L1

Aussois Lectures '09

Reinforced Random Walks

- Model introduced by Coppersmith & Diaconis ('86): person arrives in new city, first scatters randomly around the place where she arrived, but as time goes on is more likely to come back to sites / to cross again streets she has already visited ⇝ Vertex/Edge Reinforced RW: VRRW / ERRW
- depending on the nature of the interaction, asymptotic behaviour greatly differs ⇝ will focus here on <u>localization phenomena</u>

1) Introduction
2) VRRW on $\mathbb{Z}$, first localization results
3) Rubin continuous-time construction, coupling technique
4) "Short" proof of conjecture of Pemantle and Volkov
[5) Dynamical systems approach on arbitrary graphs

1) Introduction

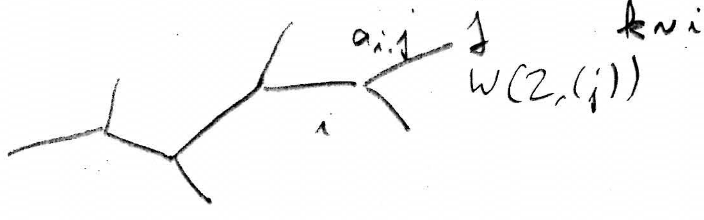
$$\left.\begin{matrix}\text{Vertex}\\ \text{Edge}\end{matrix}\right\rangle \text{ Self-Interacting RW } \left(\begin{matrix}\text{VSIRW}\\ \text{ESIRW}\end{matrix}\right)$$

- $(G, \sim)$ nonoriented locally finite graph
- $a := (a_{i,j})_{i,j \in G} \in \mathbb{R}_+^{G^2}$ reinforcement matrix with positive entries, $a_{i,j} > 0 \Leftrightarrow i \sim j$.
- $W : \mathbb{N}_0 \to \mathbb{R}_+ \setminus \{0\}$ weight function
- $(X_n)_{n \in \mathbb{N}}$ random process taking values in G
- $\mathcal{F}_n := \sigma(X_0, \ldots X_n)$
- $\forall v \in G,\ n \in \mathbb{N} \cup \{\infty\},\ Z_n(v) := \sum_{k=0}^{n} \mathbb{1}_{\{X_k = v\}} + 1$

  nb visits plus one up to time n.

$(X_n)_{n \in \mathbb{N}}$ VSIRW with starting point v_0, reinforcement matrix a and weight function W if $X_0 = v_0$ and, for all $n \in \mathbb{N}$, if $X_n = i$, then

$$\mathbb{P}(X_{n+1} = j \mid \mathcal{F}_n) = \mathbb{1}_{i \sim j} \frac{a_{i,j} W(Z_n(j))}{\sum_{k \sim i} a_{i,k} W(Z_n(k))}$$



Edge Self-Interacting RW (ESIRW): replace $Z_n(j)$

by $Z_n(\{i,j\}) := \sum_{k=1}^{n} \left(\mathbb{1}_{\{X_{k-1}=i,\, X_k=j\}} + \mathbb{1}_{\{X_{k-1}=j,\, X_k=i\}} \right)$

$\left.\begin{array}{l}\text{Edge}\\ \text{Vertex}\end{array}\right\}$ Reinforced RW : $W(n)$ affine $:= an+b$, $a, a+b >$

$G = \mathbb{Z}$, $a_{i,j} = \mathbb{1}_{i\sim j}$, $W(n) = n$

VRRW

ERRW

$$\mathbb{P}(\to \text{right}) = \frac{r}{l+r}$$

$R := \{v \in G \text{ s.t. } Z_\infty(v) > 1\}$ (range)

$R' := \{v \in G \text{ s.t. } Z_\infty(v) = \infty\}$ (asymptotic range)

Thm (Pemantle & Volkov, '99)

$$|R'| = \begin{cases} 5 & \text{with positive proba} \\ < \infty & \text{a.s.} \end{cases}$$

Thm (T, '04)

$|R'| = 5$ a.s

Thm (Diaconis, '

$R' = \mathbb{Z}$ a.s.

