The empirical collision probability of a population of interacting ants

Jean Bérard

April 2, 2004

Ex-Laboratoire de Probabilités, Combinatoire et Statistique

Université Claude Bernard, Lyon I

50, avenue Tony Garnier 69366 Lyon Cedex 07

jean.berard@univ-lyon1.fr

Abstract

We study a stochastic model of trail-following, based on a population of mutually reinforced random walks. A representation of the model via a random walk in space-time random environment is derived. Using this representation, we establish several asymptotic results (with respect to both the population size and time), that illustrate the effect of reinforcement upon the empirical characteristics of the population. In the large population and large time limit, reinforcement has no first-order effect on the distribution of the positions of the walkers, but it modifies other quantities such as the empirical collision probability.

KEY-WORDS: random walk, random environment, particle systems, reinforcement.

A.M.S. SUBJECT CLASSIFICATION: 60F05, 60J10, 60K35, 60K37.

1 Introduction

Ant trails, as observed in nature, illustrate the fascinating ability of social insects (such as termites, wasps and bees) to build complex macroscopic patterns. Experiments have shown that ants communicate with each other by dropping a chemical substance, called pheromone, on the substrate they move on, that the motion of an ant is influenced by the local pheromone concentration, and that moving along the chemical trail left by other ants of the same colony (or by the ant itself) is more likely than moving away from it ([18]). Based on this idea, several simplified models of ant behavior have been devised, and computer simulations of these models demonstrated their ability to reproduce some of the empirically observed features of real ant trails in various situations (see e.g. [26, 6, 23]). However, to our knowledge, none but very coarsened versions of these models turn out to be analytically tractable (see e.g. [1]), so that it is not unfair to say that understanding the properties of ant models on a rigorous basis represents a largely unsolved problem. This paper is devoted to a rigorous analysis of a stochastic model, still simplified when compared to more realistic ant models, but which shares two key features with the more complex models that are usually investigated by simulation only :

- a formulation of the model at the individual level; that is, the model describes how each ant in the population moves and interacts with its environment;
- an explicit treatment of space, modeled by a lattice grid.

Broadly speaking, our model relies on the family of stochastic processes known as reinforced random walks (see [22] for a review of the subject). Introduced by Coppersmith and Diaconis, edge-reinforced random walks describe the random motion of a particle along the edges of a graph. At each discrete time, the particle moves away from its current position by taking a step along a randomly chosen adjacent edge, the probability of choosing a particular edge being proportional to its current weight. Initially, all the weights are set equal to 1. Then, the weights are updated as the particle moves, the weight of an edge being increased by a fixed amount $\Delta \geq 0$ each time the particle crosses this particular edge (this is known as standard additive reinforcement scheme, and Δ is called the reinforcement strength parameter). The edge-reinforced random walk accounts in a very simple way for the tendency to follow a trail induced by previous moves, so it may serve as a much simplified basis for modeling pheromone trail following by ants (with the noticeable restriction that there is no evaporation of the trail in this model, as opposed to real ants [18]). Indeed, our model is based on the same kind of mechanism, with the difference that we consider a population of ants that interact through reinforcement, instead of a single particle that interacts with its own past history. Note that, even in the context of a single particle, the mathematical treatment of reinforced random walks is far from straightforward. On finite graphs and on infinite trees (including $\mathbb{Z}$), the behavior of the edge-reinforced random walk is reasonably well understood [21, 13, 10, 19], but, for instance, on $\mathbb{Z}^2$, questions as basic as recurrence vs. transience of the walk are still open. Even on $\mathbb{Z}$, the basic conjecture concerning the behavior of the vertex-reinforced random walk has been solved only recently [25]. As for models including a population of interacting reinforced particles, the only work we are aware of is that of Othmer and Stevens [20], in a slightly different context, so our study seems to be one of the first examples dealing with a tractable model of interacting reinforced particles.

The model we study may be quickly described as follows (a formal definition is given in the next section). Let us denote by $\mathbb{N}$ the set of non-negative integers $\mathbb{N} = \{0, 1, 2, \ldots\}$. We consider a sequence of ants denoted by $\mathfrak{a}_1, \mathfrak{a}_2, \ldots$ starting at the origin and performing directed reinforced random walks on the square lattice grid. In concrete terms, one may think of our model as describing an ant column moving towards a roughly fixed direction (North, say), but whose spatial spread (between North-West and North-East) is governed by a random reinforcement process modeling pheromone interaction between successive ants. At each discrete time $i = 1, 2, \ldots$, ant $\mathfrak{a}_i$ leaves the origin (the nest) and performs a discrete–time nearest-neighbor directed random walk on $\mathbb{Z} \times \mathbb{N}$ that "feels" the reinforcement induced by the trajectories of ants $\mathfrak{a}_1, \ldots, \mathfrak{a}_{i-1}$ (the first ant leaving the nest, $\mathfrak{a}_1$, just performs a simple directed random walk). More precisely, the position of $\mathfrak{a}_i$ after n steps outside the nest may be denoted by $\left(S_n^{(i)}, n\right) \in \mathbb{Z} \times \mathbb{N}$, the sequence $(S_n^{(i)})_{n \geq 0}$ satisfying, for every $n \geq 0$, $S_{n+1}^{(i)} = S_n^{(i)} \pm 1$. At each step, the $\mathbb{N}-$coordinate deterministically increases by one (whence the term "directed"), and the $\mathbb{Z}-$coordinate $S_n^{(i)}$ may either increase or decrease by one. Assuming that $S_n^{(i)} = x \in \mathbb{Z}$, the conditional probability of each of the two alternatives $S_{n+1}^{(i)} = x + 1$ and $S_{n+1}^{(i)} = x - 1$ depends on the number of transitions $(x, n) \to (x + 1, n + 1)$ and $(x, n) \to (x - 1, n + 1)$ that have occurred among the $i - 1$ previous ants $\mathfrak{a}_1, \ldots, \mathfrak{a}_{i-1}$, through a standard additive reinforcement scheme with strength parameter $\Delta \geq 0$. Note that, in our description, when the $\mathbb{N}-$coordinate of $\mathfrak{a}_i$ reaches the value n, all the $\mathbb{N}-$coordinates of $\mathfrak{a}_1, \ldots, \mathfrak{a}_{i-1}$ are above n, so the conditional probabilities defining how $\mathfrak{a}_i$ moves do not look into the future of the process, but depend only on the past decisions made by $\mathfrak{a}_1, \ldots \mathfrak{a}_{i-1}$. Our model on $\mathbb{Z} \times \mathbb{N}$ may seem less natural than its undirected analog on $\mathbb{Z} \times \mathbb{Z}$, in which ants would be free to move in each of the four directions of the square lattice. This last model is of course appealing, but its mathematical treatment seems out of reach for the moment. Moreover, assuming some directedness in the motion of ants is not unrealistic from a biological point of view, and several models have been devised under such an assumption [6]. We stress the interpretation of our model as describing an (idealized) ant column moving towards a roughly fixed direction. Our main focus in this paper is to study how reinforcement affects the cohesion of the column. For instance, does reinforcement lead to a spatial spread that is significantly lower than the one we would observe in the absence of reinforcement (that is, with ants performing independent simple random walks) ? To investigate such questions, that appear both interesting in the applications and mathematically challenging, we shall focus on the empirical properties of $\left(S_n^{(1)}, \ldots, S_n^{(p)}\right)$ for large values of p and n. In other words, we shall consider the list of the positions of the p first ants after their n first steps outside the nest, as an indicator of the spatial spread of the column. Perhaps surprisingly, our results show that reinforcement has no first order effect on the distribution of the positions of ants in the large p and large n limit. The effect of reinforcement is observed on other quantities, such as the empirical collision probability between two ants (see Section 3 below for precise statements of these facts).

Mathematically, it turns out that our model is equivalent, in a precise sense, to a random walk in space-time random environment model, so that our main task is to prove new asymptotic results for the random walk model, a translation of these results in terms of the original ant model being rather straightforward.

The paper is organized as follows. In Section 2, we give a formal definition of the model under study, and we define the main notations. Section 3 contains the statement of our main results along with some comments. In Section 4, we state and prove several basic properties of the model that are of constant use in the sequel. In Section 5, we collect some of the more technical results, part of the proofs being deferred to the Appendix. In Section 6 we prove the main theorems, building on the results of the previous sections. A heuristic explanation for the validity of the key result (Theorem 3 below) is presented.

2 Definition of the model and notations

2.1 The model

Our model describes the joint law of a family of $\mathbb{Z}$-valued random variables $S_n^{(i)}$ with $n \in \mathbb{N}$ and $i \geq 1$. The only parameter is the reinforcement strength Δ, which is assumed to be a fixed non-negative real number. The probability measure L governing the random variables $S_n^{(i)}$ is defined inductively. First, the sequence $(S_n^{(1)})_{n\geq 0}$ is a simple symmetric nearest-neighbor random walk on $\mathbb{Z}$ started at zero. Then, for every $i \geq 2$, conditional upon $(S_n^{(1)})_{n\geq 0}, \dots, (S_n^{(i-1)})_{n\geq 0}$, the sequence of random variables $(S_n^{(i)})_{n\geq 0}$ is a nearest-neighbor time-inhomogeneous Markov chain on $\mathbb{Z}$ started at zero and with the following transition probabilities: for $e = \pm 1$,

$$L\left[S_{k+1}^{(i)} = S_k^{(i)} + e \,\middle|\, S_k^{(i)} = x,\ (S_n^{(1)})_{n\geq 0}, \ldots, (S_n^{(i-1)})_{n\geq 0}\right] = \lambda_i(x,k,e),$$

where, for all $k \geq 0$, $x \in \mathbb{Z}$, and $e = \pm 1$,

$$\lambda_i(x,k,e) = \frac{1 + \Delta \times N_i(x,k,e)}{2 + \Delta \times (N_i(x,k,e) + N_i(x,k,-e))},$$

and

$$N_i(x,k,e) = \#\left\{j \in \{1,\ldots,i-1\}\ :\ (S_k^{(j)}, S_{k+1}^{(j)}) = (x, x+e)\right\}.$$

Note that the above expressions make sense even for $i = 1$ and are consistent with the definition of $(S_n^{(1)})_{n\geq 0}$ as a simple symmetric random walk.

In the sequel, we shall consider the restriction of the model to the p first ants, that we sometimes call the particle system with p ants, defined as

$$(S_{p,n})_{n\geq 0} = (S_n^{(1)}, \ldots, S_n^{(p)})_{n\geq 0},$$

and the (random) empirical measure of $S_{p,n}$, defined as

$$\mu_{p,n} = \frac{1}{p}\sum_{i=1}^{p} \delta_{S_n^{(i)}}.$$

Note that it is not obvious from our definition that $(S_{p,n})_{n\geq 0}$ forms a Markov chain. This will be proved in Proposition 2 in the next section.

2.2 A random walk in random environment

We now define a completely different model whose relation with the original ant model will become clear later. Consider a family of i.i.d. random variables $q = (q(x,n),\ (x,n) \in \mathbb{Z} \times \mathbb{N})$ with common law Beta$(1/\Delta, 1/\Delta)$, and denote by Q the probability measure governing these random variables. Conditional upon a given realization of q, we define a family of time-inhomogeneous nearest-neighbor transition probabilities on $\mathbb{Z}$ as follows:

$$\begin{cases} p_{q,n}(x, x-1) = q(x,n), \\ p_{q,n}(x, x+1) = 1 - q(x,n). \end{cases}$$

Next, we consider an i.i.d. family of time-inhomogeneous Markov chains

$$X^{(1)} = (X_n^{(1)})_{n\geq 0}, X^{(2)} = (X_n^{(2)})_{n\geq 0}, \ldots$$

started at zero and evolving according to the transition probabilities $p_{q,n}$ we have just defined. In other words, conditional upon a given realization of q, we have a probability measure P_q and Markov chains $X^{(1)}, X^{(2)}, \ldots$ on $\mathbb{Z}$ governed by P_q such that

- for all $i \geq 1$, $X_0^{(i)} = 0$,
- for $e = \pm 1$,

$$P_q \left[X_{n+1}^{(i)} = x + e \,\middle|\, X_n^{(i)} = x \right] = p_{q,n}(x, x+e),$$

- the trajectories $X^{(1)}, X^{(2)}, \ldots$ are mutually independent.

Conditional on q, we have thus defined the joint law of a family of random variables

$$(X_n)_{n\geq 0} = (X_n^{(1)}, X_n^{(2)}, \ldots)_{n\geq 0}.$$

We shall also consider the restriction of the model to the p first trajectories, defined as

$$(X_{p,n})_{n\geq 0} = (X_n^{(1)}, \ldots, X_n^{(p)})_{n\geq 0}.$$

Since the environment in which the random walks evolve is random with respect to both time and space, we shall call this model a family of random walks in space-time random environment.

For fixed $n \geq 0$, the common distribution of the $X_n^{(i)}$ under P_q will be of special interest in the sequel, and we denote it by $\nu_{n,q}$:

$$\nu_{n,q} = \text{distribution of } X_n^{(1)} \text{ under } P_q.$$

Note that $\nu_{n,q}$ is a probability distribution on $\mathbb{Z}$ that is random with respect to Q. Measurability issues are trivial here since, for all q, $\nu_{n,q}$ is supported on the finite set $\{-n, \ldots, n\}$.

