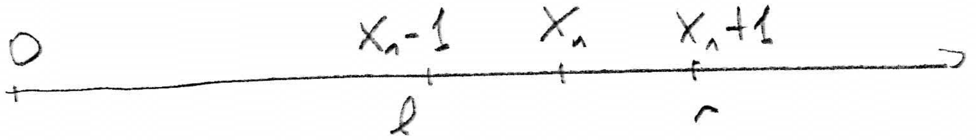
Summary from last lecture

$(X_n)_{n\geq 0}$ VRRW on $\mathbb{Z}$: $Z_n(v) := \sum_{k=0}^{n} \mathbb{1}_{\{X_k = v\}}$, $\mathcal{F}_n := \sigma(X_0, \dots X_n)$

$X_0 = 0$, $\mathbb{P}(X_{n+1} = X_n \pm 1 \mid \mathcal{F}_n) = \mathbb{1}_{y \sim X_n} \, \alpha_n^{\pm}(X_n)$



$$\alpha_n^{\pm}(x) := \frac{Z_n(x-1)}{Z_n(x+1)+Z_n(x-1)}$$

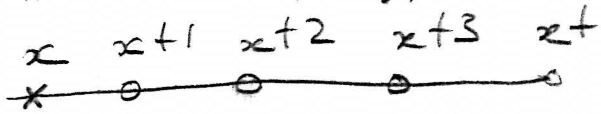
Subset

$$\Upsilon(x) := \{ x \text{ "seldom visited" } \}$$
$$= \{ Y_\infty^{\pm}(x) < \infty \}$$

where
$$Y_n^{\pm}(x) := \sum_{k=1}^{n} \frac{\mathbb{1}_{\{X_{k-1} = x, X_k = x \pm 1\}}}{Z_{k-1}(x \pm 1)} .$$

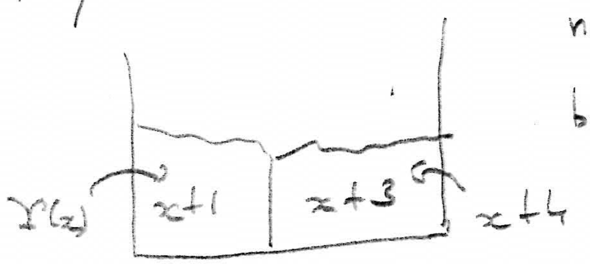
A) "Local" martingales based on weighted nbs of visits between two neighbours (for instance $Y_n^+(x) - Y_n^-(x)$)

B) $\Upsilon(x) \subseteq \{ \exists\, \alpha_\infty^-(x+2) := \lim_{n\to\infty} \alpha_n^-(x+2) \in [0,1] \}$,



$\Upsilon(x) \cap \{ \alpha_\infty^-(x+2) > 0 \} \subseteq \Upsilon(x+4)$

Pólya urn intuition



not perturbed by visits $x \rightsquigarrow x+1$,

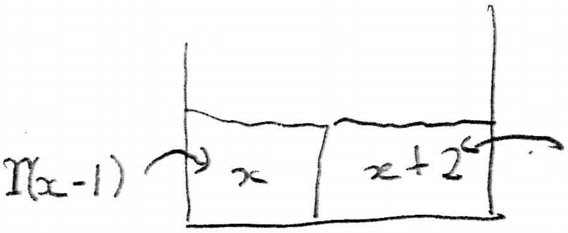
by maybe by visits $x+4 \rightsquigarrow x+3$

C)

$$\Upsilon(x-1) \cap \{ Z_\infty(x) = \infty \} \cap \{ \limsup \alpha_n^-(x+2) \leq \gamma < 1 \}$$

$$\subseteq \Upsilon(x-1, x) \cap \{ \log Z_n(x) \sim \alpha_n^-(x+2) \log Z_n(x+2) \}$$

Friedman urn intuition



add $\approx \dfrac{1}{\alpha_n^-(x+2)}$ balls

D) Consequences

- $|R'| \geq 5$ or $|R'| = 0$ a.s. (Pemantle & Volkov, '99)
- asymptotic behaviour conditional to localization on five sites (Bienvenüe, '99)

E) (Skipped)

Starting from "good" configuration at time n, lower bound of $\mathbb{P}(R' = \{ x-2, x-1, x, x+1, x+2 \} | \mathcal{F}_n)$

$\Rightarrow |R| < \infty$ a.s. (Pemantle & Volkov, '99)

Today: Rubin construction and coupling techniques, in order to sketch proof of $|R'| = 5$ a.s.