VRRW on arbitrary graphs, $(a_{i,j})$ symmetric

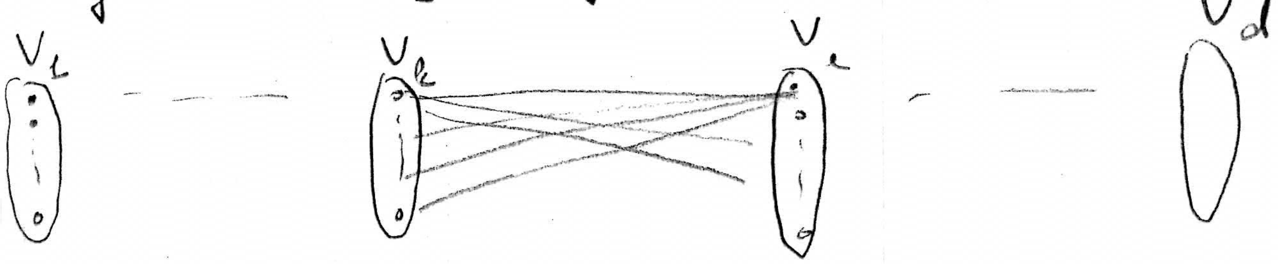
<u>Def</u> For all $d \geq 1$, subgraph $(S, \sim)$ called complete d-partite graph with possible loops [i.e. it is a complete d-partite graph with possible added loops, i.e.] if

$$S = V_1 \cup \dots \cup V_d,$$

with

(i) $\forall i \in \{1, \dots, d\}, \forall \alpha, \beta \in V_i, \alpha \neq \beta \Rightarrow \alpha \nsim \beta$

(ii) $\forall i, j \in \{1, \dots, d\}, i \neq j, \forall \alpha \in V_i, \beta \in V_j, \alpha \sim \beta$.



<u>Thm</u> Volkov ('01): $a_{i,j} = \mathbb{1}_{i \sim j}$

Benaïm & T ('08): (a_{ij}) general (symmetric).

Assume S is a complete d-partite graph and satisfies some (in general non explicit) conditions,

CVRRW localizes with positive prob on $G' = S \cup \partial S$, i.e.

$$\mathbb{P}(R' = S \cup \partial S) > 0.$$

ESIRW on G connected, W nondecreasing, $\sum_{n\in\mathbb{N}} \frac{1}{W(n)} = \infty$

Prop $\{|R'| \neq 0\} = \{R' = G\}$

Proof makes use of

Conditional Borel-Cantelli Lemma

$\mathbb{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ filtration, (ξ_n) $\mathbb{F}$-adapted a.s. bdd sequence of r.v.s taking values in $\mathbb{R}_+$ (or, more generally s.t. $\exists$ a.s. $C>0$ s.t., for all $k\in\mathbb{N}$, $\mathbb{E}(\xi_{k+1}^2 | \mathcal{F}_k) \leq C\, \mathbb{E}(\xi_{k+1} | \mathcal{F}_k)$

Then

$$\left\{ \sum_{k=1}^{\infty} \xi_k < \infty \right\} = \left\{ \sum_{k=1}^{\infty} \mathbb{E}(\xi_k | \mathcal{F}_{k-1}) < \infty \right\}$$

Rmk generally used with $\xi_k := 1\!\!1_{\Gamma_k}$, $\xi_k := \zeta_{k-1} 1\!\!1_{\Gamma_k}$

Pf $M_n := \sum_{k=1}^{n} (\xi_k - \mathbb{E}(\xi_k | \mathcal{F}_{k-1}))$ martingale, and

$$\langle M \rangle_n \leq \sum_{k=1}^{n} \mathbb{E}(\xi_k^2 | \mathcal{F}_{k-1}) = O\left(\sum_{k=1}^{n} \mathbb{E}(\xi_k | \mathcal{F}_{k-1}) \right)$$

Now, either $\langle M \rangle_\infty < \infty$ or $\frac{M_n}{\langle M \rangle_n} \xrightarrow[n\to\infty]{} 0$,

which yields the result.

Proof of Prop

Let $t_n := t_n(x)$ n-th visit time to x, then

$$\sum_{z \sim x} Z_{t_n(x)}(\{x,z\}) = 2n - \mathbb{1}_{\{X_0 = x\}},$$

Hence, for all $z \sim x$ and $n \in \mathbb{N}$

$$\mathbb{P}(X_{t_n+1} = z \mid \mathcal{F}_{t_n}) \geq Cst(a_{ij}) \frac{W(1)}{W(2n)}$$

using that W is non decreasing.

