

# Stabilité d'états d'équilibre non-constants de systèmes d'Euler-Maxwell

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# 1 Presentation of problems

Euler-Maxwell equations in magnetized plasma modeling for electrons

$$\begin{cases} \partial_t n + \operatorname{div}(nu) = 0 \\ \partial_t(nu) + \operatorname{div}(nu \otimes u) + \nabla p(n) = -n(E + u \times B) - nu \\ \partial_t E - \nabla \times B = nu, \quad \operatorname{div} E = b - n \\ \partial_t B + \nabla \times E = 0, \quad \operatorname{div} B = 0 \end{cases}$$

initial conditions in a torus  $\mathbb{T}^3$

$$t = 0 : \quad (n, u, E, B) = (n^0, u^0, E^0, B^0)$$

$$\operatorname{div} E^0 = b - n^0, \quad \operatorname{div} B^0 = 0$$

- $n$  and  $u$  : density and velocity
- $E$  and  $B$  : electric and magnetic fields
- $p$  : pressure function,  $p'(n) > 0$ ,  $\forall n > 0$
- $b = b(x)$  is a given smooth periodic function,  $b \geq \text{const.} > 0$

(a) Equivalent momentum equation with velocity dissipation for  $n > 0$  :

$$\partial_t u + (u \cdot \nabla) u + \nabla h(n) = -(E + u \times B) - u$$

the enthalpy  $h$  :

$$h'(n) = p'(n)/n > 0, \quad \forall n > 0$$

(b) All physical parameters are set equal to 1.

Otherwise, perform asymptotic analysis with small parameters

- B. Texier (2005-2007)
  - convergence of Euler-Maxwell to Zakharov equation
  - to Davey-Stewartson equation
- Peng - S. Wang (2008-2009)
  - convergence of Euler-Maxwell to incompressible Euler equations
  - to e-MHD equations

## (1) Local existence of solutions

Symmetrizable hyperbolic system :

$$\partial_t w + \sum_{j=1}^d A_j(w) \partial_{x_j} w = g(w), \quad w(0, x) = w^0(x), \quad x \in \mathbb{R}^d$$

- (a)  $\exists$  symmetrizer  $A_0(w)$ , symmetric positive definite matrix
- (b)  $\tilde{A}_j(w) \stackrel{def}{=} A_0(w)A_j(w)$  is symmetric for all  $1 \leq j \leq d$

Consequence : energy estimate

$$\frac{d}{dt} \int A_0(w) w \cdot w dx = \int (\operatorname{div}_{t,x} \vec{A} w \cdot w + 2A_0(w)g(w) \cdot w) dx$$

where

$$\int A_0(w) w \cdot w dx \approx \|w\|_{L^2}^2$$

$$\operatorname{div}_{t,x} \vec{A} = \partial_t A_0(w) + \sum_{j=1}^d \partial_{x_j} \tilde{A}_j(w)$$

## Theorem (T. Kato 1975)

Let  $s > d/2 + 1$  be an integer,  $\Omega = \mathbb{R}^d$  or  $\Omega = \mathbb{T}^d$ ,  $w^0 \in H^s(\Omega)$ .

There exist  $T > 0$  and a unique smooth solution

$$w \in C^1([0, T]; H^{s-1}(\Omega)) \cap C([0, T]; H^s(\Omega))$$

Regularity :

$$w \in \bigcap_{k=0}^s C^k([0, T]; H^{s-k}(\Omega))$$

The Euler-Maxwell system is symmetrizable hyperbolic for  $n > 0$

$$w = (n, u, E, B)^T, \quad d = 3 \implies s \geq 3$$

Then we have local existence of smooth solutions

## (2) Steady states solutions with zero velocity

$$\bar{w} = (\bar{n}(x), 0, \bar{E}(x), \bar{B}(x))^T$$

$$\begin{cases} \partial_t n + \operatorname{div}(nu) = 0 \\ \partial_t u + (u \cdot \nabla)u + \nabla h(n) = -(E + u \times B) - u \\ \partial_t E - \nabla \times B = nu, \quad \operatorname{div} E = b - n \\ \partial_t B + \nabla \times E = 0, \quad \operatorname{div} B = 0 \end{cases}$$

