $d \geqslant 9$, and $\mu \in [0, 1]$. For each λ sufficiently large, can find a $\beta_0(\mu, d, \lambda)$ so that the speed is 0. For each $d \geqslant 12$ $\beta_0(\mu, d, \lambda)$ is unique. The same is true with the roles of λ and β reversed.

sketch proof of lemma:

Fix $d \geqslant 2$ and $\mu > 0$.
Prove that $\limsup_{n\to\infty} n^{-1} X_n^{[1]} < \frac{1}{3} E[X_3^{[1]}]$, Q-almost surely:

- Explicitly write down

$$\mathbb{Q}_\omega(X_{n+3}^{[1]} - X_n^{[1]} = 3 | \vec{X}_n = \vec{x}_n)$$

$$\mathbb{Q}_\omega(X_{n+3}^{[1]} - X_n^{[1]} = 2 | \vec{X}_n = \vec{x}_n)$$

$$\mathbb{Q}_\omega(X_{n+3}^{[1]} - X_n^{[1]} = 1 | \vec{X}_n = \vec{x}_n)$$

  also -1,-2,-3 (and 0)
- the first two increase if you switch on a cookie
- so does the sum of all three
- reverse is true for negative terms

sketch proof cont.

- Take expectations w.r.t. $\mathbb{Q}$, get quantities bounded by $Q(X_3^{[1]} = j)$
- By coupling, X_n is left of walk with environmental regeneration every 3 steps
- the latter has speed $\frac{1}{3}E[X_3^{[1]}]$
  - continuous in $(\beta, \lambda) \in [0, 1]^2$
  - $< -\epsilon(d, \mu)$ when $\beta\lambda = 0$.

□

Monotonicity for RWpRE

- annealed velocity of RWRE NOT monotone increasing in the expected drift at the origin.
- if only one coordinate of environment is random, shifting probability from left steps to right steps increases speed to the right
- if more than one ???? even when components of environment are independent (or completely dependent)

Monotonicity theorem

Suppose

- $d = d_0 + d_1$ and $\lambda \leqslant d_0$
- $$\sum_{i=1}^{d_0} (\omega_o(e_i) + \omega_o(-e_i)) = \delta$$
- $c_i > 0$, are constants for $i \leqslant \lambda$.

Theorem: (H.) Let $d_1 \geqslant 7$ and X ~Bernoulli(β). Suppose that $\omega_o(e_j)$ and $\omega_o(-e_j)$ are independent of X (and β) for each $j > \lambda$ and $\mathbb{Q}$-almost surely,

$$\omega_o(e_i) = c_i X, \text{ and } \omega_o(-e_i) = c_i(1 - X), \qquad \text{for } i \leqslant \lambda,$$

$$\omega_o(u) = \frac{1-\delta}{2d_1}, \qquad \text{for } u \in \{\pm e_{d_0+1} \dots, \pm e_d\}.$$

Then for δ sufficiently small, $v^{[1]}(\beta)$ is continuous and strictly increasing in β.

Transition probabilities

Let $p_\omega(x, y) = \omega_x(y - x)$ be the probability of a transition from x to y in environment ω. Annealed trans. prob. $p^{\vec{\eta}_n}(\eta_n, \eta_n + u)$ is

- $\frac{1-\delta}{2d_1}$ for $u \in E_d \setminus E_{d_0}$,
- for $i \leqslant \lambda$,

$$p^{\vec{\eta}_n}(\eta_n, \eta_n + e_i) = c_i \left(I_{\{L_n^+ > 0, L_n^- = 0\}} + \beta I_{\{L_n^+ = 0, L_n^- = 0\}} \right)$$

$$p^{\vec{\eta}_n}(\eta_n, \eta_n - e_i) = c_i - p^{\vec{\eta}_n}(\eta_n, \eta_n + e_i).$$

- something independent of β otherwise.

The derivatives

$$\left|\frac{\partial}{\partial\beta} p^{\vec{\eta}_n}(\eta_n, \eta_n \pm e_i)\right| \leqslant c_i I_{\{L_n^+ = 0 = L_n^-\}} I_{\{i \leqslant \lambda\}} \leqslant \bar{c} I_{\{i \leqslant \lambda\}}.$$

$$|\Delta_i| \leqslant \delta \sum_{r_i=0}^{j_{i-1}} I_{\{x_{j_i}^{(i)} = x_{r_i}^{(i-1)}\}} I_{\{x_{j_i+1}^{(i)} - x_{j_i}^{(i)} \in E_{d_0}\}} I_{\{x_{r_i+1}^{(i-1)} - x_{r_i}^{(i-1)} \in E_{d_0}\}},$$

$$\left|\frac{\partial}{\partial\beta} \Delta_i\right| \leqslant \delta \sum_{r_i=0}^{j_{i-1}} I_{\{x_{j_i}^{(i)} = x_{r_i}^{(i-1)}\}} I_{\{x_{j_i+1}^{(i)} - x_{j_i}^{(i)} \in E_\lambda\}} I_{\{x_{r_i+1}^{(i-1)} - x_{r_i}^{(i-1)} \in E_\lambda\}}.$$

Now proceed with similar kind of argument as for the excited models.

Other models?

- excitement in two coordinates with $(\beta^{[1]}, \beta^{[2]})$: monotonicity of $\nu^{[1]}$ in $\beta^{[2]}$?
- once-reinforced random walk on a tree?
- variance of a random walk with partial once-reinforcement?
- the orthant model, one of the more interesting examples of random walks in (i.i.d.) degenerate random environments in H., Salisbury
  - do site percolation in $\mathbb{Z}^d$
  - from each occupied site lay down the arrows $\{+e_i, i = 1, \dots, d\}$
  - from each vacant site lay down the arrows $\{-e_i, i = 1, \dots, d\}$
  - run a random walk (that chooses uniformly from available steps) in this random environment
- once reinforced random walk in high dimensions????? (would require a tremendous advance in our understanding and use of the recursion equation)