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TOPOLOGICAL INVARIANTS FOR PROJECTION METHOD PATTERNS

Alan Forrest[1], John Hunton[2], Johannes Kellendonk[3]

[1]Department of Mathematics,
National University of Ireland,
Cork, Republic of Ireland
Current Email: alan_forest@base3.com

[2]The Department of Mathematics and Computer Science,
University of Leicester, University Road,
Leicester, LE1 7RH, England
Email: J.Hunton@mcs.le.ac.uk

[3]Fachbereich Mathematik, Sekr. MA 7-2,
Technische Universität Berlin,
10623 Berlin, Germany
Email: kellen@math.tu-berlin.de
Current Email: kellendonkj@Cardiff.ac.uk

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Abstract

This memoir develops, discusses and compares a range of commutative and non-commutative invariants defined for projection method tilings and point patterns. The projection method refers to patterns, particularly the quasiperiodic patterns, constructed by the projection of a strip of a high dimensional integer lattice to a smaller dimensional Euclidean space. In the first half of the memoir the acceptance domain is very general – any compact set which is the closure of its interior – while in the second half we concentrate on the so-called canonical patterns. The topological invariants used are various forms of K-theory and cohomology applied to a variety of both C^*-algebras and dynamical systems derived from such a pattern.

The invariants considered all aim to capture geometric properties of the original patterns, such as quasiperiodicity or self-similarity, but one of the main motivations is also to provide an accessible approach to the the K_0 group of the algebra of observables associated to a quasicrystal with atoms arranged on such a pattern.

The main results provide complete descriptions of the (unordered) K-theory and cohomology of codimension 1 projection patterns, formulæ for these invariants for codimension 2 and 3 canonical projection patterns, general methods for higher codimension patterns and a closed formula for the Euler characteristic of arbitrary canonical projection patterns. Computations are made for the Ammann-Kramer tiling. Also included are qualitative descriptions of these invariants for generic canonical projection patterns. Further results include an obstruction to a tiling arising as a substitution and an obstruction to a substitution pattern arising as a projection. One corollary is that, generically, projection patterns cannot be derived via substitution systems.

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General Introduction

Of the many examples of aperiodic tilings or aperiodic point sets in Euclidean space found in recent years, two classes stand out as particularly interesting and æsthetically pleasing. These are the *substitution* tilings, tilings which are self-similar in a rather strong sense described in [**GS**] [**R1**] [**S1**] [**AP**], and the tilings and patterns obtained by the method of cut and projection from higher dimensional periodic sets described in [**dB1**] [**KrNe**] [**KD**]. In this memoir we consider the second class. However, some of the best studied and most physically useful examples of aperiodic tilings, for example the Penrose tiling [**Pe**] and the octagonal tiling (see [**Soc**]), can be approached as examples of either class. Therefore we study specially those tilings which are in the overlap of these two classes, and examine some of their necessary properties.

Tilings and patterns in Euclidean space can be compared by various degrees of equivalence, drawn from considerations of geometry and topology [**GS**]. Two tilings can be related by simple geometric tranformations (shears or rotations), topological distortions (bending edges), or by more radical adaptation (cutting tiles in half, joining adjacent pairs etc). Moreover, point patterns can be obtained from tilings in locally defined ways (say, by selecting the centroids or the vertices of the tiles) and vice versa (say, by the well-known Voronoi construction). Which definition of equivalence is chosen is determined by the problem in hand.

In this paper, we adopt definitions of equivalence (pointed conjugacy and topological conjugacy, I.4.5) which allow us to look, without loss of generality, at sets of uniformly isolated points (point patterns) in Euclidean space. In fact these patterns will typically have the Meyer property [**La1**] (see I.4.5). Therefore in this introduction, and often throughout the text, we formulate our ideas and results in terms of point patterns and keep classical tilings in mind as an implicit example.

The current rapid growth of interest in projection method patterns started with the discovery of material quasicrystals in 1984 [**SBGC**], although these patterns had been studied before then. Quasicrystaline material surprised the physical world by showing sharp Bragg peaks under X-ray scattering, a phenomenon usually associated only with periodic crystals. Projection method patterns share

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this unusual property and in recent studies they have become the prefered model of material quasicrystals **[1]** **[2]**. This model is not without criticism, see e.g. **[La2]**.

Whatever the physical significance of the projection method consruction, it also has great mathematical appeal in itself: it is elementary and geometric and, once the acceptance domain and the dimensions of the spaces used in the construction are chosen, has a finite number of degrees of freedom. The projection method is also a natural generalization of low dimensional examples such as Sturmian sequences **[HM]** which have strong links with classical diophantine approximation.

General approach of this book In common with many papers on the topology of tilings, we are motivated by the physical applications and so are interested in the properties of an individual quasicrystal or pattern in Euclidean space. The topological invariants of the title refer not to the topological arrangement of the particular configuration as a subset of Euclidean space, but rather to an algebraic object (graded group, vector space *etc.*) associated to the pattern, and which in some way captures its geometric properties. It is defined in various equivalent ways as a classical topological invariant applied to a space constructed out of the pattern. There are two choices of space to which to apply the invariant, the one C^*-algebraic, the other dynamical, and these reflect the two main approaches to this subject, one starting with the construction of an operator algebra and the other with a topological space with $\mathbb{R}^d$ action.

The first approach, which has the benefit of being closer to physics and which thus provides a clear motivation for the topology, can be summarized as follows. Suppose that the point set $\mathcal{T}$ represents the positions of atoms in a material, like a quasicrystal. It then provides a discrete model for the configuration space of particles moving in the material, like electrons or phonons. Observables for these particle systems, like energy, are, in the absence of external forces like a magnetic field, functions of partial translations. Here a partial translation is an operator on the Hilbert space of square summable functions on $\mathcal{T}$ which is a translation operator from one point of $\mathcal{T}$ to another combined with a range projection which depends only on the neighbouring configuration of that point. The appearance of that range projection is directly related to the locality of interaction. The norm closure $\mathcal{A}_\mathcal{T}$ of the algebra generated by partial translations can be regarded as the C^*-algebra of observables. The topology we are interested in is the non-commutative topology of the C^*-algebra of observables and the topological invariants of $\mathcal{T}$ are the invariants of $\mathcal{A}_\mathcal{T}$. In particular, we shall be interested in the K-theory of $\mathcal{A}_\mathcal{T}$. Its K_0-group has direct relevance to physics through the gap-labelling. In fact, Jean Bellissard's K-theoretic formulation of the gap-labelling stands at the beginning of this approach [**B1**].

The second way of looking at the topological invariants of a discrete point set $\mathcal{T}$ begins with the construction of the continuous hull of $\mathcal{T}$. There are various ways of defining a metric on a set of patterns through comparison of their local configurations. Broadly speaking, two patterns are deemed close if they coincide on a large window around the origin $0 \in \mathbb{R}^d$ up to a small discrepancy. It is the way the allowed discrepancy is quantified which leads to different metric topologies and we choose here one which has the strongest compactness properties, though we have no intrinsic motivation for this. The continuous hull of $\mathcal{T}$ is the closure, $M\mathcal{T}$, of the set of translates of $\mathcal{T}$ with respect to this metric. We use the notation $M\mathcal{T}$ because it is essentially a mapping torus construction for a generalized discrete dynamical system: $\mathbb{R}^d$ acts on $M\mathcal{T}$ by translation and the set $\Omega_\mathcal{T}$ of all elements of $M\mathcal{T}$ which (as point sets) contain 0 forms an abstract transversal called the discrete hull. If d were 1 then $\Omega_\mathcal{T}$ would give rise to a Poincaré section, the intersection points of the flow line of the action of $\mathbb{R}$ with $\Omega_\mathcal{T}$ defining an orbit of a $\mathbb{Z}$ action in $\Omega_\mathcal{T}$, and $\mathcal{M}\mathcal{T}$ would be the mapping torus of that discrete dynamical system $(\Omega_\mathcal{T}, \mathbb{Z})$. For larger d one cannot expect to get a $\mathbb{Z}^d$ action on $\Omega_\mathcal{T}$ in a similar way but finds instead a generalized discrete dynamical system which can be summarized in a groupoid $\mathcal{G}\mathcal{T}$ whose unit space is the discrete hull $\Omega_\mathcal{T}$. Topological invariants for $\mathcal{T}$ are therefore the topological invariants of $M\mathcal{T}$ and of $\mathcal{G}\mathcal{T}$ and we shall be interested in particular in their cohomologies. We define the cohomology of $\mathcal{T}$ to be that of $\mathcal{G}\mathcal{T}$. Under a finite type condition, namely that for any given r there are only a finite number of translational congruence classes of subsets which fit inside a window of diameter r, the algebra $\mathcal{A}_\mathcal{T}$ sketched above is isomorphic to the groupoid C^*-algebra of $\mathcal{G}\mathcal{T}$. This links the two approaches.

Having outlined the general philosophy we hasten to remark that we will not explain all its aspects in the main text. In particular, we have nothing to say there about the physical aspects of the theory and the description of the algebra of observables, referring here the reader to [**BZH**] [**KePu**], or to the more original literature [**B1**] [**B2**] [**K1**]. Instead, our aim in this memoir is to discuss and compare the different commutative and non-commutative invariants, to demonstrate their applicability as providing obstructions to a tiling arising as a substitution, and finally to provide a practical method for computing them; this we illustrate with a number of examples, including that of the Ammann-Kramer ('3 dimensional Penrose') tiling. Broadly speaking, one of the perspectives of this memoir is that non-commutative invariants for projection point patterns can be successfully computed by working with suitable commutative analogues.

The subject of this book We work principally with the special class of point sets (possibly with some decoration) obtained by cut and projection from the integer lattice $\mathbb{Z}^N$ which is generated by an orthonormal base of $\mathbb{R}^N$. Reserving detail and elaborations for later, we call a *projection method pattern* $\mathcal{T}$ on $E = \mathbb{R}^d$ a pattern of points (or a finite decoration of it) given by the orthogonal projection onto E of points in a strip $(K \times E) \cap \mathbb{Z}^N \subset \mathbb{R}^N$, where E is a subspace of $\mathbb{R}^N$ and $K \times E$ is the so-called *acceptance strip*, a fattening of E in $\mathbb{R}^N$ defined by some suitably chosen region K in the orthogonal complement $E^\perp$ of E in $\mathbb{R}^N$. The pattern $\mathcal{T}$ thus depends on the dimension N, the positioning of E in $\mathbb{R}^N$ and the shape of the *acceptance domain* K. When this construction was first made **[dB1]** **[KD]** the domain K was taken to be the projected image onto $E^\perp$ of the unit cube in $\mathbb{R}^N$ and this choice gives rise to the so-called *canonical projection method patterns*, but for the first three chapters we allow K to be any compact subset of $E^\perp$ which is the closure of its interior (so, with possibly even fractal boundary, a case of current physical interest **[BKS]** **[Sm]** **[Z]** **[GLJJ]**).

It is then very natural to consider not only $\mathcal{T}$ but also all point patterns which are obtained in the same way but with $\mathbb{Z}^N$ repositioned by some vector $u \in \mathbb{R}^N$, *i.e.*, $\mathbb{Z}^N$ replaced by $\mathbb{Z}^N + u$. Completing certain subsets of positioning vectors u with respect to an appropriate pseudo-metric gives us the continuous hull $M\mathcal{T}$. This analysis shows in particular that $M\mathcal{T}$ contains another transversal $X_\mathcal{T}$ which gives rise to d independant commuting $\mathbb{Z}$ actions and hence to a genuine discrete dynamical system $(X_\mathcal{T}, \mathbb{Z}^d)$ whose mapping torus is also $M\mathcal{T}$. This is a key point in relating the K-theory of $\mathcal{A}_\mathcal{T}$ with the cohomology of $\mathcal{T}$; in the process, the latter is also identified with the Čech cohomology of $M\mathcal{T}$ and with the group cohomology of $\mathbb{Z}^d$ with coefficients in the continuous integer valued functions over $X_\mathcal{T}$.

The space $X_\mathcal{T}$ arises in another way. Let V be a connected component of the euclidian closure of $\pi^\perp(\mathbb{Z}^N)$, where $\pi^\perp$ denotes the orthoprojection onto $E^\perp$. We first disconnect V along the boundaries of all $\pi^\perp(\mathbb{Z}^N)$-translates of K (we speak loosely here, but make the idea precise in Chapter I). Then $X_\mathcal{T}$ can be understood as a compact quotient of the disconnected V with respect to a proper isometric free abelian group action. In the case of canonical projection method patterns, on which we concentrate in the last two chapters, the boundaries which disconnect V are affine subspaces and so define a directed system of locally finite CW decompositions of Euclidean space. With respect to this CW-complex, the integer valued functions over $X_\mathcal{T}$ appear as continuous chains in the limit. This makes the group cohomology of the dynamical action of $\mathbb{Z}^d$ on $X_\mathcal{T}$ accessible through the standard machinery (exact sequences and spectral sequences) of algebraic topology.