$$\begin{cases} \nabla h(\bar{n}) = -\bar{E} \\ \nabla \times \bar{B} = 0, \quad \operatorname{div} \bar{B} = 0 \implies \bar{B} \text{ is a constant} \\ \nabla \times \bar{E} = 0, \quad \operatorname{div} \bar{E} = b - \bar{n} \end{cases}$$

$$\operatorname{div} \bar{E} = b - \bar{n} \implies -\Delta h(\bar{n}) = b - \bar{n}$$

Let

$$\bar{\phi} = h(\bar{n}), \quad q = h^{-1}$$

Then  $\bar{\phi}$  satisfies a semilinear monotone elliptic equation in  $\mathbb{T}^3$  :

$$-\Delta \bar{\phi} = b - q(\bar{\phi}), \quad \bar{n} = q(\bar{\phi})$$

Consequence : there is a unique steady state periodic smooth solution

$$b \geq \text{const.} > 0 \implies \bar{n} \geq \text{const.} > 0$$

$$b = 1 \implies \bar{n} = 1, \bar{E} = 0$$

**Stability problem** :  $\|\cdot\|_m$  is a norm of  $H^m(\mathbb{T}^3)$

$$\forall s \geq 3, \quad \|w^0 - \bar{w}\|_s \text{ is small}$$

$\implies$  global existence of solution  $w$  and stability estimate

$$\|w(t, \cdot) - \bar{w}\|_s \leq C \|w^0 - \bar{w}\|_s, \quad \forall t > 0$$

## 2 Global existence near constant states

### (1) Euler-Maxwell system for $b \equiv 1$

The unique steady state solution is constant  $\bar{w} = (1, 0, 0, \bar{B})^T$

Denote

$$U = (n - 1, u)^T, \quad W = w - \bar{w} = (n - 1, u, E, B - \bar{B})^T$$

When  $\|w^0 - \bar{w}\|_s$  is small, a classical energy estimate yields

$$\frac{d}{dt} \|W(t)\|_s^2 + C_0 \|u(t)\|_s^2 \leq C \|U(t)\|_s^2 \|W(t)\|_s$$

Next, using the system and  $p'(n) > 0$  yields

$$\|n(t) - 1\|_s^2 \leq C \|u(t)\|_s^2 + C \|U(t)\|_s^2 \|W(t)\|_s + \frac{de}{dt}, \quad |e| \leq C_2 \|U\|_s^2$$

Therefore, for  $\varepsilon > 0$  small

$$\frac{d}{dt} (\|W(t)\|_s^2 - \varepsilon e) + C_1 \|U(t)\|_s^2 \leq C \|U(t)\|_s^2 \|W(t)\|_s, \quad t > 0$$

For  $\|W\|_s$  small, we obtain

$$\|W(t)\|_s^2 + \int_0^t \|U(\tau)\|_s^2 d\tau \leq C\|W(0)\|_s^2, \quad \forall t > 0$$

which yields global existence of solutions

**Theorem** (Peng - S. Wang - Q.L. Gu, 2010)

Let  $s \geq 3$  be an integer. If  $\|W(0)\|_s$  is sufficiently small, the Euler-Maxwell system admits a unique global solution

$$W \in C^1(\mathbb{R}^+; H^{s-1}(\mathbb{T}^3)) \cap C(\mathbb{R}^+; H^s(\mathbb{T}^3))$$

**Theorem** (Y. Ueda - S. Kawashima, 2011)

Let  $s \geq 6$ . If  $\|W(0)\|_{H^s(\mathbb{R}^3)}$  is small, then

$$\|W(t)\|_{H^{s-2k}(\mathbb{R}^3)} \leq C\|W(0)\|_{H^s(\mathbb{R}^3)}(t+1)^{-k/2}, \quad \forall 0 \leq k \leq [s/2]$$