2.3 Notations

To every probability measure ν with finite support on $\mathbb{Z}$, we associate its average:

$$Av(\nu) = \sum_{x\in\mathbb{Z}} x\nu(x),$$

and its collision probability, that is, the hitting probability of the diagonal by two independent copies of ν:

$$Coll(\nu) = \sum_{x\in\mathbb{Z}} \nu(x)^2,$$

In the special case of an empirical distribution, that is, if

$$\nu = \frac{1}{p}\sum_{i=1}^{p} \delta_{x_i},$$

the collision probability reads:

$$Coll(\nu) = \frac{1}{p^2}\#\left\{1 \le i,j \le p \ : x_i = x_j\right\}.$$

The notation $\beta_i(U_1,\dots,U_l)$ denotes a quantity that may depend on the parameters $U_1,\dots,U_l$, but that is implicitly assumed to be a constant with respect to the other parameters.

Here are our general notational conventions for Markov chains. When a sequence of random variables $(O_n)_{n\ge 0}$ defined on a common probability space $(\Omega,\mathcal{F},B)$ and taking their values in a discrete space χ forms a time-homogeneous Markov chain, we say that B is the probability measure that governs the chain, and we denote by $b(\cdot,\cdot)$ its transition kernel, that is

$$b(x,y) = B(O_{n+1} = y|O_n = x)$$

for all $x,y \in \chi$. Usually, we use E_B to denote the expectation with respect to the probability measure B. Given two transition kernels a and b on χ, we denote by $a \times b$ the product kernel obtained by applying b then a:

$$(a \times b)(x,y) = \sum_{z\in\chi} b(x,z)a(z,y), \quad \text{for all } x,y \in \chi.$$

For iterated kernels, we use an exponential notation:

$$b^n = \underbrace{b \times \cdots \times b}_{n\ times},$$

with the convention $b^0 = Id_{\mathcal{X}}$.

3 Statement of the main results and some comments

The first theorem concerns the relationship between the ant model $(S_n)_{n\geq 0}$ and the family of random walks $(X_n)_{n\geq 0}$ in random environments defined in Section 2.

Theorem 1 *For all $p \geq 1$, the distribution of $(S_{p,n})_{n\geq 0}$ coincides with the distribution of $(X_{p,n})_{n\geq 0}$ under P_q averaged with respect to Q. This identity of distributions extends to $(S_n)_{n\geq 0}$ and $(X_n)_{n\geq 0}$.*

As a consequence, we obtain an alternative representation of our original model: instead of considering random trajectories that interact through reinforcement, we may consider independent random trajectories evolving in a random environment whose distribution is known explicitly.

An immediate corollary of this representation is the following.

Corollary 1 *Almost surely with respect to L, for all $n \geq 0$, as p goes to infinity,*

$$\mu_{p,n} \xrightarrow{d} \mu_{\infty,n},$$

where $\mu_{\infty,n}$ is a random probability disribution on $\mathbb{Z}$ with the same law as the random distribution $\nu_{n,q}$ with respect to Q.

We now state our main results on the model, in terms of the asymptotic properties of $\mu_{\infty,n}$ as n goes to infinity. In other words, we first take the large p limit at fixed n, and then the large n limit. Non-asymptotic bounds for the discrepancy between $\mu_{p,n}$ and $\mu_{\infty,n}$ could be easily deduced from Theorem 1 together with standard estimates on empirical measures of i.i.d. random variables, so the following results admit more refined versions that apply to $\mu_{p,n}$ instead of $\mu_{\infty,n}$. However, we found it more convenient to state the results in terms of the distribution of $\mu_{\infty,n}$ with respect to L, and we leave to the interested reader the task of stating the corresponding results. Note that, according to the above corollary, the results below hold without change for the distribution of $\nu_{n,q}$ with respect to Q, and indeed, they will be proved in this last context.

Theorem 2 *Almost surely with respect to L, for all $x \in \mathbb{R}$,*

$$\lim_{n \to +\infty} \mu_{\infty,n}(]-\infty, x\sqrt{n}]) = \int_{-\infty}^{x} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}.$$

Theorem 3 *With respect to L, as n goes to infinity:*

$$n^{1/2} Coll(\mu_{\infty,n}) \stackrel{prob.}{\longrightarrow} \frac{1}{\sqrt{\pi}}(1 + \Delta/2).$$

Theorem 4 *With respect to L, as n goes to infinity,*

$$n^{-1/4} Av(\mu_{\infty,n})$$

converges in distribution to a centered normal distribution with variance σ_Δ^2 defined by $\sigma_\Delta^2 = \frac{\Delta}{\sqrt{\pi}}$.

We proved Theorem 2 above in a broader context in our earlier work [2], so its proof will not be given in the present paper. Roughly speaking, it shows that, in the large n limit, no first order effect of reinforcement can be seen on the empirical distribution of the positions of the ants, since we obtain the same limiting law that would prevail in the absence of reinforcement i.e. a standard gaussian distribution on the scale $n^{1/2}$. In this sense, the standard additive reinforcement scheme we have used in the definition of the model is not sufficient to ensure long-range cohesion of the ant column[1]. However, according to Theorem 3 above, the effect of reinforcement can be seen on the asymptotic empirical collision probability $Coll(\mu_{\infty,n})$, a quantity that goes to zero as n goes to infinity, but with a leading order behavior that depends

[1]These results leave open the interesting question of what kind of reinforcement scheme should be used in order to achieve such a long-range cohesion.

explicitly upon the reinforcement parameter Δ. Reinforcement is also responsible for the $n^{1/4}$ scaling of the asymptotic average empirical position $Av(\mu_{\infty,n})$, according to Theorem 4. Without reinforcement, $Av(\mu_{\infty,n}) = 0$ for all n, so the existence of fluctuations on the scale $n^{-1/4}$ can be seen as a (small) non self-averaging property of the positions of the ants that is due to reinforcement.

Note that the random walk in space-time random environment $(X_n)_{n\geq 0}$ to which all the above results in fact apply, can be defined for more general environments than just i.i.d. Beta-distributed ones, without any reference to an underlying reinforcement model. The study of such random walk models is of independent interest, and several results have already been obtained in this context by different authors. In a perturbative setting, that reads in our context as

$$q(x,i) = 1/2 + \text{small random term},$$

results analogous to Theorems 2, 3, 4 above have been proved in [4, 3, 5], and, free of perturbative assumptions, a result weaker than Theorem 2 has been proved in [7]. Thus, our Theorems 2, 3, 4 can be seen as extending those of [4, 3, 5] to a special non-perturbative context : that of Beta$(1/\Delta, 1/\Delta)$-distributed environments. We point out that the key estimates in the present work are obtained by quite different methods than those used in [4, 3, 5], whose applicability, according to the authors themselves, relies crucially upon the small randomness assumption. Broadly speaking, our proofs are based on generic Gaussian bounds for reversible Markov chains, and on generating functions tools.

4 Basic properties of the model

We mentioned earlier in Section 2 that it may not be obvious at first sight that the stochastic process $(S_{p,n})_{n\geq 0}$ is indeed a Markov chain. Proposition 2 below summarizes the Markov chain properties of $(S_{p,n})_{n\geq 0}$. But before we can state the proposition, a few definitions are needed.

4.1 Definitions

Definition 1 *Let* Ξ_p *denote the subset of* $p-$*tuples in* $\mathbb{Z}^p$ *whose components are all equal modulo 2, that is:*

$$\Xi_p = \left\{ (x^{(1)}, \dots, x^{(p)}) \in \mathbb{Z}^p \ : \ x^{(i)} - x^{(j)} \in 2\mathbb{Z} \text{ for all } i, j \right\}.$$

We endow Ξ_p *with a graph structure by defining* $x, y \in \Xi_p$ *to be neighbors when, for all* $i \in \{1, \dots, p\}$, $|y_i - x_i| = 1$.

It is easily checked that $S_{p,n} \in \Xi_p$ for all $n \geq 0$, and that the only transitions allowed for $S_{p,n}$ are along the edges of Ξ_p as defined above.

Definition 2 *For all* $x \in \Xi_p$, *denote by* $z^1, \dots, z^d$ *the distinct integers in the* $p-$*tuple* $x = (x^{(1)}, \dots, x^{(p)})$ *numbered in an arbitrary way. For every pair of neigbors* $x, y \in \Xi_p$, *define*

$$g(i) = \# \left\{ 1 \leq j \leq p \ : \ y^{(j)} = z^{(i)} + 1 \right\},$$

$$d(i) = \# \left\{ 1 \leq j \leq p \ : \ y^{(j)} = z^{(i)} - 1 \right\},$$

and let

$$\ell_p(x, y) = \prod_{l=1}^{d} \frac{[1(1+\Delta) \cdots (1 + (g(l)-1)\Delta)] \times [1(1+\Delta) \dots (1 + (d(l)-1)\Delta)]}{2(2+\Delta) \cdots (2 + (g(l) + d(l) - 1)\Delta)}.$$

One easily checks that the above formulæ for $g(i), d(i)$ and $\ell_p(x, y)$ do not depend on the numbering of $z^1, \dots, z^d$, so the definitions make sense.

Definition 3 *For every* $x = (x^{(1)}, \dots, x^{(p)}) \in \Xi_p$ *and* $1 \leq i \leq p$, *let* $\varphi_i(x)$ *stand for the number of distinct integers that appear exactly* i *times in the* $p-$*tuple* $(x^{(1)}, \dots, x^{(p)})$. *More formally:*

$$\varphi_i(x) = \# \left\{ a \in \mathbb{Z} \ : \ \#\{j \ : \ x_j = a\} = i \right\}.$$

Definition 4 *Let* η_p *be the measure on* Ξ_p *defined by:*

$$\eta_p(x) = \prod_{i=1}^{p} \left[\prod_{k=0}^{i-1} (2 + k\Delta) \right]^{\varphi_i(x)}.$$

For instance, if $x = (-8, 10, 6, 2, 4, 4, -8, 4, 6, 2)$, $\varphi_1(x) = 1$, $\varphi_2(x) = 3$, $\varphi_3(x) = 1$, and $\varphi_j(x) = 0$ for $4 \leq j \leq 10$. Thus, $\eta_{10}(x) = 2^5(2+\Delta)^4(2+2\Delta)$.

Proposition 2 *For every $p \geq 1$, $(S_{p,n})_{n\geq 0}$ is an irreducible time-homogeneous Markov chain on Ξ_p whose transition probabilities are $\ell_p(\cdot,\cdot)$. Moreover, $(S_{p,n})_{n\geq 0}$ is reversible with respect to the measure η_p.*

4.2 Proof of Proposition 2

We first prove that $(S_{p,n})_{n\geq 0}$ is a Markov chain with transition probabilities $\ell_p(\cdot,\cdot)$. The proof is by induction on p. For $p = 1$, $(S_{p,n})_{n\geq 0}$ reduces to a simple symmetric random walk on $\mathbb{Z}$ so the assertion is trivial. Assume that $p \geq 2$, and that we are done for $p-1$. Fix an integer $n \geq 1$ and a path $0 = x_0 \to \cdots \to x_n$ in Ξ_p, and define the events

$$B_{p,n} = \{S_{p,t} = x_t \text{ for all } \ 0 \leq t \leq n\},$$

$$B'_{p,n} = \left\{S_{p,t}^{(p)} = x_t^{(p)} \text{ for all } 0 \leq t \leq n\right\}$$

and

$$B''_{p,n} = \left\{S_t^{(i)} = x_t^{(i)} \text{ for all } 1 \leq i \leq p-1,\ 0 \leq t \leq n\right\},$$

so that $B_{p,n} = B'_{p,n} \cap B''_{p,n}$. By the very definition of $(S_{p,n})_{n\geq 0}$,

$$L\left(B'_{p,n}\,\middle|\,(S_t^{(1)})_{t\geq 0}, \ldots, (S_t^{(p-1)})_{t\geq 0}\right) = \prod_{t=0}^{n-1} \lambda_k\left(x_t^{(p)}, t, x_{t+1}^{(p)} - x_t^{(p)}\right).$$

Note that, by definition again,

$$\lambda_k\left(x_t^{(p)}, t, x_{t+1}^{(p)} - x_t^{(p)}\right) = f_p\left[(S_t^{(1)}, \ldots, S_t^{(p-1)}, x_t^{(p)}), (S_{t+1}^{(1)}, \ldots, S_{t+1}^{(p-1)}, x_{t+1}^{(p)})\right],$$

where, for all $y, z \in \Xi_p$,

$$f_p[y,z] = \frac{1 + \Delta \times \#\{j \in \{1,\ldots,p-1\} \ :\ (y^{(j)}, z^{(j)}) = (y^{(p)}, z^{(p)})\}}{2 + \Delta \times \#\{j \in \{1,\ldots,p-1\} \ :\ y^{(j)} = y^{(p)}\}}.$$

The above identity extends to the case $p = 1$, with the convention that $f_1 = 1/2$. As a consequence,

$$L\left(B'_{p,n}\,\middle|\,(S_t^{(1)})_{t\geq 0}, \ldots, (S_t^{(p-1)})_{t\geq 0}\right) = \prod_{t=0}^{n-1} f_p[x_t, x_{t+1}].$$

