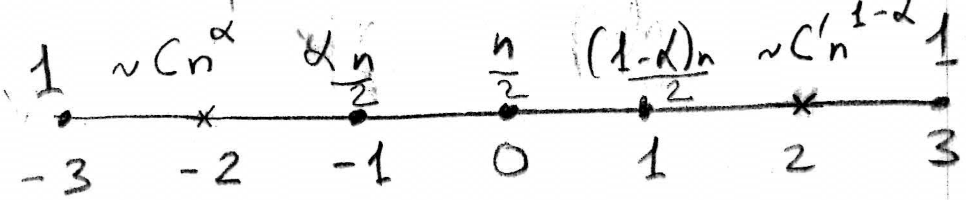
2] VRRW on $\mathbb{Z}$, first localization results

Assume $W(n) = n$ for convenience, $a_{i,j} = 1\!\!1_{i \sim j}$.

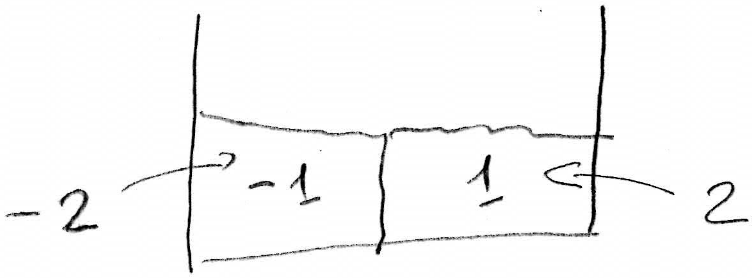
Seven points, at time k, $X_k = 1$ and



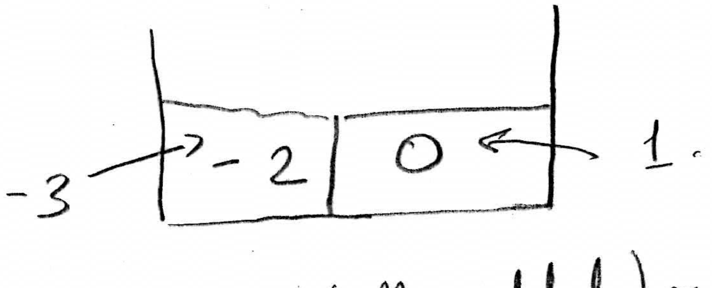
$\alpha \in (0,1)$

VRRW : $\mathbb{P}(X_{k+2} = 3 \mid \mathcal{F}_k) \sim \frac{Cn^\alpha}{n/2} \cdot \frac{1}{\alpha n/2} \sim \frac{Cst(C}{n^{2-}}$

[ERRW : $\mathbb{P}(X_{k+2} = 3 \mid \mathcal{F}_k) \sim \frac{2Cn^\alpha}{\alpha n} \cdot \frac{1}{2Cn^\alpha} \sim \frac{1}{\alpha n}$



Pólya urn



Friedman urn

$\mathbb{E}(\text{nb } 0 \text{ balls added}) \approx \frac{1}{\alpha}$ $\qquad Z_m(-2) \underset{m\to\infty}{\sim} C Z_m$

three events have to hold together:

1) no visits to -3 and 3 $\rightsquigarrow$ Borel-Cantelli
2) same proportion of balls -1 / 1 $\rightsquigarrow$ Pólya urn
3) small nb of visits to -2 and 2 $\rightsquigarrow$ Friedman urn

Cf. Sections 1, 2 & 3 of T'(04). VRRW on $\mathbb{Z}$

eventually gets stuck on five points

Techniques: local martingales introduced in T.'(04)

sections 1 to 3

[similar holds on general graphs].

