

Quelques résultats mathématiques et numériques pour une équation d'onde monodimensionnelle soumise à une condition limite unilatérale

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- 1 Mathematical formulation
- 2 Existence and uniqueness results
- 3 Numerical approximation
- 4 Time integration methods
- 5 Outlook

1 Mathematical formulation

2 Existence and uniqueness results

3 Numerical approximation

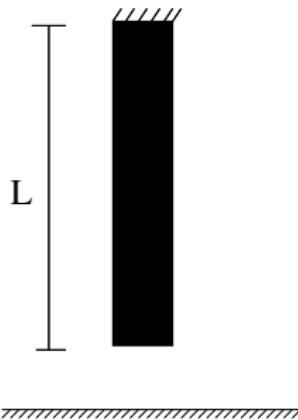
4 Time integration methods

5 Outlook

Mathematical formulation

State variable and applied field

- $u : (0, L) \times (0, T) \rightarrow \mathbb{R}$: displacement
- $f(x, t)$: density forces



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- $f(x, t)$: density forces

Dynamic problem (DP)

$$u_{tt}(x, t) - u_{xx}(x, t) = f(x, t) \quad \text{Wave equ.}$$

$$0 \leq u(0, t) \perp u_x(0, t) \leq 0 \quad \text{Signorini cond.}$$

$$u(L, t) = 0 \quad \text{Dirichlet cond.}$$

$$u(x, 0) = u^0(x) \quad \text{and} \quad u_t(x, 0) = v^0(x) \quad \text{Initial cond.}$$

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- $f(x, t)$: density forces

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$$0 \leq u(0, t) \perp u_x(0, t) \leq 0 \quad \text{Signorini cond.}$$

$$u(L, t) = 0 \quad \text{Dirichlet cond.}$$

$$\begin{aligned} u(x, 0) &= \underbrace{u^0(x)}_{\in H^1(0, L)} \quad \text{and} \quad u_t(x, 0) = \underbrace{v^0(x)}_{\in L^2(0, L)} \\ &\quad \text{Initial cond.} \end{aligned}$$

Mathematical formulation

Notation : $K := \{u \in H^1((0, L) \times (0, T)) : u(L, \cdot) = 0 \text{ and } u(0, \cdot) \geq 0\}$

Weak formulation (WF)

Find $u \in K$ such that for all $v \in K$ and for almost every $t \in [0, T]$

$$\begin{aligned} & \int_0^L (u_t(v - u))|_0^T dx - \int_0^T \int_0^L (u_t(v_t - u_t)) dx dt \\ & + \int_0^T \int_0^L (u_x(v_x - u_x)) dx dt \geq \int_0^T \int_0^L f(v - u) dx dt \end{aligned}$$

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Existence and uniqueness results

Previous works :

- Existence and uniqueness results [Schatzman, J. Differential Equations, '80].
- Existence and uniqueness results [Lebeau, Schatzman, J. Differential Equations, '84].
- Existence and uniqueness results [Dabaghi et al, J. M2AN '13].
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Theorem

Let $u^0(x) \in H^1(0, L)$ and $v^0(x) \in L^2(0, L)$, then (WF) admits a unique solution

Existence and uniqueness results

- Characteristics Method

Notation : $\xi := x + t$ and $\eta := x - t$.

Let

$$u(x, t) := p(x + t) + q(x - t).$$

We consider

Characteristics Method(CM)

$$0 \leq p(t) + q(-t) \perp p'(t) + q'(-t) \leq 0, \quad \text{Signorini cond.}$$

$$p(x) + q(x) = u_0(x) \quad \text{and} \quad p'(x) - q'(x) = v_0(x) \quad \text{Initial cond.}$$

$$\text{For all } t \in [0, T] : \quad p(L + t) + q(L - t) = 0 \quad \text{Dirichlet cond.}$$

Existence and uniqueness results

Notations :

- $\Psi(t) := -p(t) - q(-t)$

Where $p(t) = \frac{u_0(t)}{2} + \frac{1}{2} \int_0^t v_0(t) dt$ for all $t \in (0, L)$.

- Multivalued function $J_N : \mathbb{R} \longrightarrow \mathcal{P}(\bar{\mathbb{R}}) \setminus \emptyset$:

$$J_N(x) := \begin{cases} \{0\} & \text{if } x < 0, \\ [0, +\infty) & \text{if } x = 0, \\ \emptyset & \text{if } x > 0 \end{cases}$$

Then (CM) is equivalent to (CP)

Cauchy problem (CP)

$$\Psi'(t) \in -J_N(\Psi(t)) - 2p'(t) \quad \text{a.e.} \quad t \in (0, L)$$

$$\Psi(0) = -u_0(0)$$

Existence and uniqueness results

Assumption : $u_0 \in K$ and $v_0 \in L^2(0, L)$

Proposition 1

The unique solution to problem (CP) is a weak solution (WF)

Proposition 2

The weak formulation (WF), is a solution to Cauchy problem (CP).

➡ CP is equivalent to WF

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Numerical approximation

Notations :

- $V_h := \{v^h \in C^0([0, L]) : v^h|_{[a_i, a_{i+1}]} \in P_1, i = 0, \dots, n-1, v^h(L) = 0\}$
- $\varphi_i(x) := \begin{cases} 1 - \frac{|x-a_i|}{h} & \text{if } x \in [a_{\max(i-1, 0)}, a_{i+1}] \\ 0 & \text{otherwise} \end{cases}$

We approximate u by $u_h(x, t) := \sum_{j=0}^{n-1} u_j(t) \varphi_j(x)$

Approximated Problem (AP)

Find $u_h : [0, T] \rightarrow V_h, \lambda : [0, T] \rightarrow \mathbb{R}$ such that for all $v_h \in V_h$

$$\int_0^L u_{h,tt} v_h \, dx + \int_0^L u_{h,x} v_{h,x} \, dx = \int_0^L f v_h \, dx - \lambda v_h(0)$$

$$0 \leq u_h(0, \cdot) \perp \lambda \leq 0 \quad \text{a.e.} \quad t \in [0, T]$$

$$u_h(., 0) = u_h^0 \quad \text{and} \quad u_{h,t}(., 0) = v_h^0$$

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Notations :

- $U_h := (u_0, \dots, u_{n-1})^\top$
- $e_0 := (1, 0, \dots, 0)^\top$
- $M_{ij} := \int_0^L \varphi_i \varphi_j \, dx, \quad i, j = 0, \dots, n-1$
- $S_{ij} := \int_0^L \varphi'_i \varphi'_j \, dx, \quad i, j = 0, \dots, n-1$
- $F_i := \int_\Omega f \varphi_i \, dx$
- U_h^0, V_h^0 : initial displacement & velocity

Algebraic formulation (AF)