Since $B''_{p,n}$ is measurable with respect to $(S_t^{(1)})_{t\geq 0}, \ldots, (S_t^{(p-1)})_{t\geq 0}$, we deduce that

$$L(B_{p,n}) = \left(\prod_{t=0}^{n-1} f_p[x_t, x_{t+1}]\right) \times L\left(S_t^{(i)} = x_t^{(i)} \text{ for all } 1 \leq i \leq p-1,\ 0 \leq t \leq n\right).$$

Define, for all $d-$tuple $(x^{(1)}, \ldots, x^{(d)})$ with $d \geq l$, the projection π_l on the first l coordinates:

$$\pi_l(x^{(1)}, \ldots, x^{(d)}) = (x^{(1)}, \ldots, x^{(l)}).$$

With this notation, using the induction hypothesis and the above identity, we have

$$L\left(S_t^{(i)} = x_t^{(i)} \text{ for all } 1 \leq i \leq p-1,\ 0 \leq t \leq n\right) = \prod_{t=0}^{n-1} \ell_{p-1}\left(\pi_{p-1}(x_t), \pi_{p-1}(x_{t+1})\right),$$

so

$$L(B_{p,n}) = \prod_{t=0}^{n-1} f_p[x_t, x_{t+1}] \times \ell_{p-1}\left(\pi_{p-1}(x_t), \pi_{p-1}(x_{t+1})\right).$$

It is routine to check that, for every pair of neighbors $x, y \in \Xi_p$,

$$\ell_p(x, y) = f_p[x, y] \times \ell_{p-1}\left(\pi_{p-1}(x), \pi_{p-1}(y)\right), \tag{1}$$

so we are done with the proof that $(S_{p,n})_{n\geq 0}$ is a homogeneous Markov chain with transition probabilities $\ell_p(\cdot,\cdot)$. The fact that $(S_{p,n})_{n\geq 0}$ is also irreducible on Ξ_p is obvious, so the only fact that remains to be proved is the reversibility with respect to η_p. For every pair of neighbors $x, y \in \Xi_p$, define

$$g_p(x, y) = 1 + \Delta \times \#\{j \in \{1, \ldots, p-1\} \ :\ (x^{(j)}, y^{(j)}) = (x^{(p)}, y^{(p)})\}$$

and

$$h_p(x) = 2 + \Delta \times \#\{j \in \{1, \ldots, p-1\} \ :\ x^{(j)} = x^{(p)}\},$$

so that

$$f_p[x, y] = \frac{g_p(x, y)}{h_p(x)}$$

(with the convention $g_1 = 1$ and $h_1 = 2$). Observe that g_p is symmetric, that is : $g_p(x, y) = g_p(y, x)$. As a consequence, for every pair of neighbors $x, y \in \Xi_p$,

$$h_p(x) f_p[x, y] = h_p(y) f_p[y, x].$$

This, in turn, entails that :

$$\prod_{k=1}^{p} h_k(\pi_{k-1}(x)) f_k[\pi_{k-1}(x), \pi_{k-1}(y)] = \prod_{k=1}^{p} h_k(\pi_{k-1}(y)) f_k[\pi_{k-1}(y), \pi_{k-1}(x)].$$

According to Equation (1) above, we have

$$\ell_p(x,y) = \prod_{k=1}^{p} f_k[\pi_{k-1}(x), \pi_{k-1}(y)],$$

so we have proved that:

$$\left(\prod_{k=1}^{p} h_k(\pi_{k-1}(x))\right) \ell_p(x,y) = \left(\prod_{k=1}^{p} h_k(\pi_{k-1}(y))\right) \ell_p(y,x). \tag{2}$$

Now, it is routine to check that, for every $x \in \Xi_p$,

$$\eta_p(x) = \prod_{k=1}^{p} h_k(\pi_{k-1}(x)),$$

so Equation (2) just above is the detailed balance condition for $\ell_p(\cdot,\cdot)$ with respect to η_p. □

Remark 3 *According to the above Proposition 2, the main object of our study is a reversible Markov chain whose reversible measure is explicitly known, so it may seem to the reader that the asymptotic behavior of $S_{p,n}$ for large n could be easily derived. However, in the sequel, precise estimates for $(S_{p,n})_{n\geq 0}$ will be needed, that do not seem to follow readily from the basic properties stated above.*

4.3 Proof of Theorem 1

We now proceed to the proof of Theorem 1 of Section 3 concerning the representation of the law of the particle system via random walks in random environment. This type of representation is common when dealing with reinforced random walks on acyclic graphs, and, for instance, it has been used in [21] to study the behavior of edge-reinforced random walks on trees. The difference here is that we consider a population of interacting particles instead of a single particle interacting with its own past history.

Let us first describe things intuitively. After its first n steps, ant $\mathfrak{a}_i$ can either go to $S_n^{(i)}+1$ or $S_n^{(i)}-1$. Let us fix $(x,n) \in \mathbb{Z} \times N$, and call $i_1 < i_2 < \ldots$ the (random) successive indices of ants $\mathfrak{a}_i$ satisfying $S_n^{(i)} = x$. By the very definition of the reinforcement process, the list of successive choices made by the ants $\mathfrak{a}_{i_1}, \mathfrak{a}_{i_2}, \ldots$ immediately after leaving (x, n) coincides with the successive draws from a Pólya's urn $U(x, n)$ whose two "colors" are $x+1$ and $x-1$, with initial composition $(1, 1)$ and reinforcement weight Δ (see [21, 15]). According to the definition of $(S_n)_{n\geq 0}$, conditional on i_k, on the trajectories of ants $\mathfrak{a}_1$ to $\mathfrak{a}_{i_k-1}$ and on the n first steps of $\mathfrak{a}_{i_k}$, the edge chosen by ant $\mathfrak{a}_{i_k}$ when leaving (x, n) depends only on the first $k-1$ draws of $U(x, n)$. As a consequence, the draws from urn $U(x, n)$ are independent from the draws from all other urns $U(y, k)$ with $(y, k) \neq (x, n)$ (the only minor technical difficulty is to write down properly the random time change $i_1, i_2, \ldots$), and we have a collection of i.i.d. Pólya's urns $(U(x, n),\ (x, n) \in \mathbb{Z} \times \mathbb{N})$. Now, the successive draws [15, 21] from a Pólya's urn may be represented as an i.i.d. family of Bernoulli draws conditional on their parameter $q(x, n)$, $q(x, n)$ itself being chosen at random in $[0, 1]$ according to a Beta distribution $(1/\Delta, 1/\Delta)$. Turning back to the interpretation of draws form urn $U(x, n)$ as successive edges chosen by ants when leaving (x, n), we get the conclusion of the theorem. Here is a more explicit (and computational) proof.

Fix an integer $p \geq 1$, and a path $x_0 \to \cdots \to x_n$ in Ξ_p. We want to prove that :

$$L\left(S_{p,t} = x_t \text{ for all } \ 0 \leq t \leq n\right) = E_Q\left[P_q\left(X_{p,t} = x_t \text{ for all } \ 0 \leq t \leq n\right)\right],$$

where E_Q stands for the expectation with respect to the choice of the random environment q. To begin with, using the Markov property of $(S_{p,n})_{n\geq 0}$, we have

$$L\left(S_{p,t} = x_t \text{ for all } \ 0 \leq t \leq n\right) = \prod_{t=0}^{n-1} \ell_p\left(x_t, x_{t+1}\right).$$

Then, using the fact that, conditional on q, $(X^{(1)}), \ldots, (X^{(p)})$ are independent random walks,

$$E_Q\left[P_q\left(X_{p,t} = x_t \text{ for all } \ 0 \leq t \leq n\right)\right] = E_Q\left[\prod_{t=0}^{n-1}\left(\prod_{i=1}^{p} p_{q,t}\left(x_t^{(i)}, x_{t+1}^{(i)}\right)\right)\right].$$

Taking into account the mutual independence of the $q(y, j)$ with respect to Q, we see that the r.h.s. of the above equality is equal to

$$\prod_{t=0}^{n-1} E_Q \left[\left(\prod_{i=1}^{p} p_{q,t} \left(x_t^{(i)}, x_{t+1}^{(i)} \right) \right) \right].$$

As a consequence, the theorem will be proved if we show that

$$\ell_p \left(x_t, x_{t+1} \right) = E_Q \left[\left(\prod_{i=1}^{p} p_{q,t} \left(x_t^{(i)}, x_{t+1}^{(i)} \right) \right) \right]. \tag{3}$$

Using the notations of Definition (2) with $x = x_t$ and $y = x_{t+1}$, we see that, thanks to the mutual independence of the $q(y, j)$ with respect to Q,

$$E_Q \left[\left(\prod_{i=1}^{p} p_{q,t} \left(x_t^{(i)}, x_{t+1}^{(i)} \right) \right) \right] = \prod_{l=1}^{d} E_Q \left[q(z_l, t)^{g(l)} (1 - q(z_l, t)^{d(l)}) \right].$$

From Lemma 1 in [21], the $l-$th term of the product in the r.h.s. above equals

$$\frac{[1(1+\Delta) \cdots (1 + (g(l) - 1)\Delta)] \times [1(1+\Delta) \ldots (1 + (d(l) - 1)\Delta)]}{2(2+\Delta) \cdots (2 + (g(l) + d(l) - 1)\Delta)}.$$

Equation (3) follows readily from the above equality and the definition of ℓ_p.

That the identity of distributions between $(S_{p,n})_{n \geq 0}$ and $(X_{p,n})_{n \geq 0}$ extends to $(S_n)_{n \geq 0}$ and $(X_n)_{n \geq 0}$ is immediate. □

The following corollary is an immediate but useful consequence of Theorem 1.

Corollary 4 *For every subset $I = \{i_1, \ldots, i_k\}$ of $\{1, \ldots, p\}$ (where the i_js are assumed to be pairwise distinct), the sequence*

$$\left(S_n^{(i_1)}, \ldots, S_n^{(i_k)} \right)_{n \geq 0}$$

is a Markov chain on Ξ_k with transition probabilities $\ell_k(\cdot, \cdot)$.

5 Technical results

5.1 Gaussian upper bound

The following proposition provides a Gaussian upper bound for the distribution of $S_{p,n}$. Although more refined estimates are needed in the proof of our results, Proposition 5 below is a crucial step toward obtaining these. Moreover, Proposition 5 has the nice feature that it can be proved using little specific information relative to $(S_{p,n})_{n\geq 0}$, by relying on a general machinery that has been developed for Markov chains with diffusive behavior.

Proposition 5 *For all $x \in \Xi_p$, and all $n \geq 0$, the following inequality holds:*

$$\ell_p^n(0,x) \leq \frac{\beta_1(p,\Delta)}{(n+1)^{p/2}} \exp\left[-\beta_2(p,\Delta)\frac{||x||^2}{n+1}\right],$$

where $||x|| = \left[(x^{(1)})^2 + \cdots + (x^{(p)})^2\right]^{1/2}$ denotes the standard Euclidean norm.

Proof:

Much work has been devoted to finding general conditions ensuring that a Markov chain (in discrete or continuous time) on a graph satisfies Gaussian upper and lower bounds. The discrete-time result we use here is due to Delmotte [11] (Theorem 1.7), (see also [8] and [9] for related results and references). Let us quote a version of the result suited for our purposes. Assume that $(A_n)_{n\geq 0}$ is a time-homogeneous Markov chain with transition probabilities $a(\cdot,\cdot)$ on a countable locally finite graph Γ, that $a(\cdot,\cdot)$ is reversible with respect to a measure m on Γ, and that the following three conditions are met (extra notations are defined below):

1. volume doubling: there exists a constant $b > 0$ such that, for all $x \in \Gamma$ and all $r > 0$,

$$V(x,2r) \leq bV(x,r)$$

2. Poincaré inequality: there exist two constants $C_1 > 0$ and $C_2 > 1$ such that, for every function $f \in \mathbb{R}^\Gamma$, and for all $x_0 \in \Gamma$ and $r > 0$,

$$\sum_{x\in B(x_0,r)} m(x)|f(x) - f_r(x)|^2 \leq C_1 r^2 \sum_{x,y\in B(x_0,C_2 r)} m(x)a(x,y)(f(y) - f(x))^2$$

3. every $x \in \Gamma$ is a neighbor of itself, and there exists a constant $\rho > 0$ such that, for every couple of neighbors $(x,y) \in \Gamma^2$,

$$a(x,y) \geq \rho.$$

Then, the following inequality holds: for every $(x, y) \in \Gamma^2$,

$$a_k(x,y) \leq \frac{C_3 m(y)}{V(x,\sqrt{k})} \exp\left[-C_4 \frac{d(x,y)^2}{k+1}\right].$$

In the above, $d(\cdot,\cdot)$ denotes the usual graph distance on Γ, $B(x,r)$ is the ball centered at x and of radius r with respect to d, $V(x,r) \stackrel{def}{=} m(B(x,r))$ is the volume of the ball $B(x,r)$ with respect to the measure m, and $f_r(x)$ is defined by

$$f_r(x) = \frac{1}{V(x,r)} \sum_{y \in B(x,r)} f(y)m(y).$$

In the view of condition 3 above, we cannot hope to apply the theorem directly to $(S_{p,n})_{n\geq 0}$, since, from the definition, it is clear that $(S_{p,n})_{n\geq 0}$ has period 2. Instead, we apply the theorem separately to $(S_{p,2n})_{n\geq 0}$ and $(S_{p,2n+1})_{n\geq 0}$, the result on $(S_n)_{n\geq 0}$ being an easy consequence. More precisely, we view $(S_{p,2n})_{n\geq 0}$ (resp. $(S_{p,2n+1})_{n\geq 0}$) as a Markov chain on $\Xi_p^0 \stackrel{def}{=} 2\mathbb{Z}^p$ (resp. $\Xi_p^1 \stackrel{def}{=} (1,\ldots,1) + 2\mathbb{Z}^p$), endowed with the graph structure for which x and y are neighbors iff $|x^{(i)} - y^{(i)}| \leq 2$ for all $i \in \{1,\ldots,p\}$, and whose transition probabilities are the restriction of $\ell_p^2(\cdot,\cdot)$ to Ξ_p^0 (resp. Ξ_p^1). From now on, we focus on $(S_{p,2n})_{n\geq 0}$, the case of $(S_{p,2n+1})_{n\geq 0}$ being similar. With our definition, it is easily checked that condition 3 above is met for small enough ρ, and that the restriction of η_p to Ξ_p^0 is reversible with respect to the restriction of $\ell_p^2(\cdot,\cdot)$ to Ξ_p^0. Note that the graph distance on Ξ_p^0 is equivalent to the standard Euclidean distance (in the sense of equivalent distances), and, in particular, the volume doubling condition is trivially met, since the restriction of η_p to Ξ_p^0 has a density with respect to the counting measure on Ξ_p^0 that is uniformly bounded and bounded away from zero.