For all $n \in \mathbb{N}$, $x \in \mathbb{Z}$, let

$$Y_n^{\pm}(x) := \sum_{k=1}^{n} 1\!\!1_{\{X_{k-1}=x,\, X_k = x\pm 1\}} \cdot \frac{1}{Z_{k-1}(x\pm 1)}$$

$$Y_n(x) := \sum_{k=1}^{n} \frac{1\!\!1_{\{X_{k-1}=x\}}}{Z_{k-1}(x-1)+Z_{k-1}(x+1)}$$

previs

$$\hat{Y}_n^{\pm}(x) := Y_n^{\pm}(x) - Y_n(x)$$

marting

part in Doob decomposition of $Y_n^{\pm}(x)$.

$$Y_\infty^{\pm}(x) := \lim_{n\to\infty} Y_n^{\pm}(x) \quad , \quad Y_\infty(x) := \lim Y_n(x).$$

Notation

a_n, b_n random processes in $\mathbb{R}$,

$a_n \equiv b_n \iff a_n - b_n$ converges a.s.

$Cst(x_1, \dots x_n) = Cst$ dependent only on $x_1, \dots x_n$

Proposition 2.1 For all $x \in \mathbb{Z}$,

(a) $\hat{Y}_n^{\pm}(x) = Y_n^{\pm}(x) - Y_n(x)$ martingale, convergin[g] a.s. and in L^2.

(b) $Y_n^{\pm}(x) \equiv Y_n(x)$

(c) $\mathbb{E}\left((\hat{Y}_n^{\pm}(x) - \hat{Y}_\infty^{\pm}(x))^2 \mid \mathcal{F}_n \right) \leq \dfrac{1}{Z_n(x \pm 1) - 1}$

(d) $Y_n^{+}(x-1) + Y_n^{-}(x+1) = h(Z_n(x)) - h(1 + \mathbb{1}_{\{X_0 = x\}})$

$\equiv \log(Z_n(x))$

[Rmk $x-1$, x, $x+1$ case]

Pf $\hat{Y}_n^{\pm}(x)$ martingale from definitions

$$\operatorname{Var}\left(\hat{Y}_{n+1}^{\pm}(x) \mid \mathcal{F}_n\right) = \operatorname{Var}\left(Y_{n+1}^{\pm}(x) \mid \mathcal{F}_n\right)$$

$$\leq \mathbb{E}\left((Y_{n+1}^{\pm}(x) - Y_n^{\pm}(x))^2 \mid \mathcal{F}_n \right)$$

$$= \mathbb{E}\left(\frac{\mathbb{1}_{\{X_n = x,\, X_{n+1} = x \pm 1\}}}{Z_{n+1}(x \pm 1)^2} \mid \mathcal{F}_n \right)$$

Hence, for all $m \geq n$,

$$\mathbb{E}\left((\hat{Y}_n^{\pm}(x) - \hat{Y}_m^{\pm}(x))^2 \mid \mathcal{F}_n \right) \leq \mathbb{E}\left(\sum_{k=n}^{\infty} \frac{\mathbb{1}_{\{X_k = x,\, X_{k+1} = x \pm 1\}}}{Z_k(x \pm 1)^2} \mid \mathcal{F}_n \right)$$

$$\leq \sum_{\ell = Z_n(x \pm 1)}^{\infty} \frac{1}{\ell^2} \leq \frac{1}{Z_n(x \pm 1) - 1}$$

implies (a) - (c); (d) from definitions

Back to Polya urn

$$\Upsilon(x) := \{ Y_\infty^{\pm}(x) < \infty \} = \{ Y_\infty(x) < \infty \}$$
$$= \{ \text{"x seldom visited"} \}.$$

"$\Longleftrightarrow$" "x neutral with respect to its neighbours.

$$\Upsilon((x_i)_i) := \bigcap_i \Upsilon(x_i)$$

Corollary 2.1

For all $x \in \mathbb{Z}$, $n \geq m$, $a \geq Cst$,

(a) $\Upsilon(x) \subseteq \{ \exists\, \alpha_\infty^-(x+2) := \lim \alpha_n^-(x+2) \in [0,1)$