More precisely, will enable us to show that

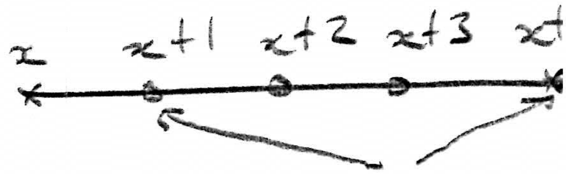
(1) $\quad \Upsilon(x) \subseteq \{ Z_\infty(x-1) < \infty \} \cup \{ Z_\infty(x+1) < \infty \}$ a.s.

Heuristics: "competition" between $x-1$ and $x+1$.

Rubin construction can be seen as an analysis of the randomness involved in the visits starting from x.

Similar method (but more involved technically, skipped here) enables one to show that

$$(2) \qquad \Upsilon(x) \subseteq \Upsilon(x+1) \cup \Upsilon(x+4) \quad \text{a.s.}$$



Again competition, between $x+1$ and $x+4$: one of them "wins" eventually

Already proved $\Upsilon(x) \cap \{\alpha_\infty^-(x+2) > 0\} \subseteq \Upsilon(x+4)$.

Lemma 2.3 in T'(04) shows that

$$\Upsilon(x) \cap \{\alpha_\infty^-(x+2) = 0\} \subseteq \Upsilon(x+1) \quad \text{a.s.}$$

Then

$$\{Z_\infty(x-1) < \infty\} \cap \{Z_\infty(x) = \infty\}$$
$$\subseteq \Upsilon(x-1, x, x+1) \cup \Upsilon(x-1, x, x+4) \quad \text{a.s.}$$

But
$$\Upsilon(x-1, x+1) = \{Y_\infty^+(x-1) < \infty\} \cap \{Y_\infty^-(x+1) < \infty\}$$
$$= \{h(Z_\infty(x)) - h(1 + \mathbb{1}_{\{X_0 = x\}}) + Y_\infty^+(x-1) + Y_\infty^-(x+1) < \infty\}$$
$$= \{Z_\infty(x) < \infty\} \qquad \text{(Prop 2.1 (d))}$$

(at most two consecutive seldom & infinitely often visited sites) and

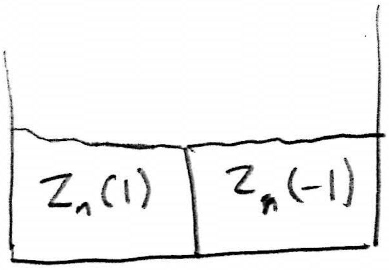
$$\Upsilon(x-1,x,x+4)\cap\{Z_\infty(x-1)<\infty\}\cap\{Z_\infty(x)=\infty\}$$

$$\subseteq \Upsilon(x+4)\cap\{Z_\infty(x+3)=\infty\}\subseteq\{Z_\infty(x+5)<\infty\} \text{ a.s.}$$

3) Rubin continuous-time construction

Further reading: What is the difference between a squa the difference between a squa and a triangle? (Limic & T, Oc

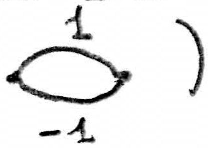
3.1) W-urn, $\sum \frac{1}{W(k)} < \infty$



$Z_n(x) :=$ nb balls colour x at time n

Time $n \to n+1$, proba $\dfrac{W(Z_n(\pm 1))}{W(Z_n(1))+W(Z_n(-1))}$

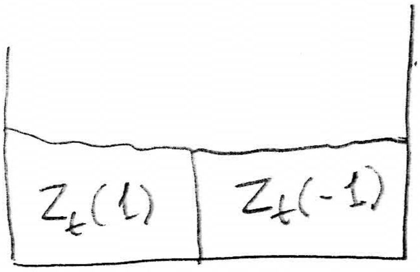
to pick a ball of colour ± 1

($\Leftrightarrow$ VSIRW on  $\Leftrightarrow$ ESIRW on )

Goal: prove that $Z_\infty(1)<\infty$ or $Z_\infty(-1)$ a.s.