As mentioned, the interest in physics in the non-commutative topology of tilings and point sets is based on the observation that $\mathcal{A}_\mathcal{T}$ is the C^*-algebra of observables for particles moving in $\mathcal{T}$. In particular, any Hamilton operator which describes this motion has the property that its spectral projections on energy intervals whose boundaries lie in gaps of the spectrum belong to $\mathcal{A}_\mathcal{T}$ as well and thus define elements of $K_0(\mathcal{A}_\mathcal{T})$. Therefore, the ordered K_0-group (or its image on a tracial state) may serve to 'count' (or label) the possible gaps in the spectrum the Hamilton operator **[B2]** **[BBG]** **[K1]**. One of the main results of this memoir is the determination in Chapter V of closed formulae for the ranks of the K-groups corresponding to canonical projection method patterns with small codimension (as one calls the dimension of V). These formulæ apply to all common tilings including the Penrose tilings, the octagonal tilings and three dimensional icosahedral tilings. Unfortunately our method does not as yet give full information on the order of K_0 or the image on a tracial state.

Further important results of this memoir concern the structure of a K-group of a canonical projection method pattern. We find that its K_0-group is generically infinitely generated. But when the rank of its rationalization is finite then it has to be free abelian. We observe in Chapters III and IV that both properties are obstruction to some kinds of self-similarity. More precisely, infinitely generated rationalized cohomology rules out that the tiling is a substitution tiling. On the other hand, if we know already that the tiling is substitutional then its K-group must be free abelian for it to be a canonical projection method tiling.

No projection method pattern is known to us which has both infinitely generated cohomology and also allows for local matching rules in the sense of [**Le**]. Furthermore, all projection method patterns which are used to model quasicrystals seem to have a finitely generated K_0-group. We cannot offer yet an interpretation of the fact that some patterns produce only finitely many generators for their cohomology whereas others do not, but, if understood, we hope it could lead to a criterion to single out a subset of patterns relevant for quasicrystal physics from the vast set of patterns which may be obtained from the projection method.

We have mentioned above the motivation from physics to study the topological theory of point sets or tilings. The theory is also also of great interest for the theory of topological dynamical systems, since in $d = 1$ dimensions the dynamical systems mentioned above have attracted a lot of attention. In [**GPS**] the meaning of the non-commutative invariants for the one dimensional case has been analysed in full detail. Furthermore, substitution tilings give rise to hyperbolic $\mathbb{Z}$-actions with expanding attractors (hyperbolic attractors whose topological dimensions are that of their expanding direction) [**AP**] a subject of great interest followed up recently by Williams [**W**] who conjectures that continuous hulls of substitution tilings (called tiling spaces in [**W**]) are fiber bundles over tori with the Cantor set as fiber. We have not put emphasis on this question but it may be easily concluded from our analysis of Section I.10 that the continuous hulls of projection method tilings are always Cantor set fiber bundles over tori (although these tilings are rarely substitutional and therefore carry in general no obvious hyperbolic $\mathbb{Z}$-action). Anderson and Putnam [**AP**] and one of the authors [**K2**] have employed the substitution of the tiling to calculate topological invariants of it.

Organization of the book The order of material in this memoir is as follows. In Chapter I we define and describe the various dynamical systems mentioned above, and examine their topological relationships. These are compared with the pattern groupoid and its associated C^* algebra in Chapter II where these latter objects are introduced. Also in Chapter II we set up and prove the equivalence of our various topological invariants; we end the chapter by demonstrating how these invariants provide an obstruction to a pattern being self-similar.

The remaining chapters offer three illustrations of the computability of these invariants. In Chapter III we give a complete calculation for all 'codimension one' projection patterns – patterns arising from the projection of slices of $\mathbb{Z}^{d+1}$ to $\mathbb{R}^d$ for more or less arbitrary acceptance domains. In Chapter IV we give descriptions of the invariants for generic projection patterns arising from arbitrary projections $\mathbb{Z}^N$ to $\mathbb{R}^d$ but with canonical acceptance domain. Here, applying the result at the end of Chapter II, we prove the result mentioned above that almost all canonical projection method patterns have infinitely generated cohomology and so fail to be substitution tilings. In Chapter V we examine the case of canonical projection method patterns with finitely generated cohomology, such as would arise from a substitution system. We develop a systematic approach to the calculation of these invariants and use this to produce closed formulæ for the cohomology and K-theory of projection patterns of codimension 1, 2 and 3: in principle the procedure can be iterated to higher codimensions indefinitely, though in practice the formulæ would soon become tiresome. Some parameters of these formulae allow for a simple description in arbitrary codimension, as e.g. the Euler characterisitc (V.2.8). We end with a short description of the results for the Ammann-Kramer tiling.

There is a separate introduction to each chapter where relevant classical work is recalled and where the individual sections are described roughly. We adopt the following system for crossreferences. The definitions, theorems etc. of the same chapter are cited e.g. as Def. 2.1 or simply 2.1. The definitions, theorems etc. of the another chapter are cited e.g. as Def. II.2.1 or simply II.2.1.

A note on the writing of this book Originally this memoir was conceived by the three authors as a series of papers leading to the results now in Chapters IV and V, aiming to found a calculus for projection method tiling cohomology. These papers are currently available as a preprint-series *Projection Quasicrystals I-III* **[FHKI-III]** covering most of the results in this memoir. The authors' collaboration on this project started in 1997 and, given the importance of the subject and time it has taken to bring the material to its current state, it is inevitable that some results written here have appeared elsewhere in the literature during the course of our research. We wish to acknowledge these independent developments here, although we will refer to them again as usual in the body of the text.

The general result of Chapter I, that the tiling mapping torus is also a discrete dynamical mapping torus, and that the relevant dynamics is an almost 1-1 extension of a rotation on a torus, has been known with varying degrees of precision and generality for some time and we mention the historical developments in the introduction to Chapter I. Our approach constructs a large topological space from which the pattern dynamical system is formed by a quotient and so we follow most closely the idea pioneered by Le [**Le**] for the case of canonical projection tilings. The "Cantorization" of Euclidean space by corners or cuts, as described by Le and others (see [**Le**] [**H**] etal.), is produced in our general topological context in sections I.3, I.4 and I.9. In this, we share the ground with Schlottmann [**Sch**] and Herrmann [**He**] who have recently established the results of Chapter I in such (and even greater) generality, Schlottmann in order to generalize results of Hof and describe the unique ergodicity of the underlying dynamical systems and Herrmann to draw a connection between codimension 1 projection patterns and Denjoy homeomorphisms of the circle. We mention this relation at the end of chapter III.

Bellissard, Contensou and Legrand [**BCL**] compare the C^*-algebra of a dynamical groupoid with a C^*-algebra of operators defined on a class of tilings obtained by projection, the general theme of Chapter II. Using a Rosenberg Shochet spectral sequence, they also establish, for 2-dimensional canonical projection tilings, an equation of dynamical cohomology and C^* K-theory in that case. It is the first algebraic topological approach to projection method tiling K-theory found in the literature. We note, however, that the groupoid they consider is not always the same as the tiling groupoid we consider, nor do the dynamical systems always agree; the Penrose tiling is a case in point, where we find that K_0 of the spaces considered in [**BCL**] is $\mathbb{Z}^\infty$. The difference may be found in the fact that we consider a given projection method tiling or pattern and its translates, while they consider a larger set of tilings, two elements of which may sometimes be unrelated by approximation and translation parallel to the projection plane.

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I Topological Spaces and Dynamical Systems

1 Introduction

In this chapter our broad goal is to study the topology and associated dynamics of projection method patterns, while imposing only few restrictions on the freedom of the construction. From a specific set of projection data we define and examine a number of spaces and dynamical systems and their relationships; from these constructions, in later chapters, we set up our invariants, defined via various topological and dynamical cohomology theories. In Chapter II we shall also compare the commutative spaces of this chapter with the non-commutative spaces considered by other authors in *e.g.* **[B2]** **[BCL]** **[AP]** **[K2]**.

Given a subspace, E, acceptance domain, K, and a positioning parameter u, we distinguish two particular $\mathbb{R}^d$ dynamical systems constructed by the projection method, $(MP_u, \mathbb{R}^d)$ and $(M\widetilde{P}_u, \mathbb{R}^d)$, the first automatically a factor of the second. This allows us to define a *projection method pattern* (with data (E, K, u)) as a pattern, $\mathcal{T}$, whose dynamical system, $(M\mathcal{T}, \mathbb{R}^d)$, is intermediate to these two extreme systems. Sections 3 to 9 of this chapter provide a complete description of the spaces and the extension $M\widetilde{P}_u \longrightarrow MP_u$, showing, under further weak assumptions on the acceptance domain, that it is a finite isometric extension. In section 7 we conclude that this restricts $(M\mathcal{T}, \mathbb{R}^d)$ to one of a finite number of possibilities, and that any projection method pattern is a finite decoration of its corresponding point pattern P_u (2.1). The essential definitions are to be found in Sections 2 and 4.

In section 10 we describe yet another dynamical system connected with a projection method pattern, this time a $\mathbb{Z}^d$ action on a Cantor set X, whose mapping torus is the space of the pattern dynamical system. It will be this dynamical system that, in chapters 3, 4 and 5, will allow the easiest computation and discussion of the behaviour of our invariants. For the canonical case with $E^\perp \cap \mathbb{Z}^N = 0$ this is the same system as that constructed in **[BCL]**.

All the dynamical systems produced in this memoir are almost 1-1 extensions of an action of $\mathbb{R}^d$ or $\mathbb{Z}^d$ by rotations (Def. 2.15) on a torus (or torus extended by a finite abelian group). In each case the dimension of the torus and the generators of the action can be computed explicitly. This gives a clear picture of the orbits of non-singular points in the pattern dynamical system. A precursor to our description of the pattern dynamical system can be found in the work of Robinson [**R2**], who examined the dynamical system of the Penrose tiling and showed that it is an almost 1-1 extension of a minimal $\mathbb{R}^2$ action by rotation on a 4-torus. Although Robinson used quite special properties of the tiling, Hof [**H**] has noted that these techniques are generalizable without being specific about the extent of the generalization.

Our approach is quite different from that of Robinson and, by constructing a larger topological space from which the pattern dynamical system is formed by a quotient, we follow most closely the approach pioneered by Le [**Le**] as noted in the General Introduction. The care taken here in the topological foundations seems necessary for further progress and to allow general acceptance domains. Even in the canonical case, Corollary 7.2 and Proposition 8.4 of this chapter, for example, require this precision despite being direct generalizations of Theorem 3.8 in [**Le**]. Also, as mentioned in the General Introduction, many of the results of this Chapter are to be found independently in [**Sch**].

2 The projection method and associated geometric constructions

We use the construction of point patterns and tilings given in Chapters 2 and 5 of Senechal's monograph [**Se**] throughout this paper, adding some assumptions on the acceptance domain in the following definitions.

Definitions 2.1 Consider the lattice $\mathbb{Z}^N$ sitting in standard position inside $\mathbb{R}^N$ (i.e. it is generated by an orthonormal basis of $\mathbb{R}^N$). Suppose that E is a d dimensional subspace of $\mathbb{R}^N$ and $E^\perp$ its orthocomplement. For the time being we shall make no assumptions about the position of either of these planes.

Let π be the projection onto E and $\pi^\perp$ the projection onto $E^\perp$.

Let $Q = \overline{E + \mathbb{Z}^N}$ (Euclidean closure). This is a closed subgroup of $\mathbb{R}^N$.

Let K be a compact subset of $E^\perp$ which is the closure if its interior (which we write $IntK$) in $E^\perp$. Thus the boundary of K in $E^\perp$ is compact and nowhere dense. Let $\Sigma = K + E$, a subset of $\mathbb{R}^N$ sometimes refered to as *the strip with acceptance domain K*.

A point $v \in \mathbb{R}^N$ is said to be *non-singular* if the boundary, $\partial\Sigma$, of Σ does not intersect $\mathbb{Z}^N + v$. We write NS for the set of non-singular points in $\mathbb{R}^N$. These points are also called *regular* in the literature.

Let $\widetilde{P}_v = \Sigma \cap (\mathbb{Z}^N + v)$, the *strip point pattern.*

Define $P_v = \pi(\widetilde{P}_v)$, a subset of E called the *projection point pattern.*

In what follows we assume E and K are fixed and suppress mention of them as a subscript or argument.