Find $U_h : [0, T] \rightarrow \mathbb{R}^n$ and $\lambda : [0, T] \rightarrow \mathbb{R}$ such that

$$MU_{h,tt} + SU_h = F - \lambda e_0 \quad \text{a.e.} \quad t \in [0, T]$$

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→ (AF) is ill-posed problem^{1, 2}

1. Paoli, R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci., '01
2. Khenous, Laborde, Renard, Eur. J. Mech. A Solids, '08

Numerical approximation

Mass redistribution method (MOD)³

Notations :

- $M_{ij}^{\text{mod}} := \int_h^L \varphi_i \varphi_j \, dx \quad i, j = 0, \dots, n - 1$
- $\bar{M}_{ij} := M_{i+1,j+1}, \quad i, j = 1, \dots, n - 2$
- Modified mass matrix : $M^{\text{mod}} := \begin{pmatrix} 0 & 0 \\ 0 & \bar{M} \end{pmatrix}$

Semi-discrete approximation with mass redistribution method (MA)

Find $U : [0, T] \rightarrow \mathbb{R}^n$ and $\lambda : [0, T] \rightarrow \mathbb{R}$ such that

$$M^{\text{mod}} U_{tt} + S U = F + \Lambda \quad \Lambda := (-\lambda, 0)^\top$$

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Discrete Potential energy relation associated to (MA)

$$\mathcal{E}_h(t) = \frac{1}{2} (U_{h,t}^\top M^{\text{mod}} U_{h,t} + U_h^\top S U_h) - U_{h,t}^\top F$$

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⇒ (MA) is well-posed

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⇒ (MA) is well-posed

⇒ The solution of (MA) is energy conservative

Find $U : [0, T] \longrightarrow \mathbb{R}^n$ and $\lambda : [0, T] \longrightarrow \mathbb{R}$ such that

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⇒ (MA) is well-posed

⇒ The solution of (MA) is energy conservative

Theorem⁴

The solution of (MA) converges to the unique solution to (WF)

Numerical approximation

Notation : $a(u, v) = \int_0^L u'v' dx$

Variational formulation associated to MA(VM)

Find $u^h : [0, T] \rightarrow V^h$ s.t. for all $v^h \in K^h$

$$\int_0^T \left(\int_h^L \ddot{u}^h(v^h - u^h) dx + a(u^h, v^h - u^h) \right) dt \geq 0,$$

$$u^h(x, 0) = u^{h0} \quad \text{and} \quad \dot{u}^h(x, 0) = v^{h0}$$

Assumption : $\lim_{h \rightarrow 0} (\|u^h(x, 0) - u_0\|_{H^1(0, L)} + \|\dot{u}^h(x, 0) - v_0\|_{L^2(0, L)}) = 0$

Theorem (Convergence)

The solution of approximation problem (VM) converge to the unique solution of (WF)

Idea of proof : Energy estimate implies

$$\left. \begin{aligned} u^h &\rightharpoonup u & \text{in } L^\infty(0, T; V) & \text{weak *} \\ \dot{u}^h &\rightharpoonup \dot{u} & \text{in } L^\infty(0, T; H) & \text{weak *} \end{aligned} \right\} \implies u^h \rightarrow u \quad \text{in } C^{0,\alpha}(Q_T) \quad \text{for all } \alpha < \frac{1}{2}$$

Numerical approximation

Notations :

- $K^h := K \cap V^h$
- $\Phi(t) := \begin{cases} 1 & \text{if } t \in [0, T - \frac{\epsilon}{2}] \\ 0 & \text{if } t \in [T - \frac{\epsilon}{4}, T] \end{cases}$
- Q^h the projection onto V^h

We observe that the test function

$$v^h := u^h(t) + Q^h(v^\eta(t) - u(t)) \in K^h \quad \text{for all } t \in [0, T],$$

where

$$v^\eta(x, t) := \begin{cases} u(x, t) + \frac{1}{\eta} \int_t^{t+\eta} (v - u)(x, s) ds + k(\eta)(L - x)\Phi(t) & \text{if } t \leq T - \eta, \\ u(x, t) & \text{if } t \geq T - \eta. \end{cases}$$

$$\text{and } k(\eta) := \frac{5}{3L} C \|u\|_{\mathbb{H}_\infty} \sqrt{\eta} \quad \text{with} \quad \mathbb{H}_\infty := \{u \in L^\infty(0, T; V) : \dot{u} \in L^\infty(0, T; H)\}$$

Schatzman, Bercovier, Math. Comp. 1989.

Numerical approximation

The test function v^h is substituted in (VM), we obtain

$$\begin{aligned} & - \int_0^L \dot{u}^h(0) Q^h(v^\eta(0) - u(0)) dx + \int_0^h \dot{u}^h(0) Q^h(v^\eta(0) - u(0)) dx \\ & - \int_0^L \int_0^T \dot{u}^h(t) Q^h(\dot{v}^\eta(t) - \dot{u}(t)) dt dx + \int_0^h \int_0^T \dot{u}^h(t) Q^h(\dot{v}^\eta(t) - \dot{u}(t)) dt dx \\ & + \int_0^T a(u^h(t), Q^h(v^\eta(t) - u(t))) dt \geq 0. \end{aligned}$$

Therefore we may pass to the limit when h tends to 0, we get

$$\begin{aligned} \dot{u}^h(0) &\rightarrow v_0, \\ Q^h(\dot{v}^\eta(t) - \dot{u}(t)) &\rightarrow \dot{v}^\eta(t) - \dot{u}(t), \\ Q^h(v^\eta(t) - u(t)) &\rightarrow v^\eta(t) - u(t) \end{aligned}$$

Then we pass to the limit with respect to η and we get (WF)

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Time integration methods

Newmark method

$$\begin{aligned}U_h^{n+1} &= U_h^n + \Delta t U_{h,t}^n + \left(\frac{1}{2} - \beta\right) \Delta t^2 U_{h,tt}^n + \beta \Delta t^2 U_{h,tt}^{n+1} \\U_{h,t}^{n+1} &= U_{h,t}^n + (1 - \gamma) \Delta t U_{h,tt}^n + \gamma \Delta t U_{h,tt}^{n+1}\end{aligned}$$

(NP) : Application of Newmark method to (AF) $\gamma \geq \frac{1}{2}$, $\beta \geq \frac{1}{4}(\frac{1}{2} + \gamma)^2$

Find $U_h^{n+1} : [0, T] \rightarrow \mathbb{R}^n$ and $\lambda^{n+1} : [0, T] \rightarrow \mathbb{R}$ such that

$$\left(\frac{M}{\beta \Delta t^2} + S\right) U_h^{n+1} = \frac{M}{\beta \Delta t^2} (U_h^n + \Delta t U_{h,t}^n) + \frac{1 - 2\beta}{2\beta} M U_{h,tt}^n + F - \lambda^{n+1} e_0$$

$$0 \leq u_0^{n+1} \perp \lambda^{n+1} \leq 0$$

$$U_h(0) = U_h^0 \quad \text{and} \quad U_{h,t}(0) = V_h^0$$

Time integration methods

Newmark method

$$U_h^{n+1} = U_h^n + \Delta t U_{h,t}^n + \left(\frac{1}{2} - \beta \right) \Delta t^2 U_{h,tt}^n + \beta \Delta t^2 U_{h,tt}^{n+1}$$
$$U_{h,t}^{n+1} = U_{h,t}^n + (1 - \gamma) \Delta t U_{h,tt}^n + \gamma \Delta t U_{h,tt}^{n+1}$$