The Poincaré inequality is also easy to establish, using the now classical argument of [14, 9] based on paths. To every pair $(x,y) \in B(a,l)^2$, we associate a path $\mathcal{C}(x,y)$ in Ξ_p^0:

$$x = [y,x]_1 \longrightarrow [y,x]_2 \longrightarrow \cdots \longrightarrow [y,x]_k \longrightarrow \cdots \longrightarrow [y,x]_{p+1} = y,$$

where, for all $i \in \{1,\ldots,p\}$,

$$[y,x]_i = (y_{(1)},\ldots,y^{(i-1)},x^{(i)},\ldots,x^{(p)}),$$

and where $[y,x]_i \longrightarrow [y,x]_{i+1}$ stands for the path linking $[y,x]_i$ to $[y,x]_{i+1}$ in $(1/2)|x^{(i)} - y^{(i)}|$ steps by adding $2 \times \mathrm{sgn}(x^{(i)} - y^{(i)})$ to the $i-$th coordinate at each step, the other coordinates remaining unchanged. Note that $\mathcal{C}(x,y)$ is contained in $B(a,2l)$, and that its length (in number of steps) cannot exceed $2lp$. Moreover, it is easily checked that a given edge (u,v) cannot belong to more than $(2l)^{p+1}$ paths of the form $\mathcal{C}(x,y)$, $(x,y) \in B(a,l)^2$.

Using these remarks and the convexity of $x \mapsto x^2$, we obtain the following inequality, for all $(x,y) \in B(a,l)^2$:

$$|f(x) - f(y)|^2 \leq 2lp \sum_{(u,v)\in\mathcal{C}(x,y)} |f(u) - f(v)|^2.$$

Using the fact that η_p is bounded from below by 1, and that, for every edge (u,v) in Ξ_p^0, $\ell_p^2(u,v) \geq \rho$, we deduce from the above inequality that:

$$\sum_{(x,y)\in B(a,l)^2} |f(x) - f(y)|^2 \leq (2lp)(2l)^{p+1}\rho^{-1} \sum_{u,v} \eta_{p,\Delta}(u)\ell_p^2(u,v)|f(u) - f(v)|^2, \quad (4)$$

where the last sum runs over all pairs (u,v) of neighbors in $B(a,2l)^2$. On the other hand, using the convexity of $x \mapsto x^2$, and the fact that η_p is bounded from above, we see that:

$$\sum_{x\in B(x_0,r)} \eta_{p,\Delta}(x)|f(x) - f_r(x)|^2 \leq (\sup \eta_{p,\Delta}) \sum_{x\in B(x_0,r)} \frac{1}{V(x,r)} \sum_{y\in B(x,r)} |f(x) - f(y)|^2,$$

whence

$$\sum_{x\in B(x_0,r)} \eta_{p,\Delta}(x)|f(x) - f_r(x)|^2 \leq (\sup \eta_{p,\Delta}) \sum_{(x,y)\in B(x_0,2r)^2} \frac{1}{V(x,r)}|f(x) - f(y)|^2.$$

Combining the above inequality with Inequality (4), and using the fact that $V(x,r) \propto r^p$, we obtain the Poincaré inequality. $\square$

5.2 Generating functions estimates

The rest of this section is devoted to studying the probability

$$L\left(S_k^{(i)} = S_k^{(j)}\right),$$

and other closely related quantities, upon which precise control will be needed in the proofs of the main theorems. To state things properly, let us define the set A as follows

Definition 5

$$A = \left\{ (s^{(1)}, s^{(2)}) \in \Xi_2 \ : \ s^{(1)} = s^{(2)} \right\}.$$

According to Corollary 4 above, we have, for all i, j and k,

$$L\left(S_k^{(i)} = S_k^{(j)}\right) = \ell_2^k(0, A).$$

More generally, we investigate in this section the behavior of $\ell_2^k(x, A)$ for an arbitrary $x \in \Xi_2$. The leading order behavior of $\ell_2^k(0, A)$ is stated in the following proposition.

Proposition 6 *As k goes to infinity,*

$$\ell_2^k(0, A) \sim b(\Delta) k^{-1/2},$$

where $b(\Delta) = \frac{1}{\sqrt{\pi}}(1 + \Delta/2)$.

Note that this is the simplest kind of asymptotic behavior that we may expect for the hitting time probability of a set of codimension 1 by a diffusive Markov chain on Ξ_2.

Consider now the general situation where the starting point is not zero. According again to the diffusive behavior of the chain, we may reasonably expect that, starting from two neighboring points $y, s \in \Xi_2$, the hitting probabilities $\ell_2^k(y, A)$ and $\ell_2^k(s, A)$ are close one to each other when k is large. The following proposition provides a precise formulation of this fact.

Proposition 7 *For all $y, s \in \Xi_2$ such that $\left|y^{(1)} - s^{(1)}\right| \leq 2$ and $\left|y^{(2)} - s^{(2)}\right| \leq 2$, and all $k \geq 0$, the following estimate holds:*

$$\left|\ell_2^k(y, A) - \ell_2^k(s, A)\right| \leq \frac{\beta_3(\Delta)}{k+1} \exp\left[-\beta_4(\Delta) \frac{|y^{(1)} - y^{(2)}|^2}{k+1}\right].$$

Note that the $1/(k+1)$ order in the above estimate is not surprising, since a similar estimate also holds, for instance, for the simple random walk on $\mathbb{Z}^2$. Broadly speaking, Propositions 6 and 7 are nothing but precise statements of regularity properties that are expected to be rather generic for diffusive Markov chains. However, the proofs presented here are quite specific, and rely on an explicit computation of generating functions. Such a computation is made possible in our context by expressing the quantities of interest in terms of a tractable one-dimensional random walk. For all $k \geq 0$, define T_k as the difference

$$T_k = S_k^{(1)} - S_k^{(2)}.$$

It turns out that $(T_k)_{k\geq 0}$ is a Markov chain on $2\mathbb{Z}$, as stated in the following lemma, whose proof is immediate.

Lemma 1 *The sequence $(T_n)_{n\geq 0}$ is a Markov chain on $2\mathbb{Z}$, whose transition probabilities $t(\cdot,\cdot)$ are given by:*

- *if $x \neq 0$, $t(x,x) = 1/2$, $t(x,x-2) = t(x,x+2) = 1/4$,*
- *if $x = 0$, $t(0,0) = \frac{1+\Delta}{2+\Delta}$, $t(0,-2) = t(0,2) = \frac{1}{2(2+\Delta)}$.*

Now, it is immediate from the definition of T_k that, for all $(x^{(1)}, x^{(2)}) \in \Xi_2$ and $k \geq 0$,

$$\ell_2^k(x,A) = t^k\left(x^{(1)} - x^{(2)}, 0\right).$$

As a consequence, computing the generating function of the sequence $(\ell_2^k(x,A))_{k\geq 0}$ amounts to computing the generating function of the sequence $\left(t^k\left(x^{(1)} - x^{(2)}, 0\right)\right)_{k\geq 0}$.

Let us give precise definitions. The generating function of $(\ell_2^k(x,A))_{k\geq 0}$ is defined by

Definition 6 *For all $x \in \Xi_2$ and all $|z| < 1$, we put*

$$H_x(z) = \sum_{k\geq 0} \ell_2^k(x,A) z^k.$$

To express $H_x(z)$ under a concise form, we introduce two additional notations.

Definition 7

- *The parameter u is defined by:*

$$u = \frac{2+2\Delta}{2+\Delta}.$$

- *For all $x \in \Xi_2$, we put*

$$\delta(x) = \left| x^{(1)} - x^{(2)} \right|$$

The explicit formula for the generating function $H_x(z)$ is stated in the following proposition.

Proposition 8 *For all $|z| < 1$, and all $x \in \Xi_2$,*

$$H_x(z) = \frac{1}{(u-1)(1-z) + (2-u)\sqrt{1-z}} \times \left(\frac{1-\sqrt{1-z}}{\sqrt{z}} \right)^{\delta(x)}.$$

The proof of Proposition 8 above relies on another result proved in earlier work (Proposition 1 in [2]), that we quote as Proposition 9. The proof of Proposition 9 is not reproduced here, and we just mention that it is based on a one-dimensional computation involving $t(\cdot,\cdot)$.

Proposition 9 *For all $|z| < 1$,*

$$H_0(z) = \frac{1}{(u-1)(1-z) + (2-u)\sqrt{1-z}}.$$

Proof of Proposition 8:

As announced, we start from the obvious identity:

$$\ell_2^k(x, A) = t^k \left(x^{(2)} - x^{(1)}, 0 \right).$$

Taking into account the symmetry of t with respect to sign reversal, we have in fact:

$$\ell_2^k(x, A) = t^k \left(\delta(x), 0 \right).$$

Note that the above identity explains why the generating function $H_x(z)$ depends on x only through $\delta(x)$. The case $\delta(x) = 0$ is treated by proposition 9, so we assume in the sequel that $\delta(x) > 0$. We have to study the behavior of the Markov chain $(T_n)_{n \geq 0}$ started at $\delta(x) > 0$. Let us define (with the convention that $\inf \emptyset = +\infty$)

$$\tau_0(T) = \inf\{ j \geq 0 \ : \ T_j = 0 \}.$$

On the other hand, let us consider a simple symmetric random walk $(W_k)_{k \geq 0}$ on $\mathbb{Z}$, that is

$$W_k = \delta(x) + \epsilon_1 + \cdots + \epsilon_k,$$

where the ϵ_i are i.i.d. with common distribution $\frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$, and define

$$\tau_0(W) = \inf\{j \geq 0 \ : \ W_j = 0\}.$$

We note that, for every $h > 0$, the transition probabilities $t(h, \cdot)$ of T starting from h, coincide with those of $(W_{2k})_{k\geq 0}$. Moreover, for parity reasons, $W_{2k+1} \neq 0$ for every $k \geq 0$. As a consequence, we have the following identity in law:

$$\tau_0(W) \stackrel{law}{=} 2\tau_0(T).$$

But the law of $\tau_0(W)$ is well-known, and its generating function is given, for all $|z| < 1$, by (see e.g. [16] p. 351)

$$\sum_{k=1}^{+\infty} P\left(\tau_0(W) = 2k\right) z^{2k} = \left(\frac{1-\sqrt{1-z^2}}{z}\right)^{\delta(x)}$$

As a consequence:

$$\sum_{k=1}^{+\infty} P\left(\tau_0(T) = k | T_0 = \delta(x)\right) z^k = \left(\frac{1-\sqrt{1-z}}{\sqrt{z}}\right)^{\delta(x)}.$$

Thanks to the strong Markov property of $(T_n)_{n\geq 0}$, we check that:

$$t^n(\delta(x), 0) = \sum_{k=1}^{n} P(\tau_0(T) = k | T_0 = \delta(x)) \times t^{n-k}(0,0),$$

whence, taking generating functions on both sides:

$$\left(\sum_{k=1}^{+\infty} t^k(\delta(x), 0) z^k\right) = \left(\sum_{k=1}^{+\infty} P\left(\tau_0(T) = k | T_0 = \delta(x)\right) z^k\right) \left(\sum_{k=0}^{+\infty} t^k(0,0) z^k\right).$$

Replacing the generating function by its expression from Proposition 9, we obtain the desired formula for $H_x(z)$. □

Remark 10 *The presence of u in the expression of the above generating functions accounts for the effect of reinforcement in the model. Without reinforcement ($\Delta = 0$), we would get the classical formulæ that hold for simple random walks.*

Precise information on the terms $\ell_2^k(x, A)$ can now be extracted from the above explicit formulæ for generating functions, using standard complex analysis techniques (see e.g. [17]), and this is how we prove Propositions 6 and 7. Since the proofs are of a rather technical nature, we defer them to the appendix.