Lemma 2.2 *With the notation above,*

i/ NS is a dense G_δ subset of $\mathbb{R}^N$ invariant under translation by E.

ii/ If $u \in NS$, then $NS \cap (Q + u)$ is dense in $Q + u$.

iii/ If $u \in NS$ and F is a vector subspace of $\mathbb{R}^N$ complementary to E, then $NS \cap (Q + u) \cap F$ is dense in $(Q + u) \cap F$.

Proof i/ Note that $\mathbb{R}^N \setminus NS$ is a translate of the set $\cup_{v \in \mathbb{Z}^N} (\partial K + E + v)$ (where the boundary is taken in $E^\perp$) and our conditions on K complete the proof.

ii/ $NS \cap (Q + u) \supset E + \mathbb{Z}^N + u$.

iii/ $\overline{(E + \mathbb{Z}^N + u) \cap F} = (Q + u) \cap F$. □

Remark 2.3 The condition on the acceptance domain K is a topological version of the condition of [**H**]. We note that our conditions include the examples of acceptance domains with fractal boundaries which have recently interested quasicrystalographers [**BKS**] [**Sm**] [**Z**] [**GLJJ**].

In the original construction [**dB1**] [**KD**] $K = \pi^\perp([0,1]^N)$. We call this the *canonical acceptance domain* and we reserve the name *canonical projection method pattern* for the patterns P_u produced from this acceptance domain. Sometimes this is shortened to *canonical pattern* for convenience.

This is closely related to the *canonical projection tiling*, defined by [**OKD**] formed by a canonical acceptance domain, $u \in NS$ and projecting onto E those d-dimensional faces of the lattice $\mathbb{Z}^N + u$ which are contained entirely in Σ. We write this tiling $\mathcal{T}_u$.

The following notation and technical lemma makes easier some calculations in future sections.

Definition 2.4 If X is a subspace of Y, both topological spaces, and $A \subset X$, then we write $Int_X A$ to mean the interior of A in the subspace topology of X.

Likewise we write $\partial_X A$ for the boundary of A taken in the subspace topology of X.

Lemma 2.5 *a/ If $u \in NS$, then $(Q+u) \cap IntK = Int_{(Q+u)\cap E^\perp}((Q+u) \cap K)$ and $(Q+u) \cap \partial_{E^\perp} K = \partial_{(Q+u)\cap E^\perp}((Q+u) \cap K)$.*

b/ If $u \in NS$, then $((Q+u) \cap E^\perp) \setminus NS = \partial_{(Q+u)\cap E^\perp}((Q+u) \cap K) + \pi^\perp(\mathbb{Z}^N)$.

Proof a/ To show both facts, it is enough to show that $(\partial_{E^\perp} K) \cap (Q+u)$ has no interior as a subspace of $(Q+u) \cap E^\perp$.

Suppose otherwise and that U is an open subset of $\partial K \cap (Q+u)$ in $(Q+u) \cap E^\perp$. By the density of $\pi^\perp(\mathbb{Z}^N)$ in $Q \cap E^\perp$, we find $v \in \mathbb{Z}^N$ such that $u \in U + \pi^\perp(v)$. But this implies that $u \in \partial K + \pi^\perp(v)$ and so $u \notin NS$ - a contradiction.

b/ By defintion the left-hand side of the equation to be proved is equal to $(\partial_{E^\perp} K + \pi^\perp(\mathbb{Z}^N)) \cap (Q+u)$ which equals $(\partial_{E^\perp} K \cap (Q+u)) + \pi^\perp(\mathbb{Z}^N)$ since $\pi^\perp(\mathbb{Z}^N)$ is dense in $Q \cap E^\perp$. By part a/ therefore we obtain the right-hand side of the equation. $\square$

Condition 2.6 We exclude immediately the case $(Q+u) \cap IntK = \emptyset$ since, when $u \in NS$, this is equivalent to $P_u = \emptyset$.

Examples 2.7 We note the parameters of two well-studied examples, both with canonical acceptance domain (2.3).

The *octagonal tiling* [**Soc**] has $N = 4$ and $d = 2$, where E is a vector subspace of $\mathbb{R}^4$ invariant under the action of the linear map which maps orthonormal basis vectors $e_1 \mapsto e_2$, $e_2 \mapsto e_3$, $e_3 \mapsto e_4$, $e_4 \mapsto -e_1$. Its orthocomplement, $E^\perp$, is the other invariant subspace. Here $Q = \mathbb{R}^4$ and so many of the distinctions made in subsequent sections are irrelevant to this example.

The *Penrose tiling* [**Pe**] [**dB1**] has $N = 5$ and $d = 2$ (although we note that there is an elegant construction using the root lattice of A_4 in $\mathbb{R}^4$ [**BJKS**]). The linear map which maps $e_i \mapsto e_{i+1}$ (indexed modulo 5) has two 2 dimensional and one 1 dimensional invariant subspaces. Of the first two subspaces, one is chosen as E and the other we name V. Then in fact $Q = E \oplus V \oplus \widetilde{\Delta}$, where $\widetilde{\Delta} = \frac{1}{5}(e_1 + e_2 + e_3 + e_4 + e_5)\mathbb{Z}$, and Q is therefore a proper subset of $\mathbb{R}^5$, a fact which allows the construction of *generalized Penrose tilings* using a parameter $u \in NS \setminus Q$.

Note that we speak of tilings and yet only consider point patterns. In both examples, the projection tiling [**OKD**] is conjugate to both the corresponding strip point pattern and projection point pattern, a fact proved in greater generality in section 8.

We develop these geometric ideas in the following lemmas. The next is Theorem 2.3 from [**Se**].

Theorem 2.8 *Suppose that $\mathbb{Z}^N$ is in standard position in $\mathbb{R}^N$ and suppose that $\phi: \mathbb{R}^N \longrightarrow \mathbb{R}^n$ is a surjective linear map. Then there is a direct sum decomposition $\mathbb{R}^n = V \oplus W$ into real vector subspaces such that $\phi(\mathbb{Z}^N) \cap V$ is dense in V, $\phi(\mathbb{Z}^N) \cap W$ is discrete and $\phi(\mathbb{Z}^N) = (V \cap \phi(\mathbb{Z}^N)) + (W \cap \phi(\mathbb{Z}^N))$.* $\square$

We proceed with the following refinement of Proposition 2.15 of [**Se**].

Lemma 2.9 *Suppose that $\mathbb{Z}^N$ is in standard position in $\mathbb{R}^N$ and suppose that $\phi: \mathbb{R}^N \longrightarrow F$ is an orthogonal projection onto F a subspace of $\mathbb{R}^N$. With the decomposition of F implied by Theorem 2.8, $(F \cap \mathbb{Z}^N) + (V \cap \phi(\mathbb{Z}^N)) \subset \phi(\mathbb{Z}^N)$ as a finite index subgroup.*

Also, the lattice dimension of $F \cap \mathbb{Z}^N$ equals $\dim F - \dim V$ and the real vector subspace generated by $F \cap \mathbb{Z}^N$ is orthogonal to V.

Proof Suppose that U is the real linear span of $\Delta = F \cap \mathbb{Z}^N$. Since Δ is discrete, the lattice dimension of Δ equals the real space dimension of U.

The argument of the proof of Proposition 2.15 in [**Se**] shows that each element of $F \cap \mathbb{Z}^N$ is orthogonal to V. Therefore we have $\dim_{\mathbb{R}}(U) \leq \dim_{\mathbb{R}}(F) - \dim_{\mathbb{R}}(V)$ immediately.

Consider the rational vector space $\mathbb{Q}^N$, contained in $\mathbb{R}^N$ and containing $\mathbb{Z}^N$, both in canonical position. Let U' be the rational span of Δ and note that $U' = U \cap \mathbb{Q}^N$ and that $\dim_{\mathbb{Q}}(U') = \dim_{\mathbb{R}}(U)$. Let $U'^{\perp}$ be the orthocomplement of U' with respect to the standard inner product in $\mathbb{Q}^N$ so that, by simple rational vector space arguments, $\mathbb{Q}^N = U' \oplus U'^{\perp}$. Thus $(U' \cap \mathbb{Z}^N) + (U'^{\perp} \cap \mathbb{Z}^N)$ forms a discrete lattice of dimension N.

Extending to the real span, we deduce that $(U \cap \mathbb{Z}^N) + (U^{\perp} \cap \mathbb{Z}^N)$ is a discrete sublattice of $\mathbb{Z}^N$ of dimension N, hence a subgroup of finite index. Also the lattice dimension of $U \cap \mathbb{Z}^N$ and $U^{\perp} \cap \mathbb{Z}^N$ are equal to $\dim_{\mathbb{R}}(U)$ and $\dim_{\mathbb{R}}(U^{\perp})$ respectively.

Let $L = (U^{\perp} \cap \mathbb{Z}^N)$ be considered as a sublattice of $U^{\perp}$. It is integral (with respect to the restriction of the inner product on $\mathbb{R}^N$)

and of full dimension. The projection ϕ restricts to an orthogonal projection $U^\perp \longrightarrow U^\perp \cap F$ and, by construction, $U^\perp \cap F \cap L = 0$. Therefore Proposition 2.15 of [**Se**] applies to show that $\phi(L)$ is dense in $U^\perp \cap F$ and that ϕ is 1-1 on L.

However $\phi(L) \subset \phi(\mathbb{Z}^N)$ and so, by the characterisation of Theorem 2.2, we deduce that $U^\perp \cap F \subset V$. However, since $U^\perp \supset V$, we have $U^\perp \cap F = V$.

We have $U \cap \mathbb{Z}^N = F \cap \mathbb{Z}^N$ and $\phi(U^\perp \cap \mathbb{Z}^N) = \phi(\mathbb{Z}^N) \cap V$ automatically. Therefore $(\phi(\mathbb{Z}^N) \cap V) + (F \cap \mathbb{Z}^N) = \phi((U^\perp \cap \mathbb{Z}^N) + (U \cap \mathbb{Z}^N))$. As proved above, this latter set is the image of a finite index subgroup of the domain, $\mathbb{Z}^N$, and therefore it is a finite index subgroup of the image $\phi(\mathbb{Z}^N)$ as required.

The remaining properties follow quickly from the details above.
□

Definition 2.10 Let $\Delta = E^\perp \cap \mathbb{Z}^N$ and $\widetilde{\Delta} = U \cap \overline{\pi^\perp(\mathbb{Z}^N)}$ where U is the real vector space generated by Δ.

Note that the discrete group Δ defined here is not the real vector space $\Delta(E)$ defined in [**Le**], but it is a cocompact sublattice and so the dimensions are equal.

Corollary 2.11 *With the notation of Theorem 2.8 and taking $\phi = \pi^\perp$, $\overline{\pi^\perp(\mathbb{Z}^N)} = V \oplus \widetilde{\Delta}$ and $Q = E \oplus V \oplus \widetilde{\Delta}$ are orthogonal direct sums. Moreover, Δ is a subgroup of $\widetilde{\Delta}$ with finite index.* □

Example 2.12 For example the octagonal tiling has $\Delta = 0$ and the Penrose tiling has $\Delta = (e_1 + e_2 + e_3 + e_4 + e_5)\mathbb{Z}$, a subgroup of index 5 in $\widetilde{\Delta}$.

And finally a general result about isometric extensions of dynamical systems.

Definition 2.13 Suppose that $\rho\colon (X, G) \longrightarrow (Y, G)$ is a factor map of topological dynamical systems with group, G, action. If every fibre $\rho^{-1}(y)$ has the same finite cardinality, n, then we say that (X, G) is an n-to-1 extension.

The structure of such extensions, a special case of isometric extensions, is well-known [**F**].

Lemma 2.14 *Suppose that* $\rho\colon (X,G) \longrightarrow (Y,G)$ *is an* n*-to-1 extension and that* (X,G) *is minimal. Suppose further that there is an abelian group* H *which acts continuously on* X*, commutes with the* G *action, preserves* ρ *fibres and acts transitively on each fibre. If* $(X,G) \xrightarrow{\rho'} (Z,G) \xrightarrow{\rho''} (Y,G)$ *is an intermediate factor, then* (Z,G) *is an* m*-to-1 extension where* m *divides* n*, and we can find a subgroup,* H' *of* H*, so that*

i/ H/H' *acts continuously on* Z*, commutes with the* G *action, preserves* ρ'' *fibres and acts transitively on each fibre and*

ii/ H' *acts on* X *as a subaction of* H*, preserving* ρ' *fibres and acting transitively on each fibre.*

Proof Given $h \in H$, consider $X_h = \{x \mid \rho'(x) = \rho'(hx)\}$ which is a closed G-invariant subset of X. Therefore, by minimality, $X_h = \emptyset$ or X. Let $H' = \{h \in H \mid X_h = X\}$ which can be checked is a subgroup of H. The properties claimed follow quickly. $\square$

Definitions 2.15 We will call an extension which obeys the conditions of Lemma 2.14 a *finite isometric extension.*

We say an *almost 1-1 extension* of topological dynamical systems $\rho\colon (X,G) \longrightarrow (Y,G)$ is one in which the set $\rho^{-1}(y)$ is a singleton for a dense G_δ of $y \in Y$. In the case of minimal actions, it is sufficient to find just one point $y \in Y$ for which $\rho^{-1}(y)$ is a singleton.