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$$U_h(0) = U_h^0 \quad \text{and} \quad U_{h,t}(0) = V_h^0$$

Time integration methods : Newmark method

Notations :

$$\bullet \quad V_h^n := U_h^{n+1} - U_h^n \quad \bullet \quad \Lambda^n := \lambda^{n+1} - \lambda^n \quad \bullet \quad \Lambda_\gamma^n := (1-\gamma)\lambda^n + \gamma\lambda^{n+1}$$

Energy evolution associated to (NP)

$$\begin{aligned}\Delta \mathcal{E}_h^n = & (1-2\gamma)(V_h^n)^\top S V_h^n + \Delta t \left(\beta - \frac{\gamma}{2} \right) (V_{h,t}^n)^\top S V_h^n \\ & + \Delta t \left(\beta - \frac{\gamma}{2} \right) (V_{h,t}^n)^\top \Lambda^n e_0 - (V_h^n)^\top \Lambda_\gamma^n e_0\end{aligned}$$

$$\bullet \quad (\beta, \gamma) = \left(\frac{1}{4}, \frac{1}{2} \right) \implies \begin{cases} \text{Crank-Nicholson} \\ \Delta \mathcal{E}_h^n = -(V_h^n)^\top \Lambda_{\frac{1}{2}}^n e_0 \end{cases}$$

$$\bullet \quad (\beta, \gamma) = \left(\frac{1}{2}, 1 \right) \implies \begin{cases} \text{Energy dissipative} \\ \Delta \mathcal{E}_h^n = -(V_h^n)^\top S V_h^n - (V_h^n)^\top \Lambda_1^n e_0 \end{cases}$$

Time integration methods : Newmark method

Notations :

- $V_h^n := U_h^{n+1} - U_h^n$
- $\Lambda^n := \lambda^{n+1} - \lambda^n$
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- $(\beta, \gamma) = (\frac{1}{4}, \frac{1}{2}) \Rightarrow \begin{cases} \text{Crank-Nicholson} \\ \Delta \mathcal{E}_h^n = -(V_h^n)^\top \Lambda_{\frac{1}{2}}^n e_0 \end{cases}$

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Crank-Nicholson

Comparison of the analytical solution u and the approximated solutions U_h^n by using standard mass matrix (left) and modified mass matrix (right) in the contact

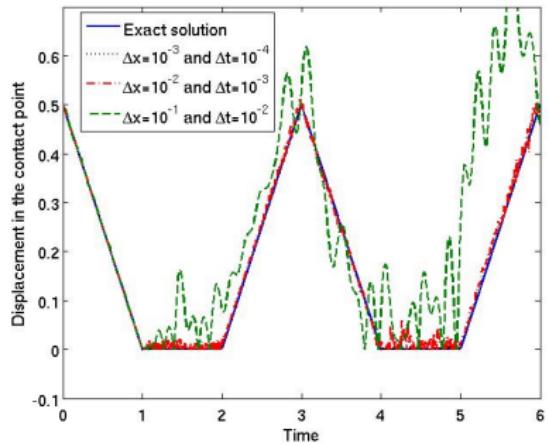


FIGURE: STD

FIGURE: MOD

Crank-Nicholson

Comparison of the convergence curves of displacement obtained by using standard and modified mass matrices

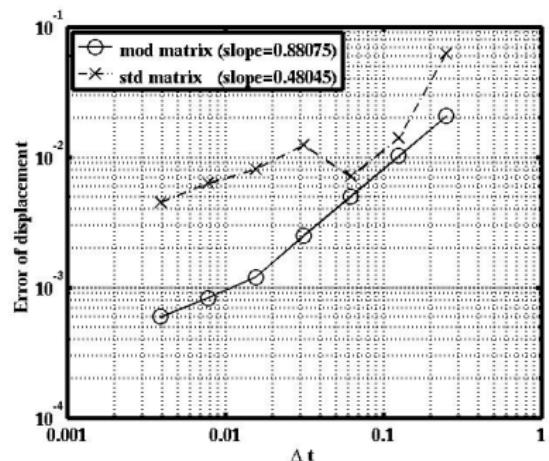


FIGURE: $\|U_h^n - u\|_{L^\infty(0, T; L^2(0, L))}$

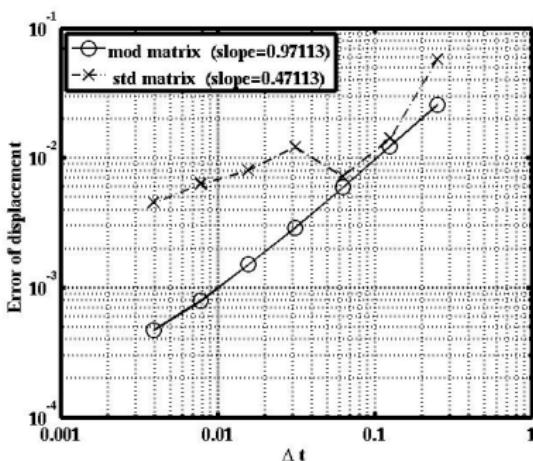


FIGURE: $\|U_h^n - u\|_{L^2(0, T; L^2(0, L))}$

Crank-Nicholson

Comparison of the convergence curves of displacement obtained by using standard and modified mass matrices

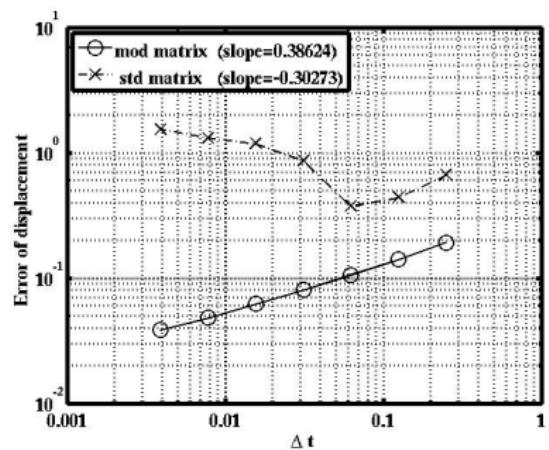


FIGURE: $\|U_h^n - u\|_{L^\infty(0, T; H^1(0, L))}$

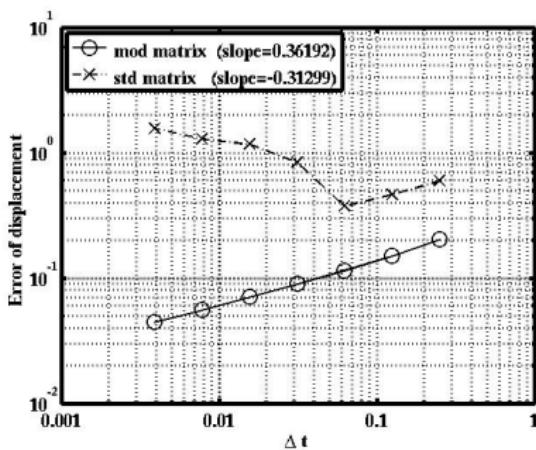


FIGURE: $\|U_h^n - u\|_{L^2(0, T; H^1(0, L))}$

Crank-Nicholson

Comparison of the analytical Lagrange multiplier λ and the approximated Lagrange multiplier λ^n by using standard mass matrix (left) and modified mass matrix (right) in the contact