Rubin construction: continuous-time process $(Z_t(1), Z_t(-1))_{t\in\mathbb{R}}$ taking values in $\mathbb{N}^2$, which will be equal in law to $(Z_n(1), Z_n(-1))_{n\in\mathbb{N}_0}$ after renormalization in time

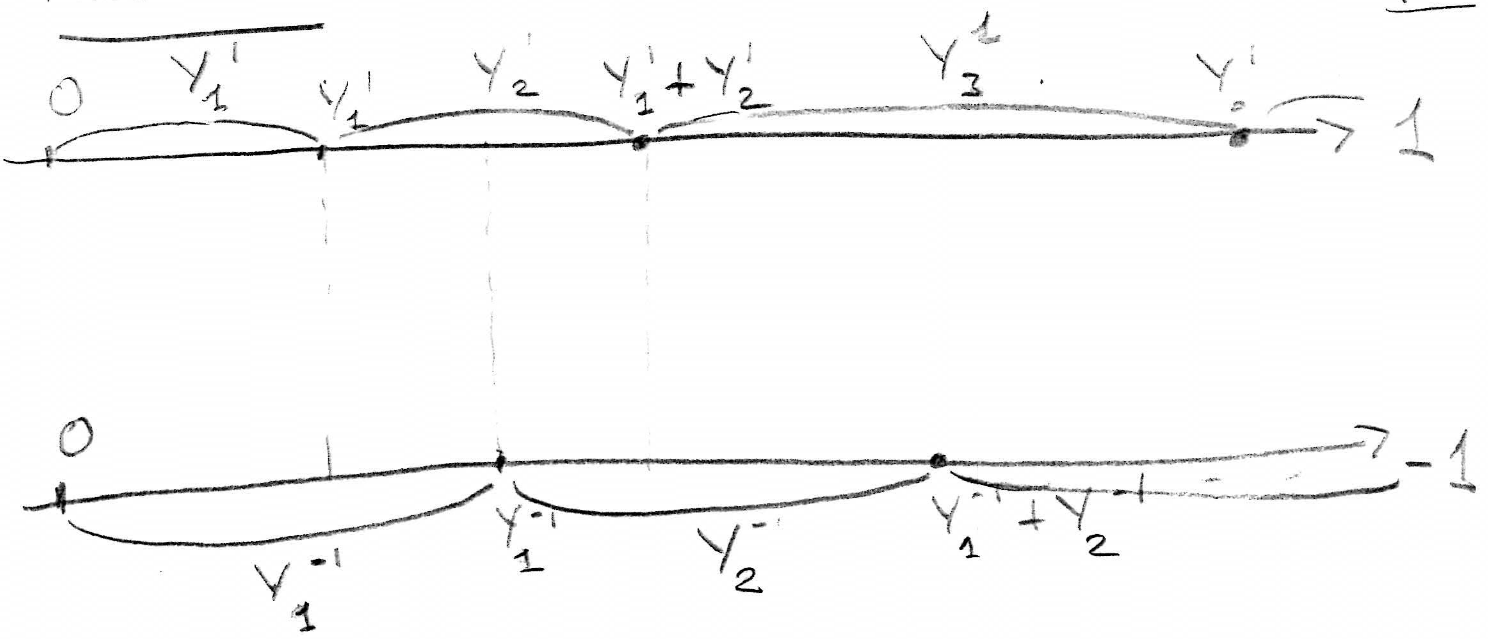
(seen from times of jumps)



add ball of colour ε at rate $W(Z_t(\varepsilon))$.

- Let $(Y_k^{1})_{k\in\mathbb{N}}$ and $(Y_k^{-1})_{k\in\mathbb{N}}$ be collections of indep. r.vs of exponential law with $\mathbb{E}\, Y_k^{\nu} = W(k)^{-1}$, indep. from each other
- Each of the colours 1 and -1 has a clock with an alarm, set initially to Y_1^{1} and Y_1^{-1} resp.
- Each time an alarm rings, we add a ball of the corresponding colour, say ε. The other clock $-\varepsilon$ keeps running, while we set a new alarm with ε, at time distance Y_k^{ε} if there are k balls of colour ε in the urn