We say that an abelian topological group, G, acting on a compact abelian topological group, Z say, acts *by rotation* if there is a continuous group homomorphism, $\psi : G \longrightarrow Z$ such that $gz = z + \psi(g)$ for all $z \in Z$ and $g \in G$.

3 Topological spaces for point patterns

When v is non-singular, P_v forms an almost periodic pattern of points in the sense that each spherical window, whose position is shifted over the infinite pattern, reveals the same configuration at a syndetic (relatively dense) set of positions [**Se**]. We may formulate this fact precisely in terms of minimal dynamics in a well-known process. Here we note the relevant constructions and lemmas.

Definition 3.1 Let $B(r)$ be the closed ball in E, centre 0 and of radius r with boundary $\partial B(r)$. Given a closed subset, A, of $\mathbb{R}^N$, define $A[r] = (A \cap B(r)) \cup \partial B(r)$, a closed subset of $B(r)$. Consider the Hausdorff metric d_r defined among closed subsets of $B(r)$ and define a metric (after **[R1]**, **[S2]**) on closed subsets of the plane by

$$D(A, A') = \inf\{1/(r+1) \mid d_r(A[r], A'[r]) < 1/r\}.$$

We are grateful to Johansen for pointing out that the topology induced by D on subsets of E is precisely the topology induced when E is embedded canonically in its one-point compactification, the sphere of dimension $\dim E$, and the Hausdorff metric is used to compare subsets of this sphere. Such an observation proves quickly the following Proposition which appears first in **[Ru]** (see also **[R1]** and **[Ra]**).

Proposition 3.2 *If $u \in NS$, then the sets $\{P_v \mid v \in NS\}$ and $\{P_v \mid v \in u+E\}$ are precompact with respect to D.* □

Definition 3.3 Define

$$MP = \overline{\{P_v \mid v \in NS\}}$$

and

$$MP_u = \overline{\{P_v \mid v \in u+E\}},$$

the completions of the above sets with respect to D. The symbol M is used throughout this paper to indicate a construction such as this: a "Mapping Torus" or *continuous hull*.

Remark 3.4 The term "continuous hull" (of P_u with respect to D) simply refers to the fact that MP_u is the D-closure of the orbit of P_u under the continuous group E. A similar construction starts not with a point set or a tiling but with an operator on a Hilbert space **[B2]**, this is where the name came from. See **[BZH]** for details and a comparison.

Note that $\Delta = 0$ if and only if $MP = MP_u$ for all $u \in NS$, which happens if and only if $MP = MP_u$ for some $u \in NS$.

Also P_v forms a Delone set (see **[S2]**), so we deduce that, for $w \in E$ and $\|w\|$ small enough, $D(P_v, P_{v+w}) = \|w\|/(1+\|w\|)$.

Proposition 3.5 *Suppose that $w \in E$, then the map $P_v \mapsto P_{v+w}$, defined for $v \in NS$, may be extended to a homeomorphism of MP, and the family of homeomorphisms defined by taking all choices of $w \in E$ defines a group action of $\mathbb{R}^d \cong E$ on NS.*

Also for each $u \in NS$, MP_u is invariant under this action of E and E acts minimally on MP_u. $\square$

The dynamical system MP_u with the action by $E \cong \mathbb{R}^d$ is the dynamical system associated with the point pattern P_u, analogous to that constructed by Rudolf [**Ru**] for tilings. We modify this to an action by E on a non-compact cover of MP_u as follows.

Definition 3.6 For $v, v' \in \mathbb{R}^N$, write $\overline{D}(v,v') = D(P_v, P_{v'}) + \|v - v'\|$; this is clearly a metric. Let Π be the completion of NS with respect to this metric.

The following lemma starts the basic topological description of these spaces.

Lemma 3.7 *a/ The canonical injection $NS \longrightarrow \mathbb{R}^N$ extends to a continuous surjection $\mu\colon \Pi \longrightarrow \mathbb{R}^N$. Moreover, if $v \in NS$, then $\mu^{-1}(v)$ is a single point.*

b/ The map $v \mapsto P_v$, $v \in NS$, extends to a continuous E-equivariant surjection, $\eta\colon \Pi \longrightarrow MP$, which is an open map.

c/ The action by translation by elements of E on NS extends to a continuous action of $\mathbb{R}^d \cong E$ on Π.

d/ Similarly the translation by elements of $\mathbb{Z}^N$ is $\overline{D}$-isometric and extends to a continuous action of $\mathbb{Z}^N$ on Π. This action commutes with the action of E found in part c/.

e/ If $a \in MP$ and $b \in \mathbb{R}^N$, then $|\eta^{-1}(a) \cap \mu^{-1}(b)| \leq 1$.

Proof a/ The only non-elementary step of this part is the latter sentence.

We must show that if $v \in NS$ then for all $\epsilon > 0$ there is a $\delta > 0$ such that $\|w - v\| < \delta$ and $w \in NS$ implies that $D(P_w, P_v) < \epsilon$. However, we know that if B is a ball in $\mathbb{R}^N$ of radius much bigger than $1/(2\epsilon)$, then $(\mathbb{Z}^N + v) \cap B$ is of strictly positive distance, say at least 2δ with $\delta > 0$ chosen $< \epsilon/2$, from $\partial\Sigma$. Therefore, whenever $\pi(v - w) = 0$ and $\|v - w\| < \delta$, we have $P_v \cap B = P_w \cap B$ and hence $D(P_v, P_w) < \delta$. On the other hand, if $\pi(v - w) \neq 0$ but $\|v - w\| < \delta$ then we may replace w by $w' = w + \pi(v - w)$, a displacement by less than δ. By the remark (3.4), we deduce that $D(P_w, P_{w'}) < \delta$ and so we have $D(P_w, P_v) < 2\delta < \epsilon$ in general, as required.

b/ The extension to η, and the equivariance and surjectivity, are immediate. The open map condition is quickly confirmed using remark (3.4).

c/ follows from the uniform action of E noted in Remark 3.4. d/ follows similarly where uniform continuity is immediate from the isometry.

Note that e/ is a direct consequence of the definition of the metric $\overline{D}$. □

Definition 3.8 For $u \in NS$, let Π_u be the completion of $E + \mathbb{Z}^N + u$ with respect to the $\overline{D}$ metric.

Lemma 3.9 *For $u \in NS$, Π_u is a closed $E + \mathbb{Z}^N$-invariant subspace of Π. If $x \in \Pi_u$, then $(E + \mathbb{Z}^N)x$, the orbit of x under the E and $\mathbb{Z}^N$ actions, is dense in Π_u. Consequently*

a/ The injection, $E + \mathbb{Z}^N + u \longrightarrow \mathbb{R}^N$ extends to a continuous map, equal to the restriction of μ to Π_u, $\mu_u\colon \Pi_u \longrightarrow \mathbb{R}^N$, whose image is $Q + u$.

b/ By extending the action by translation by elements of $E + \mathbb{Z}^N$ on $E + \mathbb{Z}^N + u$, $E + \mathbb{Z}^N$ acts continuously and minimally on Π_u. This is the restriction of the action of Lemma 3.7 c/ and d/.

c/ The map $v \mapsto P_v$, $v \in E + \mathbb{Z}^N + u$, extends to an open continuous E-equivariant surjection, $\eta_u\colon \Pi_u \longrightarrow MP_u$, which is the restriction of η.

d/ If $x \in \Pi_u$ and $v \in E + \mathbb{Z}^N$ acts on Π_u fixing x, then in fact $v = 0$.

Proof The first sentence is immediate since, by definition, Π_u is the closure of an $E + \mathbb{Z}^N$ orbit in Π.

Suppose that $x \in \Pi_u$ and that $y \in E + \mathbb{Z}^N + u$ which we consider as a subset of Π_u. Then there are $x_n \in E + \mathbb{Z}^N + u$ such that $x_n \to x$ in the $\overline{D}$ metric. Write $\beta_n\colon \Pi_u \longrightarrow \Pi_u$ for the translation action by $-x_n$ and write α for the translation action by y. Then we have $\mu(\beta_n(x)) \to 0$ and so $\mu(\alpha\beta_n(x)) = y + \mu(\beta_n(x)) \to y$.

But, since μ is 1-1 at $y \in NS$ by Lemma 3.7 a/, we deduce that $\overline{D}(\alpha\beta_n(x), y) \to 0$ and so y is in the closure of the $E + \mathbb{Z}^N$ orbit of x. However the orbit of y is dense and so we have the density of the x orbit as well.

The lettered parts follow quickly from this. □

By the results of parts b/ and c/ of Lemma 3.9, we may drop the suffix u from the maps μ_u and η_u without confusion, and this is what we do unless it is important to note the domain explicitly.

The aim of the next few sections is to fill in the fourth corner of the commuting square

$$\begin{tikzcd} \Pi_u \arrow[r, "\eta"] \arrow[d, "\mu"] & MP_u \arrow[d, "?"] \\ (Q+u) \arrow[r, "?"] & ? \end{tikzcd}$$

in a way which illuminates the underlying structure.

4 Tilings and Point Patterns

We now connect the original construction of projection tilings due to Katz and Duneau [**KD**] with the point patterns that we have been considering until now. We refer to [**OKD**] and [**Se**] for precise descriptions of the construction; we extract the points essential for our argument below.

We note two developments of the D metric (3.1) which will be used ahead. The first development is also E.A.Robinson's original application of D [**R1**].

Definition 4.1 We suppose that we have a finite set of pointed compact subsets of E which we call the *units*, and we suppose that we have a uniformly locally finite subset of E, a *point pattern.* A *pattern*, $\mathcal{T}$, in E (with these units and underlying point pattern) is an arrangement of translated copies of the units in E, the distinguished point of each copy placed over a point of the point pattern, no point of the point pattern uncovered and no point of the point pattern covered twice. Sometimes, an underlying point pattern is not mentioned explicitly.

For example we could take a tiling of E and let the pattern consist of the boundaries of the tiles with superimposed decorations, *i.e.* small compact sets, in their interior giving further asymmetries or other distinguishing features. Or we could take a point pattern, perhaps replacing each point with one of a finite number of decorations. See [**GS**] for a thorough discussion of this process in general.

By taking the union of all the units of the pattern, we obtain a locally compact subset $P(\mathcal{T})$ of E which can be shifted by elements of E, $P(\mathcal{T}) \mapsto P(\mathcal{T})+v$, and these various subsets of E can be compared using D literally as defined (3.1) (the addition of further decorations can also solve the problem of confusing overlap of adjacent units of the pattern, a complication which we ignore therefore without loss of generality). Under natural conditions (see **[Ru]** **[S2]**), which are always satisfied in our examples, the space $\{P(\mathcal{T}) + v \mid v \in E\}$ is precompact with respect to the D metric and its closure, the *continuous hull* of $\mathcal{T}$ written $M\mathcal{T}$ here, supports a natural continuous E action. The *pattern dynamical system* of $\mathcal{T}$ is this dynamical system $(M\mathcal{T}, E)$.

Definition 4.2 The second development adapts D to compare subsets of Σ. Recall the notation $B(r)$ for the closed Euclidean r-ball in E (3.1). Let $C(r) = \pi^{-1}(B(r)) \cap \Sigma$ and let $dC(r) = \pi^{-1}(\partial B(r)) \cap \Sigma$.

Given a subset, A, of Σ define $A[r] = (A \cap C(r)) \cup dC(r)$. Let d'_r be the Hausdorff metric defined among closed subsets of $C(r)$ and define a metric on subsets of Σ by

$$D'(A, A') = \inf\{1/(r+1) \mid d'_r(A[r], A'[r]) < 1/r\}$$

Let $\overline{D}'(v, w) = D'(\widetilde{P}_v, \widetilde{P}_w) + \|v - w\|$, where we recall that $\widetilde{P}_v = \Sigma \cap (\mathbb{Z}^N + v)$.

Let $M\widetilde{P}_u$ be the D'-closure of the space $\{\widetilde{P}_v \mid v \in E+u\}$, and let $\widetilde{\Pi}_u$ be the $\overline{D}'$ completion of $NS \cap (Q+u)$. Let $M\widetilde{P}$ be the D' closure of the space $\{\widetilde{P}_v \mid v \in NS\}$.

