Deux équations surcritiques d'ordre quatre

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1. Selected results on the classical Gelfand problem
2. Selected results on the fourth-order Gelfand problem
3. Moser's iteration method for the fourth-order Gelfand problem

   joint work with [M. Ghergu, O. Goubet, G. Warnault, Arch. Rational Mech. Anal., 2013]

4. Fleming's blow-down method for the fourth-order Lane-Emden problem

   joint work with [J. Dávila, K. Wang, J. Wei, see ArXiv, submitted]

The Gelfand problem

Take a parameter $\lambda \geq 0$ and B the unit ball of $\mathbb{R}^N$, $N \geq 1$.

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } B, \\ u = 0 & \text{on } \partial B. \end{cases}$$

$N = 2$ [Liouville, J. Math. Pures Appl., 1853]

SUR L'ÉQUATION AUX DIFFÉRENCES PARTIELLES

$$\frac{d^2 \log \lambda}{du\, dv} \pm \frac{\lambda}{2a^2} = 0;$$

PAR J. LIOUVILLE.

(Extrait des *Comptes rendus de l'Académie des Sciences*, tome XXXVI. — Séance du 28 février 1853.)

En m'occupant (dans une des Notes de l'*Application de l'analyse à la Géométrie*, par Monge, 5e édition, page 597) de la recherche des surfaces pour lesquelles la mesure de courbure en chaque point est constante, j'ai été conduit à l'équation aux différences partielles

$$\frac{d^2 \log \lambda}{du\, dv} \pm \frac{\lambda}{2a^2} = 0, \tag{1}$$

Liouville was interested in the construction of surfaces of constant Gaussian curvature. He proves that every real-valued solution to

$$-\Delta u = 2Ke^u$$

can be represented as

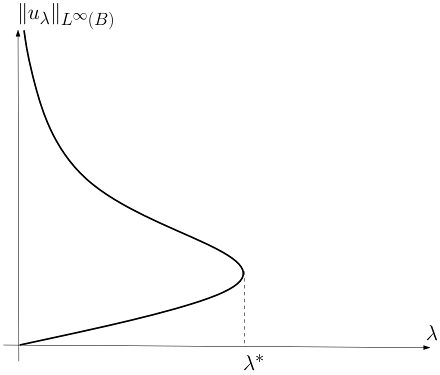
$$u = \ln \frac{|f'|^2}{(1 + (K/4)|f|^2)^2},$$

where f is, apart from simple poles, any complex analytic f'n.

In particular, if $\lambda > 2$ (resp. $\lambda = 2, \lambda < 2$), the Gelfand problem has 0 (resp. $1, 2$) solutions, explicitly given by

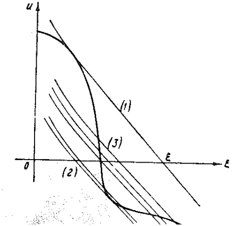
$$u_\lambda(r) = \ln \frac{8b_-}{(1+\lambda b_- r^2)^2}, \qquad U_\lambda(r) = \ln \frac{8b_+}{(1+\lambda b_+ r^2)^2}$$

where $b_\pm = \frac{4-\lambda \pm \sqrt{16-8\lambda}}{\lambda^2}$, $r \in [0,1]$.

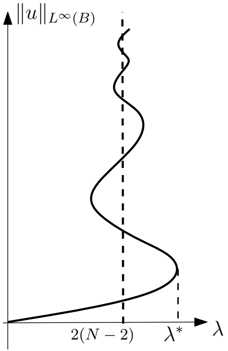


$N = 3$ [Barenblatt, AMS Transl., 1959]

Barenblatt was interested in a simplified model in combustion theory : the exp. nonlinearity is related to the Arrhenius law and models the reaction, while the Laplace operator corresponds to standard diffusion of heat when the system has reached a steady state. Barenblatt discovers that, in dimension $N = 3$, the equation has **infinitely** many solutions for $\lambda = 2$.