Timelines



More precisely let, for $\varepsilon \in \{1,-1\}$,

- $S_\varepsilon := \left\{ \sum_{i=1}^{n} Y_i^\varepsilon \,,\, n \in \mathbb{N} \right\}$

  $S := S_1 \cup S_{-1}$

- Let ξ_i be the i-th smallest element in S

- $Z_t(\varepsilon) := \sup \{ k \geq 0 \text{ st. } \sum_{i=1}^{k} Y_k^\varepsilon \leq t \} + 1$

- $\tilde{Z}_n(\varepsilon) := \sum_{k=1}^{n} \mathbb{1}_{\{\xi_k \in S_\varepsilon\}} + 1$

  $= Z_{\xi_n}(\varepsilon)$

<u>Lemma</u> The processes $(\tilde{Z}_n(1), \tilde{Z}_n(-1))_{n\in\mathbb{N}_0}$ and $(Z_n(1), Z_n(-1))_{n\in\mathbb{N}_0}$ (from U-urn) are equal in law.

<u>Pf</u> based on

- memoryless property of exponentials: if U is an exponential variable, then $\mathcal{L}[U-a \mid U>a] = \mathcal{L}[U]$
- if U and V are two indep. exponential r.vs with $\mathbb{E}U = u^{-1}$ and $\mathbb{E}V = v^{-1}$, then $\mathbb{P}[U<V] = \frac{u}{u+v}$

Let $n\in\mathbb{N}$. Assume for instance $\xi_n \in S_1$, with

$$\xi_n = \sum_{i=1}^{p-1} Y_i^1, \text{ and } \sum_{i=1}^{q-1} Y_i^{-1} \leq \xi_n < \sum_{i=1}^{q} Y_i^{-1}$$

Then $\tilde{Z}_n(1) = p$, $\tilde{Z}_n(-1) = q$, and

$$\xi_{n+1} - \xi_n = \min\left(Y_p^1, \sum_{i=1}^{q} Y_i^{-1} - \xi_n\right)$$

Now

$$\mathcal{L}\left(\sum_{i=1}^{q} Y_i^{-1} - \xi_n \,\middle|\, \sum_{i=1}^{q-1} Y_i^{-1} \leq \xi_n < \sum_{i=1}^{q} Y_i^{-1}\right)$$

$$= \mathcal{L}(Y_q^{-1})$$

and

$$\mathbb{P}\left(\tilde{Z}_{n+1}(1) = \tilde{Z}_n(1) + 1 \,\middle|\, \xi_n = \sum_{i=1}^{p-1} Y_i^1, \sum_{i=1}^{q-1} Y_i^{-1} \leq \xi_n \leq \sum_{i=1}^{q} Y_i^{-1}\right)$$

$$= \mathbb{P}(Y_p^1 \leq Y_q^{-1}) = \frac{W(p)}{W(p)+W(q)} = \frac{w(\tilde{Z}_n(1))}{w(\tilde{Z}_n(1)) + w(\tilde{Z}_n(-1))}$$

□

Now, if $\sum \frac{1}{w(k)} < \infty$, then

$$\mathbb{P}\left(\tilde{Z}_\infty(1) = \tilde{Z}_\infty(-1) = \infty \right)$$

$$= \mathbb{P}\left(\sum_{i=1}^{\infty} Y_i^{1} = \sum_{i=1}^{\infty} Y_i^{-1} \right)$$

$$= \mathbb{P}\left(Y_1^{1} = \sum_{i=1}^{\infty} Y_i^{-1} - \sum_{i=2}^{\infty} Y_i^{1} \right) = 0$$

(Y_1^1 has continuous density, indep from

$$\sum_{i=1}^{\infty} Y_i^{-1} - \sum_{i=2}^{\infty} Y_i^{1} \text{).} \qquad \square$$

3.2) ESIRW on locally finite graph $(G, \sim)$, $\sum \frac{1}{w(k)} < \infty$

$(X_n)_{n \geq 0}$ random process taking values in vertices of $(G, \sim)$

$X_0 := x_0$,

$Z_n(e) :=$ nb visits to (nonoriented) edge e $+1$

$$\mathbb{P}(X_{n+1} = y) = \frac{W(Z_n(\{X_n, y\}))}{\sum_z W(Z_n(\{X_n, z\}))}$$

$E(G)$ (non oriented) edges of G

- $(Y_i^e)_{e \in E(G), i \in \mathbb{N}}$ collection of indep r.v.s of exponential law with $\mathbb{E}\, Y_i^e = W(i)^{-1}$.