6 Proofs of theorems 3 and 4

6.1 Proof of theorem 3

According to Theorem 1, the collision probability $Coll(\nu_{n,q})$ satisfies:

$$E_Q(Coll(\nu_{n,q})) = E_Q(P_q(X_n^{(1)} = X_n^{(2)})) = L(S_n^{(1)} = S_n^{(2)}) = L(S_{2,n} \in A) = \ell_2^n(0, A).$$

Moreover, according to Proposition 6, as n goes to infinity,

$$\ell_2^n(0, A) \sim b(\Delta) n^{-1/2}.$$

Thus, we may rephrase the statement of the theorem we want to prove as follows: as n goes to infinity, $n^{1/2} Coll(\nu_{n,q})$ converges in probability (with respect to Q) to the limit of its expectation. As a consequence, to prove the theorem, it is enough to bound the variance of $Coll(\nu_{n,q})$ in the following way:

$$E_Q \left[Coll(\nu_{n,q})^2 \right] - \left(E_Q[Coll(\nu_{n,q})] \right)^2 = o(n^{-1}).$$

In fact, we shall prove the following more precise result: as n goes to infinity,

$$E_Q \left[Coll(\nu_{n,q})^2 \right] - \left(E_Q[Coll(\nu_{n,q})] \right)^2 = O\left(\frac{\log n}{n^{3/2}} \right). \tag{5}$$

We have just seen that $E_Q[Coll(\nu_{n,q})]$ can be expressed in terms of the law of the particle system with two ants $S_{2,n}$, since $E_Q(Coll(\nu_{n,q})) = \ell_2^n(0, A)$. Now, thanks to Theorem 1 again, we can express $E_Q \left[Coll(\nu_{n,q})^2 \right]$ in terms of the law of the particle system with four ants $S_{4,n}$:

$$\begin{aligned}
E_Q \left[Coll(\nu_{n,q})^2 \right] &= E_Q \left[P_q(X_n^{(1)} = X_n^{(2)})^2 \right] \\
&= E_Q \left[P_q(X_n^{(1)} = X_n^{(2)}) \times P_q(X_n^{(3)} = X_n^{(4)}) \right] \\
&= E_Q \left[P_q(X_n^{(1)} = X_n^{(2)}, X_n^{(3)} = X_n^{(4)}) \right] \\
&= L \left[S_n^{(1)} = S_n^{(2)}, S_n^{(3)} = S_n^{(4)} \right] \\
&= L(S_{4,n} \in J) = \ell_4^n(0, J),
\end{aligned}$$

where

$$J = \left\{ (s^{(1)}, s^{(2)}, s^{(3)}, s^{(4)}) \in \Xi_4 \ : \ s^{(1)} = s^{(2)}, \ s^{(3)} = s^{(4)} \right\},$$

and where we have used the fact that, for fixed n, the random variables $X_n^{(1)}, \dots, X_n^{(4)}$ are i.i.d. with respect to P_q. As a consequence, the difference we want to bound may be rewritten as

$$\ell_4^n(0,J) - \left(\ell_2^n(0,A)\right)^2.$$

The idea of the proof is now to introduce an auxiliary Markov chain $(Y_n)_{n\geq 0}$ on Ξ_4 that is in some sense close to $(S_{4,n})_{n\geq 0}$ and such that $\left(\ell_2^n(0,A)\right)^2$ appears as the hitting probability of J by Y_n. An obvious choice is to define $(Y_n)_{n\geq 0}$ as the concatenation of two independent copies $(Y_n^{(1)}, Y_n^{(2)})_{n\geq 0}$ and $(Y_n^{(3)}, Y_n^{(4)})_{n\geq 0}$ of $(S_n^{(1)}, S_n^{(2)})_{n\geq 0}$. More formally, we consider the Markov chain $(Y_n)_{n\geq 0}$ on Ξ_4 defined by $Y_0 = 0$ and by the transition probabilities $r = \ell_2 \otimes \ell_2$:

$$r(a,b) = \ell_2\left((a^{(1)}, a^{(2)}), (b^{(1)}, b^{(2)})\right) \times \ell_2\left((a^{(3)}, a^{(4)}), (b^{(3)}, b^{(4)})\right).$$

We call R the probability measure governing $(Y_n)_{n\geq 0}$. With these definitions:

$$\ell_4^n(0,J) - \left(\ell_2^n(0,A)\right)^2 = \ell_4^n(0,J) - r^n(0,J),$$

and our aim is to bound this difference. To study the above expression, we rewrite it as a sum of progressive differences:

$$r^n(0,J) - \ell_4^n(0,J) = \sum_{k=0}^{n-1} \left[\left(r^{k+1} \times \ell_4^{n-k-1}\right)(0,J) - \left(r^k \times \ell_4^{n-k}\right)(0,J)\right].$$

Let us fix $0 \leq k \leq n-1$. Making explicit the step from time $n-k-1$ to time $n-k$ in the above expressions, we see that

$$\left(r^{k+1} \times \ell_4^{n-k-1}\right)(0,J) - \left(r^k \times \ell_4^{n-k}\right)(0,J)$$

is equal to

$$\sum_{y,s\in\Xi_4} \left[r^k(s,J) \times r(y,s) \times \ell_4^{n-k-1}(0,y) - r^k(s,J) \times \ell_4(y,s) \times \ell_4^{n-k-1}(0,y)\right].$$

Extracting the common factor $\ell_4^{n-k-1}(0,y)$, we see that the above expression is equal to

$$\sum_{y\in\Xi_4} \ell_4^{n-k-1}(0,y) \left[\sum_{s\in\Xi_4} r^k(s,J) \times r(y,s) - r^k(s,J) \times \ell_4(y,s)\right].$$

Now, for every $y \in \Xi_4$,

$$\sum_{s \in \Xi_4} r^k(s,J)r(y,s) - r^k(s,J)\ell_4(y,s) = E_R(r^k(Y_1,J)|Y_0 = y) - E_L(r^k(S_{4,1},J)|S_{4,0} = y). \tag{6}$$

To alleviate notations, define

$$D_1 = \ell_2^k\left[(S_1^{(1)}, S_1^{(2)}), A\right] - \ell_2^k\left[(y^{(1)}, y^{(2)}), A\right],$$

$$D_2 = \ell_2^k\left[(S_1^{(3)}, S_1^{(4)}), A\right] - \ell_2^k\left[(y^{(3)}, y^{(4)}), A\right],$$

$$D_3 = \ell_2^k\left[(Y_1^{(1)}, Y_1^{(2)}), A\right] - \ell_2^k\left[(y^{(1)}, y^{(2)}), A\right],$$

$$D_4 = \ell_2^k\left[(Y_1^{(3)}, Y_1^{(4)}), A\right] - \ell_2^k\left[(y^{(3)}, y^{(4)}), A\right],$$

and put

$$\sigma = \ell_2^k\left[(y^{(1)}, y^{(2)}), A\right], \ \sigma' = \ell_2^k\left[(y^{(3)}, y^{(4)}), A\right].$$

By the definition of the transition kernel $r = \ell_2 \otimes \ell_2$, we have

$$r^k(S_{4,1}, J) = \ell_2^k\left[(S_1^{(1)}, S_1^{(2)}), A\right] \times \ell_2^k\left[(S_1^{(3)}, S_1^{(4)}), A\right]$$

and

$$r^k(Y_1, J) = \ell_2^k\left[(Y_1^{(1)}, Y_1^{(2)}), A\right] \times \ell_2^k\left[(Y_1^{(3)}, Y_1^{(4)}), A\right].$$

Using the random variables D_i we have just defined, the two above identities imply that

$$E_R(r^k(Y_1,J)|Y_0 = y) - E_L(r^k(S_{4,1},J)|S_{4,0} = y)$$

is equal to

$$E_R\left[\left(\ell_2^k\left[(y^{(1)}, y^{(2)}), A\right] + D_3\right) \times \left(\ell_2^k\left[(y^{(3)}, y^{(4)}), A\right] + D_4\right) \middle| Y_0 = y\right] -$$
$$E_L\left[\left(\ell_2^k\left[(y^{(1)}, y^{(2)}), A\right] + D_1\right) \times \left(\ell_2^k\left[(y^{(3)}, y^{(4)}), A\right] + D_2\right) \middle| S_{4,0} = y\right].$$

Let us now expand the products in the above expectations. We get the following expression:

$$E_R(\sigma\sigma' + \sigma D_3 + \sigma' D_4 + D_3 D_4 | Y_0 = y) - E_L(\sigma\sigma' + \sigma D_1 + \sigma' D_2 + D_1 D_2 | S_{4,0} = y).$$

We first notice that the $\sigma\sigma'$ terms cancel out. Moreover, and this is the crucial point in this computation, the terms of degree 1 with respect to the D_is also cancel out. Indeed, by the very definition of the transition kernel r, the law of D_1 conditional on $S_{4,0} = y$ and the law of D_3 conditional on $Y_0 = y$ coincide, and similarly the law of D_2 conditional on $S_{4,0} = y$ coincide with the law of D_4 conditional on $Y_0 = y$. We finally obtain that:

$$E_R(r^k(Y_1, J)|Y_0 = y) - E_L(r^k(S_{4,1}, J)|S_{4,0} = y) = E_R(D_3 D_4|Y_0 = y) - E_L(D_1 D_2|S_{4,0} = y). \tag{7}$$

We now estimate separately all the D_is. Conditional on $S_{4,0} = y$, we have $|S_1^{(j)} - y^{(j)}| \leq 2$ for all $1 \leq j \leq 4$, and, similarly, conditional on $Y_0 = y$, we have $|Y_1^{(j)} - y^{(j)}| \leq 2$ for all $1 \leq j \leq 4$. As a consequence, we can apply proposition 7 to uniformly estimate D_1 and D_2 conditional on $S_{4,0} = y$, and D_3 and D_4 conditional on $Y_0 = y$. Specifically, conditional on $S_{4,0} = y$,

$$|D_1| \leq \frac{\beta_3(\Delta)}{k+1} \exp\left[-\beta_4(\Delta)\frac{|y^{(1)} - y^{(2)}|^2}{k+1}\right]$$

and

$$|D_2| \leq \frac{\beta_3(\Delta)}{k+1} \exp\left[-\beta_4(\Delta)\frac{|y^{(3)} - y^{(4)}|^2}{k+1}\right],$$

and conditional on $Y_0 = y$,

$$|D_3| \leq \frac{\beta_3(\Delta)}{k+1} \exp\left[-\beta_4(\Delta)\frac{|y^{(1)} - y^{(2)}|^2}{k+1}\right]$$

and

$$|D_4| \leq \frac{\beta_3(\Delta)}{k+1} \exp\left[-\beta_4(\Delta)\frac{|y^{(3)} - y^{(4)}|^2}{k+1}\right].$$

These estimates, together with the identities (6) and (7) finally lead to the upper bound

$$\left|\sum_{s\in\Xi_4} r^k(s, J)r(y, s) - r^k(s, J)\ell_4(y, s)\right| \leq 2\left(\frac{\beta_3(\Delta)}{k+1}\right)^2 \exp\left[-\beta_4(\Delta)\frac{|y^{(1)} - y^{(2)}|^2 + |y^{(3)} - y^{(4)}|^2}{k+1}\right]. \tag{8}$$

For every subset $\mathcal{B} \subset \Xi_4$, define

$$d_k(\mathcal{B}) = \left|\sum_{y\in\mathcal{B}} \ell_4^{n-k-1}(0, y)\left[\sum_{s\in\Xi_4} r^k(s, J) \times r(y, s) - r^k(s, J) \times \ell_4(y, s)\right]\right|,$$

and recall that our aim is to bound from above the expression $d_k(\Xi_4)$. Note that, for every $y \in \Xi_4$ such that $\{y^{(1)}, y^{(2)}\} \cap \{y^{(3)}, y^{(4)}\} = \emptyset$, $r(y,s) = \ell_4(y,s)$ for all s, by the definition of r. As a consequence, putting

$$\mathcal{K} = \left\{ y \in \Xi_4 \ : \ \exists i \in \{2,3\},\ \exists j \in \{3,4\},\ y^{(i)} = y^{(j)} \right\},$$

we have the identity

$$d_k(\Xi_4) = d_k(\mathcal{K}).$$

Now, define

$$\mathcal{K}_{i,j} = \left\{ y \in \Xi_4 \ : \ y^{(i)} = y^{(j)} \right\},$$

and note that

$$\mathcal{K} = \mathcal{K}_{1,3} \cup \mathcal{K}_{1,4} \cup \mathcal{K}_{2,3} \cup \mathcal{K}_{2,4}.$$

As a consequence:

$$d_k(\Xi_4) \leq d_k(\mathcal{K}_{1,3}) + d_k(\mathcal{K}_{1,4}) + d_k(\mathcal{K}_{2,3}) + d_k(\mathcal{K}_{2,4}).$$

Now, using the bound (8) and Proposition 5, we obtain that, for all $y \in \mathcal{K}_{i,j}$, $d_k(\{y\})$ is at most

$$\frac{\beta_1(4,\Delta)}{(n-k)^2} \exp\left[-\beta_2(4,\Delta) \frac{||y||^2}{n-k} \right] 2 \left(\frac{\beta_3(\Delta)}{k+1} \right)^2 \exp\left[-\beta_4(\Delta) \frac{|y^{(1)} - y^{(2)}|^2 + |y^{(3)} - y^{(4)}|^2}{k+1} \right].$$