## (2) A more general framework

Consider a quasilinear symmetrizable hyperbolic system

$$\partial_t w + \sum_{j=1}^d \partial_{x_j} f_j(w) = g(w), \quad w(0, x) = w^0(x), \quad x \in \mathbb{R}^d$$

$$w(t, x) \in \mathbb{R}^N, \quad \bar{w} = 0 \quad \text{and} \quad g(0) = 0$$

$$\text{Entropy-flux } (E, F) : \quad F'(w) = E'(w)(f'_1(w), \dots, f'_d(w))$$

Two stability conditions

(i) partial dissipation : there exist a strictly convex entropy  $E(w)$  and a change of variables  $w \mapsto (u, v)^T$ ,  $u \in \mathbb{R}^{N-r}$ ,  $v \in \mathbb{R}^r$ , such that

$$(E'(w) - E'(0))g(w) \leq -c_0|v|^2, \quad |\cdot| \text{ is a norm of } \mathbb{R}^r$$

which implies

$$\|w(t)\|_s^2 + \int_0^t \|v(\tau)\|_s^2 d\tau \leq C\|w^0\|_s^2 + C \int_0^t \|w(\tau)\|_s (\|\nabla u(\tau)\|_{s-1}^2 + \|v(\tau)\|_s^2) d\tau$$

(ii) Shizuta-Kawashima condition :

$$\forall \nu \in S^{d-1}, \quad \forall \lambda \in \mathbb{C}, \quad \mathcal{N}(\lambda I_N - A(\bar{w}, \nu)) \cap \mathcal{N}(g'(\bar{w})) = \{0\}$$

$$A(w, \nu) = \sum_{j=1}^d \nu_j A_j(w), \quad A_j(w) = f'_j(w), \quad \nu = (\nu_1, \dots, \nu_d) \in S^{d-1}$$

This condition implies

$$\int_0^t \|\nabla u(\tau)\|_{s-1}^2 d\tau \leq C \|w^0\|_s^2 + C \int_0^t \|w(\tau)\|_s (\|\nabla u(\tau)\|_{s-1}^2 + \|v(\tau)\|_s^2) d\tau$$

Thus, for small solutions, we have

$$\|w(t)\|_s^2 + \int_0^t (\|\nabla u(\tau)\|_{s-1}^2 + \|v(\tau)\|_s^2) d\tau \leq C \|w^0\|_s^2, \quad t > 0$$

which yields global existence of solutions when  $\|w^0\|$  is small

- B. Hanouzet - R. Natalini (2003)  
global existence 1-d
- W.A. Yong (2004)  
global existence  $d \geq 1$
- S. Bianchini - B. Hanouzet - R. Natalini (2007)  
algebraic decay of solutions  $O(t^{-\mu})$ ,  $\mu > 0$
- K. Beauchard - E. Zuazua (2011)  
refined results  $O(t^{-\mu})$

**Remark** (ii) is not fulfilled by the Euler-Maxwell system

### (3) Euler-Maxwell system without velocity dissipation

$$\begin{cases} \partial_t n + \operatorname{div}(nu) = 0 \\ \partial_t u + (u \cdot \nabla)u + \nabla h(n) = -(E + u \times B) \\ \partial_t E - \nabla \times B = nu, \quad \operatorname{div} E = 1 - n \\ \partial_t B + \nabla \times E = 0, \quad \operatorname{div} B = 0 \end{cases}$$

- P. Germain - N. Masmoudi (preprint 2011)

$$\begin{aligned} \partial_t(B - \nabla \times u) &= \nabla \times (u \times (B - \nabla \times u)) \\ B^0 - \nabla \times u^0 &= 0 \implies B - \nabla \times u = 0 \end{aligned}$$