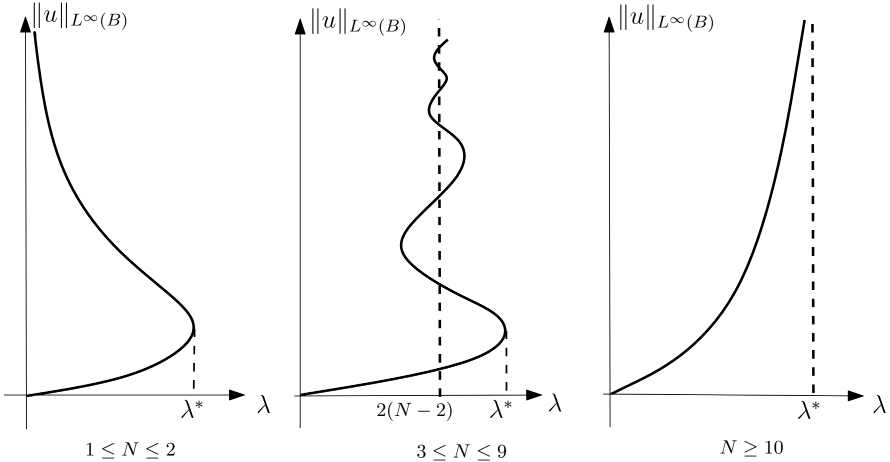


However, if $\lambda \leq \bar{\xi}_1$, then the construction of the set of stationary solutions will be more complicated than in the preceding cases. Namely, if $\lambda = \lambda_0 = \bar{\xi}$, then the solution will be correct if $\bar{\xi}_3 < \lambda < \lambda_0$, where $\bar{\xi}_3$ is the point of intersection of the third envelope with the abscissa axis, then there will be two solutions, and at $\lambda = \xi_3$ there will be three solutions, etc; finally, when $\lambda = \bar{\xi}_\infty = 1$ there will be an infinite number of solutions. The num-

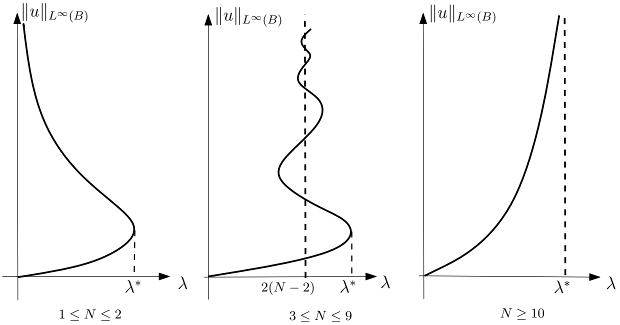


$3 \leq N \leq 9$

$N \geq 4$ [Joseph-Lundgren, Arch. Rational Mech. Anal., 1972]



[Nagasaki-Suzuki, Math. Ann., 1994]



The solutions can be classified according to their Morse index, which increases by one unit, every time we pass a turning point.

[Dancer-Farina, Proc. Amer. Math. Soc., 2009]

Theorem

Assume $3 \leq N \leq 9$. Every solution to

$$-\Delta u = e^u \quad \text{in } \mathbb{R}^N$$

has infinite Morse index.

Using blow-up analysis and bifurcation theory, they obtain

Corollary

Assume $3 \leq N \leq 9$, $\Omega \subset \mathbb{R}^N$ a smoothly bounded domain. Consider

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

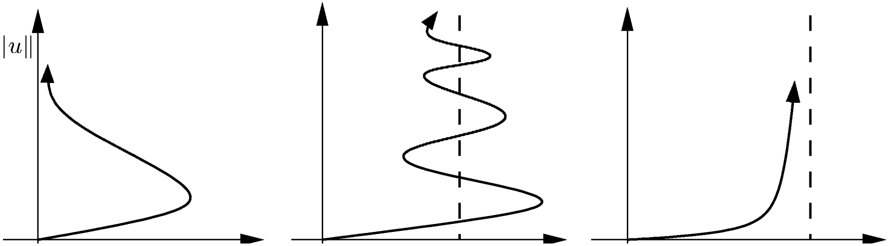
There exists an unbounded piecewise analytic curve of solutions. Solutions are nondegenerate, except at infinitely many isolated points, which are either turning points or secondary bifurcations, and for any solution u,

$$\|u\|_{L^\infty(\Omega)} \leq C(N, \Omega, \lambda, \mathit{ind}\, u).$$

The fourth-order Gelfand problem

[Dávila-Flores-Guerra, J. Differential Equations, 2009]

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{in } B, \\ u = |\nabla u| = 0 & \text{on } \partial B. \end{cases}$$



$1 \le N \le 4$ $\quad$ $5 \le N \le 12$ $\quad$ $N \ge 13$

The result remains true for Navier boundary conditions
$u = \Delta u = 0$.

- Earlier results : [Arioli-Gazzola-Grunau-Mitidieri, SIAM J. Math. Anal., 2005], [Arioli-Gazzola-Grunau, J. Differential Equations, 2006], [Davila-D-Guerra-Montenegro, SIAM J. Math. Anal., 2007].
- The proof is more involved than the second order case, since the phase-space analysis must be carried out in four dimensions.
- The Dirichlet problem on general domains seems difficult due to the failure of the comparison principle
- The Navier problem is a good toy-model for the study of systems (in particular the Lane-Emden system)
- Do the results of Dancer-Farina remain true in the biharmonic setting ?