Now $$\sum_{n \geq 1} \frac{1}{W(2n)} \geq \sum_{n \geq 0} \frac{1}{W(2n+1)},$$

so that $$\sum_{n \geq 1} \frac{1}{W(2n)} \geq \frac{1}{2} \sum_{n \geq 1} \frac{1}{W(n)} = \infty.$$

But, by conditional Borel-Cantelli Lemma,

$$\{Z_\infty(x) = \infty\} \subseteq \left\{ \sum_{n \in \mathbb{N}} \mathbb{P}(X_{t_n+1} = z \mid \mathcal{F}_{t_n}) \mathbb{1}_{\{t_n < \infty\}} = \infty \right\}$$

$$= \left\{ \sum_{n \in \mathbb{N}} \mathbb{1}_{\{X_{t_n+1} = z\}} \mathbb{1}_{\{t_n < \infty\}} = \infty \right\}$$

$$\subseteq \{Z_\infty(z) = \infty\} \quad \text{a.s}$$

Rmk W not nondecreasing, possible to have

$\sum \frac{1}{w(k)} = \infty$ together with $\{|R'| < \infty\}$.

Sellke ('94): $\sum \frac{1}{w(2k)} = \infty$ and $\sum \frac{1}{w(2k+1)} < \infty$,

$a_{i,j} = \mathbb{1}_{i \sim j}$, $G = \mathbb{R}^d$, then

$\mathbb{P}(X_{2n} = 0$ for all sufficiently large $n) > 0$.

ESIRW, $a_{i,j} = \mathbb{1}_{i \sim j}$, $\sum \frac{1}{w(k)} < \infty$

Thm (Davis, '90)

$G = \mathbb{Z}$: $|R'| = 2$ a.s.

using Rubin construction

Thm (Sellke, '94)

On any graph of bdd degree without odd cycles,

$|R'| = 2$ a.s.

Triangle conjecture: same on any graph of bounded degree?

Thm (Limic, '03)

Yes if $w(k) = (k+1)^p$, $p > 1$

thm (Limic & T, '07)

Yes if W is nondecreasing.

Two figures : $a_{i,j} = \mathbb{1}_{i\sim j}$

VSIRW on $\mathbb{Z}$

$W(n) = n^p, p<1$ $\quad$ $W(n) = n$ $\quad$ $\sum \frac{1}{W(n)} < \infty$ $\quad$ reinforcem

$|R'| \in \{0,\infty\}$ a.s. (Volkov, '06) $\quad$ $|R'| = 5$ a.s (T. '04) $\quad$ $|R'| = 2$ a.s (Bienvenüe, '99)

ESIRW on $\mathbb{Z}$

$W(n) = \exp(-\beta n^{\varkappa})$, $\beta > 0$ $\quad$ $W(n) = n^{\alpha}$ $\quad$ reinf.

$\beta = 1$ skaling $\simeq t^{\frac{\varkappa+1}{\varkappa+2}}$ (Tóth, 94) $\quad$ $\beta = 0$ $\quad$ $\simeq t^{1/2}$ (Tóth, '97) $\quad$ $\alpha =$

Tóth & Werner : "true" self-repelling motion

$W(n) = n^{\alpha}$ $\quad$ $\sum \frac{1}{W(n)} < \infty$ $\quad$ reinf.

$\alpha = 0$ $\quad$ $\approx t^{\frac{1-\alpha}{2-\alpha}}$ (Tóth, '97) $\quad$ $\alpha = 1$ recurrent (Diaconis, '86) $\le C \log t$ $\quad$ $|R'| = 2$ a.s. (Davis, '90).

$W(n) = n$ on general graphs : recurrence / transience

questions.

Applications, related questions

- Myxobacteria, ants, Othmer & Stevens ('97)

Global localization behaviour emerging from short range interactions

- Reinforcement learning, Erev & Roth ('95)

m people play repeatedly same game, each has d possible strategies

Time n

propensity $q^i_{j,n}$ for player i to play

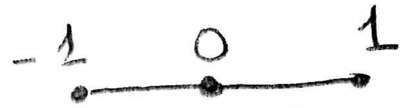
probability $\dfrac{q^i_{j,n}}{\sum_k q^i_{k,n}}$ strategy j

$$q^i_{j,n+1} = q^i_{j,n} + \text{payoff of } i \text{ if he played } j \text{ at time } n.$$

Synchronization, signaling models: Pemantle & Skyrms

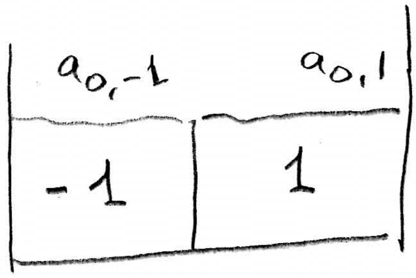
Argiento, Pemantle, Skyrms & Volkov ('09).