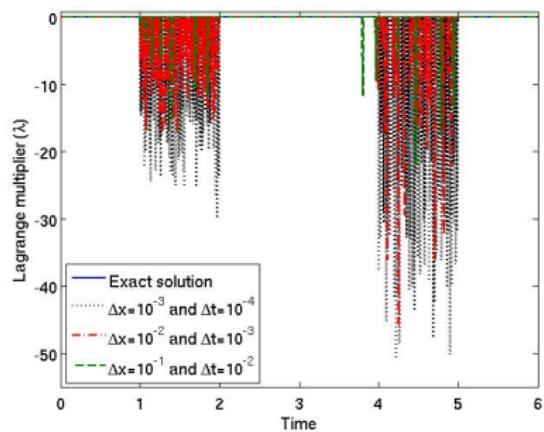


FIGURE: STD

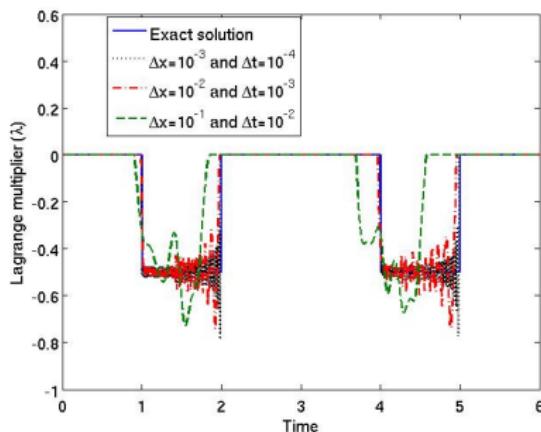


FIGURE: MOD

Comparison of the convergence curves of contact force obtained by using standard and modified mass matrices

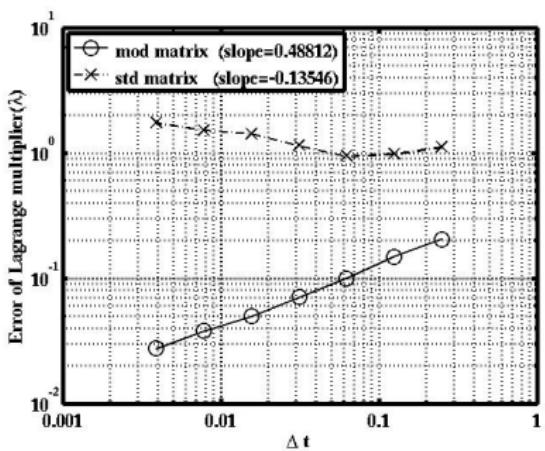


FIGURE: $\|\lambda^n - \lambda\|_{L^2(0,T)}$

Crank-Nicholson

Comparison of the energy associated to the analytical solution and the energy associated to the approximated ones for standard (left) and modified (right) mass matrices

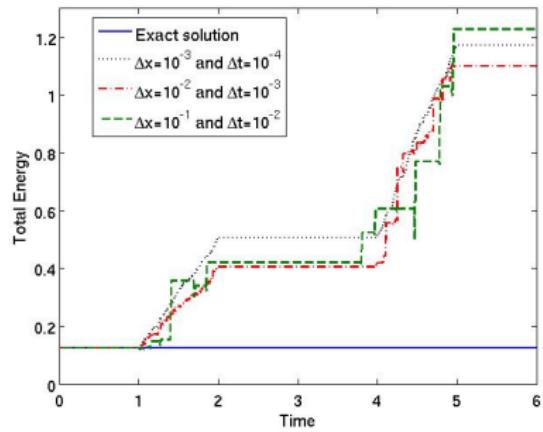


FIGURE: STD

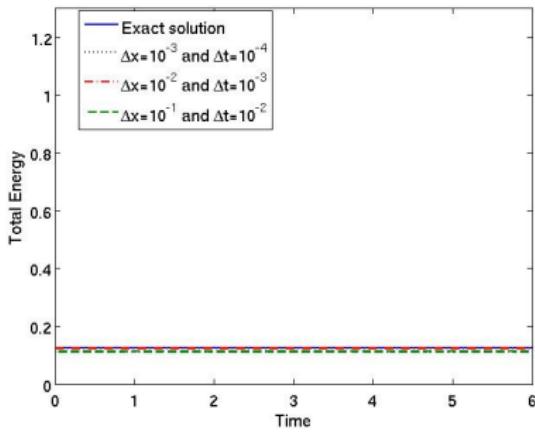


FIGURE: MOD

Crank-Nicholson

Comparison of the convergence curves of the energy obtained by using standard and modified mass matrices

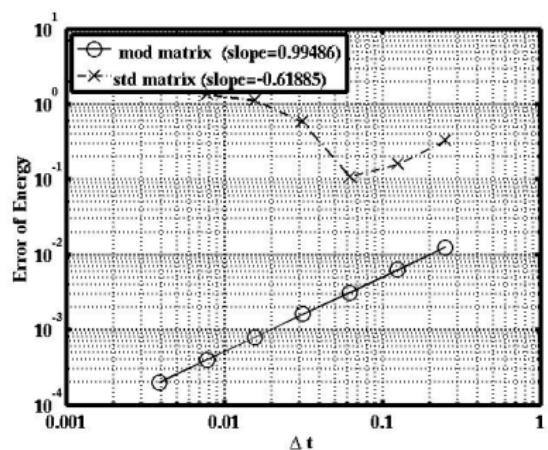


FIGURE: $\|\mathcal{E}_h - \mathcal{E}\|_{L^\infty(0,T)}$

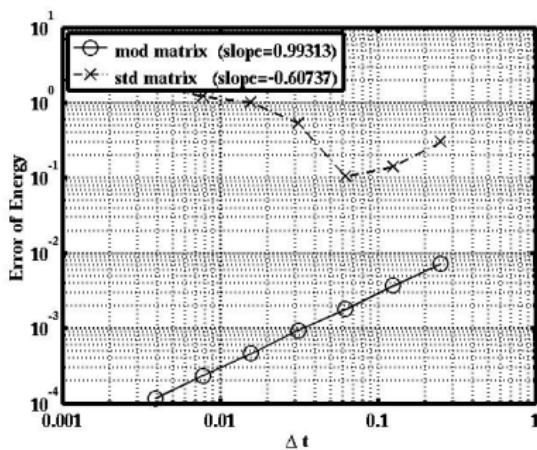


FIGURE: $\|\mathcal{E}_h - \mathcal{E}\|_{L^2(0,T)}$

Newmark method with $(\beta, \gamma) = (\frac{1}{2}, 1)$

Comparison of the analytical solution u and the approximated solutions U_h^n by using standard mass matrix (left) and modified mass matrix (right) in the contact