- (a) global existence near constant states
- (b) linearized system around constant states is of Klein-Gordon type

time decay  $O(t^{-\frac{3}{2}})$

- (c) link with Euler-Poisson system for potential flows (Y. Guo, 1998)

$$B = 0 \implies \nabla \times u = 0$$

### 3 Global existence near non constant states

**Notation :**

$$\|\cdot\| = \|\cdot\|_0, \quad \text{a norm of } L^2(\mathbb{T}^3)$$

For  $m \in \mathbb{N}$  and  $v \in \cap_{k=0}^m C^k([0, T]; H^{m-k}(\mathbb{T}^3))$ , define

$$\|v(t, \cdot)\|_m = \left( \sum_{k+|\alpha| \leq m} \|\partial_t^k \partial_x^\alpha v(t, \cdot)\|^2 \right)^{\frac{1}{2}}, \quad t \in [0, T]$$

$$\alpha \in \mathbb{N}^3, \quad |\alpha| = \alpha_1 + \alpha_2 + \alpha_3$$

$\implies \| \cdot \|_m$  is a norm and  $\|\cdot\|_m \leq \| \cdot \|_m$

known:  $w = (n, u, E, B)^T$

steady state solution:  $\bar{w}(x) = (\bar{n}(x), 0, \bar{E}(x), \bar{B}(x))^T$

initial data  $w^0$

**Theorem** (Peng, preprint 2013) Let  $s \geq 3$ . If  $\|w^0 - \bar{w}\|_s$  is small, the periodic problem admits a unique global smooth solution :

$$w - \bar{w} \in \bigcap_{k=0}^s C^k \left( \mathbb{R}^+; H^{s-k}(\mathbb{T}^3) \right)$$

$$\|w(t) - \bar{w}\|_s^2 + \int_0^t \| (n(\tau) - \bar{n}, u(\tau)) \|_s^2 \leq C \|w^0 - \bar{w}\|_s^2, \quad t > 0$$

Moreover, if

$$\int_{\mathbb{T}^3} B^0(x) dx = \bar{B}$$

then

$$\lim_{t \rightarrow +\infty} \|w(t) - \bar{w}\|_{s-1} = 0$$

## Energy estimates

Let

$$N = n - \bar{n}, \quad F = E - \bar{E}, \quad G = B - \bar{B}$$

The Euler-Maxwell system is

$$\begin{cases} \partial_t N + u \cdot \nabla N + n \operatorname{div} u + \nabla \bar{n} \cdot u = 0 \\ \partial_t u + (u \cdot \nabla) u + \nabla(h(n) - h(\bar{n})) + u \times G + (u + F + u \times \bar{B}) = 0 \\ \partial_t F - \nabla \times G = (N + \bar{n})u, \quad \operatorname{div} F = -N \\ \partial_t G + \nabla \times F = 0, \quad \operatorname{div} G = 0 \end{cases}$$

where

$$\nabla(h(n) - h(\bar{n})) = h'(n)\nabla N + \nabla h'(\bar{n})N + r$$

$$r = (h'(N + \bar{n}) - h'(\bar{n}) - h''(\bar{n})N)\nabla \bar{n} = O(N^2)$$

$$U = (n - \bar{n}, u)^T, \quad W = w - \bar{w}$$

Then Euler equations are written as

$$\partial_t U + \sum_{j=1}^3 A_j(n, u) \partial_{x_j} U + L(x)U + M(W) = f$$

$$f = - \begin{pmatrix} 0 \\ r + u \times G \end{pmatrix} = O(U) O(W)$$

$$M(W) = \begin{pmatrix} 0 \\ u + F + u \times \bar{B} \end{pmatrix}$$

In  $M(W)$ ,

- $u$  stands for velocity dissipation
- $F$  can be treated together with Maxwell equations
- $u \cdot (u \times \bar{B}) = 0$

$$A_j(n, u) = \begin{pmatrix} u_j & ne_j^T \\ h'(n)e_j & u_j \mathbf{I}_3 \end{pmatrix}, \quad L(x) = \begin{pmatrix} 0 & (\nabla \bar{n})^T \\ \nabla h'(\bar{n}) & 0 \end{pmatrix}$$