The analogues of Proposition 3.5 and Lemma 3.9 with respect to $\widetilde{P}$, $M\widetilde{P}$, $\widetilde{\Pi}_u$, $M\widetilde{P}_u$ and $Q+u$, continue to hold and so we define maps $\widetilde{\mu}\colon \widetilde{\Pi}_u \longrightarrow Q+u$ and $\widetilde{\eta}\colon \widetilde{\Pi}_u \longrightarrow M\widetilde{P}_u$.

We use the projection π to compare the strip pattern with the projection pattern. It will turn out that π is a homeomorphism between Π_u and $\widetilde{\Pi}_u$, but that the definition of $\widetilde{\Pi}_u$ will be more convenient than that of Π_u. Using π we may work with either space.

Theorem 4.3 *There are E-equivariant maps π_* induced by the projection π which complete the commuting square*

$$\begin{array}{ccc}
\widetilde{\Pi}_u & \xrightarrow{\pi_*} & \Pi_u \\
\Big\downarrow{\scriptstyle \widetilde{\eta}} & & \Big\downarrow{\scriptstyle \eta} \\
M\widetilde{P}_u & \xrightarrow{\pi_*} & MP_u
\end{array}$$

Furthermore we have the following commuting square

$$\begin{tikzcd} \widetilde{\Pi}_u \arrow[r, "\pi_*"] \arrow[d, "\widetilde{\mu}"] & \Pi_u \arrow[d, "\mu"] \\ Q+u \arrow[r, equal] & Q+u \end{tikzcd}$$

in which all the labelled maps are 1-1 on NS. □

Consider the example of the canonical projection tiling, $\mathcal{T}_u$ (2.3). If we know $\widetilde{P}_u$ then we have all the information needed to reconstruct $\mathcal{T}_u$ by its definition. Conversely, the usual assumption that the projected faces are non-degenerate (see **[Le]** (3.1)) allows us to distinguish the orientation of the lattice face (in $\mathbb{Z}^N$) from which a given tile came. Piecing together all the faces defined this way obtains $\widetilde{P}_u$. So the canonical projection tiling is conjugate (in the sense defined ahead in 4.5) to $\widetilde{P}_u$.

On the other hand, the well-known Voronoi or Dirichlet tiling **[GS]** obtained from a point pattern in E is a tiling conjugate to the original point pattern provided we decorate each tile with the point which generates it.

With these two examples of tiling in mind, we consider the pattern $\widetilde{P}_u$ to represent the most elaborate tiling or pattern that can be produced by the projection method, without imposing further decorations not directly connected with the geometry of the construction, and at the other extreme, the point pattern, P_u, represents the least decorated tiling or pattern which can be produced by the projection method.

Definition 4.4 For a given E and K as in (2.1), we include in the *class of projection method patterns* all those patterns, $\mathcal{T}$, of $\mathbb{R}^d$ such that there is a $u \in NS$ and two E-equivariant surjections

$$M\widetilde{P}_u \longrightarrow M\mathcal{T} \longrightarrow MP_u$$

whose composition is π_*.

We call (E, K, u) the *data* of the projection method and by presenting these data we require tacitly that K has the properties of Definition 2.1, that $u \in NS$ and that $(Q + u) \cap IntK \neq \emptyset$ (2.6).

Thus the tilings of [**OKD**] and the Voronoi tilings discussed above are examples from this class when $K = \pi^\perp([0,1]^N)$. In order to compare these two constructions, or to consider projection method patterns in the general sense of (4.4), we aim to describe $\pi_*: M\widetilde{P}_u \longrightarrow MP_u$.

First we adopt the following definitions which possibly duplicate notions already existing in the literature.

Definitions 4.5 Adapting a definition of Le [**Le**], we say that two patterns, $\mathcal{T}, \mathcal{T}'$, in E are *topologically conjugate* if there is an E-equivariant homeomorphism, $M\mathcal{T} \leftrightarrow M\mathcal{T}'$.

Say that the two patterns, $\mathcal{T}, \mathcal{T}'$, are *pointed conjugate* if there is an E-equivariant homeomorphism, $M\mathcal{T} \leftrightarrow M\mathcal{T}'$, which maps $\mathcal{T}$ to $\mathcal{T}'$.

A pattern $\mathcal{T}'$ is a *finite decoration* of a pattern $\mathcal{T}$ if there are real numbers r and s so that the following happens: i/ $\mathcal{T}$ may be constructed from $\mathcal{T}'$ by a transformation which alters the unit of $\mathcal{T}$ at a point $v \in \mathbb{R}^d$ according only to how $\mathcal{T}$ appears in the ball $v+B(r)$: and ii/, conversely, if, for any choice of $w \in \mathbb{R}^d$ we know what $\mathcal{T}'$ looks like in the ball $w + B(s)$, then we can construct the remainder of $\mathcal{T}'$ from $\mathcal{T}$ by a transformation (depending perhaps on the appearance of $\mathcal{T}'$ in the ball $w + B(r)$) which alters the unit of $\mathcal{T}'$ at a point $v \in \mathbb{R}^d$ according only to how $\mathcal{T}'$ appears in the ball $v + B(r)$.

Finally, we say that a pattern is a *Meyer pattern* if the underlying point pattern is a Meyer set, that is a set M for which we can find R and r so that $M - M$ intersects every R-ball in at least one point and intersects every r ball in at most one point.

Remark 4.6 Note that all the patterns we consider in this paper are (pointed conjugate to) Meyer patterns. This is the starting point for Schlottmann's analysis of the projection method [**Sch**].

To tie these definitions in to the exisiting literature, we note that topological conjugacy is strictly weaker than local isomorphism (as in [**Le**] for example) and strictly stronger than equal quasicrystal type [**R1**]. Pointed conjugacy is strictly stronger than mutual local derivability [**BSJ**] and topological equivalence [**K3**], but has no necessary relation with local isomorphism and quasicrystal type. Finite decoration is strictly weaker than local derivability [**BSJ**].

However, we have the following, an immediate application of the definitions to the fact that an n-to-1 factor map (see 2.13) is an open map [**F**].

Lemma 4.7 *Suppose we have two Meyer patterns, $\mathcal{T}, \mathcal{T}'$, in E and* *a continuous E-equivariant surjection $M\mathcal{T}' \longrightarrow M\mathcal{T}$ which is n-to-1, sending $\mathcal{T}'$ to $\mathcal{T}$. Then $\mathcal{T}'$ is a finite decoration of $\mathcal{T}$.* $\square$

5 Comparing Π_u and $\widetilde{\Pi}_u$

We start by examining $\pi_*: \widetilde{\Pi}_u \longrightarrow \Pi_u$ from (4.3) and seek conditions under which it is a homeomorphism. As the section proceeds we shall find that the conditions can be whittled away to the minimum possible. Recall the space V, one of the orthocomponents of the decomposition of Q in Corollary 2.11. It is the connected component of $\overline{\pi^\perp(\mathbb{Z}^N)}$ containing 0.

Lemma 5.1 *Suppose that $u \in NS$ and that, for all $v \in Q+u$ such that $v \in \partial((V+v)\cap IntK)$ (the boundary taken in $V+v$), we have $(\Delta+v)\cap K = \{v\}$; then $\pi_*: \widetilde{\Pi}_u \longrightarrow \Pi_u$ is an E-equivariant homeomorphism.*

Proof We ask under what circumstances could we find $x \in \Pi_u$ with two preimages under π_* in $\widetilde{\Pi}_u$? We would need two sequences $v_n, w_n \in (Q+u)\cap NS$ both converging to x in the $\overline{D}$ metric such that $\widetilde{P}_{v_n}$ and $\widetilde{P}_{w_n}$ have different $\overline{D}'$ limits, say A and B respectively. From this we see that $A\Delta B \subset \partial\Sigma$ (symmetric difference) and yet $\pi(A) = \pi(B)$.

Let $p \in \pi(A\Delta B)$ and consider the set $(A\Delta B)\cap\pi^{-1}(p)$. As noted above, this set is a subset of the boundary of $\Sigma\cap\pi^{-1}(p) \equiv K$ and each pair of elements is separated by some element of Δ.

Suppose that $a \in (A \setminus B)\cap\pi^{-1}(p)$. By construction, there are $a_n \in (Q+u)\cap NS\cap\widetilde{P}_{v_n}$ converging to a implying that $a \in \partial((Q+u)\cap IntK)$. But by hypothesis, we deduce $B\cap\pi^{-1}(p) = \emptyset$ - a contradiction to the fact that $p \in \pi(A) = \pi(B)$.

A symmetric argument produces a contradiction from $b \in (B \setminus A)\cap\pi^{-1}(p)$. $\square$

Note that if $\Delta = 0$ or, more generally, if $K\cap(K+\delta) = \emptyset$ whenever $\delta \in \Delta$, $\delta \neq 0$, then the hypothesis of the Lemma is satisfied trivially.

Corollary 5.2 *If $\Delta = 0$, then $\pi_*: \widetilde{\Pi}_u \longrightarrow \Pi_u$ is an E-equivariant homeomorphism.* $\square$

In special cases the hypothesis is satisfied less trivially. We give a slightly more special condition here.

Lemma 5.3 *Suppose that J is the closure of a fundamental domain for Δ in $E^\perp$, and that $J = \overline{IntJ}$ in $E^\perp$. If K is contained in some* *translate of J, then $\pi_*: \widetilde{\Pi}_u \longrightarrow \Pi_u$ is a homeomorphism. In particular, if $K = \pi^\perp([0,1]^N)$, then $\pi_*: \widetilde{\Pi}_u \longrightarrow \Pi_u$ is a homeomorphism.*

Proof For the first part, suppose that $a, b \in K$ and $0 \neq a-b = \delta \in \Delta$, then, by construction, a and b sit one in each of two hyperplanes orthogonal to δ between which K lies. Note that then these hyperplanes are therefore both parallel to V and each intersects K only in a subset of ∂K. Therefore, $a, b \in \partial K$ and further, since $V+a$ and $V+b$ are contained one in each of the hyperplanes, we have $a \notin \partial((V+a) \cap IntK)$ (boundary in $V+a$) and $b \notin \partial((V+b) \cap IntK)$ (*similis*). Therefore the conditions of Lemma 5.1 are fulfilled vacuously.

In the second part, suppose the domain $K = \pi^\perp([0,1]^N)$ and that $\delta = (\delta_1, \delta_2, ..., \delta_N) \in \Delta$, $\delta \neq 0$ (the case $\Delta = 0$ is easy). Consider the set $I = \{\langle \delta, t \rangle \mid t \in K\}$, where $\langle ., . \rangle$ is the inner product on $\mathbb{R}^N$. This is a closed interval. Also, since δ is fixed by the orthonormal projection $\pi^\perp$, $I = \{\langle \delta, s \rangle \mid s \in [0,1]^N\}$, from which we deduce that the length of I is $\sum_j |\delta_j|$. But since $|\delta_j| < 1$ implies that $\delta_j = 0$, we have $\langle \delta, \delta \rangle = \sum |\delta_j|^2 \geq \sum |\delta_j|$ and so K can be fitted between two hyperplanes orthogonal to δ and separated by δ.

Therefore K is contained in a translate of

$$\bigcap_{\delta \in \Delta, \delta \neq 0} \{v \in E^\perp \mid |\langle v, \delta \rangle| \leq (1/2)\langle \delta, \delta \rangle \},$$

which in turn is contained in the closure of a fundamental domain for Δ. So we have confirmed the conditions of the first part. $\square$

Remark 5.4 Using Lemma 2.5, the condition of Proposition 5.3 is equivalent to the following condition: $((IntK) - (IntK)) \cap \Delta = \{0\}$, where we write $A - A = \{a - b \mid a, b \in A\}$ for the arithmetic (self-)difference of A, a subset of an abelian group. Compare with 8.2 ahead.

All of these results say that if K is small enough relative to Δ then π_* is a homeomorphism. In fact, we can dispense with all such conditions, and the following construction gives a procedure to reduce the size of a general acceptance domain appropriately.

Theorem 5.5 *Suppose that E and K are as in definition 2.1, and that $u \in NS$. Then $\pi_* : \widetilde{\Pi}_u \longrightarrow \Pi_u$ is an E-equivariant homeomorphism.*

Proof Suppose that E, K and u are chosen as required and that $\Delta \neq 0$. The case $\Delta = 0$ is covered by Corollary 5.2.

Suppose that J is the closure of a fundamental domain for Δ in $E^{\perp}$, such that $J = \overline{IntJ}$ in $E^{\perp}$ and suppose, as we always can by shifting J if necessary, that $\partial J \cap (Q + u) = \emptyset$.