Basic properties of the equation

The equation

$$-\Delta u = e^u \qquad \text{in } \mathbb{R}^N$$

is invariant under the scaling transformation

$$u_\lambda(x) = u(\lambda x) + 2 \ln \lambda, \quad x \in \mathbb{R}^N, \lambda > 0,$$

So, up to rescaling, there exists **a unique regular radial solution.**

For

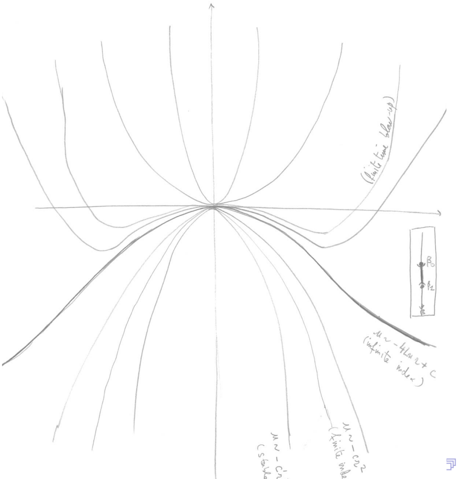
$$\Delta^2 u = e^u \qquad \text{in } \mathbb{R}^N$$

we have the same scale invariance (replace 2 by 4). In particular, up to rescaling, there exists **a one-parameter family of regular radial solutions, parametrized e.g. by** $\beta = -\Delta u(0)$**.**

The radial solutions for $5 \leq N \leq 12$

Assume $u(0) = 0$.Thanks to [Arioli-Gazzola-Grunau, J. Differential Equations, 2006], [Bercchio-Ferrero-Farina-Gazzola, J. Differential Equations, 2012] & [D-Ghergu-Goubet-Warnault, Arch. Rational Mech. Anal., 2013],

- no entire solution for $\beta < \beta_0$,
- an infinite Morse index sol. for $\beta = \beta_0$ ($u(r) \sim -4 \ln r + c$),
- finite Morse index sol. for $\beta_0 < \beta < \beta_1$ ($u(r) \sim -r^2$),
- stable sol. for $\beta \geq \beta_1$ ($u(r) \sim -r^2$)



- So, the Dancer-Farina result cannot hold in our setting
- Perhaps the only stable solutions are radially symmetric about some point ?

Theorem ([D-Ghergu-Goubet-Warnault, Arch. Rational Mech. Anal., 2013])

Assume $N \geq 5$. Take a point $x^0 = (x_1^0, \ldots, x_N^0) \in \mathbb{R}^N$, parameters $\alpha_1, \ldots, \alpha_N > 0$, and let

$$p(x) = \sum_{i=1}^{N} \alpha_i (x_i - x_i^0)^2.$$

Then, there exists a solution u such that

$$u(x) = -p(x) + C + \mathcal{O}(|x|^{4-N}) \quad \text{as } |x| \to \infty,$$

In particular, u has finite Morse index (resp. is stable, if $\min_{i=1,\ldots,N} \alpha_i$ is large enough) and u is not radial about any point if the coefficients α_i are not all equal.

All stable solutions that we have encountered so far have quadratic behavior at infinity. In particular, letting

$$v = -\Delta u \quad \text{and} \quad \overline{v}(r) = \fint_{\partial B_r} v \, d\sigma,$$

these solutions satisfy $\overline{v}(\infty) > 0$, where

$$\overline{v}(\infty) := \lim_{r \to +\infty} \overline{v}(r).$$

This motivates the following Liouville-type result.

Theorem ([D-Ghergu-Goubet-Warnault, Arch. Rational Mech. Anal., 2013])
Assume $5 \leq N \leq 12$. Let u be a solution such that $\overline{v}(\infty) = 0$. Then, u has infinite Morse index.

Regularity of stable solutions

Let $N \geq 1$ and let Ω be a smoothly bounded domain of $\mathbb{R}^N$. Consider

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

By standard arguments, one can prove that there exists a (unique) curve of smooth stable solutions for $\lambda < \lambda^* < +\infty$, which converges to a weak stable solution u^*, as $\lambda \nearrow \lambda^*$.

Is the extremal solution u^* smooth ?

Regularity of stable solutions

Theorem ([D-Ghergu-Goubet-Warnault, Arch. Rational Mech. Anal., 2013])

Let u^* be the extremal solution.

- *If $1 \leq N \leq 12$, then $u^* \in C^\infty(\overline{\Omega})$.*
- *If $N \geq 13$, then $u^* \in C^\infty(\Omega \setminus \Sigma)$, where Σ is a closed set whose Hausdorff dimension is bounded above by*

$$\mathcal{H}_{dim}(\Sigma) \leq N - 4p^*$$

and $p^* > 3$ is the largest root of the polynomial $(X - \frac{1}{2})^3 - 8(X - \frac{1}{2}) + 4$.

Solutions of bounded Morse index

Theorem ([D-Ghergu-Goubet-Warnault, Arch. Rational Mech. Anal., 2013])
Let $5 \leq N \leq 12$. Assume Ω convex. Let $u \in C^4(\overline{\Omega})$ a solution and $v = -\Delta u$. There exists a compact subdomain $\omega \subset \Omega$ such that if

$$\int_{B_r(x_0)} v\,dx \leq K r^{N-2}, \tag{1}$$

for every ball $B_r(x_0) \subset \omega$ and for some constant $K > 0$, then,

$$\|u\|_{L^\infty(\Omega)} \leq C(N, \Omega, \lambda, ind\,u, K).$$

If u is stable, then (1) holds for some constant K depending only on Ω, N, and ω. We do not know whether this remains valid for solutions of bounded Morse index. Also, how does C depend on the Morse index of u ?