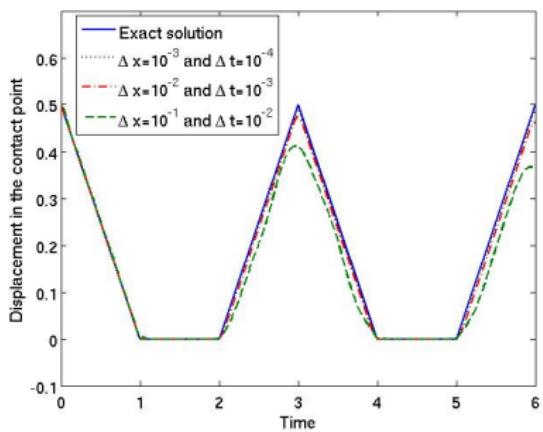


FIGURE: STD

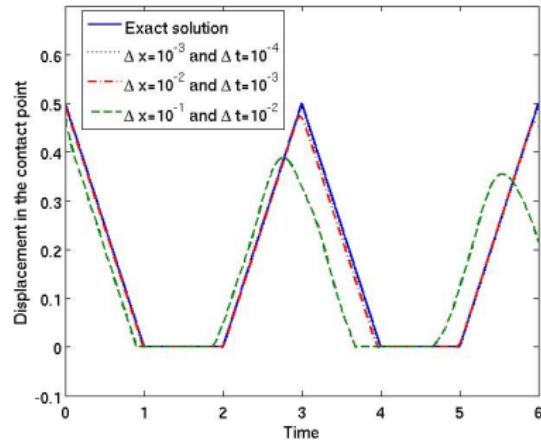


FIGURE: MOD

Newmark method with $(\beta, \gamma) = (\frac{1}{2}, 1)$

Comparison of the convergence curves of displacement obtained by using standard and modified mass matrices

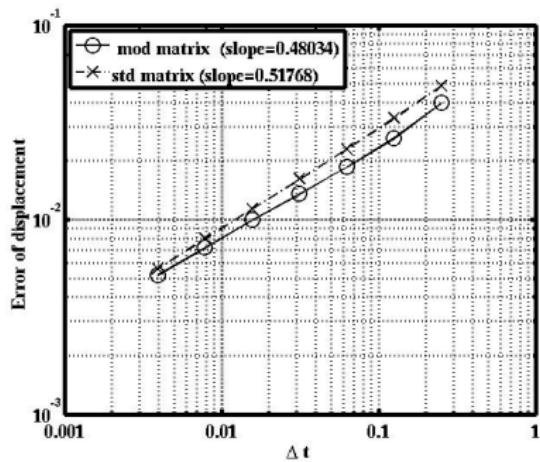
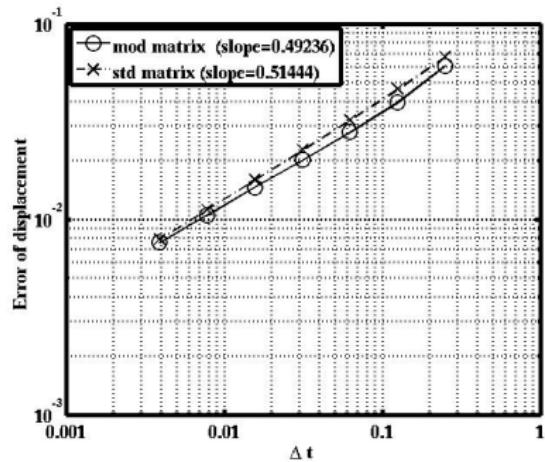


FIGURE: $\|U_h^n - u\|_{L^\infty(0, T; L^2(0, L))}$

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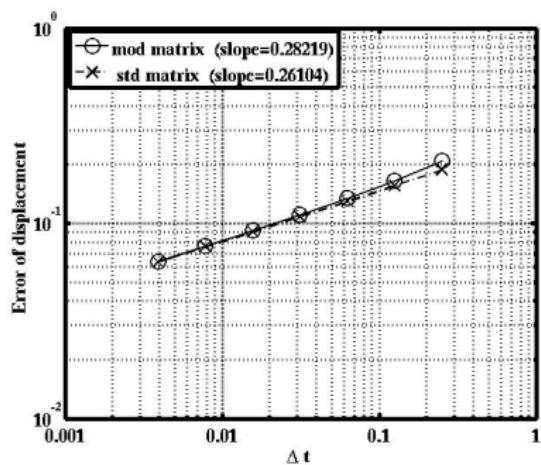


FIGURE: $\|U_h^n - u\|_{L^\infty(0, T; H^1(0, L))}$

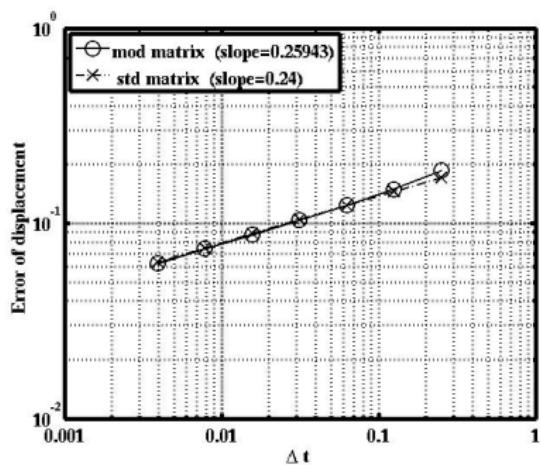


FIGURE: $\|U_h^n - u\|_{L^2(0, T; H^1(0, L))}$

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Comparison of the analytical Lagrange multiplier λ and the approximated Lagrange multiplier λ^n by using standard mass matrix (left) and modified mass matrix (right) in the contact