Process $(X_t)_{t \in \mathbb{R}_+}$, starts at $X_0 := x_0$.

- each edge e has its own clock, which only runs when the process (X_t) is adjacent to e
- each time edge e has just been crossed (resp. at time 0) its clock sets up an alarm at distance Y_k^e if e has been crossed k times so far (resp Y_1^e at time 0)
- each time an edge e sounds an alarm, X_t crosses this edge instantaneously

Let $\tau_n :=$ n-th jump time of $(X_t)_{t \in \mathbb{R}_+}$

$$\tilde{X}_n := X_{\tau_n}, \quad \tilde{X}_0 = x_0$$

<u>Lemma</u> (Davis '90, Sellke '94)

The processes $(\tilde{X}_n)_{n \in \mathbb{N}_0}$ and $(X_n)_{n \in \mathbb{N}_0}$ have the same distribution.

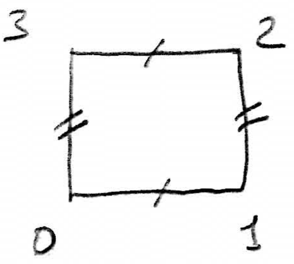
Let $\mathcal{G}_\infty := \{ e \in E(G) \text{ s.t. } Z_\infty(e) = \infty \}$

Lemma

$\mathcal{G}_\infty$ contains no even cycle.

Proof For simplicity, let us denote an even cycle by $\mathbb{Z}/\ell\mathbb{Z}$, ℓ even.

Ex: square



Let, for all $k \in \mathbb{Z}/\ell\mathbb{Z}$,

$$T_k := \sum_{i=1}^{\infty} Y_i^{\{k,k+1\}}$$

Then
$$\{\mathbb{Z}/\ell\mathbb{Z} \subseteq \mathcal{G}_\infty\} \subseteq \Big\{\sum_{x\in\mathbb{Z}/\ell\mathbb{Z}} (-1)^x T^x = 0\Big\}$$

Now $\sum_{x\in\mathbb{Z}/\ell\mathbb{Z}} (-1)^x T^x \neq 0$ a.s., which implies

that $\mathbb{P}(\mathbb{Z}/\ell\mathbb{Z} \subseteq \mathcal{G}_\infty) = 0$

$\square$

Rmk Technique carries over to show (Sellke '94) that on graphs of bdd degree with odd cycles (i.e. bipartite graphs), $|R'| = 2$ a.s., but does not apply to the triangle.

3.3 VSIRW on locally finite graph G

$Z_n(y) :=$ nb visits to y at time n $+1$

$$\mathbb{P}(X_{n+1} = y \mid \mathcal{F}_n) = \frac{W(Z_n(y))\, \mathbb{1}_{X_n \sim y}}{\sum_{z \sim X_n} W(Z_n(z))}.$$

Rubin construction (Bienvenüe, '99): replace "each edge e by "each vertex v", "crossed" by "visited", $(Y_i^e)_{e \in E(G), i \in \mathbb{N}}$ by $(Y_i^v)_{v \in V(G), i \in \mathbb{N}}$, $V(G)$ vertices of G.

Implies 1) asymptotic behaviour of VRRW conditional to localization to five points $(W(n) = n)$

2) Assuming $\sum \frac{1}{W(n)} < \infty$,

- $\mathbb{P}(R' = \mathbb{Z}/\ell\mathbb{Z},\ \ell$ multiple of $4) = 0$

Pf: $\left.\begin{matrix} S_a \\ T_a \end{matrix}\right\}$ time spent $\left\{\begin{matrix} \text{in } a \\ \text{adjacent to } a \end{matrix}\right.$ then $T_a = S_{a-1} + S_{a+1}$

$$\sum_{i=4k \in \mathbb{Z}/\ell\mathbb{Z}} T_i = \sum_{i=2k+1 \in \mathbb{Z}/\ell\mathbb{Z}} S_i = \sum_{i=4k+2 \in \mathbb{Z}/\ell\mathbb{Z}} T_i$$