Stability

The energy associated to our equation is

$$\mathcal{E}_\Omega(u) = \frac{1}{2}\int_\Omega |\Delta u|^2\,dx - \int_\Omega e^u\,dx$$

Its second variation is the quadratic form

$$Q_u(\varphi) = \int_\Omega |\Delta\varphi|^2\,dx - \int_\Omega e^u\varphi^2\,dx$$

Since we are working with Navier boundary conditions, we say that u is stable if

$$Q_u(\varphi) \geq 0 \qquad \text{for all } \varphi \in H_0^1(\Omega) \cap H^2(\Omega),$$

resp. u has finite Morse index m if m is the maximal dimension of any subspace on which Q_u remains negative.

An interpolation lemma

Lemma ([D-Ghergu-Goubet-Warnault, Arch. Rational Mech. Anal., 2013])
If u is stable, then for every $s \in (0,1]$, $\varphi \in H_0^1(\Omega) \cap H^2(\Omega)$,

$$\int_\Omega |(-\Delta)^s \varphi|^2 \, dx - \int_\Omega e^{su} \varphi^2 \, dx \geq 0$$

Proof (case $\Omega = \mathbb{R}^N$): by Plancherel, stability is

$$\int_{\mathbb{R}^N} e^u \varphi^2 \, dx \leq \int_{\mathbb{R}^N} |\Delta \varphi|^2 \, dx = (2\pi)^{-N} \int_{\mathbb{R}^N} |\xi|^4 |\mathcal{F}(\varphi)|^2 \, d\xi.$$

In other words, $\|\mathcal{F}^{-1}\|_{\mathcal{L}(X_1,Y_1)} \leq 1$, where X_s, Y_s given by

$$X_s = L^2((2\pi)^{-N} |\xi|^{4s} d\xi)\,, \quad Y_s = L^2(e^{su} dx).$$

Also, $\|\mathcal{F}^{-1}\|_{\mathcal{L}(X_0,Y_0)} = 1$. Apply complex interpolation :
$\|\mathcal{F}^{-1}\|_{\mathcal{L}(X_s,Y_s)} \leq 1$ for all $0 \leq s \leq 1$.

In particular, for $s = 1/2$, we recover the following identity previously observed by [D-Farina-Sirakov, Geometric PDEs, to appear] and [Cowan-Ghoussoub, Cal. Var. PDE, to appear]:

$$\int_\Omega |\nabla\varphi|^2\, dx = \int_\Omega |(-\Delta)^{\frac{1}{2}}\varphi|^2\, dx \geq \int_\Omega e^{\frac{u}{2}}\varphi^2\, dx$$

Now, write the equation as a system :

$$\begin{cases} -\Delta u = v & \text{in } \Omega, \\ -\Delta v = e^u & \text{in } \Omega, \\ \quad u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

Test the equation

$$\begin{cases} -\Delta u = v & \text{in } \Omega, \\ -\Delta v = e^u & \text{in } \Omega, \\ \quad u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

Fix $\alpha > \frac{1}{2}$ and multiply the first equation by $e^{\alpha u} - 1$.

$$\int_\Omega (e^{\alpha u} - 1)\, v\, dx = \alpha \int_\Omega e^{\alpha u} |\nabla u|^2\, dx = \frac{4}{\alpha} \int_\Omega \left| \nabla \left(e^{\frac{\alpha u}{2}} - 1 \right) \right|^2 dx.$$

Apply stability

$$\int_\Omega e^{\frac{u}{2}} \varphi^2 \, dx \leq \int_\Omega |\nabla \varphi|^2 \, dx$$

So,

$$\int_\Omega e^{\frac{u}{2}} \left(e^{\frac{\alpha u}{2}} - 1 \right)^2 \, dx \leq \int_\Omega \left| \nabla \left(e^{\frac{\alpha u}{2}} - 1 \right) \right|^2 \, dx.$$

eq.+stability

Combining

$$\int_\Omega (e^{\alpha u} - 1)\, v\, dx = \frac{4}{\alpha} \int_\Omega \left| \nabla \left(e^{\frac{\alpha u}{2}} - 1 \right) \right|^2 dx.$$

and

$$\int_\Omega e^{\frac{u}{2}} \left(e^{\frac{\alpha u}{2}} - 1 \right)^2 dx \le \int_\Omega \left| \nabla \left(e^{\frac{\alpha u}{2}} - 1 \right) \right|^2 dx,$$

we deduce that

$$\int_\Omega e^{\frac{u}{2}} \left(e^{\frac{\alpha u}{2}} - 1 \right)^2 dx \le \frac{\alpha}{4} \int_\Omega (e^{\alpha u} - 1)\, v\, dx.$$

Interpolate again

We just proved

$$\int_\Omega e^{\frac{u}{2}}\left(e^{\frac{\alpha u}{2}}-1\right)^2\,dx \le \frac{\alpha}{4}\int_\Omega (e^{\alpha u}-1)\,v\,dx.$$