Define

$$H_{n,k}(y) = \beta_2(4,\Delta) \frac{||y||^2}{n-k} + \beta_4(\Delta) \frac{|y^{(1)} - y^{(2)}|^2 + |y^{(3)} - y^{(4)}|^2}{k+1}.$$

The above bound for $d_k(\{y\})$ implies that:

$$d_k(\mathcal{K}_{i,j}) \leq \sum_{y \in \mathcal{K}_{i,j}} d_k(\{y\}) \leq \frac{\beta_{11}(\Delta)}{(k+1)^2 (n-k)^2} \sum_{y \in \mathcal{K}_{i,j}} \exp(-H_{n,k}(y)).$$

Since $H_{n,k}(y)$ is invariant under swapping the first two or the last two coordinates of y, we see that, in the above expression, the terms $\sum_{y \in \mathcal{K}_{i,j}} (\cdots)$ do not depend on i,j, and, as a consequence, it is enough to study the sum over $\mathcal{K}_{2,3}$ (for instance). So let us assume that $y^{(2)} = y^{(3)}$, and set $\delta_1 = y^{(1)} - y^{(2)}$ and $\delta_2 = y^{(4)} - y^{(3)}$. Expanding $H_{n,k}(y)$, and using the fact that $y^{(2)} = y^{(3)}$, we see that $H_{n,k}(y)$ equals

$$\frac{\beta_2(4,\Delta)}{(n-k)} \left[(2y^{(2)} + 1/2(\delta_1 + \delta_2))^2 - \left(\frac{\delta_1 + \delta_2}{2} \right)^2 + (\delta_1^2 + \delta_2^2) \right] + \frac{\beta_4(\Delta)}{k+1} (\delta_1^2 + \delta_2^2).$$

Thanks to the convexity inequality:

$$\left(\frac{\delta_1+\delta_2}{2}\right)^2 \leq 1/2(\delta_1^2+\delta_2^2),$$

we see that $H_{n,k}(y)$ is bounded from below by $G_{n,k}(y)$, where

$$G_{n,k}(y) = \frac{\beta_2(4,\Delta)}{(n-k)}\left(2y^{(2)}+1/2(\delta_1+\delta_2)\right)^2 + \left(\frac{\beta_2(4,\Delta)}{2(n-k)}+\frac{\beta_4(\Delta)}{k+1}\right)(\delta_1^2+\delta_2^2).$$

As a consequence,

$$d_k(\Xi_4) \leq \frac{4\beta_{11}(\Delta)}{(k+1)^2(n-k)^2}\sum_{y\in\mathcal{K}_{2,3}}\exp(-G_{n,k}(y)).$$

Changing variables in the following way: $y \leftrightarrow (y^{(2)},\delta_1,\delta_2)$, leads to a bijection between $\mathcal{K}_{2,3}$ and $\mathbb{Z}\times(2\mathbb{Z})\times(2\mathbb{Z})$, so that

$$\sum_{y\in\mathcal{K}_{2,3}}\exp(-G_{n,k}(y))$$

is bounded from above by

$$\sum_{\delta_1,\delta_2\in 2\mathbb{Z}}\exp\left[-\left(\frac{\beta_2(4,\Delta)}{2(n-k)}+\frac{\beta_4(\Delta)}{k+1}\right)(\delta_1^2+\delta_2^2)\right]\sum_{y^{(2)}\in\mathbb{Z}}\exp\left[-\frac{\beta_2(4,\Delta)}{(n-k)}\left(2y^{(2)}+1/2(\delta_1+\delta_2)\right)^2\right]$$

Using the following inequality, valid for all $b>0$ and $c\in\mathbb{R}$:

$$\sum_{a\in\mathbb{Z}}\exp(-b(a+c)^2) \leq 2+\frac{\sqrt{\pi}}{\sqrt{b}},$$

we deduce that

$$\sum_{y\in\mathcal{K}_{2,3}}\exp(-G_{n,k}(y)) \leq \left(2+\frac{\beta_{12}(\Delta)}{\sqrt{\frac{1}{n-k}+\frac{1}{k+1}}}\right)^2 \times \left(2+\beta_{12}(\Delta)\sqrt{n-k}\right).$$

To simplify the above expression, note that, since , for all $0\leq k\leq n-1$, $\sqrt{n-k}\geq 1$,

$$\left(2+\beta_{12}(\Delta)\sqrt{n-k}\right) \leq \beta_{13}(\Delta)\sqrt{n-k},$$

and, similarly, since, for all $0\leq k\leq n-1$, $\frac{1}{k+1}+\frac{1}{n-k}\leq 2$,

$$\left(2+\frac{\beta_{12}(\Delta)}{\sqrt{\frac{1}{n-k}+\frac{1}{k+1}}}\right)^2 \leq \left(\frac{\beta_{15}(\Delta)}{\sqrt{\frac{1}{n-k}+\frac{1}{k+1}}}\right)^2 = \beta_{15}(\Delta)^2\frac{(n-k)(k+1)}{n+1}.$$

As a consequence,

$$\sum_{y\in\mathcal{K}_{2,3}}\exp(-G_{n,k}(y)) \leq \beta_{16}(\Delta)(n-k)^{3/2}(k+1)(n+1)^{-1},$$

whence

$$d_k(\Xi_4) \leq \beta_{17}(\Delta)(k+1)^{-1}(n-k)^{-1/2}(n+1)^{-1}.$$

Now, it is easy to check that

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (k+1)^{-1}(n-k)^{-1/2}(n+1)^{-1} = O\left(\frac{\log n}{n^{3/2}}\right)$$

and that

$$\sum_{k=\lfloor n/2 \rfloor+1}^{n} (k+1)^{-1}(n-k)^{-1/2}(n+1)^{-1} = O\left(\frac{1}{n^{3/2}}\right).$$

As a consequence,

$$\sum_{k=0}^{n-1} d_k(\Xi_4) = O\left(\frac{\log n}{n^{3/2}}\right).$$

The conclusion of the theorem follows from the above estimate, since

$$|r^n(0,J) - \ell_4^n(0,J)| \leq \sum_{k=0}^{n-1} d_k(\Xi_4).$$

□

6.2 Heuristic explanation for Theorem 3

Although the above argument may seem a bit technical, it is possible to give some intuition about why the result should hold and why the properties we used should play a role in its proof. Informally, proving the theorem amounts to proving that, to the first order, the events $S_n^{(1)} = S_n^{(2)}$ and $S_n^{(3)} = S_n^{(4)}$ are independent when n is large. Note that, except when $S_{4,n} \in \mathcal{K}$, $(S_n^{(1)}, S_n^{(2)})$ and $(S_n^{(3)}, S_n^{(4)})$ evolve independently from one another. That the perturbating effect of visits to $\mathcal{K}$ on the independence of $(S_n^{(1)}, S_n^{(2)})$ and $(S_n^{(3)}, S_n^{(4)})$ may be neglected when computing $L(S_n^{(1)} = S_n^{(2)}, S_n^{(3)} = S_n^{(4)}) = L(S_{4,n} \in J)$ to the leading order, is justified (at least at a heuristic level) by the following facts. First, according to our Gaussian upper bound for $S_{p,n}$ (Proposition 5), the law of $S_{p,n}$ is "evenly spread" over Ξ_4 when n is large, so visits to the small (codimension 1) set $\mathcal{K}$ are rare events. Moreover, visits to $\mathcal{K}$ typically occur far away from J, since $\mathcal{K} \cap J$ has codimension 3. Then, we know from Proposition 7 that the hitting probability of J by S_n homogeneizes quickly as time goes by, so we may hope that the perturbating effect of visits to $\mathcal{K}$ is smoothed by the (typically long) time interval separating a visit to $\mathcal{K}$ and a visit to J. Note that making such heuristic statements precise requires special care in this context, since we are dealing all along with asymptotically small probabilities of various degrees of smallness. We think that the above heuristic argument is of some value, since it provides an intuitive explanation in terms of Markov chains for the concentration of the (random) collision probability $Coll(\nu_{n,q})$ around its mean value when n is large, a result about random walks in random environments for which we have no immediate explanation. We stress the fact that the significance for the model of the hitting time probabilities of the particle system with two or four ants relies heavily upon the representation induced by Theorem 1. On the other hand, it should be noted that the two key ingredients in the proof (Propositions 5 and 7) are expected to be rather generic properties of diffusive Markov chains, so that our strategy of proof might be applied to other situations. In particular, in the case of a random walk in a general space-time random i.i.d. environment (not necessarily Beta-distributed), an analog of Proposition 7 holds. Unfortunately, the corresponding Markov chain $(S_{4,n})$ is in general non-reversible, and its invariant measure is unknown, so proving an analog of Proposition 5 may require extra work.

6.3 Proof of Theorem 4

The proof is analogous to the one in [3] in the perturbative setting. It relies upon the central limit theorem for martingales, Theorem 3 above giving the required estimates on the asymptotic variance.

Let us introduce the filtration $(\mathcal{F}_n)_{n\geq 0}$, defined by

$$\mathcal{F}_n = \sigma\left(q(x,i),\ x\in\mathbb{Z},\ 0\leq i\leq n\right).$$

(We set $\mathcal{F}_{-1} = \{\emptyset, \Omega\}$.) We shall prove that the sequence (ξ_n), defined for all $n\geq 0$ by

$$\xi_n = Av(\nu_{n+1,q}) - Av(\nu_{n,q})$$

is a martingale-difference with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$. Note that, by the definition of $\nu_{n,q}$,

$$\xi_n = E_{P_q}(X_{n+1}^{(1)}) - E_{P_q}(X_n^{(1)}) = E_{P_q}(X_{n+1}^{(1)} - X_n^{(1)}). \tag{9}$$

Thanks to the Markov property of $X_n^{(1)}$ conditional on q:

$$\xi_n = \sum_{a=-n}^{n} P_q(X_n^{(1)} = a) w_{n,q}(a), \tag{10}$$

where $w_{n,q}(a)$ denotes the average bias of the walk at point a and at time n, conditional on q. More precisely:

$$w_{n,q}(a) = E_{P_q}\left(X_{n+1}^{(1)} - X_n^{(1)} \,\middle|\, S_n = a\right) = 2q(a,n) - 1.$$

We observe that, for all $n \geq 0$, ξ_n is mesurable with respect to $\mathcal{F}_n$ since every $P_q(X_n^{(1)} = a)$ and every $w_{n,q}(a)$s can be expressed as a polynomial in $(q(x,i)\ , 0 \leq i \leq n)$. Moreover, noting that every $P_q(X_n^{(1)} = a)$ may in fact be expressed in terms of $(q(x,i)\ , 0 \leq i \leq n-1)$ only, and that every $w_{n,q}(a)$ is a random variable independent from $(q(x,i)\ , 0 \leq i \leq n-1)$, and centered since $E_Q(q(x,i)) = 1/2$ for all x and i, we deduce that

$$E_Q\left(\xi_n | \mathcal{F}_{n-1}\right) = 0,\ Q - \text{a.s.}$$

As a consequence, $(\xi_n)_{n\geq 0}$ is a martingale-difference with respect to $(\mathcal{F}_n)_{n\geq 0}$.

According to the central limit theorem for martingales in [24] (theorem 4, p. 543), we only have to check the following two conditions:

1. (Lindeberg condition) Under Q, as n goes to infinity, for all $\epsilon > 0$,

$$n^{-1/2} \sum_{k=0}^{n} E_Q\left(\xi_k^2 \mathbf{1}_{\{|\xi_k| \geq \epsilon n^{1/4}\}} | \mathcal{F}_{k-1}\right) \xrightarrow{prob.} 0.$$

2. Under Q, as n goes to infinity,

$$n^{-1/2} \sum_{k=0}^{n} E_Q\left(\xi_k^2 | \mathcal{F}_{k-1}\right) \xrightarrow{prob.} \frac{1}{\sqrt{\pi}} \Delta.$$

The sequence $(|\xi_k|)_{k\geq 0}$ being uniformly bounded by 1, as equation (9) shows, due to the fact that $|X_{k+1}^{(1)} - X_k^{(1)}| \leq 1$ for all k, the Lindeberg condition is trivially met.

It only remains to check the second condition. According to Equation (10),

$$\xi_n^2 = \sum_{a_1,a_2=-n}^{n} P_q(X_n^{(1)} = a_1) P_q(X_n^{(1)} = a_2) w_{n,q}(a_1) w_{n,q}(a_2).$$

Taking the conditional expectation (under Q) of the above expression with respect to $\mathcal{F}_{n-1}$, all the terms such that $a_1 \neq a_2$ cancel out, since $w_{n,q}(a_1)$ and $w_{n,q}(a_2)$ are independent, independent from $\mathcal{F}_{n-1}$, and centered, whereas every $P_q(X_n^{(1)} = a_i)$ is mesurable with respect to $\mathcal{F}_{n-1}$. As a consequence, using again the fact that every $w_{n,q}(a)$ is independent from $\mathcal{F}_{n-1}$, and the fact that $E_Q(w_{n,q}(a)^2) = \frac{\Delta}{2+\Delta}$, we deduce that

$$E_Q\left(\xi_k^2 | \mathcal{F}_{k-1}\right) = \sum_{a=-n}^{n} P_q(X_n^{(1)} = a)^2 E_Q(w_{n,q}(a)^2) = \left(\frac{\Delta}{2+\Delta}\right) Coll(\nu_{n,q}).$$

From the proof of Theorem 3, $n^{1/2} Coll(\nu_{n,q})$ converges to $b(\Delta)$ not only in probability under Q, but also in $\mathcal{L}^2(Q)$. We can thus apply the following easy lemma with $x_k = k^{1/2} Coll(\nu_{k,q})$ and $l = b(\Delta)$.