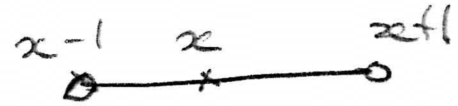
- similarly, on $\mathbb{Z}$, $\mathbb{P}(|R'| \text{ odd}) = 0$
- (if $R' = \{0, \dots 2n\}$), then $T_0 = S_1$, $T_{2n} = S_{2n-1}$,

$$T_a = S_{a-1} + S_{a+1}, \quad 1 \le a \le 2n-1 \;)$$

4) "Short" proof of conjecture of Pemantle and Volkov, coupling technique

Rubin construction would not apply directly, since $\sum \frac{1}{k} = \infty$.

Aim: show that $\Upsilon(x) \subseteq \{Z_\infty(x-1)<\infty\} \cup \{Z_\infty(x+1)<\infty\}$ a.s.



Rubin construction of Section 3.3:

- $t_n(x) \in \mathbb{R}_+$ n-th visit time to x
- $Z_n(x) :=$ time spent in x between times $t_n(x)$ and $t_{n+1}(x)$ $(:= 0$ if x visited $< n)$.
- $T_x :=$ total time spent at $x := \sum_{n=1}^{\infty} Z_n$

$(Z_{n+1}(x) \mid \mathcal{F}_{t_n}, t_n(x) < \infty)$ exponential r.v. of parameter

$Z_{t_n}(x-1) + Z_{t_n}(x+1)$

$$\Upsilon(x) = \{Y_\infty(x) < \infty\}$$

$$= \left\{ \sum_{n \in \mathbb{N},\, t_n(x)<\infty} \frac{1}{Z_{t_n}(x-1) + Z_{t_n}(x+1)} < \infty \right\}$$

$$= \left\{ T_x = \sum_{n=1}^{\infty} Z_n < \infty \right\}.$$

by conditional Borel-Cantelli Lemma.

But $\sum_{n:t_n(x)<\infty} Y^{x+1}_{Z_{t_n}(x+1)}$ and $\sum_{n:t_n(x)<\infty} Y^{x-1}_{Z_{t_n}(x-1)}$

are not anymore sums of iid random variables, so

$\mathbb{P}$(equality) not necessarily 0.

Need for another Rubin construction, with exponential random times indep of numbers of visits and defined on oriented edges

Goal: be able to create coupling with modified VRRW.

Alternative Rubin construction on $\mathbb{Z}$

- $\vec{E}$ oriented edges of $\mathbb{Z}$, $e=(x,y)$ then $\underline{e}=x$, $\bar{e}=y$. $\sigma(e)=(y,x)$
- $(Y_i^e)_{e\in\vec{E}, i\in\mathbb{N}}$ collection of indep r.v.s of exponential law with parameter one

Process $(X_t)_{t\in\mathbb{R}_+}$, starts at $X_0 := x_0$.

- each (oriented) edge e has own clock, only runs when $X_t = \underline{e}$ $\overset{e}{\longrightarrow}$
- time t where $\sigma(e)$ has just been crossed (resp at time 0) clock of e sets up alarm at distance $Y_k^e / Z_t(\bar{e})$ (resp Y_1^e at time 0)
- each time edge e sounds an alarm, X_t crosses it instantaneously.

$x-1 \quad x \quad x+1$

$(X_t)_{t\in\mathbb{R}_+}$ called walk $\mathcal{M}$.

walk $\mathcal{M}'_{n,r}$ defined similarly except that we add r to $Y_n^{(e)}$, $e=(x,x-1)$: process $(X'_t)_{t\in\mathbb{R}_+}$.

Def $(X_t)_{t\in\mathbb{R}_+}$, $(X'_t)_{t\in\mathbb{R}_+}$ $\rightsquigarrow$ discrete time walks

$\tilde{X}_k$, $\tilde{X}'_k$, $Z_k(v)$, $Z'_k(v)$, $\mathcal{H}' = \mathcal{H}_{n,r}$

. $i\in\mathbb{N}$, $j\in\mathbb{Z}$, $n_{i,j}$ = i-th visit time to j $\Big\}$ discrete-time

$e\in\vec{E}$, $n_{i,e}$:= " e $\Big\}$ walk

similarly $n'_{i,j}$, $n'_{i,e}$ for $(\tilde{X}'_n)$.