Similarly,

$$\int_\Omega e^{\frac{u}{2}} v^{2\alpha}\,dx \le \frac{\alpha^2}{2\alpha-1}\int_\Omega e^u v^{2\alpha-1}\,dx.$$

Interpolate the RHS (Hölder)

$$\int_\Omega e^u v^{2\alpha-1}\,dx \le \left(\int_\Omega e^{\frac{u}{2}} v^{2\alpha}\,dx\right)^{\frac{2\alpha-1}{2\alpha}}\left(\int_\Omega e^{\frac{u}{2}} e^{\alpha u}\,dx\right)^{\frac{1}{2\alpha}} \quad \text{and}$$

$$\int_\Omega e^{\alpha u} v\,dx \le \left(\int_\Omega e^{\frac{u}{2}} v^{2\alpha}\,dx\right)^{\frac{1}{2\alpha}}\left(\int_\Omega e^{\frac{u}{2}} e^{\alpha u}\,dx\right)^{\frac{2\alpha-1}{2\alpha}}.$$

Deduce

$$\left(\int_\Omega e^{\frac{u}{2}} v^{2\alpha}\, dx\right)^{\frac{1}{2\alpha}} \le \frac{\alpha^2}{2\alpha-1}\left(\int_\Omega e^{\frac{u}{2}} e^{\alpha u}\, dx\right)^{\frac{1}{2\alpha}} \quad \text{and}$$

$$\int_\Omega e^{\frac{u}{2}}\left(e^{\frac{\alpha u}{2}}-1\right)^2 dx \le \frac{\alpha}{4}\left(\int_\Omega e^{\frac{u}{2}} v^{2\alpha}\, dx\right)^{\frac{1}{2\alpha}}\left(\int_\Omega e^{\frac{u}{2}} e^{\alpha u}\, dx\right)^{\frac{2\alpha-1}{2\alpha}}.$$

Multiply

$$\int_\Omega e^{\frac{u}{2}}\left(e^{\frac{\alpha u}{2}}-1\right)^2 dx \le \frac{\alpha^3}{8\alpha-4}\int_\Omega e^{(\alpha+\frac{1}{2})u}\, dx$$

so that

$$\left(1-\frac{\alpha^3}{8\alpha-4}\right)\int_\Omega e^{\frac{u}{2}} e^{\alpha u}\, dx \le 2\int_\Omega e^{\frac{\alpha+1}{2}u}\, dx.$$

Apply Hölder again:

$$\int_\Omega e^{\frac{\alpha+1}{2}u}\, dx \le \left(\int_\Omega e^{\frac{2\alpha+1}{2}u}\, dx\right)^{\frac{\alpha+1}{2\alpha+1}} |\Omega|^{\frac{\alpha}{2\alpha+1}}$$

and so

$$\left(1-\frac{\alpha^3}{8\alpha-4}\right)\left(\int_\Omega e^{\frac{2\alpha+1}{2}u}\, dx\right)^{\frac{\alpha}{2\alpha+1}} \le 2\,|\Omega|^{\frac{\alpha}{2\alpha+1}}.$$

Liouville theorem

As in the bounded domain case, multiply the first equation by $e^{\alpha u}\varphi^2$ and the second by $v^{2\alpha-1}\varphi^2$ to get

$$\frac{\sqrt{2\alpha-1}}{\alpha}||\nabla(v^\alpha\varphi)||_{L^2(\Omega)} \le \|e^{\frac{u}{2}}v^{\alpha-\frac{1}{2}}\varphi\|_{L^2(\Omega)} + C||v^\alpha\nabla\varphi||_{L^2(\Omega)}.$$

and

$$\frac{2}{\sqrt{\alpha}}||\nabla(e^{\frac{\alpha}{2}u}\varphi)||_{L^2(\Omega)} \le \|e^{\frac{\alpha}{2}u}v^{\frac{1}{2}}\varphi\|_{L^2(\Omega)} + C||e^{\frac{\alpha}{2}u}\nabla\varphi||_{L^2(\Omega)}.$$

Problem (to be expected): the first error term cannot be controlled by interpolation. Still, by stability, for $\alpha < 2.5^+$, either

$$\int_\Omega |\nabla(v^\alpha\varphi)|^2\,dx \le C\int_\Omega v^{2\alpha}|\nabla\varphi|^2\,dx,$$

or

$$\int_\Omega |\nabla(e^{\frac{\alpha}{2}u}\varphi)|^2\,dx \le C\int_\Omega e^{\alpha u}|\nabla\varphi|^2\,dx,$$

Now, apply Sobolev's inequality instead of stability to set up a Moser-like iteration scheme.