Let $K' = (K + \Delta) \cap J$. Then K' is a subset of $E^{\perp}$ which obeys the conditions required in the original definition of (2.1). Also the placement of J ensures that the points in $Q + u$, in particular u itself, which are non-singular with respect to K are also non-singular with respect to K'.

Moreover, if we define $\Sigma' = K' + E$, then, by construction, $\pi(\Sigma' \cap (v + \mathbb{Z}^N)) = \pi(\Sigma \cap (v + \mathbb{Z}^N)) = P_u$ for all $v \in \mathbb{R}^N$. Therefore, working with Σ' instead of Σ, we can retrieve the projection point pattern, P_u. So, by Lemma 5.3 and the fact that $K' \subset J$, we see that π_* is a homeomorphism between the spaces Π_u and $\widetilde{\Pi}_u(\Sigma')$, the construction of (4.2) with respect to Σ'.

However, for any $v \in E + \mathbb{Z}^N + u$, we have the equalities: $\Sigma' \cap (v+\mathbb{Z}^N) = \Sigma' \cap ((\Sigma \cap (v+\mathbb{Z}^N)) + \Delta)$ and $\Sigma \cap (v+\mathbb{Z}^N) = \Sigma \cap ((\Sigma' \cap (v+\mathbb{Z}^N)) + \Delta)$. Therefore the set $\Sigma' \cap (v+\mathbb{Z}^N)$ can be constructed from $\Sigma \cap (v + \mathbb{Z}^N) = \widetilde{P}_v$ and vice versa. Moreover, this correspondence defines a $\overline{D}$ metric isometry, between $\widetilde{\Pi}_u(\Sigma')$ and $\widetilde{\Pi}_u$, which intertwines π_*. Completing the correspondence gives an isometry between $\widetilde{\Pi}_u(\Sigma')$ and $\widetilde{\Pi}_u$ which intertwines π_*, as required. $\square$

Remark 5.6 We note a second process of reduction without loss of generality. Until now we have assumed nothing about the rational position of E, but it is convenient to assume and is often required in the literature that $E \cap \mathbb{Z}^N = 0$: the *irrational* case.

If we do not assume this then we can always reduce to the irrational case by quotienting out the rational directions. A simple argument allows us to find in the most general case of projection method pattern, P say, an underlying irrational projection method pattern, P', with, $MP = MP' \times \mathbb{T}^k$ for some value of k; and this torus factor splits naturally with respect to the various constructions and group actions we find later. We leave the details to the reader.

6 Calculating $M\widetilde{P}_u$ and MP_u

We now describe $M\widetilde{P}_u$ and MP_u as quotients of $\widetilde{\Pi}_u$ and Π_u respectively. Although the last section established an equivalence between $\widetilde{\Pi}_u$ and Π_u, we find it useful to distinguish the two constructions.

First we examine $M\widetilde{P}_u$ and prove a generalisation of (3.8) of [**Le**].

Proposition 6.1 *Suppose that $u \in NS$, then there is an isometric action of $\mathbb{Z}^N$ on $\widetilde{\Pi}_u$, which factors by $\widetilde{\mu}$ to the translation action by $\mathbb{Z}^N$ on $Q+u$, and $M\widetilde{P}_u = \widetilde{\Pi}_u/\mathbb{Z}^N$. Thus we obtain a commutative square of E equivariant maps*

$$\begin{array}{ccc}
\widetilde{\Pi}_u & \xrightarrow{\widetilde{\eta}} & M\widetilde{P}_u \\
\Big\downarrow \widetilde{\mu} & & \Big\downarrow \widetilde{\mu} \\
Q+u & \longrightarrow & (Q+u)/\mathbb{Z}^N.
\end{array}$$

The left vertical map is 1-1 precisely at the points in $NS \cap (Q+u)$. The right vertical map is 1-1 precisely on the same set, modulo the action of $\mathbb{Z}^N$.

Proof The action of $\mathbb{Z}^N$ on $\widetilde{\Pi}_u$, as an extension of the action on $Q+u$ by translation, is easy to define since the maps are $\overline{D}'$-isometries.

If $v, w \in NS$ then it is clear that $\widetilde{P}_v = \widetilde{P}_w$ if and only if $v - w \in \mathbb{Z}^N$. Moreover, there is $\delta > 0$ so that $\|v-w\| < \delta$ implies that $D'(\widetilde{P}_v, \widetilde{P}_w) \geq \|v-w\|/2$.

From this we see that, if $\widetilde{P}_v = \widetilde{P}_w$ and $\widetilde{P}_{v'} = \widetilde{P}_{w'}$ and $\|v-v'\| < \delta/2$ and $\|w-w'\| < \delta/2$, then $v - w = v' - w'$. The uniformity of δ irrespective of the choice of v, w, v' and w' shows that the statement $\widetilde{\eta}(v) = \widetilde{\eta}(w)$ implies $\widetilde{\mu}(v) - \widetilde{\mu}(w) \in \mathbb{Z}^N$, which is true for $v, w \in NS \cap (Q+u)$, is in fact true for all pairs in $\widetilde{\Pi}_u$, the $\overline{D}'$ closure.

To show the 1-1 properties for the map on the left, suppose that $v \in Q+u$ and that $p \in \partial\Sigma \cap (\mathbb{Z}^N + v)$, i.e. $v \notin NS$. Then since K is the closure of its interior and since NS is dense in $\mathbb{R}^N$ (Lemma 2.2), there are two sequences $v_n, v'_n \in NS$ both converging to v in Euclidean topology and such that $p + (v_n - v) \in \Sigma$ and $p + (v'_n - v) \notin \Sigma$. This implies that any D' limit point of $\widetilde{P}_{v_n}$ contains p and any D' limit point of $\widetilde{P}_{v'_n}$ does not contain p. But both such limit points (which exist by compactness of $M\widetilde{P}$) are in $\widetilde{\mu}^{-1}(v)$ which is a set of at least two elements therefore.

The 1-1 property for the map on the right follows directly from this and the commuting diagram. □

The space $(Q+u)/\mathbb{Z}^N$ and its E action, which is being compared with $M\widetilde{P}_u$, also has a simple description.

Lemma 6.2 *With the data above, $(Q+u)/\mathbb{Z}^N$ is a coset of the closure of E mod $\mathbb{Z}^N$ in $\mathbb{R}^N/\mathbb{Z}^N \equiv \mathbb{T}^N$. Therefore $(Q+u)/\mathbb{Z}^N$ with its E action is isometrically conjugate to a minimal action of $\mathbb{R}^d$ by rotation on a torus of dimension $N - \dim \Delta$.*

Proof The space Q mod $\mathbb{Z}^N$ is equal to the closure of E mod $\mathbb{Z}^N$ and its translate by u mod $\mathbb{Z}^N$ is an isometry which is E equivariant. The action of E on its closure is isometric and transitive, hence minimal, and is by translations. E is a connected subgroup of $\mathbb{T}^N$ and so also is the closure of E, which is therefore equal to a torus of possibly smaller dimension. The codimension of this space agrees with the codimension of $V+E$ (the continuous component of Q) in $\mathbb{R}^N$ which, by Lemma 2.9 and Corollary 2.11, equals $\dim \Delta$ as required. $\square$

Now we turn to a description of MP_u which is similar in form to that of $M\widetilde{P}_u$, but as to be shown in examples 8.7 and 8.8, need not be equal to $M\widetilde{P}_u$.

Lemma 6.3 *Suppose that $u \in NS$. If $v, w \in NS \cap (Q+u)$ and $P_v = P_w$ then there are $v^* \in v + \mathbb{Z}^N$ and $w^* \in w + \mathbb{Z}^N$ such that $v^*, w^* \in \Sigma$ and $\pi(v^*) = \pi(w^*)$, and with this choice $\widetilde{P}_v + \Delta - \pi^\perp(v^*) = \widetilde{P}_w + \Delta - \pi^\perp(w^*)$.*

Proof Fix $p_o \in P_v = P_w$ and let $v^* \in \widetilde{P}_v$ be chosen so that $\pi(v^*) = p_o$ and similarly, let $w^* \in \widetilde{P}_w$ be chosen so that $\pi(w^*) = p_o$. Clearly v^* and w^* obey the conditions required. Also $\widetilde{P}_w - \pi^\perp(w^*)$ and $\widetilde{P}_v - \pi^\perp(v^*)$ are both contained in $p_o + \mathbb{Z}^N$ and project under π to the same set P_v. Thus the difference of two points, one in $\widetilde{P}_w - \pi^\perp(w^*)$ and the other in $\widetilde{P}_v - \pi^\perp(v^*)$, and each with the same image under π, is an element of Δ as required. $\square$

Proposition 6.4 *Suppose that $x, y \in \Pi_u$ and that $\eta(x) = \eta(y)$, then there is a $v \in Q$ and a $\overline{D}$ isometry $\phi\colon \Pi_u \longrightarrow \Pi_u$ so that $\phi(x) = \phi(y)$ and the following diagram commutes*

$$\begin{array}{ccc} \Pi_u & \xrightarrow{\phi} & \Pi_u \\ \Big\downarrow \mu & & \Big\downarrow \mu \\ Q+u & \xrightarrow{w \mapsto w+v} & Q+u \end{array}$$

and $\eta_u\phi = \eta_u$ (here the restriction to Π_u is important to note, c.f. 3.9). In this case we deduce $v+u \in NS$.

Conversely, if we have such an isometry in such a diagram and if $P_{u+v} = P_u$, then $v+u \in NS$ and $\eta_u\phi = \eta_u$ automatically.

Proof Suppose $w \in E+\mathbb{Z}^N$ and that $\alpha_w : \Pi_u \longrightarrow \Pi_u$ is the map completed from the map $z \mapsto z+w$ defined first for $z \in NS \cap (Q+u)$ (see Proposition 3.5). Then, since $\eta(x) = \eta(y)$ and η is $(E+\mathbb{Z}^N)$-equivariant, we have $\eta(\alpha_w(x)) = \eta(\alpha_w(y))$ for all $w \in E+\mathbb{Z}^N$. So, by definition, the map $\alpha_w(x) \mapsto \alpha_w(y)$ defined point-by-point for $w \in E+\mathbb{Z}^N$ is a $\overline{D}$ isometry from the $(E+\mathbb{Z}^N)$-orbit of x onto the $(E+\mathbb{Z}^N)$-orbit of y (By Lemma 3.9 d/ the mapping is well-defined). These two orbits being dense (Lemma 3.9) in Π_u, this map extends as an isometry onto, $\phi : \Pi_u \longrightarrow \Pi_u$.

Since μ is $(E+\mathbb{Z}^N)$-equivariant, we deduce the intertwining with translation by $v = \mu(y) - \mu(x)$. Also since $\eta\phi = \eta$ on the $E+\mathbb{Z}^N$ orbit of x, the E-equivariance of η extends this equality over all of Π_u.

Conversely suppose we have an isometry which intertwines the translation by v on $Q+u$. Then for general topological reasons the cardinality of the μ preimage of a point in $Q+u$ is preserved by translation by v and we deduce that $NS \cap (Q+u)$ is invariant under the translation by v. In particular $u+v \in NS$. The equation follows since it applies, by hypothesis and Lemma 6.3, at u and therefore, by equivariance, at all points in $E+\mathbb{Z}^N+u$, a dense subset. □

Definition 6.5 For $u \in NS$, let $R_u = \{v \in Q \mid v+u \in NS, P_{u+v} = P_u\}$.

Corollary 6.6 *Suppose that $u \in NS$ and $w \in NS \cap (Q+u)$, then $R_w = R_u$. Therefore, if $v \in R_u$, then $v+w \in NS \cap (Q+u)$ for all $w \in NS \cap (Q+u)$.*

Proof By Lemma 3.9, we know that $\Pi_w \xrightarrow{\eta_w} Q+w$ equals $\Pi_u \xrightarrow{\eta_u} Q+u$ and so any isometry of Π_u which factors by η through to a translation by v also does the same for Π_w. Proposition 6.4 completes the equivalence.

The second sentence follows directly from the definition of R_w. □

Remarks 6.7 It would be natural to hope that the condition $u+v \in NS$ could be removed from the definition of R_u. We have been unable to do this in general. But since NS is a dense G_δ set (2.2) and, anticipating Theorem 7.1, R_u is countable, we see that for a dense G_δ set of $u \in NS$ (*generically*) we can indeed equate $R_u = \{v \in Q : P_{u+v} = P_u\}$.

This is bourne out in Corollary 6.6 where we see that R_u is defined independently of the choice of u generically, and R_u can be thought of as an invariant of Π_u. This result also shows that R_u is a subset of the translations of $\mathbb{R}^N$ which leave $NS \cap (Q+u)$ invariant.