Lemma 2 *Let $(\mathcal{E}, |\cdot|_{\mathcal{E}})$ be a normed linear space, and let $(x_k)_{k\geq 0}$ be a sequence of elements of $\mathcal{E}$. Assume that*

$$\lim_{k\to+\infty} x_k = l.$$

Then

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n} \frac{x_k}{\sqrt{k}} = 2l.$$

We deduce from the lemma that, as n goes to infinity,

$$n^{-1/2} \sum_{k=0}^{n} Coll(\nu_{k,q})$$

converges to $2b(\Delta)$ in $\mathcal{L}^2(Q)$, whence in probability under Q. The result follows. □

7 Acknowledgements

We gratefully acknowledge Didier Piau and Jean-Baptiste Gouéré for useful comments and suggestions.

References

[1] M. Beekman, D.J. Sumpter, F.L. Ratnieks. Phase transition between disordered and ordered foraging in Pharaoh's ants. *Proc. Natl. Acad. Sci.* 98(17):9703–9706, 2001.

[2] J. Bérard. Almost sure Central Limit Theorem for one-dimensional nearest-neighbor random walk in space-time random environment. *Journal of Applied Probability.* 41:1–10, 2004.

[3] M.S. Bernabei. Anomalous behaviour for the random corrections to the cumulants of random walks in fluctuating random media. *Probab. Theory Related Fields.* 119(3):410–432, 2001.

[4] M.S. Bernabei, C. Boldrighini, R.A. Minlos and A. Pellegrinotti. Almost-sure Central Limit Theorem for a Model of Random Walk in Fluctuating Random Environment. *Markov Processes Relat. Fields.* 4(3):381–393, 1998.

[5] C. Boldrighini and A. Pellegrinotti. $T^{-1/4}$-noise for random walks in dynamic environment on $\mathbb{Z}$. *Moscow Mathematical Journal.* 1(3):365–380, 2001.

[6] E. Bonabeau, G. Théraulaz. Auto-organisation et comportements collectifs : la modélisation des sociétés d'insectes. In G. Théraulaz et F. Spitz coord. *Auto-organisation et comportement.* Hermès 1997. (In French)

[7] J.G. Conlon, R. Song. Gaussian limit theorems for diffusion processes and an application. *Stochastic Process. Appl.* 81(1):103–128, 1999.

[8] T. Coulhon, A. Grigoryan. Random walks on graphs with regular volume growth. *Geom. Funct. Anal.* 8(4):656–701, 1998.

[9] T. Coulhon, L. Saloff-Coste. Isopérimétrie pour les groupes et les variétés. *Rev. Mat. Iberoamericana.* 9(2):293–314, 1993.

[10] B. Davis. Reinforced random walk. *Probab. Theory Related Fields.* 84(2):203–229, 1990.

[11] T. Delmotte. Parabolic Harnack inequality and estimates of Markov chains on graphs. *Rev. Mat. Iberoamericana.* 15(1):181–232, 1999.

[12] A. Dembo, O. Zeitouni. Large deviations techniques and applications. Second Edition. Springer-Verlag, 1998.

[13] P. Diaconis. Recent progress on de Finetti's notions of exchangeability. In J.M. Bernardo, M.H. DeGroot, D.V. Lindley and A.F.M. Smith, editors, *Bayesian Statistics.* Vol. 3, 111-125. Oxford University Press. 1988.

[14] P. Diaconis, D. Stroock. Geometric bounds for eigenvalues of Markov chains. *Ann. Appl. Probab.* 1(1):36–61, 1991.

[15] W. Feller. An Introduction to Probability Theory and Its Applications. Vol. 2. Wiley, 1966.

[16] W. Feller. An Introduction to Probability Theory and Its Applications. Vol. 1. Third Edition. Wiley, 1968.

[17] P. Flajolet, A. Odlyzko. Singularity analysis of generating functions. *SIAM J. Discrete Math.* 3 (2):216–240, 1990.

[18] B. Hölldobler, E.O. Wilson. The Ants. Belknap, Cambridge, USA, 1990.

[19] M. Keane, S. Rolles. Edge-reinforced random walk on finite graphs. In P. Clement, F. den Hollander, J. van Neerven, and B. de Pagter, editors, *Infinite dimensional stochastic analysis*, 217–234. K. Ned. Akad. Wet. Verh. Afd. Natuurkd. 1. Reeks. K., 2000.

[20] H. Othmer and A. Stevens. Aggregation, blowup and collapse: the ABC's of taxis and reinforced random walk. *SIAM J. Appl. Math.* 57:1044–1081. 1997.

[21] R. Pemantle. Phase transition in reinforced random walk and RWRE on trees. *Ann. Probab.* 16(3):1229–1241, 1988.

[22] R. Pemantle. Random Processes with reinforcement. *Preprint.* 2001.

[23] F. Schweitzer, K. Lao, F. Family. Active Random Walkers Simulate Trunk Trail Formation by Ants. *BioSystems,* 41:153–166, 1997.

[24] A. Shiryaev. Probability. Springer-Verlag, 1996.

[25] P. Tarrès. VRRW on $\mathbb{Z}$ eventually gets stuck at a set of five points. *Preprint.* 2002.

[26] J. Watmough, L. Edelstein-Keshet. Modelling the Formation of trail networks by foraging ants. *J. Theor. Biol.* 176:357–371, 1995.

8 Appendix

In this part, we use standard methods from complex analysis [17] to deduce estimates on the coefficients from knowldege of the generating functions. The main tools are Cauchy's formula, and integration over Hankel contours. Specifically, for all $\alpha > 0$, $0 < \theta < \pi/2$, and $K > \alpha$, we define the Hankel contour Γ as the concatenation of four pieces

$$\Gamma(\alpha, \theta, K) \stackrel{def}{=} \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$$

where

$$\begin{cases} \Gamma_1 = \{z \ : \ |z-1| = \alpha, \ |\mathrm{Arg}(z-1)| \geq \theta\}, \\ \Gamma_2 = \{z \ : \ |z-1| \geq \alpha, \ |z| \leq K, \ \mathrm{Arg}(z-1) = \theta\}, \\ \Gamma_3 = \{z \ : \ |z-1| = K, \ |\mathrm{Arg}(z-1)| \geq \theta\}, \\ \Gamma_4 = \{z \ : \ |z-1| \geq \alpha, \ |z| \leq K, \ \mathrm{Arg}(z-1) = -\theta\}. \end{cases}$$

As will become apparent in the proofs, these Hankel contours allow us to "turn around" singularities of generating functions located at $z = 1$.

8.1 Proof of Proposition 6

First note that H_0 extends to a holomorphic function on $\mathbb{C} \setminus [1, +\infty[$. Indeed, there is a unique holomorphic square root function on $\mathbb{C} \setminus]-\infty, 0]$ extending the usual square root function, and it is easy to check that, on $\mathbb{C} \setminus [1, +\infty[$, the equation

$$(u-1)(1-z) + (2-u)\sqrt{1-z} = 0$$

admits no solution.

In order to estimate $\ell_2^k(0, A)$ from the explicit formula for H_0, we use the function defined on $\mathbb{C} \setminus [1, +\infty[$ by

$$M(z) = \frac{1}{(2-u)\sqrt{1-z}} = \sum_{k=0}^{+\infty} m_k z^k,$$

whose coefficients m_k are well-known, and we estimate the difference $\ell_2^k(0, A) - m_k$ thanks to a contour argument involving $H_0 - M$. To this end, we define, for all

$z \in \mathbb{C} \setminus [1, +\infty[$, we set:

$$\phi(z) = H_0(z) - M(z).$$

The function ϕ being holomorphic on $\mathbb{C} \setminus [1, +\infty[$, Cauchy's formula entails that, for all $\alpha > 0$, $0 < \theta < \pi/2$, and $K > \alpha$,

$$\ell_2^k(0, A) - m_k = \frac{1}{2i\pi} \int_{\Gamma(\alpha,\theta,K)} \frac{\phi(z)}{z^{k+1}} dz.$$

Reducing H_0 et M to the same denominator, we obtain that

$$\phi(z) = H_0(z) - M(z) = \frac{-(u-1)(1-z)}{\left((2-u)\sqrt{1-z}\right)\left((2-u)\sqrt{1-z} + (u-1)(1-z)\right)}.$$

We first check that $\phi(z)$ has bounded modulus on every compact set included in $\mathbb{C} \setminus [1, +\infty[$. Indeed, for $z \in \mathbb{C} \setminus [1, +\infty[$ close to 1,

$$\phi(z) = \frac{-(u-1)(1-z)}{(1+o(1))(2-u)^2(1-z)} \sim -\frac{(u-1)}{(2-u)^2}. \tag{11}$$

On the other hand, the expression

$$(u-1)(1-z) + (2-u)\sqrt{1-z}$$

is holomorphic on $\mathbb{C} \setminus [1, +\infty[$, and does not vanish on this set, as we have already noted. As a consequence, $\phi(z)$ has bounded modulus on every compact set included in $\mathbb{C} \setminus [1, +\infty[$. By continuity, according to (11), we may extend ϕ to $\mathbb{C} \setminus]1, +\infty[$ by continuity, putting $\phi(1) = -\frac{(u-1)}{(2-u)^2}$. Thus extended, ϕ has bounded modulus on every compact set included in $\mathbb{C} \setminus]1, +\infty[$. We note that the contour $\Gamma(\alpha, \theta, K)$ is included in the compact set

$$\{1\} \cup \{z \in \mathbb{C} \setminus \{1\} \ : \ |z| \leq K, \ |\mathrm{Arg}(z-1)| \geq \theta\}.$$

As a consequence, for fixed θ and K, ϕ has bounded modulus on $\Gamma(\alpha, \theta, K)$ uniformly with respect to α. Now, we fix $K > 1$ and $0 < \theta < \pi/2$, and set $\alpha = 1/k$. If $z \in \Gamma_1$, $|z| \geq 1 - 1/k$, so that with our choice $|z|^{-k} = O(1)$. The function ϕ having bounded modulus on Γ uniformly over α, it is easy to check that

$$\int_{\Gamma_1} \left| \frac{H_0(z) - M(z)}{z^{k+1}} \right| dz = O(\text{perimeter of } \Gamma_1) = O(1/k).$$

Let us now study

$$\left|\int_{\Gamma_2}\frac{\phi(z)}{z^{k+1}}dz\right|.$$

By definition,

$$\int_{\Gamma_2}\frac{\phi(z)}{z^{k+1}}dz=\int_{\alpha}^{K}\frac{\phi(1+te^{i\theta})}{(1+te^{i\theta})^{k+1}}dt.$$

Changing variables in the following way: $v=t/\alpha$, we obtain that:

$$\int_{\Gamma_2}\frac{\phi(z)}{z^{k+1}}dz=\alpha\int_{1}^{K/\alpha}\frac{\phi(1+\alpha ve^{i\theta})}{(1+\alpha ve^{i\theta})^{k+1}}dv.$$

Since, for all $v>0$,

$$|1+\alpha ve^{i\theta}|\geq\sqrt{1+2\alpha v\cos\theta},$$

and, for all $0\leq x\leq 2K$, $\log(1+x)\geq\gamma(K)x$, where $\gamma(K)>0$, we deduce from the fact that ϕ has bounded modulus on Γ uniformly with respect to α that

$$\left|\int_{\Gamma_2}\frac{\phi(z)}{z^{k+1}}dz\right|=O\left(\alpha\int_{1}^{K/\alpha}\exp\left[-\gamma(K)(2\alpha v\cos\theta)((k+1)/2)\right]dv\right).$$

Using the inequality

$$\int_{1}^{K/\alpha}\exp\left[-\gamma(K)(2\alpha v\cos\theta)((k+1)/2)\right]dv\leq\frac{1}{\gamma(K)\alpha\cos\theta(k+1)}\exp\left[-\gamma(K)\alpha\cos\theta(k+1)\right],$$

we finally obtain, replacing α by its chosen value $1/k$, that

$$\left|\int_{\Gamma_2}\frac{\phi(z)}{z^{k+1}}dz\right|=O(1/k).$$

The integral over Γ_4 can be handled similarly. As regards the integral over Γ_3, we bound from above its modulus by noting that ϕ has bounded modulus on Γ_3 uniformly with respect to α, and that every $z\in\Gamma_3$ satisfies $|z|=K$, whence

$$\left|\int_{\Gamma_3}\frac{\phi(z)}{z^{k+1}}dz\right|=O(K^{-k}),$$

where $K>1$. Finally, as k goes to infinity,

$$|\ell_2^k(0,A)-m_k|=O(1/k).$$

It is well-known that $m_k\sim\frac{1}{\sqrt{\pi}}(2-u)^{-1}k^{-1/2}$, so the proposition is established. □