(b) $\Upsilon(x) \cap \{ \alpha_\infty^-(x+2) > 0 \} \subseteq \Upsilon(x+4)$

(c) $\mathbb{P}(\mathcal{E}_{m,a}(x+2) \mid \mathcal{F}_m) \leq Cst(a) \left(\frac{1}{Z_m(x+1)} + \frac{1}{Z_m(x+\ldots} \right.$

where

$$\mathcal{E}_{m,a}(y) := \Big\{ \sup_{m \leq k \leq n < \infty} \big(f(\alpha_n^-(y)) - f(\alpha_k^-(y)) - Y_{k,n}(y-2 \ldots \geq a$$

$$Y_{k,n}(y) := Y_n(y) - Y_k(y), \quad f(t) := \log\left(\frac{t}{1-t}\right)$$

Pf (a) and (b)

By Prop 1 (d), a.s. on $\Upsilon(x) = \{Y^+_\infty(x) < \infty$

$$\log Z_n(x+1) \equiv Y_n^+(x) + Y_n^-(x+2) \equiv Y_n^+(x+2)$$
$$\equiv \log Z_n(x+3) - Y_n^-(x+4),$$

so that

$$\log \frac{Z_n(x+1)}{Z_n(x+3)} \equiv - Y_n^-(x+4)$$

(c) By Prop 2.1 (d),

$$h(Z_n(x+1)) - h(1 + \mathbb{1}_{\{X_0 = x+1\}})$$
$$= Y_n^+(x) + Y_n^-(x+2)$$
$$= Y_n(x) + Y_n^+(x+2) + \hat{Y}_n^+(x) + \hat{Y}_n^-(x+2) \quad Y_n^-(x+2)$$
$$- \hat{Y}_n^+(x+2)$$

$$\overset{\text{Prop 2.1(d)}}{=} Y_n(x) + h(Z_n(x+3)) - h(1 + \mathbb{1}_{\{X_0 = x+3\}})$$
$$- Y_n^-(x+4) + \hat{Y}_n^+(x) + \hat{Y}_n^-(x+2)$$
$$- \hat{Y}_n(x+2$$

Now

$$f(\alpha_n^-(x+2)) - f(\alpha_k^-(x+2)) = \log\left(\frac{Z_n(x+1)}{Z_k(x+1)}\right) - \log\left(\frac{Z_n(x+3)}{Z_k(x+3)}\right)$$

Observe on one hand that, for large enough n, $\frac{1}{n} - \frac{1}{n^2} \leq \log\left(\frac{n+1}{n}\right) \leq \frac{1}{n}$, so that, for large $n \geq m$,

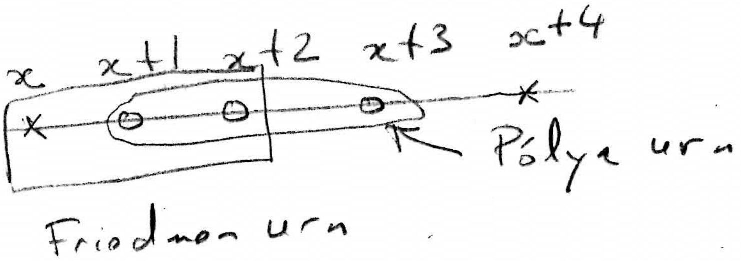
$$-\frac{1}{m-1} \leq \log\left(\frac{n}{m}\right) - \left(h(n)-h(m)\right) \leq 0.$$