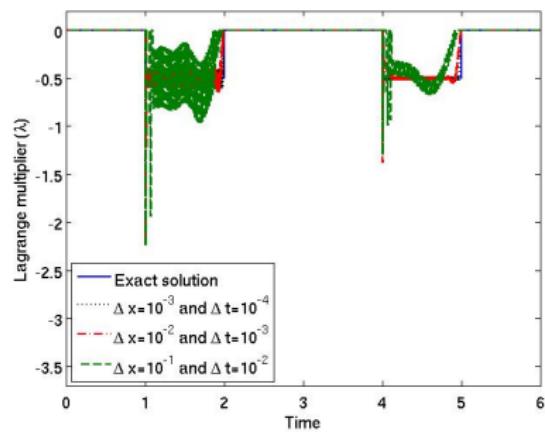


FIGURE: STD

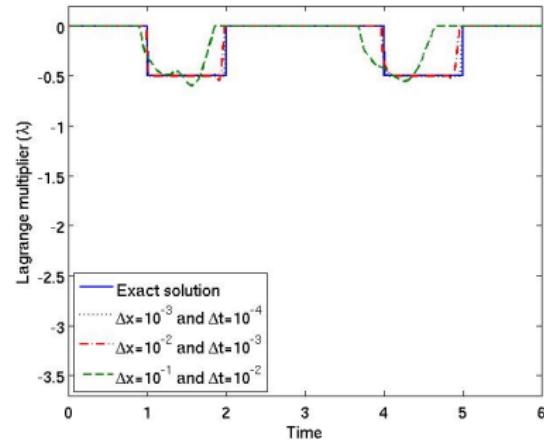


FIGURE: MOD

Newmark method with $(\beta, \gamma) = (\frac{1}{2}, 1)$

Comparison of the convergence curves of contact force obtained by using standard and modified mass matrices

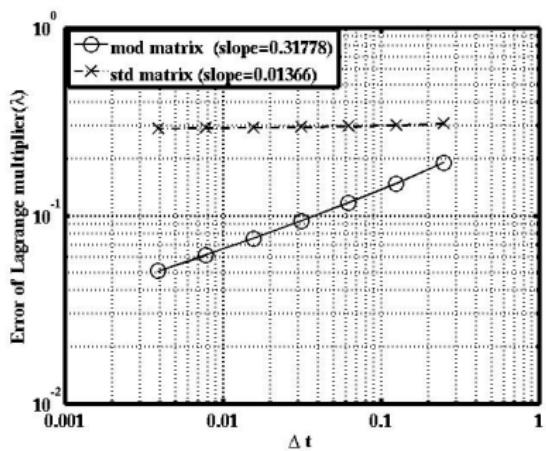


FIGURE: $\|\lambda^n - \lambda\|_{L^2(0, T)}$

Newmark method with $(\beta, \gamma) = (\frac{1}{2}, 1)$

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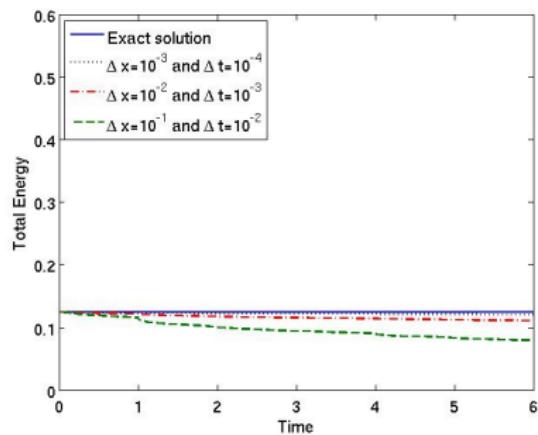


FIGURE: STD

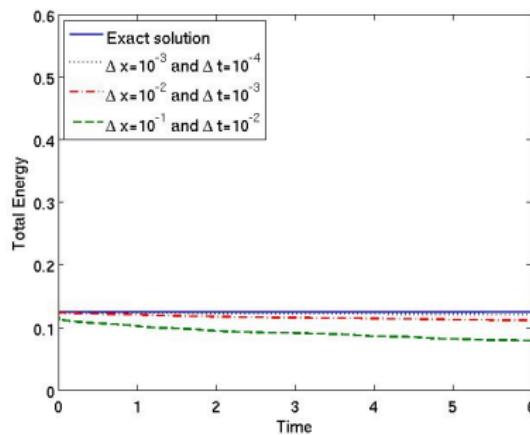


FIGURE: MOD

Newmark method with $(\beta, \gamma) = (\frac{1}{2}, 1)$

Comparison of the convergence curves of the energy obtained by using standard and modified mass matrices

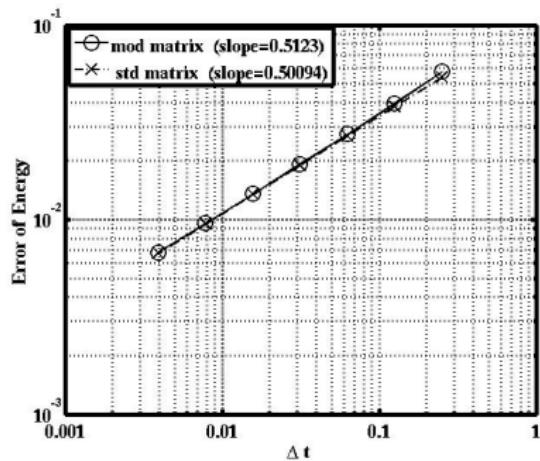


FIGURE: $\|\mathcal{E}_h - \mathcal{E}\|_{L^\infty(0,T)}$

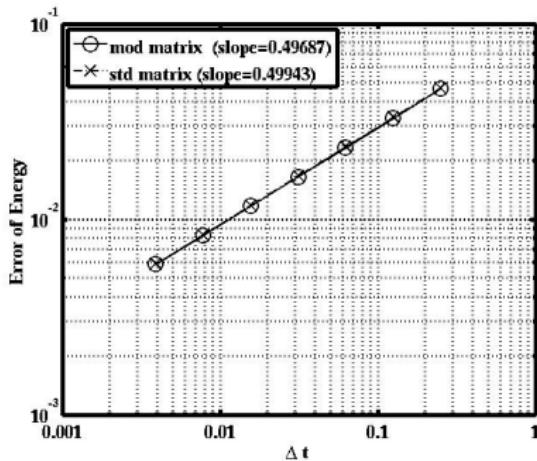


FIGURE: $\|\mathcal{E}_h - \mathcal{E}\|_{L^2(0,T)}$

Time integration methods

Paoli- Schatzman method

- Paoli- Schatzman method with $e \in [0, 1]^4$
- Newmark method with $\gamma = \frac{1}{2}$ and $\beta \in [0, \frac{1}{2}]^5$

Application of Paoli- Schatzman method to (AF)

Find $U_h^{n+1} : [0, T] \rightarrow \mathbb{R}^n$ and $\lambda^{n+1} : [0, T] \rightarrow \mathbb{R}$ such that :

$$\frac{M(U_h^{n+1} - 2U_h^n + U_h^{n-1})}{\Delta t^2} + S(\beta U_h^{n+1} + (1-2\beta)U_h^n + \beta U_h^{n-1}) = F - \lambda^{n+1} e_0$$

$$0 \leq u_0^{n,e} \perp \lambda^{n+1} \leq 0 \quad \text{with} \quad u_0^{n,e} = \frac{u_0^{n+1} + e u_0^{n-1}}{1 + e}$$