Note that, since μ is 1-1 only on NS, R_u could as well have been defined as $\{v \in Q \mid \eta\mu^{-1}(u+v) = \{P_u\}\ \}$.

It is clear that $\mathbb{Z}^N \subset R_u$.

Theorem 6.8 *If $u \in NS$, then R_u is a closed subgroup of Q. Also R_u acts by ϕ isometrically on Π_u and defines a homeomorphism $\Pi_u/R_u \equiv MP_u$. Moreover the R_u action commutes with the E-action, so the homeomorphism is E-equivariant.*

Proof The main point to observe is that R_u consists precisely of those elements v such that there is an isometry ϕ_v as in Proposition 6.4 with $\eta_u\phi_v = \eta_u$. Since the inverse of such an isometry is another such, and the composition of two such isometries produces a third, we deduce the group property for R_u immediately. The isometric action is given to us and Proposition 6.4 shows directly that $\Pi_u/R_u \equiv MP_u$.

Closure of R_u is more involved. Suppose that $v_n \in R_u$ and that $v_n \to v$ in the Euclidean topology. Then ϕ_{v_n} is uniformly Cauchy and so converges uniformly to a bijective isometry, ψ, of Π_u which intertwines the translation by v on $Q+u$. Therefore, if $\mu^{-1}(u+v)$ has at least two elements, then so also does $\psi^{-1}\mu^{-1}(u+v)$, but this set is contained in $\mu^{-1}(u)$, a contradiction since $\mu^{-1}(u)$ is a singleton. Therefore $u+v \in NS$ and $\mu^{-1}(u+v) = \psi\mu^{-1}(u) = \lim \phi_{v_n}\mu^{-1}(u) = \lim \mu^{-1}(u+v_n) = \mu^{-1}(u)$. Thus $v \in R_u$ and so R_u is closed.

The commutativity with the E action on Π_u is immediate from the corresponding commutativity on $Q+u$. $\square$

7 Comparing MP_u with $M\widetilde{P}_u$

In Section 4 we defined projection method patterns as those whose dynamical system sits intermediate to $M\widetilde{P}_u$ and MP_u. We discover in this section how closely these two spaces lie and circumstances under which they are equal.

To compare MP_u with $M\widetilde{P}_u$ we start with the fact that $\widetilde{\Pi}_u = \Pi_u$ (5.5). By Proposition 6.1 and Theorem 6.8, therefore, the problem becomes the comparison of R_u with $\mathbb{Z}^N$. Perhaps surprisingly, under general conditions we find that R_u is not much larger than $\mathbb{Z}^N$ and under normal conditions the two groups are equal.

Theorem 7.1 *For all $u \in NS$, $\mathbb{Z}^N \subset R_u$ as a finite index subgroup. In fact, with the notation of (2.10), $R_u \subset \mathbb{Z}^N + \widetilde{\Delta}$.*

Proof Suppose that $v \in R_u$. Then in particular, by (6.7), $P_{v+u} = P_u$. Therefore there is an $a \in \mathbb{Z}^N$ such that $\pi(v + u + a) = \pi(u)$ and so by translating if necessary, we may assume without loss of generality that $v \in E^\perp$; and this defines v uniquely $\mod \Delta$.

With this assumption we deduce from Lemma 6.3 that $\widetilde{P}_{u+v} + \Delta = \widetilde{P}_u + \Delta + v$ In particular, $\pi^\perp(\widetilde{P}_{u+v}) + \Delta = \pi^\perp(\widetilde{P}_u) + \Delta + v$.

Now each of $\pi^\perp(\widetilde{P}_{u+v})$ and $\pi^\perp(\widetilde{P}_u)$ is contained in K a compact set. Suppose that $\alpha \in \Delta^\perp$ and that $\langle v, \alpha\rangle \neq 0$, then there is $t \in \mathbb{Z}$ such that $|\langle tv, \alpha\rangle| > 2\|\alpha\|\ diamK$. However, since $tv \in R_u$ by Theorem 6.3, we have $\pi^\perp(\widetilde{P}_{tv+u}) + \Delta = \pi^\perp(\widetilde{P}_u) + \Delta + tv$. Applying the function $\langle ., \alpha\rangle$ to both sets produces a contradiction by construction. Thus we have $v \in U$, the space generated by Δ (see 2.9). But if $v \in R_u$ then $v \in Q$ by definition, and so we have $v \in \widetilde{\Delta}$, a group which, by Corollary 2.11, contains Δ with finite index. $\square$

We note, for use in section 10, that therefore R_u is free abelian on N generators.

Corollary 7.2 *If $\Delta = 0$, then $R_u = \mathbb{Z}^N$ and $\pi_*: M\widetilde{P}_u \leftrightarrow MP_u$.* $\square$

The following combines Propositions 6.1 and 6.8 and fits the present circumstances to the conditions of Lemma 2.14. Recall the definitions of n-to-1 extension (2.13) and finite isometric extension (2.15).

Proposition 7.3 *Suppose that $u \in NS$. The map $\pi_*: M\widetilde{P}_u \longrightarrow MP_u$ is p-to-1 where p is the index of $\mathbb{Z}^N$ in R_u. The group R_u acts isometrically on $M\widetilde{P}_u$, commuting with the E action, preserving π_* fibres and acting transitively on each fibre. This action is mapped almost 1-1 by $\widetilde{\mu}$ to an action by R_u on $(Q+u)/\mathbb{Z}^N$ by rotation, and so we complete a commuting square*

$$\begin{array}{ccc} M\widetilde{P}_u & \xrightarrow{\pi_*} & MP_u \\ \Big\downarrow \widetilde{\mu} & & \Big\downarrow \mu \\ (Q+u)/\mathbb{Z}^N & \xrightarrow{mod\ R_u/\mathbb{Z}^N} & (Q+u)/R_u. \end{array}$$

$\square$

From this and the construction of Lemma 2.14 applied to the case $G = E$ and $H = R_u$, we deduce the main theorem of the section.

Theorem 7.4 *Suppose that, E, K and $u \in NS$ are data for $\mathcal{T}$, a projection method pattern. Then there is a group $S_{\mathcal{T}}$, intermediate to $\mathbb{Z}^N < R_u$, which fits into a commutative diagram of E equivariant maps*

$$\begin{array}{ccccc}
M\widetilde{P}_u & \longrightarrow & M\mathcal{T} & \longrightarrow & MP_u \\
\downarrow{\scriptstyle \widetilde{\mu}} & & \downarrow & & \downarrow{\scriptstyle \mu} \\
(Q+u)/\mathbb{Z}^N & \xrightarrow{mod\ S_{\mathcal{T}}/\mathbb{Z}^N} & (Q+u)/S_{\mathcal{T}} & \xrightarrow{mod\ R_u/S_{\mathcal{T}}} & (Q+u)/R_u
\end{array}$$

where the top row maps are finite isometric extensions and the bottom row maps are group quotients.

Conversely, every choice of group S' intermediate to $\mathbb{Z}^N < R_u$ admits a projection method pattern, $\mathcal{T}$, fitting into the diagram above with $S' = S_{\mathcal{T}}$. □

With the considerations of section 4 (in particular using 4.7) we can count the projection method patterns up to topological conjugacy or pointed conjugacy in the following corollary.

Corollary 7.5 *With fixed projection data and the conditions of Theorem 7.4, the set of topological conjugacy classes of projection method patterns is in bijection with the lattice of subgroups of $R_u/\mathbb{Z}^N$. Moreover, each projection method pattern, $\mathcal{T}$, is pointed conjugate to a finite decoration of P_u, and $\widetilde{P}_u$ is pointed conjugate to a finite decoration of $\mathcal{T}$.* □

We deduce the following result also found by Schlottmann [**Sch**], which is in turn a generalisation of the result of Robinson [**R2**] and the topological version of the result of Hof [**H**].

Corollary 7.6 *With the conditions assumed in Theorem 7.4, the pattern dynamical system $M\mathcal{T}$ is an almost 1-1 extension (2.15) of a minimal $\mathbb{R}^d$ action by rotation on a $(N - \dim \Delta)$-torus.*

Proof It suffices to show that the central vertical arrow in the diagram of Theorem 7.4 is 1-1 at some point. But this is immediate since each of the end arrows is 1-1 at u say. $\square$

8 Examples and Counter-examples

In this section we give sufficient conditions, similar to and stronger than 5.3, under which P_u or $\mathcal{T}$ is pointed conjugate to $\widetilde{P}_u$, and show why these conjugacies are not true in general.

Definition 8.1 For data (E, K, u), define $B_u = \overline{(Q + u) \cap IntK}$ (Euclidean closure in $E^\perp$).

Proposition 8.2 *Suppose that E, K and $u \in NS$ are chosen so that $E \cap \mathbb{Z}^N = 0$ and $\Delta \cap [(B_u - B_u) - (B_u - B_u)] = \{0\}$, then $R_u = \mathbb{Z}^N$. In this case, therefore, P_u is pointed conjugate to $\widetilde{P}_u$.*

Proof This follows from the fact, deduced directly from the condition given, that if $v \in NS \cap (Q + u)$ and $a, b \in P_v$, then we can determine $w - w'$ whenever $w, w' \in \widetilde{P}_v$ are such that $a = \pi(w)$ and $b = \pi(w')$. Knowing the differences of elements of $\widetilde{P}_v$ forces the position of $\widetilde{P}_v$ in Σ by the density of $\pi^\perp(\widetilde{P}_v)$ in B_u. So we can reconstruct $\widetilde{P}_v$ uniquely from P_v and we have $R_u = \mathbb{Z}^N$. $\square$

Corollary 8.3 *In the case that the acceptance domain is canonical, the condition that the points $\pi(w) \mid w \in \{-1, 0, 1\}^N$ are all distinct is sufficient to show that $R_u = \mathbb{Z}^N$ for all $u \in NS$. In this case, therefore, P_u is pointed conjugate with $\widetilde{P}_u$.*

Proof The condition implies that $\Delta \cap [(K - K) - (K - K)] = \{0\}$ and this gives the condition in the proposition since $B_u \subset K$. $\square$

If we are interested merely in the equation between $\mathcal{T}$ and $\widetilde{P}_u$, then the case that the acceptance domain is canonical also allows simple sufficient conditions weaker than 8.3, as we show below.

We observe first that the construction of **[OKD]** can be extended to admit non-generic parameters, provided that we are comfortable with “tiles” which, although they are convex polytopes, have no interior in E and are unions of faces of the tiles with interior. We retain these degenerate tiles as components of our “tiling”, *i.e.* as units of a pattern, giving essential information about the pattern dynamics. We call such patterns *degenerate* canonical projection tilings.

We write e_j with $1 \leq j \leq N$ for the canonical unit basis of $\mathbb{Z}^N$.

Proposition 8.4 *In the case of a canonical acceptance domain, the condition that no two points from $\{\pi(e_j) \mid 1 \leq j \leq N\}$ are collinear is sufficient to show that $\mathcal{T}_u$, the canonical (but possibly degenerate) tiling, is pointed conjugate to $\widetilde{P}_u$ for all $u \in NS$.*

Proof We show that the conditions given imply that the shape of a tile (even in degenerate cases) determines from which face of the lattice cube it is projected. In fact we shall show that if $I \subset \{1, 2, ..., N\}$ then knowing $\pi(\gamma^I)$ and the cardinality of I determines I (we write $\gamma^I = \{\sum_{i \in I} \lambda_i e_i \mid 0 \leq \lambda_i \leq 1, \ \forall\, i \in I\}$).

Suppose that $\pi(\gamma^I) = \pi(\gamma^J)$ and $I, J \subset \{1, 2, .., N\}$ are of the same cardinality. It is possible always to distinguish an edge on the polyhedron $\pi(\gamma^I)$ which is parallel to a vector $\pi(e_i)$ for some $i \in I$; and i is determined from this edge by hypothesis. The same is true of this same edge with respect to J and so $i \in J$ also.

Writing $I' = I \setminus \{i\}$ and $J' = J \setminus \{i\}$ we deduce that $\pi(\gamma^{I'}) = \pi(\gamma^I) \cap (\pi(\gamma^I) - \pi(e_i)) = \pi(\gamma^J) \cap (\pi(\gamma^J) - \pi(e_i)) = \pi(\gamma^{J'})$. Now we can apply induction on the cardinality of I, and deduce that $I' = J'$ and so $I = J$. Induction starts at cardinality 1 by hypothesis.