VSIRW on three consecutive vertices



$\Longleftrightarrow$ ESIRW on 

$\Longleftrightarrow$ "W-urn"



$X_0 := 0$ for simplicity.

Let, for all $n \in \mathbb{N}$, $W^*(n) := \sum_{k=1}^{n-1} \frac{1}{W(k)}$

($:= 0$ for $n \leq 1$ by convention)

Then $M_n := a_{0,-1} W^*(Z_n(1)) - a_{0,1} W^*(Z_n(-1))$ martingale.

Indeed,

$$\mathbb{E}(M_{n+1} - M_n \mid \mathcal{F}_n)$$

$$= \frac{a_{0,1} W(Z_n(1))}{a_{0,1} W(Z_n(1)) + a_{0,-1} W(Z_n(-1))} \cdot \frac{a_{0,-1}}{W(Z_n(1))}$$

$$- \frac{a_{0,-1} W(Z_n(-1))}{a_{0,1} W(Z_n(1)) + a_{0,-1} W(Z_n(-1))} \cdot \frac{a_{0,1}}{W(Z_n(-1))} = 0$$

Particular cases

1) $a_{0,1} = a_{0,-1} = 1$

a) W linear :

$W^*(n) = \sum_{k=1}^{n-1} \frac{1}{k} =: h(n)$ harmonic series, $h(n) - \log n$ converges

(M_n) converges a.s. and in L^2 by Doob

$\rightsquigarrow \log\left(\frac{Z_n(-1)}{Z_n(1)}\right)$ converges a.s.

Pólya urn : $\frac{Z_n(-1)}{Z_n(-1)+Z_n(1)} \longrightarrow \beta \in (0,1)$

β beta distribution of parameters $(Z_1(-1), Z_1(1$

b) $\sum \frac{1}{W(k)} < \infty$

(M_n) converges a.s. (difference of nondecreasing bdd sequences).

$\{|R'| = 3\} \subseteq \{M_\infty = 0\}$

Can prove $\mathbb{P}(M_\infty = 0) = 0$ almost surely.

Limic & T: What is the difference between the square and the triangle? ('08)

c) $W(n) = n^p$, $p < 1$, martingale estimates

$$\Rightarrow \frac{Z_n(-1)}{Z_n(1)} \xrightarrow[n\to\infty]{} 1 \text{ a.s.}$$

2) $a_{0,1} \neq a_{0,-1}$, $W(n) = n$.

Study similar to 1) a): M_n converges a.s.

$\Rightarrow a_{0,1} \log(Z_n(-1)) - a_{0,-1} \log(Z_n(1))$ converg

$$\Rightarrow Z_n(-1)^{a_{0,1}} \underset{n\to\infty}{\sim} C \, Z_n(1)^{a_{0,-1}}$$

$C > 0$ random.

[Quite distinct asymptotic behaviour from equality case, one prevails].

(i)

$a_{0,-1}$	$a_{0,1}$
-1	1

$$\mathbb{P}(\text{choose } \pm 1) = \frac{a_{0,\pm1} Z_n(\pm 1)}{a_{0,-1} Z_n(-1) + a_{0,1} Z_n(1)}$$

Put back with another ball of the same colour

Friedman urn ('4
Analyzed by Friedman(

(ii)

-1	1

$\mathbb{P}(\text{choose } 1) = \frac{Z_n(}{Z_n(-1)+}$

Put back with,
in (cond) expectation
$a_{0,1}$ balls ($a_{0,-1}$ ba

Same martingale $M_n := a_{0,1} h(Z_n(-1)) - a_{0,-1} h(Z_n(1))$ associated to (i) and (ii): same asymptotics $Z_n(-1)^{a_{0,1}} \underset{n\to\infty}{\sim} C Z_n(1$