U_h^0 and V_h^0 are given

$$U_h^1 = U_h^0 + \Delta t V_h^0 + \Delta t z(\Delta t)$$

-
4. Paoli, R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci, '01
 5. Dumont et al, M2AN, '06

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-
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Time integration methods : Paoli- Schatzman method

$(e - \beta P)$: Application of Paoli- Schatzman method to (AF)

Find $U_h^{n+1} : [0, T] \rightarrow \mathbb{R}^n$ and $\lambda^{n+1} : [0, T] \rightarrow \mathbb{R}$ such that :

$$\frac{M(U_h^{n+1} - 2U_h^n + U_h^{n-1})}{\Delta t^2} + S(\beta U_h^{n+1} + (1 - 2\beta)U_h^n + \beta U_h^{n-1}) = F - \lambda^{n+1} e_0$$

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$$U_h^1 = U_h^0 + \Delta t V_h^0 + \Delta t z(\Delta t)$$

Energy evolution for $\beta = \frac{1}{4}$ $\implies \Delta \mathcal{E}_h^n = \frac{(1+e)}{2} \lambda^{n+1} u_0^{n-1}$

Paoli- Schatzman method with $(\epsilon, \beta) = (\frac{1}{2}, \frac{1}{4})$

Comparison of the analytical solution u and the approximated solutions U_h^n by using standard mass matrix (left) and modified mass matrix (right) in the contact

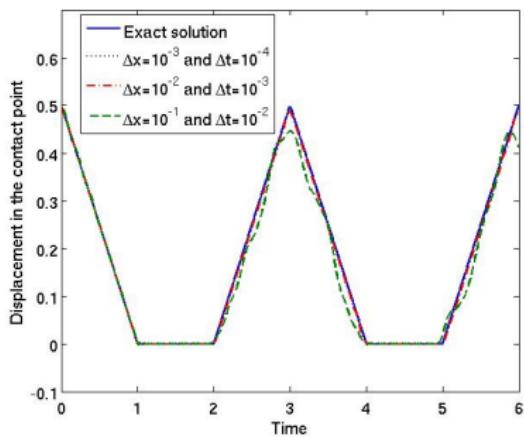


FIGURE: STD

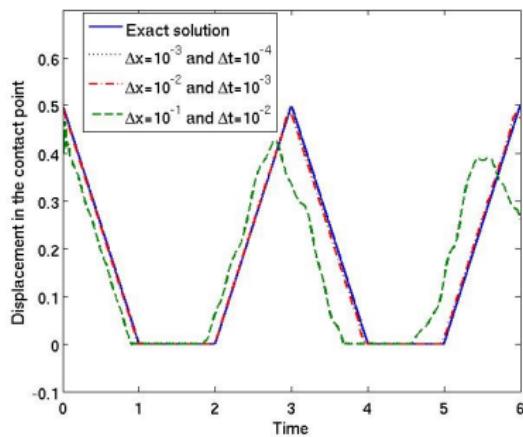


FIGURE: MOD

Paoli- Schatzman method with $(\epsilon, \beta) = (\frac{1}{2}, \frac{1}{4})$

Comparison of the convergence curves of displacement obtained by using standard and modified mass matrices

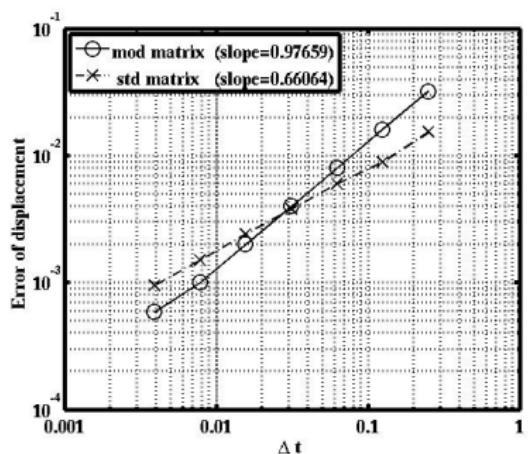


FIGURE: $\|U_h^n - u\|_{L^\infty(0, T; L^2(0, L))}$

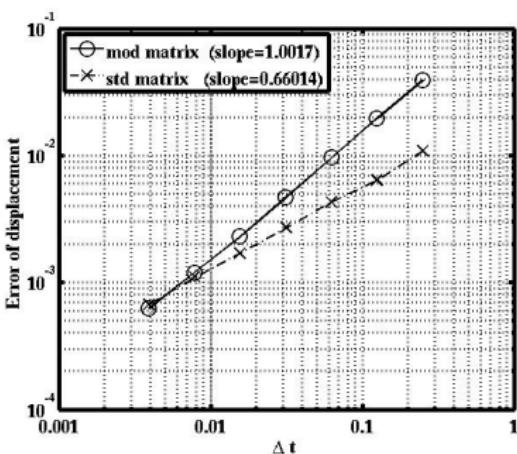


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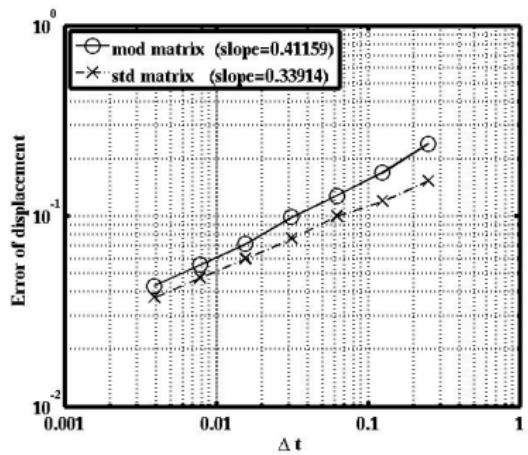


FIGURE: $\|U_h^n - u\|_{L^\infty(0, T; H^1(0, L))}$

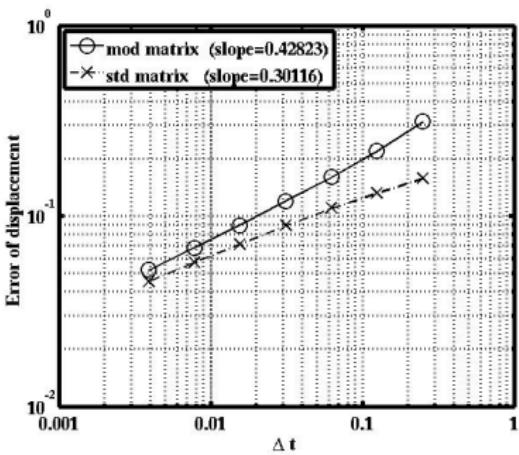


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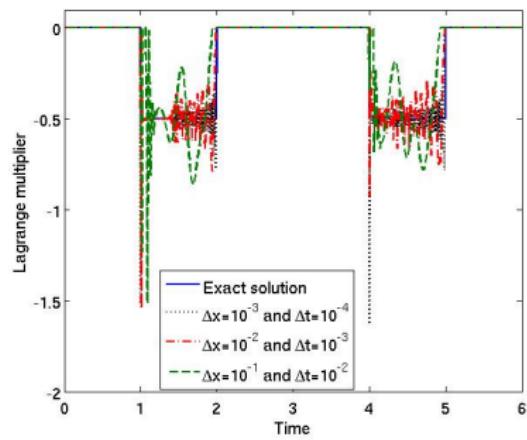