On the other hand, by Doob's inequality,

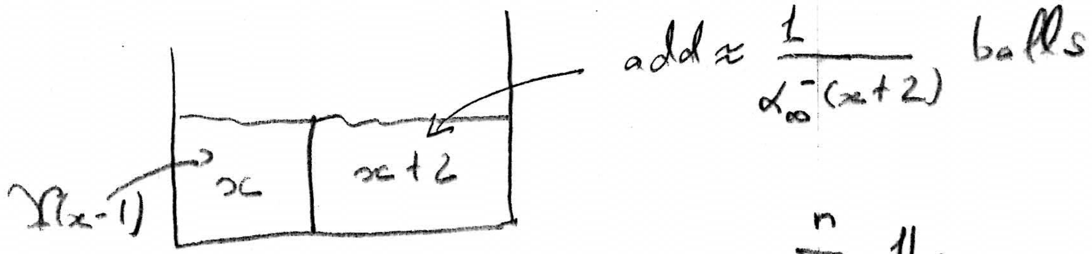
$$\mathbb{P}\left(\hat{Y}_n^+(x) - \hat{Y}_m^+(x) \geq \frac{a}{3} \,\Big|\, \mathcal{F}_m\right) \leq \frac{36}{a^2}\,\mathbb{E}\left(\left(\hat{Y}_n^+(x) - \hat{Y}_\infty^+(x)\right)^2 \right.$$

$$\leq \frac{36}{a^2} \cdot \frac{1}{Z_m(x+1)},$$

and similar inequalities occur for $\hat{Y}_n^-(x+2)$ and $\hat{Y}_n(x+2)$ $\square$



"x versus $x+2$" heuristics



$$\log Z_n(x) \equiv Y_n^-(x+1) \equiv Y_n^+(x+1) = \sum_{k=1}^{n} \frac{\mathbb{1}_{\{X_{k-1}=x+1,\, X_k = x}}{Z_k(x+2)}$$

$$\simeq \sum_{k=1}^{n} \frac{\mathbb{1}_{\{X_k = x+2,\, X_{k+1} = x+1\}}}{Z_k(x+2)} \simeq \sum_{k=1}^{n} \frac{\mathbb{1}_{\{X_k = x+2\}}}{Z_k(x+2)}\alpha_k^-(x+2)$$

$$\simeq \alpha_\infty^-(x+2) \log Z_n(x+$$

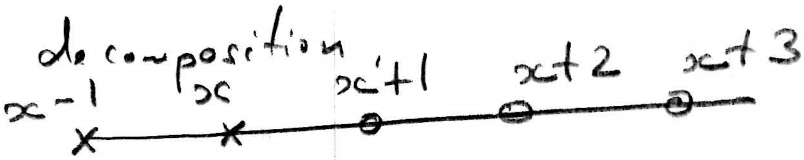
For all $x \in \mathbb{Z}$, $n \in \mathbb{N}$, let

$$\tilde{Y}_n^{\pm}(x) := \sum_{k=1}^{n} \mathbb{1}_{\{X_k = x\pm1,\, X_k = x\}} \frac{1}{Z_{k-1}(x\pm1)}$$

$$\bar{Y}_n^{\pm}(x) := \sum_{k=1}^{n} \frac{\mathbb{1}_{\{X_k = x\}}}{Z_k(x)} \alpha_k^{\pm}(x)$$

$$\left.\begin{array}{l} \check{Y}_n^{\pm}(x) := \tilde{Y}_n^{\pm}(x) - \bar{Y}_n^{\mp}(x\pm1) \\ \qquad\qquad\qquad\quad \bar{Y}_n^{\mp}(x\pm1) \end{array}\right\} \begin{array}{l} \text{martingale} \\ \text{previsible} \end{array} \Bigg\rangle$$

part of $\tilde{Y}_n^{\pm}(x)$ Doob decomposition



Proposition 2.2 For all $x \in \mathbb{Z}$, $C, \gamma > 0$, $n \in \mathbb{N}$,

(a) $n \mapsto Y_n^+(x) + \dfrac{\mathbb{1}_{\{X_n \le x\}}}{Z_{n-1}(x+1)} - \tilde{Y}_n^+(x)$ a.s. constant