Case  $b = 1$  :

$\bar{n} = 1 \implies L(x) = 0 \implies$  no linear term in the system

$$\nabla_x n = \nabla_x(n - 1) \implies \partial_x A_j(n, u) = O(\partial_x U)$$

Case  $b = b(x)$  : two main difficulties

$$(1) \quad L(x) \neq 0 \implies L(x)U = O(U)$$

$$(2) \quad \partial_x A_j(n, u) = \partial_x A_j(N + \bar{n}, u) = O(1)$$

$\implies$  difficulty in higher order energy estimates

symmetrizer  $A_0(n) = \begin{pmatrix} h'(n) & 0 \\ 0 & n\mathbf{I}_3 \end{pmatrix} \implies \langle A_0(n)U, U \rangle \approx \|U\|^2$

(1)  $L(x) \neq 0$

$L^2$ -estimate :  $L^2$ -inner product  $\langle \cdot, \cdot \rangle$

$$\begin{aligned} \frac{d}{dt} \langle A_0(n)U, U \rangle &= \langle \partial_t A_0(n)U, U \rangle + \langle Q(x, n, u)U, U \rangle \\ &\quad + 2 \langle A_0(n)f, U \rangle - 2 \langle A_0(n)M(W), U \rangle \end{aligned}$$

$$\|\partial_t n\|_\infty = \|\partial_t N\|_\infty \leq C\|U\|_s \implies \langle \partial_t A_0(n)U, U \rangle \leq C\|U\|_s^3$$

$$f = O(U)O(W) \implies 2 \langle A_0(n)f, U \rangle \leq C\|U\|_s^2 \|W\|_s$$

$$-2 \langle A_0(n)M(W), U \rangle = -2 \langle nu, u \rangle - 2 \langle nu, F \rangle$$

Energy estimates of Maxwell equations yield

$$\frac{d}{dt} (\|F\|^2 + \|G\|^2) = 2 \langle nu, F \rangle$$

Estimate for  $Q$  (Y. Guo - W. Strauss, 2006)

$$\begin{aligned}
 Q(x, n, u) &= \sum_{j=1}^3 \partial_{x_j} \tilde{A}_j(n, u) - 2A_0(n)L(x) \\
 &= \begin{pmatrix} \operatorname{div}(h'(n)u) & (\nabla p'(n) - 2h'(n)\nabla\bar{n})^T \\ \nabla p'(n) - 2n\nabla h'(\bar{n}) & \operatorname{div}(nu) \mathbf{I}_3 \end{pmatrix}
 \end{aligned}$$

is an anti-symmetric matrix at  $(n, u) = (\bar{n}, 0)$ , because

$$\nabla p'(n) - 2h'(n)\nabla n = -(\nabla p'(n) - 2n\nabla h'(n))$$

Hence,

$$|\langle Q(x, n, u)U, U \rangle| \leq C\|U\|_s^3$$

Consequence :  $L^2$ -estimate

$$\frac{d}{dt}\|W\|^2 + C_0\|u\|^2 \leq C\|U\|_s^2 \|W\|_s$$

Similar treatment for  $Q$  in higher order estimates

$$(2) \quad \partial_x A_j(n, u) = O(1)$$

Higher order estimates :  $\alpha \in \mathbb{N}^3$ ,  $|\alpha| \leq s$

$$\frac{d}{dt} \langle A_0(n) \partial_x^\alpha U, \partial_x^\alpha U \rangle = 2 \sum_{j=1}^3 \langle A_j \partial_{x_j} (\partial_x^\alpha U) - \partial_x^\alpha (A_j \partial_{x_j} U), A_0 \partial_x^\alpha U \rangle + \dots$$