Initial step

Lemma ([D-Ghergu-Goubet-Warnault, Arch. Rational Mech. Anal., 2013])

Assume $N \geq 5$ and let u be a stable entire solution such that $\overline{v}(\infty) = 0$. Then,

$$\int_{B_R} v \, dx \leq CR^{N-2} \quad \textit{for every } R > 0 \, .$$

Claim: $v > 0$. It suffices to prove that $v(0) > 0$. If not, $v(0) = \overline{v}(0) \leq 0$. Have

$$\begin{cases} -\Delta \overline{v} = \overline{e^u} & \text{in } \mathbb{R}^N, \\ -\Delta \overline{u} = \overline{v} & \text{in } \mathbb{R}^N. \end{cases} \tag{2}$$

In particular, $\overline{v}$ is decreasing and $\overline{v}(r) < 0$ for all $r > 0$. So, $\overline{u}$ is increasing, and so it is bounded below.

$$\int_{B_{2R}} e^{\overline{u}} \, dx \geq e^{u(0)} \int_{B_{2R}} dx \gtrsim R^N.$$

Apply Jensen and stability with a cut-off $\varphi(x/R)$

$$\int_{B_{2R}} e^{\overline{u}} dx \leq \int_{B_{2R}} \overline{e^u} dx \lesssim R^{N-4}$$

a contradiction. Hence, $v(0) > 0$.

Recall that stability implies

$$\int_{B_r} e^u \, dx \lesssim r^{N-4}. \tag{3}$$

From the system

$$-(r^{N-1}\overline{v}')' = r^{N-1}\overline{e^u}.$$

Integrate on $(0, r)$. By (3),

$$-r^{N-1}\overline{v}'(r) = \int_0^r t^{N-1}\overline{e^u} dt \lesssim r^{N-4}.$$

We integrate once more between R and $+\infty$. Since $\overline{v}(\infty) = 0$, we obtain

$$\overline{v}(R) \lesssim R^{-2},$$

that is

$$\int_{B_R} v \, dx \leq CR^{N-2}.$$

Bootstrap

We may now use our Moser-like iteration scheme. Set $\alpha^* = 2.5^+$. Then,

$$\int_{B_R} (e^{\alpha u} + v^{2\alpha})\, dx \le C R^{N-4\alpha}. \qquad (H_\alpha)$$

for every $\alpha < \frac{N}{N-2}\alpha^*$.
Recall that one of our alternatives was

$$\int_\Omega |\nabla(e^{\frac{\alpha}{2}u}\varphi)|^2\, dx \le C \int_\Omega e^{\alpha u} |\nabla \varphi|^2\, dx,$$

Apply once more stability with test function $e^{\frac{\alpha}{2}u}\varphi(x/R)$.

$$\int_{B_R} e^{pu}\, dx \le C R^{N-4p}, \qquad \text{for all } p < p^* := \alpha^* + \frac{1}{2},$$

$$\int_{B_R} v^q\, dx \le C R^{N-2q}, \qquad \text{for all } q < q^* := \frac{2N}{N-2}\alpha^*.$$

The fourth-order Lane-Emden problem

Basic properties of the fourth-order Lane-Emden eq.

For $p > 1$, consider the equation

$$\Delta^2 u = |u|^{p-1} u \qquad \text{in } \mathbb{R}^N$$

As in Gelfand's problem,

- there is a scale invariance:

$$u_\lambda(x) = \lambda^{\frac{4}{p-1}} u(\lambda x), \quad x \in \mathbb{R}^N, \lambda > 0,$$

- The equation is variational with energy functional given by

$$\int \frac{1}{2}(\Delta u)^2 - \frac{1}{p+1}|u|^{p+1}$$

But,

- Moser's iteration (or rather the interpolation lemma) gives partial results [Cowan, see ArXiv, 2012], [Hajlaoui-Harrabi-Ye, see ArXiv, 2012]
- The L^1 estimate always holds. Better, if u is stable, then

$$\int_{B_R} |u|^{p+1} \leq C R^{N-4\frac{p+1}{p-1}},$$

as proved by [Wei-Ye, Math. Ann., to appear].

Critical exponents

Let

$$p_S(N) = \begin{cases} +\infty & \text{if } N \le 4 \\ \dfrac{N+4}{N-4} & \text{if } N \ge 5 \end{cases}$$

and

$$p_c(N) = \begin{cases} +\infty & \text{if } N \le 12 \\ \dfrac{N+2-\sqrt{N^2+4-N\sqrt{N^2-8N+32}}}{N-6-\sqrt{N^2+4-N\sqrt{N^2-8N+32}}} & \text{if } N \ge 13 \end{cases}$$

Equivalently, for fixed $p > p_S(N)$, let N_p be the smallest dimension s.t. $p \ge p_c(N)$.

Then, as observed by [Gazzola-Grunau, Math. Ann., 2006],

$$u_s(x) = C|x|^{-4/(p-1)} \text{ is stable } \iff p \ge p_c(N) \iff N \ge N_p.$$

A Liouville theorem

Theorem ([Dávila-D-Wang-Wei, see ArXiv, submitted])

Let u be solution with finite Morse index.

- *If $p \in (1, p_c(N))$, $p \neq p_S(N)$, then $u \equiv 0$;*
- *If $p = p_S(N)$, then u has finite energy i.e.*

$$\int_{\mathbb{R}^N} (\Delta u)^2 = \int_{\mathbb{R}^N} |u|^{p+1} < +\infty.$$

If in addition u is stable, then in fact $u \equiv 0$.