(b) $\tilde{Y}_n^{\pm}(x)$ converges a.s. and in L^2, and

$$\mathbb{E}\left((\check{Y}_n^{\pm}(x) - \check{Y}_\infty^{\pm}(x))^2 \,\middle|\, \mathcal{F}_n \right) \le \frac{1}{Z_n(x\pm1) - 1}$$

(c) $\Upsilon(x-1) \cap \{Z_\infty(x) = \infty\} \cap \{\limsup \alpha_n^-(x+2) \le \gamma\}$

$\subseteq \Upsilon(x-1, x) \cap \{\log Z_n(x) \underset{n\to\infty}{\sim} \alpha_\infty^-(x+2) \log Z_n(x+2$

($\equiv$ with more effort)

(d) Assume $Z_m(x) \le C Z_m(x+2)^\alpha$, and let

$T := \inf\{n \ge m \text{ s.t. } \alpha_n^-(x+2) \ge \gamma \text{ or } X_n = x-1\}$. Then

$$\mathbb{P}\left(\sup_{T > n \ge m} \frac{Z_n(x)}{Z_n(x+2)^\gamma} \ge e^a C \,\middle|\, \mathcal{F}_m \right) \le Cst(a)\left(\frac{1}{Z_m(x)} + \frac{1}{Z_m(x$$

(a): $\overset{x}{\leftarrow}\!\!\!\overset{x+1}{\times}$ add $\dfrac{1}{Z_{n-1}(x+1)}$ to $\tilde{Y}_n^+(x)$

$\boxed{\text{time step } n-1 \text{ to } n}$ $\overset{x}{\times}\!\!\!\overset{x+1}{\rightarrow}$ add $\dfrac{1}{Z_{n-1}(x+1)}$ to $Y_n^+(x)$

(b) Similar to Prop 2.1 (c).

(c) $\log Z_n(x) \equiv Y_n^+(x-1) + Y_n^-(x+1)$

$$\equiv Y_n^+(x+1) \equiv \tilde{Y}_n^+(x+1)$$

$$\equiv \sum_{k=1}^{n} \frac{1\!\!1_{\{X_k = x+2\}}}{Z_k(x+2)} \alpha_k^-(x+2)$$

$$\leq (\gamma+\varepsilon) \log Z_n(x+2)$$

for large $n \in \mathbb{N}$ if $\limsup \alpha_k^-(x+2) \leq \gamma$ and $Z_\infty(x) =$

$$\Rightarrow \bar{Y}_\infty^-(x+1) = \sum_{k=1}^{\infty} \frac{1\!\!1_{\{X_k = x+1\}}}{Z_k(x+1)} \alpha_k^-(x+1) < \infty$$

using $\alpha_k^-(x+1) \leq (Z_n(x) + Z_n(x+2))^{\gamma+\varepsilon-1} \leq Z_n(x+1)^{\gamma+\varepsilon}$

for large $n \in \mathbb{N}$

$\rightarrow Y_\infty^+(x) < \infty \Rightarrow (\alpha_n^-(x+2))$ converges a.s.

(d) Similarly as in Corollary 2.1 (c), for all $a > 2$, outside of an even of proba $\leq Cst(a)\left(\dfrac{1}{Z_m(x)} + \dfrac{1}{Z_m(x+2)}\right)$,

$$\log\left(\frac{Z_n(x)}{Z_m(x)}\right) \leq Y_n^+(x-1) - Y_m^+(x-1) + a\left(\frac{1}{Z_m(x)} + \frac{1}{Z_m(x+2)}\right)$$

$$+ \left(\sup_{m \leq k \leq n} \alpha_k^-(x+2)\right) \log\left(\frac{Z_n(x+2)}{Z_m(x+2)}\right)$$