. T_x, T'_x total time spent in x (by continuous-time walks)

Def $i\in\mathbb{N}$, $j\in\mathbb{Z}$,

$$\text{property } \mathcal{E}_{i,j}: \begin{cases} Z_{n'_{i,j}}(j+1) \geq Z_{n_{i,j}}(j+1) \\ Z'_{n'_{i,j}}(j-1) \leq Z_{n_{i,j}}(j-1). \end{cases}$$

Lemma $\forall i\in\mathbb{N}$, $j\in\mathbb{Z}$, $\mathcal{E}_{i,j}$ holds

Pf Note that $\mathcal{E}_{i,j} \Rightarrow \tilde{X}'_{n'_{i,j}+1} \geq \tilde{X}_{n_{i,j}+1}$

Prove

$P_k = \{\forall i\in\mathbb{N}^*, j\in\mathbb{Z} : n_{i,j}\leq k, n'_{i,j}\leq k, \mathcal{E}_{i,j} \text{ holds}\}$

by induction on k; see Proof of Lemma 4.1.

T'(04)

$\square$

Def $i \in \mathbb{N}, j \in \mathbb{Z}$,

property $F_{i,j}$
$$\begin{cases} Z'_{t_i'((j+1,j))}(j+1) \geq Z_{t_i((j+1,j))}(j+1) \\ Z'_{t_i'((j-1,j))}(j-1) \leq Z_{t_i((j-1,j))}(j-1) \end{cases}$$

Lemma $\forall i \in \mathbb{N}, j \in \mathbb{Z}$, $F_{i,j}$ holds.

Pf left to the reader.

Def $S_{\mathcal{U}} := \{Z_\infty(x+1) = Z_\infty(x-1) = \infty\} \cap \{T_x < \infty\}$

$S_{\mathcal{U}'} := \{Z'_\infty(x+1) = Z'_\infty(x-1) = \infty\} \cap \{T'_x < \infty\}$

Lemma If $n \geq 2, r > 0$, then $S_{\mathcal{U}} \cap S'_{\mathcal{U}'_{n,r}} = \emptyset$

Pf If $Z_\infty(x+1) = Z_\infty(x-1) = \infty$ and $T_x < \infty$,
and $Z'_\infty(x+1) = Z'_\infty(x-1) = \infty$, $T'_x < \infty$, then

$$T_x = \sum_{k=1}^{\infty} \frac{Y_k^{(x,x+1)}}{Z_{t_k((x+1,x))}(x+1)} = \sum_{k=1}^{\infty} \frac{Y_k^{(x,x-1)}}{Z_{t_k((x-1,x))}(x-1)}$$

$$T_x' = \sum_{k=1}^{\infty} \frac{Y_k^{(x,x+1)}}{Z'_{t_k'((x+1,x))}(x+1)} \leq T_x < \sum_{k=1}^{\infty} \frac{Y_k^{(x,x-1)}}{Z'_{t_k'((x-1,x))}(x-1)} = T_x'$$

Contradiction $\square$

Let $\mathcal{F}_n = \sigma(\tilde{X}_0, \dots \tilde{X}_{n-1})$

$$\subseteq \sigma(Y_k^{(e)}, 1 \leq k \leq n-1)$$

Then $\mathbb{P}(\mathcal{S}_{\mathcal{U}}^c \cup \mathcal{S}_{\mathcal{U}'_{n,r}}^c \mid \mathcal{F}_n) = 1$.

But $\mathbb{P}(\mathcal{S}_{\mathcal{U}}^c \mid \mathcal{F}_n) \geq e^{-r}\, \mathbb{P}(\mathcal{S}_{\mathcal{U}'_{n,r}} \mid \mathcal{F}_n)$,

so that $\mathbb{P}(\mathcal{S}_{\mathcal{U}}^c \mid \mathcal{F}_n) \geq (1+e^{r})^{-1}$.

Now $\mathbb{P}(\mathcal{S}_{\mathcal{U}}^c \mid \mathcal{F}_n) \xrightarrow[n\to\infty]{} \mathbb{1}_{\mathcal{S}_{\mathcal{U}}^c}$ a.s.

so that $\mathcal{S}_{\mathcal{U}}^c$ holds almost surely.