Remark

Generalizes a similar result of Farina for the second-order case. The proof is quite different.

Regularity theory

Theorem ([Dávila-D-Wang-Wei, see ArXiv, submitted])

Let u^* be the extremal solution of

$$\begin{cases} \Delta^2 u = \lambda(1+u)^p & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

- *If $N < N_p$ (i.e. $p \in (1, p_c(N))$), then $u^* \in C^\infty(\overline{\Omega})$.*
- *If $N = N_p$ (i.e. $p = p_c(N)$), then $u^* \in C^\infty(\Omega \setminus \Sigma)$, where Σ is a discrete set.*
- *If $N > N_p$ (i.e. $p > p_c(N)$), then $u^* \in C^\infty(\Omega \setminus \Sigma)$, where Σ is a closed set whose Hausdorff dimension is bounded above by*

$$\mathcal{H}_{dim}(\Sigma) \leq N - N_p.$$

[Bernstein, Comm. Soc. Math. de Kharkov, 1915]

Theorem

Let $N \leq 7$. Assume $u \in C^2(\mathbb{R}^N; \mathbb{R})$ is a solution of the minimal surface equation in $\mathbb{R}^N$. Then, the graph of u is a hyperplane.

Remark

The original proof of Bernstein, in dimension $N = 2$, contained a gap, discovered and fixed by [Hopf, Proc. Amer. Math. Soc., 1950]. The case $N = 3$ is due to [De Giorgi, Ann. Scuola Norm. Sup. Pisa, 1965], $N = 4$ to [Almgren, Ann. of Math., 1966], $N \leq 7$ to [Simon, Ann. of Math.,1968]. A counter-example was found by [Bombieri-De Giorgi-Giusti, Invent. Math., 1969] for $N \geq 8$. An important step in the proofs is the following result due to Fleming:

Theorem ([Fleming, Rend. Circ. Mat. Palermo, 1962])

If there exists a nonplanar entire minimal graph, then there exists a singular area-minimizing hypercone.

sketch of the proof of our theorem

- Assume first that u is stable.
- Derive a monotonicity formula $E = E(r)$ for our equation
- Estimate solutions in the L^{p+1} norm (Cacciopoli or energy method, test with $u\eta^2$ [Wei-Ye, Math. Ann., to appear])
- Consider the blow-down (weak) limit

$$u^{\infty}(x) = \lim_{\lambda \to \infty} \lambda^{\frac{4}{p-1}} u(\lambda x)$$

- u^{∞} satisfies $E(r) \equiv const$. Hence, u^{∞} is a homogeneous stable solution
- Prove that such solutions are trivial if $p < p_c(n)$, by analyzing the equation on the sphere.
- Using the monotonicity formula again, prove that in fact u is trivial.
- Extend the result to solutions of finite Morse index, again by blow-down.

The monotonicity formula

The equation is variational, with energy functional given by

$$E_1(u;x,r) = \int_{B(x,r)} \frac{1}{2}(\Delta u)^2 - \frac{1}{p+1}|u|^{p+1}$$

and it is invariant under the scaling transformation

$$u^\lambda(x) = \lambda^{\frac{4}{p-1}} u(\lambda x).$$

Compute the energy of u^λ on a ball of given size:

$$E_1(u^\lambda;0,1) = \lambda^{4\frac{p+1}{p-1}-N} E_1(u;0,\lambda)$$

This suggests to look at the variations of the rescaled energy

$$E_2(u;x,r) := r^{4\frac{p+1}{p-1}-N} \int_{B(x,r)} \frac{1}{2}(\Delta u)^2 - \frac{1}{p+1}|u|^{p+1}$$

Then, $r \mapsto E_2(u;x,r)$ is constant if u is homogeneous and for any u

$$E_2(u;0,\lambda) = E_2(u^\lambda;0,1).$$

The monotonicity formula

Augmented by the appropriate boundary terms, the above quantity is in fact nonincreasing. More precisely define

$$
\begin{aligned}
E(r;x,u) \quad := \quad & r^{4\frac{p+1}{p-1}-N}\int_{B_r(x)}\frac{1}{2}(\Delta u)^2-\frac{1}{p+1}|u|^{p+1} \\
& +\frac{2}{p-1}\left(N-2-\frac{4}{p-1}\right)r^{\frac{8}{p-1}+1-N}\int_{\partial B_r(x)}u^2 \\
& +\frac{2}{p-1}\left(N-2-\frac{4}{p-1}\right)\frac{d}{dr}\left(r^{\frac{8}{p-1}+2-N}\int_{\partial B_r(x)}u^2\right) \\
& +\frac{r^3}{2}\frac{d}{dr}\left[r^{\frac{8}{p-1}+1-N}\int_{\partial B_r(x)}\left(\frac{4}{p-1}r^{-1}u+\frac{\partial u}{\partial r}\right)^2\right] \\
& +\frac{1}{2}\frac{d}{dr}\left[r^{\frac{8}{p-1}+4-N}\int_{\partial B_r(x)}\left(|\nabla u|^2-|\frac{\partial u}{\partial r}|^2\right)\right] \\
& +\frac{1}{2}r^{\frac{8}{p-1}+3-N}\int_{\partial B_r(x)}\left(|\nabla u|^2-|\frac{\partial u}{\partial r}|^2\right),
\end{aligned}
$$