Corollary 2.2

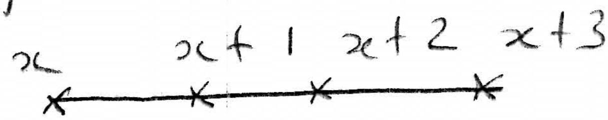
(a) $\{ R' = \{ x-2, x-1, x, x+1, x+2 \} \}$

(Bienvenüe, '99)

$$\subseteq \Upsilon(x-3, x-2, x+2, x+3)$$

$$\cap \{ \exists \alpha_\infty^-(x) := \lim_{n\to\infty} \alpha_n^-(x) \in (0,1) \}$$

$$\cap \left\{ \frac{\log Z_n(x\pm 2)}{\log Z_n(x)} \xrightarrow[n\to\infty]{} \alpha_\infty^\pm(x) \text{ a.s} \right\}$$



(b) (Pemantle, '99)

$$\{ |R| < \infty \} \subseteq \{ |R'| \geq 5 \} \text{ a.s.}$$

Pf (a) $\{ Z_\infty(x \pm 3) < \infty \} \subseteq \Upsilon(x\pm 3, x\pm 2) \cap \{ \exists \alpha_\infty^\pm(x) := \lim \alpha_n^-(x) \in [0,1)$

by Cor 1(a), and apply Prop 2.2 (c).

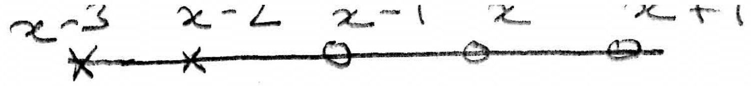
(b) $\{ R' \subseteq \{ x, x+1, x+2, x+3 \} \}$

$$\subseteq \Upsilon(x, x+3) \cap \{ \alpha_\infty^-(x+2) < 1 \}$$

$$\cap \left\{ \lim_{n\to\infty} \frac{Z_n(x+3)}{Z_n(x+1)} = 1 - \alpha_\infty^-(x+2) = 0 \right\}$$

$$= \emptyset \text{ a.s.}$$

since $\alpha_\infty^+(x+1) < 1$



Corollary 2.3

For all $x \in \mathbb{Z}$, $\gamma \in [0,1)$, $\varepsilon \in (0, 1-\gamma)$, $C > 0$, $m \in \mathbb{N}$,

assume $Z_m(x \pm 3) = 1$, $Z_m(x \pm 2) \le C\, Z_m(x)^{\gamma}$

and $\alpha_m^{\pm}(x) \le \gamma$.

Then

$$\mathbb{P}\Big(\{ Z_\infty(x \pm 3) = 1 \} \cap \{ \sup_{n \geq m} \alpha_n^{\pm}(x) \le \gamma + \varepsilon \} \,\Big|\, \mathcal{F}_m \Big)$$

$$\geq 1 - Cst(\gamma, \varepsilon, C) \big(Z_m(x)^{-\gamma} + Z_m(x \pm 1)^{\gamma + \varepsilon - 1} \big)$$

Assume $\pm := -$, and $X_m \geq x-1$ for simplicity.

Pf Let $T_1 := \inf\{ n \geq m \text{ s.t. } Z_n(x-3) > 1 \}$

$T_2 := \inf\{ n \geq m \text{ s.t. } \alpha_n^-(x) \geq \gamma + \varepsilon \}$

$T_3 := \inf\{ n \geq m \text{ s.t. } Z_n(x-2) \geq C e^{\varepsilon} Z_n(x)^{\gamma + \varepsilon} \}$

• $\mathbb{P}(T_1 < T_2 \wedge T_3 \mid \mathcal{F}_m) \le \sum_{k=m}^{\infty} \mathbb{1}_{\{X_k = x-1\}} \dfrac{C e^{\varepsilon} Z_k(x)^{\gamma+\varepsilon}}{(Z_k(x-2) + Z_k(x))} \dfrac{1}{Z_k(x\cdots}$

$$\le C e^{\varepsilon} \sum_{k = Z_m(x-1)}^{\infty} k^{\gamma + \varepsilon - 2} \le \frac{e^{\varepsilon} C (1 - (\gamma+\varepsilon))^{-1}}{(Z_m(x-1) - 1)^{1 - (\gamma+\varepsilon)}}$$