FIGURE: STD

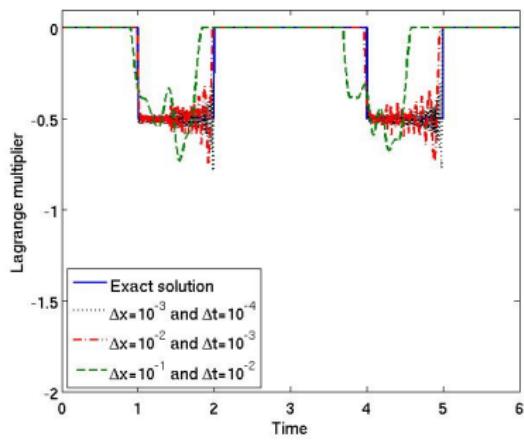


FIGURE: MOD

Paoli- Schatzman method with $(\epsilon, \beta) = (\frac{1}{2}, \frac{1}{4})$

Comparison of the convergence curves of contact force obtained by using standard and modified mass matrices

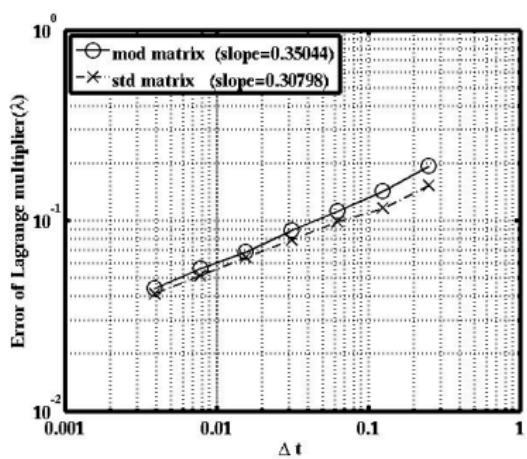


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Comparison of the energy associated to the analytical solution and the energy associated to the approximated ones for standard (left) and modified (right) mass matrices

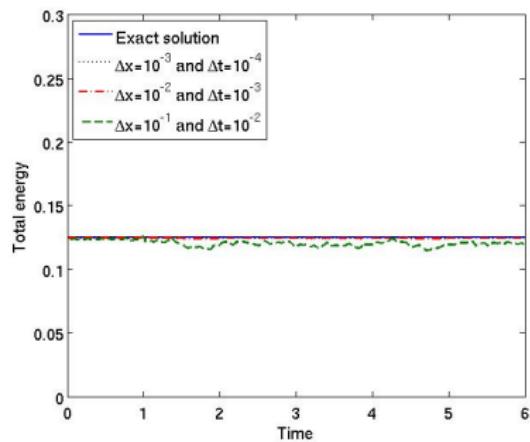


FIGURE: STD

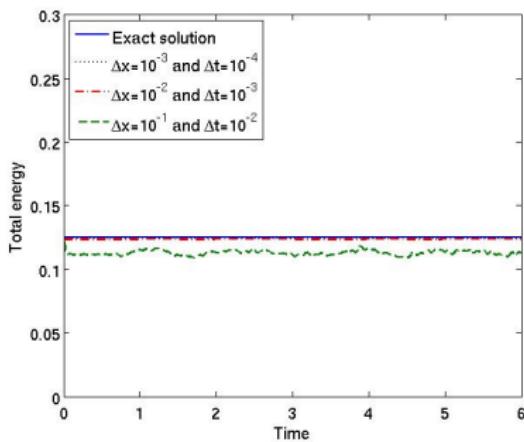


FIGURE: MOD

Paoli- Schatzman method with $(\epsilon, \beta) = (\frac{1}{2}, \frac{1}{4})$

Comparison of the convergence curves of the energy obtained by using standard and modified mass matrices

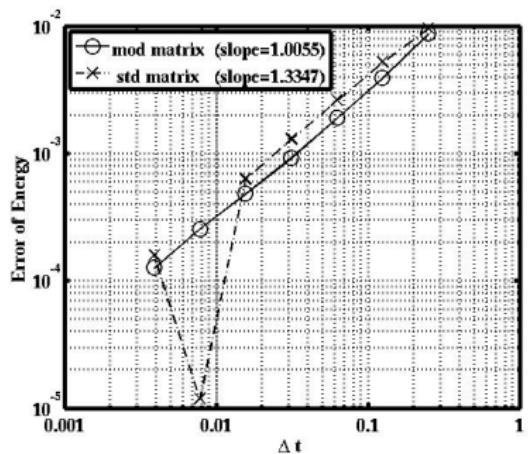


FIGURE: $\|\mathcal{E}_h - \mathcal{E}\|_{L^\infty(0,T)}$

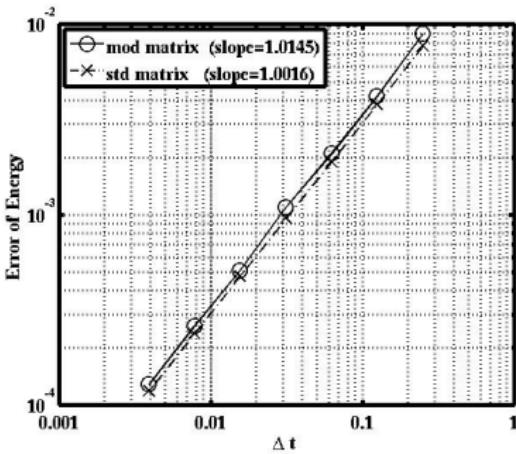


FIGURE: $\|\mathcal{E}_h - \mathcal{E}\|_{L^2(0,T)}$

Plan

- 1 Mathematical formulation
- 2 Existence and uniqueness results
- 3 Numerical approximation
- 4 Time integration methods
- 5 Outlook

- It is proved that the semi-discrete contact Problem by (MOD) is well-posed, energy conservative and its solution converges to the unique solution to (DP)
- We compared several time-integration methods by using (MOD) and (STD)
- Numerical experiments illustrated that by using (MOD)
 - The oscillations of displacement are eliminated
 - The oscillations of contact forces are reduced
 - The energies are conservative with very small oscillations
- In general, (MOD) improved the convergence rate

Future Aim : study the space semi-discretization by mass redistribution method in higher dimension

Thank you for your attention