Theorem ([Dávila-D-Wang-Wei, see ArXiv, submitted])

Assume that

$$N \geq 5, \qquad p > \frac{N+4}{N-4}.$$

Let $u \in W^{4,2}_{loc}(\Omega) \cap L^{p+1}_{loc}(\Omega)$ *be a weak solution. Then,* $E(r; x, u)$ *is non-decreasing in* $r \in (0, R)$*. Furthermore there is a constant* $c(N,p) > 0$ *such that*

$$\frac{d}{dr} E(r; 0, u) \geq c(N,p) r^{-N+2+\frac{8}{p-1}} \int_{\partial B_r} \left(\frac{4}{p-1} \frac{u}{r} + \frac{\partial u}{\partial r} \right)^2 .$$

A proof in the second order case

Assume $N \geq 3$, $p > (N+2)/(N-2)$ and

$$-\Delta u = |u|^{p-1} u \qquad \text{in } \mathbb{R}^N,$$

Consider

$$E_1(\lambda; x, u) = \lambda^{2\frac{p+1}{p-1}-N} \int_{B(x,\lambda)} \left(\frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} |u|^{p+1} \right) \, dx$$

Let

$$U(x, \lambda) = \lambda^{\frac{2}{p-1}} u(\lambda x)$$

By construction,

$$E_1 = \int_{B_1} \left(\frac{1}{2} |\nabla U|^2 - \frac{1}{p+1} |U|^{p+1} \right) \, dx.$$

So,

$$\frac{d}{d\lambda} E_1 = \int_{B_1} \nabla U \cdot \nabla U_\lambda - |U|^{p-1} U U_\lambda \, dx. = \int_{\partial B_1} U_r U_\lambda \, d\sigma$$

Now,

$$\lambda U_\lambda = \frac{2}{p-1} U + r \, U_r.$$

So,

$$\frac{d}{d\lambda} E_1 = \int_{\partial B_1} \left(\lambda U_\lambda^2 - \frac{2}{p-1} U U_\lambda \right) \, d\sigma$$

i.e.

$$\frac{d}{d\lambda} \left(E_1 + \frac{1}{p-1} \lambda^{\frac{4}{p-1}+1-N} \int_{\partial B(x,\lambda)} u^2 d\sigma \right) = \lambda^{2\frac{p+1}{p-1}-N} \int_{\partial B(x,\lambda)} \left(\frac{2}{p-1} \frac{u}{r} + u_r \right)^2 \, d\sigma$$

The blow-down limit is homogeneous

Lemma ([Dávila-D-Wang-Wei, see ArXiv, submitted])

u^∞ is homogeneous.

Proof (sketch): Take $0 < r < R < +\infty$. Since $E(r;0,u)$ is monotone, its limit at infinity exists. This limit is finite, thanks to the energy estimate of [Wei-Ye, Math. Ann., to appear]. So,

$$\lim_{\lambda\to+\infty} E(\lambda R;0,u) - E(\lambda r;0,u) = 0.$$

But

$$E(\lambda R;0,u) = E(R;0,u^\lambda) \quad \text{and} \quad E(\lambda r;0,u) = E(r;0,u^\lambda)$$

Hence (...)

$$E(R;0,u^\infty) = E(r;0,u^\infty)$$

and so

$$0 = \frac{d}{dr}E(r;0,u^\infty) \geq cr^{-N+2+\frac{8}{p-1}} \int_{\partial B_r} \left(\frac{4}{p-1} r^{-1} u^\infty + \frac{\partial u^\infty}{\partial r} \right)^2$$

Liouville for homogeneous stable solutions

Write

$$u^\infty(r,\theta) = r^{-\frac{4}{p-1}} w(\theta).$$

where

$$\Delta_\theta^2 w - J_1 \Delta_\theta w + J_2 w = w^p,$$

Stability:

$$p \int_{\mathbb{R}^N} |u^\infty|^{p-1} \varphi^2 \leq \int_{\mathbb{R}^N} |\Delta \varphi|^2$$

+ test functions optimizing the Hardy-Rellich inequality $\varphi = r^{2-N/2}\eta(r)w(\theta)$:

$$p \int_{\mathbb{S}^{N-1}} |w|^{p+1} d\theta \leq \int_{\mathbb{S}^{N-1}} |\Delta_\theta w|^2 + \frac{N(N-4)}{2} |\nabla_\theta w|^2 + \frac{N^2(N-4)^2}{16} w^2.$$

Multiply the equation by w and compare the constants: if $p < p_c(N)$, then $u^\infty \equiv 0$.