• $\mathbb{P}(T_2 < T_1 \wedge T_3 \mid \mathcal{F}_m) \le \mathbb{P}(\mathcal{E}_{m,\frac{\varepsilon}{2}}(x) \mid \mathcal{F}_m)$ (Corollary 2.1(c)).

[Assume $\mathcal{E}_{m,\frac{\varepsilon}{2}}(x)$ holds. For all $n \geq m$, let $p(n)$ be the last time k s.t. $\alpha_k^-(x) \le \gamma$. Then

$$Y_{p(n),n}(x-2) \le \gamma^{-1} \sum_{k=p(n)+1}^{n} \frac{\mathbb{1}_{\{X_k = x-2\}}}{Z_k(x)}$$

$$\le \gamma^{-1} (e^{\varepsilon} C)^{(\gamma+\varepsilon)^{-1}} \sum_{k=p(n)+1}^{n} \frac{\mathbb{1}_{\{X_{k-1} = x-2\}}}{Z_k(x-2)^{(\gamma+\varepsilon)^{-1}}} \le \frac{\varepsilon}{2}$$

if $Z_m(x-2) \geq Cst(C,\gamma,\varepsilon)$].

- $\mathbb{P}(T_3 < T_1 \wedge T_2 \mid \mathcal{F}_m) \geq \mathbb{E}\left(\mathbb{P}(T_3 < T_1 \wedge T_2 \mid \mathcal{F}_S)\right)$

  where $S := \inf\{n \geq m \text{ s.t. } Z_m(x-2) \geq c\, Z_n(x)^{\gamma}\}$

  By Prop 2.2 (d),

  $\mathbb{P}(T_3 < T_1 \wedge T_2 \mid \mathcal{F}_S) \leq Cst(\varepsilon)\left(Z_S(x-2)^{-1} + Z_S(x)^{-1}\right)$

  $\leq Cst(C,\varepsilon)\; Z_m(x)^{-\gamma}$ $\square$

Corollary 2.4 (Pemantle & Volkov, '99)

(a) For all $x \in \mathbb{Z}$, $\alpha \in (0,1)$, $\varepsilon < \alpha \wedge (1-\alpha)$,

$\mathbb{P}\left(\{R' = \{x-2, x-1, x, x+1, x+2\}\} \cap \{\alpha_\infty^-(x) \in (\alpha-\varepsilon, \alpha+\varepsilon)\}\right) > C$

(b) $|R| < \infty$ a.s.

Pf (a) $x \leq 0$ for instance

$$\mathbb{P}\left(\begin{array}{ccccc} x & x+2 & x & x+1 & x+2 \\ \times & \circ & \circ & \circ & \times \\ 1 & \simeq\frac{\alpha n_0}{2} & \frac{n_0}{2} & \simeq\frac{(1-\alpha)n_0}{2} & 2 \end{array}\right) \geq Cst(n_0)$$

and apply Corollary 2.3 twice ($\pm = +$ and $-$)

(b) For all $x \leq 0$, let us prove

$\mathbb{P}(x-2 \notin R \mid x \in R) \geq \varepsilon > 0$, which implies the conclusion.

$t(x)$ time of first visit to x

$$\mathbb{P}\left(\begin{array}{ccccc} x-2 & x-1 & x & x+1 & x+2 \\ \times & \times & \circ & \circ & \circ \\ 1 & 1 & n_0 & \geq n_0 & \end{array} \,\middle|\, \mathcal{F}_{t(x)}\right) \geq Cst(n_0)$$

and apply Corollary 2.3 to $x := x+1$. $\square$