8.2 Proof of Proposition 7

We first need a few definitions. Let us fix y et s in Ξ_2 as in the statement of the proposition, and set

$$\kappa = \delta(s) - \delta(y).$$

From our hypotheses, we have $\kappa \in \{-4, 2, 0, 2, 4\}$. Now, for all $z \in \mathbb{C} \setminus [1, +\infty[$, define

$$\zeta(z) = \frac{1 - \sqrt{1-z}}{\sqrt{z}},$$
$$\psi_2(z) = (u-1)(1-z) + (2-u)\sqrt{1-z},$$
$$\psi(z) = \frac{1}{\psi_2(z)}\sqrt{1-z} \times (\zeta(z)^\kappa - 1).$$

Now define, for all $k \geq 0$:

$$v_k = \ell_2^k(s, A) - \ell_2^k(y, A).$$

According to Proposition 8, the generating function

$$V(z) = \sum_{k \geq 0} v_k z^k$$

of the sequence $(v_k)_{k \geq 0}$ is given, for all $|z| < 1$, by

$$V(z) = \frac{1}{(u-1)(1-z) + (2-u)\sqrt{1-z}} \times \left[\zeta(z)^{\delta(s)} - \zeta(z)^{\delta(y)}\right].$$

Rewriting this expression with our notations:

$$V(z) = \zeta(z)^{\delta(y)} \times \psi(z).$$

As in the proof of Proposition 6 above, we note that V extends to a holomorphic function over $\mathbb{C} \setminus [1, +\infty[$. In order to estimate $|v_k|$ from the formula giving V, we use the same kind of contour arguments in the proof of Proposition 6 above. According to Cauchy's formula, for all $\alpha > 0$, $0 < \theta < \pi/2$, and $K > \alpha$:

$$v_k = \frac{1}{2i\pi} \int_{\Gamma(\alpha,\theta,K)} \frac{V(z)}{z^{k+1}} dz.$$

First, we check that $\psi(z)$ has bounded modulus on $\mathbb{C} \backslash (\{1\} \cup \{z \ : \ |\mathrm{Arg}(z-1)| < \theta\})$. Indeed, for $z \in \mathbb{C} \setminus [1, +\infty[$ close to 1, we have

$$\psi(z) = \frac{1}{(1+o(1))(2-u)\sqrt{1-z}}((1+o(1))\kappa\sqrt{1-z}),$$

so ψ has bounded modulus on a set $\mathcal{D}_1 = \{z \ : \ |z-1| \leq \gamma_1, \ z \notin [1,+\infty[\}$, where $\gamma_1 > 0$. On the other hand, it is easy to check that

$$\zeta(z)^\kappa - 1$$

has bounded modulus on $\mathbb{C} \setminus (\mathcal{D}_1 \cup [1,+\infty[)$.

Now, $\psi_2(z)$ is holomorphic on $\mathbb{C} \setminus [1,+\infty[$, and does not vanish on this set, as we have already noted. As a consequence, $1/\psi_2(z)$ has bounded modulus on every compact set of the form

$$\mathcal{D}_2 = \{z \ : \ \gamma_1 \leq |z-1| \leq \gamma_2, \ |\mathrm{Arg}(z-1)| \geq \theta\}.$$

At last, as $|z| \to +\infty$,

$$|\psi_2(z)| = |(u-1)(1-z) + (2-u)\sqrt{1-z}| \sim (u-1)|z|,$$

and $1/\psi_2(z)$ is bounded on the set $\mathcal{D}_3 = \{z \in \mathbb{C} \setminus [1,+\infty[: \ |z| \geq \gamma_2\}$ provided that γ_2 is large enough. We see that

$$\mathbb{C} \setminus (\{1\} \cup \{z \ : \ |\mathrm{Arg}(z-1)| < \theta\}) \subset \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3,$$

so we deduce from the above considerations that, for all $z \in \mathbb{C} \setminus (\{1\} \cup \{z \ : \ |\mathrm{Arg}(z-1)| < \theta\})$, the following inequality holds:

$$|\psi(z)| \leq \beta_5(\Delta, \kappa, \theta).$$

Turning back to V, we deduce that, for every $z \in \Gamma$,

$$|V(z)| \leq \beta_5(\Delta, \kappa, \theta) \, |\zeta(z)|^{\delta(y)},$$

whatever the chosen values for α and K.

Put

$$\alpha = \epsilon \frac{\delta(y)^2}{k^2},$$

where ϵ is a parameter at our disposal. Note that, if $\delta(y) \geq 2k+1$, the probability $\ell_2^k(y, A) = t^k(\delta(y), 0)$ must be equal to zero since the Markov chain $(T_n)_{n \geq 0}$ takes steps of length at most 2, so that it cannot reach 0 from $\delta(y)$ using k steps or less. Similarly, since we assume that $\delta(s) \geq \delta(y) - 4$, the probability $\ell_2^k(s, A) = t^k(\delta(s), 0)$ must be equal to zero when $\delta(y) \geq 2k + 5$. In this last case, we thus have $v_k = 0$, and the estimate we want to prove for $|v_k|$ holds trivially. In the sequel, we assume that $\delta(y) \leq 2k + 4$ (remember that $\delta(y)$ is an integer). As a consequence, the following inequality holds:

$$\alpha \leq 36\epsilon$$

Les us first bound the integral of $|V(z)|$ on Γ_1. Using the fact that, for all $z \in \Gamma_1$,

$$\frac{-\pi + \theta}{2} \leq \mathrm{Arg}(\sqrt{1-z}) \leq \frac{\pi - \theta}{2},$$

we check that, for every small enough α (depending on θ), every $z \in \Gamma_1$ satisfies

$$|1 - \sqrt{1-z}| \leq 1 - (1/2)\sin(\theta/2)\sqrt{|1-z|}.$$

On the other hand, every $z \in \Gamma_1$ satisfies $|z| \geq 1 - \alpha$. Putting $\gamma_4 = 1/2\sin(\theta/2)$, we see that, for every small enough α (depending on θ), and every $z \in \Gamma_1$,

$$\left|\frac{V(z)}{z^{k+1}}\right| \leq \beta_5(\Delta, \kappa, \theta)(1 - \gamma_4\sqrt{\alpha})^{\delta(y)}(1-\alpha)^{-(k+1)-\delta(y)/2}.$$

A bound on the integral follows:

$$\left|\int_{\Gamma_1} \frac{V(z)}{z^{k+1}} dz\right| \leq \beta_5(\Delta, \kappa, \theta) 2\pi\alpha (1 - \gamma_4\sqrt{\alpha})^{\delta(y)}(1-\alpha)^{-(k+1-\delta(y)/2)}.$$

For small enough ϵ, $\log(1-\alpha) \geq -2\alpha$, and, on the other hand, we always have $1 - \gamma_4\sqrt{\alpha} \leq \exp(-\gamma_4\sqrt{\alpha})$. As a consequence, for small enough ϵ (depending on θ):

$$\left|\int_{\Gamma_1} \frac{V(z)}{z^{k+1}} dz\right| \leq \beta_5(\Delta, \kappa, \theta) 2\pi\alpha \exp\left[-\delta(y)\sqrt{\alpha}\gamma_4 + (2(k+1) + \delta(y))\alpha\right].$$

Replacing α by its expression, and taking into account the inequality $\delta(y) \leq 2k + 4$, the term in the above exponential may be rewritten as:

$$-\delta(y)\gamma_4\sqrt{\epsilon}\frac{\delta(y)}{k} + (4k+6)\epsilon\frac{\delta(y)^2}{k^2}.$$

By choosing ϵ small enough (depending on θ), the first term in the above sum dominates, so that

$$\left|\int_{\Gamma_1} \frac{V(z)}{z^{k+1}} dz\right| \leq \beta_5(\Delta, \kappa, \theta) 2\pi\alpha \exp\left[-(1/2)\gamma_4\sqrt{\epsilon}\frac{\delta(y)^2}{k}\right].$$

Writing $\alpha = \epsilon\frac{\delta(y)^2}{k} \times \frac{1}{k}$, and using the fact that the function $s \mapsto se^{-s}$ is bounded on $\mathbb{R}_+$, we deduce from the above inequality that:

$$\left|\int_{\Gamma_1} \frac{V(z)}{z^{k+1}} dz\right| \leq \beta_6(\Delta, \kappa, \theta, \epsilon)\frac{1}{k} \exp\left[-\beta_7(\Delta, \kappa, \theta, \epsilon)\frac{\delta(y)^2}{k}\right].$$

Let us now study

$$\left|\int_{\Gamma_2} \frac{V(z)}{z^{k+1}} dz\right|.$$

By definition,

$$\int_{\Gamma_2} \frac{V(z)}{z^{k+1}} dz = \int_{\alpha}^{K} \frac{V(1+te^{i\theta})}{(1+te^{i\theta})^{k+1}} dt$$

Changing variables in the following way: $w = t/\alpha$, we obtain that:

$$\int_{\Gamma_2} \frac{V(z)}{z^{k+1}} dz = \alpha \int_{1}^{K/\alpha} \frac{V(1+\alpha we^{i\theta})}{(1+\alpha we^{i\theta})^{k+1}} dw.$$

We now show that, for all $0 < \theta < \pi/2$, all $\alpha > 0$ and all $w > 0$:

$$\left|\frac{1-\sqrt{-\alpha we^{i\theta}}}{\sqrt{1+\alpha we^{i\theta}}}\right| \leq 1. \tag{12}$$

Indeed, putting $z = \sqrt{-\alpha we^{i\theta}}$, the following inequality holds, since $\Re(z) > 0$,

$$\frac{|1-z|}{|1+z|} \leq 1.$$

Taking square roots, we deduce that

$$\sqrt{\frac{|1-z|}{|1+z|}} = \frac{|\sqrt{1-z}|}{|\sqrt{1+z}|} \leq 1$$

Multiplying numerator and denominator by $|\sqrt{1-z}|$, this leads to

$$\frac{|1-z|}{|\sqrt{1-z^2}|} \leq 1,$$

which is exactly Inequality 12. Turning back to V, we deduce from previous estimates that, for all $0 < \theta < \pi/2$, and all $\alpha, w > 0$:

$$|V(1+\alpha we^{i\theta})| \leq \beta_5(\Delta, \kappa, \theta).$$

As a consequence,

$$\left|\int_{\Gamma_2} \frac{V(z)}{z^{k+1}} dz\right| \leq \alpha\beta_5(\Delta, \kappa, \theta) \int_{1}^{K/\alpha} \frac{1}{|1+\alpha we^{i\theta}|^{k+1}} dw.$$

Noting that, for all $w > 0$,

$$|1+\alpha w e^{i\theta}| \geq \sqrt{1+2\alpha w\cos\theta},$$

and that, for all $0 \leq x \leq 2K$, $\log(1+x) \geq \beta_8(K)x$, with $\beta_8(K) > 0$, we deduce that:

$$\left|\int_{\Gamma_2} \frac{V(z)}{z^{k+1}}dz\right| \leq \alpha\beta_5(\Delta,\kappa,\theta)\int_1^{K/\alpha} \exp\left[-\beta_8(K)(2\alpha w\cos\theta)((k+1)/2)\right]dw$$

Using the following inequality:

$$\int_1^{K/\alpha} \exp\left[-\beta_8(K)(2\alpha w\cos\theta)((k+1)/2)\right]dw \leq \frac{1}{\beta_8(K)\alpha\cos\theta(k+1)}\exp\left[-\beta_8(K)\alpha\cos\theta(k+1)\right],$$

we obtain that

$$\left|\int_{\Gamma_2} \frac{V(z)}{z^{k+1}}dz\right| \leq \beta_9(\Delta,\kappa,\theta,K)\frac{1}{k}\exp\left(-\beta_{10}(\Delta,\kappa,\theta,\epsilon,K)\frac{\delta(y)^2}{k}\right).$$

Noting that, for all $z \in \mathbb{C}\setminus[1,+\infty[$, $|V(z)| = |V(\bar{z})|$, and that

$$\Gamma_4 = \{\bar{z} \ : \ z \in \Gamma_2\},$$

we see that the inequality we have just proved for the integral over Γ_2 applies to the integral over Γ_4.

To finish the proof, we study

$$\left|\int_{\Gamma_3} \frac{V(z)}{z^{k+1}}dz\right|.$$

It is easily checked that, when $|z|$ is large enough,

$$|\zeta(z)| \leq 1+\frac{2}{\sqrt{|z|}} \leq \exp\left(2/|z|^{1/2}\right).$$

As a consequence, when K is large enough, we have, for all $z \in \Gamma_3$,

$$\left|\frac{V(z)}{z^{k+1}}\right| \leq 2\pi K\beta_5(\Delta,\kappa,\theta)\exp\left[\delta(y)\left(2K^{-1/2}\right)-(k+1)\log K\right].$$

As a consequence, to prove the following inequality:

$$\left|\int_{\Gamma_3} \frac{V(z)}{z^{k+1}}dz\right| \leq \frac{1}{k}\exp\left(-\frac{\delta(y)^2}{k}\right),$$

we only have to find K such that

$$\delta(y)\left(2K^{-1/2}\right) - k\log K \leq -\frac{\delta(y)^2}{k} - \log k.$$

Taking into account the fact that $\delta(y) \leq 2k+4$, we see that every large enough K (uniformly with respect to k and $\delta(y)$) enjoys the above property.

Putting together the estimates of integrals over Γ_i, $i = 1, \ldots, 4$, the proposition follows. $\square$