$$\text{Moser inequality} \implies \langle A_j \partial_{x_j} (\partial_x^\alpha U) - \partial_x^\alpha (A_j \partial_{x_j} U), A_0 \partial_x^\alpha U \rangle = O(\|U\|_s^2)$$

$$\frac{d}{dt} \|W(t)\|_{|\alpha|}^2 + C_0 \|u(t)\|_{|\alpha|}^2 \leq C \|N(t)\|_{|\alpha|}^2 + C \|U(t)\|_s^2 \|W(t)\|_s + \dots$$

$$\|N(t)\|_{|\alpha|}^2 \leq C \|u(t)\|_{|\alpha|}^2 + C \|U(t)\|_s^2 \|W(t)\|_s + \dots$$

$\implies$  these energy estimates are not sufficient to conclude

**Idea** : estimates for  $\partial_t^k \partial_x^\alpha W$  and induction relations on  $k + |\alpha|$

- Time derivative estimates :  $\alpha = 0$

$$\frac{d}{dt} \|\partial_t^k W\|^2 + C_0 \|\partial_t^k u\|^2 \leq C \|U\|_s^2 \|W\|_s, \quad \forall 0 \leq k \leq s$$

The momentum equation implies

$$\|\partial_t^k N\|_1^2 \leq C \|\partial_t^k u\|^2 + C \|\partial_t^{k+1} u\|^2 + C \|U\|_s^2 \|W\|_s, \quad 0 \leq k \leq s-1$$

The density equation implies

$$\|\partial_t^s N\|^2 \leq C \|\partial_t^{s-1} u\|_1^2 + C \|U\|_s^2 \|W\|_s \quad (k=s)$$

- Time-space derivative estimates :  $\forall k + |\alpha| \leq s, |\alpha| \geq 1$

$$\begin{aligned} & \frac{d}{dt} \|\partial_t^k W\|_{|\alpha|}^2 + C_0 \|\partial_t^k U\|_{|\alpha|}^2 \\ & \leq C (\|\partial_t^k U\|_{|\alpha|-1}^2 + \|\partial_t^{k+1} u\|_{|\alpha|-1}^2) + C \|U\|_s^2 \|W\|_s \end{aligned}$$

- Induction argument : for  $k$  decreasing and  $|\alpha|$  increasing
  - $(k, |\alpha|) = (s, 0)$  :
 
$$\frac{d}{dt} \|\partial_t^s W\|^2 + C_0 \|\partial_t^s U\|^2 \leq C \|\partial_t^{s-1} u\|_1^2 + C \|U\|_s^2 \|W\|_s$$
  - $(k, |\alpha|) = (s-1, 1)$  :
 
$$\frac{d}{dt} \|\partial_t^{s-1} W\|_1^2 + C_0 \|\partial_t^{s-1} U\|_1^2 \leq C \|\partial_t^{s-1} U\|^2 + C \|\partial_t^s u\|^2 + C \|U\|_s^2 \|W\|_s$$

By induction we obtain

$$\frac{d}{dt} \|W\|_s^2 + 2C_1 \|U\|_s^2 \leq C \|U\|_s^2 \|W\|_s$$

For small solutions, we have

$$\|W(t)\|_s^2 + C_1 \int_0^t \|U(\tau)\|_s^2 d\tau \leq C \|W(0)\|_s^2, \quad \forall t > 0$$

which yields the global existence of solutions

**Long-time behavior of solutions :**

For all  $k + |\alpha| \leq s - 1$ ,

$$\partial_t^k \partial_x^\alpha U \in L^2(\mathbb{R}^+; L^2(\mathbb{T}^3)) \cap W^{1,\infty}(\mathbb{R}^+; L^2(\mathbb{T}^3))$$

which implies

$$\lim_{t \rightarrow +\infty} \| (n(t) - \bar{n}, u(t)) \|_{s-1} = 0$$

Similarly for  $E$  and  $B$