Now, given this correspondence between shape of tile and its preimage under π, we reconstruct $\widetilde{P}_u$ from $\mathcal{T}_u$ much as we did in Proposition 8.2 above. To complete the argument we must check that no other element of $M\widetilde{P}_u$ maps onto $\mathcal{T}_u$ in $M\mathcal{T}_u$. But if there were such an element, then the argument above shows that it cannot be of the form $\widetilde{P}_{u'}$ for $u' \in NS$. Also, by Theorem 7.4, we deduce that the map $M\widetilde{P}_u \longrightarrow M\mathcal{T}_u$ is p-to-1 with $p \geq 2$, and so, using Lemma 2.2ii/, we find $\mathcal{T}_v$ with two preimages of the form $\widetilde{P}_v$ and $\widetilde{P}_{v'}$. But this contradicts the principle of the previous sentence. $\square$

The conditions of this proposition include all the non-degenerate cases of the canonical projection tiling usually treated in the literature (including the Penrose tiling), so from the equation $M\mathcal{T}_u = \widetilde{\Pi}_u/\mathbb{Z}^N$, deduced from Proposition 8.4 as a consequence, we retrieve many of the results stated in section 3 of **[Le]**.

Now we turn to conditions under which R_u differs from $\mathbb{Z}^N$. We can extend the argument of 7.1 to give a geometric condition for elements of R_u, of considerable use in computing examples.

Lemma 8.5 *Suppose that $E \cap \mathbb{Z}^N = 0$, $u \in NS$ and $v \in E^\perp$. Then $v \in R_u \cap E^\perp$ if and only if $v + u \in NS$ and $v + (B_u + \Delta) = B_u + \Delta$.*

Proof Suppose that $u, v + u \in NS$. Then $\overline{\pi^\perp(\widetilde{P}_u)} = \overline{(Q + u) \cap IntK}$ and $\overline{\pi^\perp(\widetilde{P}_{u+v})} = \overline{(Q + u + v) \cap IntK}$.

If $v \in R_u \cap E^\perp$, then, by Lemma 6.3, we have $\widetilde{P}_u + \Delta + v = \widetilde{P}_{u+v} + \Delta$. Also, since by 7.1, v is in $\widetilde{\Delta}$, we have $Q + u = Q + u + v$ and $\pi^\perp(\widetilde{P}_u) + \Delta + v = \pi^\perp(\widetilde{P}_{u+v}) + \Delta$. Putting all these together gives the required equality $v + (B_u + \Delta) = B_u + \Delta$.

Conversely, if $v + (B_u + \Delta) = B_u + \Delta$, then, by the argument of 7.1, $v \in \widetilde{\Delta}$ and so, as above, $Q + u = Q + u + v$ and $\pi^\perp(\widetilde{P}_u) + \Delta + v = \pi^\perp(\widetilde{P}_{u+v}) + \Delta$. So, if $a \in \widetilde{P}_u$, then there is a $b \in \widetilde{P}_{u+v}$ such that $\pi^\perp(a) - \pi^\perp(b) \in \Delta - v$. However, since $\pi^\perp$ is 1-1 on $\mathbb{Z}^N$, we can retrieve the set $\widetilde{P}_u$ as the inverse $\pi^\perp$ image of $\pi^\perp(\widetilde{P}_u) \cap IntK$ and similarly for $\widetilde{P}_{u+v}$. This forces $a - b \in \Delta - v$ therefore, and so $\pi(a) = \pi(b)$. Thus we see that $P_u = P_{u+v}$, as is required to show that $v \in R_u$ $\square$

Example 8.6 By Corollary 7.2 and the fact that $\Delta = 0$, the octagonal tiling is pointed conjugate to both its projection and strip point patterns.

Also it is from Lemma 8.5 (and not 8.3) that we can deduce that $R_u = \mathbb{Z}^5$ for the (generalised) Penrose tiling for all choices of $u \in NS$, and so the generalised Penrose tiling is pointed conjugate to both its projection and strip point patterns.

Lemma 8.5 makes it quite easy to construct counter-examples to the possibility that $R_u = \mathbb{Z}^N$ always.

Example 8.7 We start with a choice of E for which $\Delta \neq 0$ and $\widetilde{\Delta}$ contains Δ properly. The E used to construct the Penrose tiling is such an example. As in the proof of Lemma 2.6, let U be the real span of Δ and V the orthocomplement of U in $E^\perp$. Choose a closed unit disc, I, in V and let J be the closure of a rectangular fundamental domain for Δ in U. Let $K = I + J \subset E^\perp$ and note that K has all the propeties required of an acceptance domain in this paper and that, by Lemma 5.1, we have $\Pi_u = \widetilde{\Pi}_u$.

With $u \notin V + \widetilde{\Delta}$ (equivalently $u \in NS$), the rectilinear construction of K ensures that $((Q + u) \cap IntK) + \Delta$ is invariant under the translation by any element, v, of $\widetilde{\Delta}$. Also $v + u \in NS$ since the boundary of $((Q + u) \cap IntK) + \Delta$ in $Q + u$ is invariant under translations by $\widetilde{\Delta}$. So, with the characterisation of Lemma 8.5, this shows that $R_u = \widetilde{\Delta} + \mathbb{Z}^N$ which is strictly larger than $\mathbb{Z}^N$.

By varying the shape of J in this example, we can get $R_u/\mathbb{Z}^N$ equal to any subgroup of $(\widetilde{\Delta} + \mathbb{Z}^N)/\mathbb{Z}^N$, and we can make it a non-constant function of $u \in NS$ as well.

Example 8.8 Take $N = 3$ with unit vectors e_1, e_2 and e_3. Let L be the plane orthonormal to $e_1 - e_2$ and let E be a line in L placed so that $E \cap \mathbb{Z}^3 = \{0\}$. Then $E^\perp$ is a plane which contains $e_1 - e_2$ and we have $\Delta = \{n(e_1 - e_2) \mid n \in \mathbb{Z}\}$.

Write $e_1^\perp$, $e_2^\perp$, and $e_3^\perp$ for the image under $\pi^\perp$ of e_1, e_2 and e_3 respectively. Then $e_1^\perp + e_2^\perp$ and $e_3^\perp$ are collinear in $E^\perp$ and they are both contained in V (the continuous subspace of $V + \widetilde{\Delta} = \overline{\pi^\perp(\mathbb{Z}^3)}$). Δ is orthogonal to V and $e_1^\perp - e_2^\perp = e_1 - e_2$. However $\widetilde{\Delta} = \{n(e_1 - e_2)/2 \mid n \in \mathbb{Z}\}$ which contains Δ as an index 2 subgroup.

The set $K = \pi^\perp([0,1]^3)$ is a hexagon in $E^\perp$ with a centre of symmetry. It is contained in the closed strip defined by two lines, $V + a$ and $V + b$, where $b - a = e_1 - e_2$, and it is in fact reflectively symmetric around an intermediate line, $V + c$ where $c - a = (e_1 - e_2)/2$. The boundary of the hexagon on each of $V + a$ and $V + b$ is an interval congruent to $e_3^\perp$ (*i.e.* a translate of $\{te_3^\perp \mid 0 \le t \le 1\}$). The four other sides are intervals congruent to $e_1^\perp$ or $e_2^\perp$, two of each. The vertices of the hexagon are on $V + a$, $V + b$ or $V + c$, two on each.

The point of all this is that there is a choice of non-singular u (in $E^\perp$ without loss of generality) such that $B_u = \overline{(Q + u) \cap IntK}$ consists of the two intervals $K \cap (V + a')$ and $K \cap (V + b')$, where $2a' = a + c$ and $2b' = b + c$ (we can choose $u \in (V + a') \cap NS$ for example), and these intervals are a translate by $\pm(e_1 - e_2)/2$ of each other. Thus we deduce that $B_u + \Delta = B_u + \Delta + v$ for all $v \in \widetilde{\Delta}$.

Upon confirming that $v + u \in NS$ for all $v \in \widetilde{\Delta}$ as well, we use 8.5 to show that $R_u = \mathbb{Z}^3 + \widetilde{\Delta}$, which contains $\mathbb{Z}^3$ with index 2.

Remark 8.9 We note that Example 8.8 is degenerate and Proposition 8.4 shows why this must be the case. However, under any circumstances, there exist projection method tilings, in the sense of 4.4, pointed conjugate to P_u or to $\widetilde{P}_u$. The point here is that these tilings will not necessarily be constructed by the special method of Katz and Duneau.

Also, leaving the details to the reader, we mention that Example 8.8 and its analogues in higher dimensions are the only counter-examples to the assertion $R_u = \mathbb{Z}^N$ when the acceptance domain is canonical and Δ is singly generated (and here we find always that $R_u/\mathbb{Z}^N$ is at most a cyclic group of 2 elements). When Δ is higher dimensional we have no concise description of the exceptions allowed.

9 The topology of the continuous hull

Sections 6 and 7 tell us that the continuous hulls MP_u and $M\widetilde{P}_u$ are quotients of the same space $\Pi_u \equiv \widetilde{\Pi}_u$. One advantage of this equality is that the topology of $\widetilde{\Pi}_u$ is more easily described than Π_u *a priori*.

Definition 9.1 Let F be a plane complementary, but not necessarily orthonormal, to E and let π' be the skew projection (idempotent map) onto F parallel to E.

Let $K' = \pi'(K)$.

Set $F_u^o = NS \cap (Q+u) \cap F$ and let F_u be the $\overline{D}'$-closure (Def. 4.2) of F_u^o in $\widetilde{\Pi}_u$.

Note that, since $R_u \subset Q$, F_u^o is invariant under translation by $\pi'(r)$, $r \in R_u$ and by extension F_u supports a continuous R_u action. R_u acts freely on F_u when $E \cap \mathbb{Z}^N = 0$ (i.e. with $x \in F_u$, the equation $gx = x$ implies $g = 0$).

Similarly, R_u acts on E by translation by $\pi(r)$, $r \in R_u$.

Lemma 9.2 *With the data above, $F_u = \widetilde{\mu}^{-1}(F \cap (Q+u))$ (c.f. (7.3)) and there is a natural equivalence $\widetilde{\Pi}_u \equiv F_u \times E$ and a surjection $\nu\colon F_u \longrightarrow ((Q+u) \cap F)$ which fits into the following commutative square*

$$\begin{array}{ccc}
\widetilde{\Pi}_u & \longleftrightarrow & F_u \times E \\
\Big\downarrow{\scriptstyle \widetilde{\mu}} & & \Big\downarrow{\scriptstyle \nu \times id} \\
Q+u & \longleftrightarrow & ((Q+u) \cap F) \times E.
\end{array}$$

Moreover these maps are E-equivariant where we require that E acts trivially on F_u. The set $\nu^{-1}(v)$ is a singleton whenever $v \in NS \cap F \cap (Q+u)$.

The canonical action of R_u on $\widetilde{\Pi}_u$ is represented in this equivalence as the direct sum (i.e. diagonal) of the action of R_u on F_u and E described in (9.1).

Proof This follows quickly from the observation that there is a natural, $\overline{D}'$-uniformly continuous and E equivariant equivalence $NS \cap (Q+u) = F_u^o + E \equiv F_u^o \times E$, which can be completed. $\square$

Definitions 9.3 Let $\mathcal{A}_u$ be the algebra of subsets (*i.e.* closed under finite union, finite intersection and symmetric difference) of F_u^o generated by the sets $(NS \cap (Q+u) \cap K') + \pi'(v)$ as v runs over $\mathbb{Z}^N$. It is clear that this algebra is countable and invariant under the action of R_u.

Write $C^*(\mathcal{A}_u)$ for the smallest C^* algebra which contains the indicator functions of the elements of $\mathcal{A}_u$.

Let $\mathbb{Z}\mathcal{A}_u$ be the ring (pointwise addition and multiplication) generated by this same collection of indicator functions.

Let CF_u be the group of continuous integer valued functions compactly supported on F_u.

These three algebraic objects support a canonical R_u action induced by the action of R_u on F_u described in (8.1) and so we define three $\mathbb{Z}[R_u]$ modules. As $\mathbb{Z}^N$ sits inside R_u, this action can be restricted to a canonical subaction by $\mathbb{Z}^N$ thereby defining three $\mathbb{Z}[\mathbb{Z}^N]$ modules.

Let $\mathcal{B}_u = \{\overline{A} \mid A \in \mathcal{A}_u\}$ where the bar refers to $\overline{D}'$ closure in F_u.

And finally, we give the main theorem of this section which is of utmost importance for the remaining chapters.

Theorem 9.4 *With data (E, K, u) and $\widetilde{\Pi}_u = \Pi_u$,*

a/ The collection $\mathcal{B}_u$ is a base of clopen neighbourhoods which generates the topology of F_u.

b/ We have the $*$-isomorphisms of C^*-algebras $C_o(F_u) \cong C^*(\mathcal{A}_u)$ and $C_o(\widetilde{\Pi}_u) \cong C^*(\mathcal{A}_u) \otimes C_o(E)$ which respect the maps defined in Lemma 8.2.

c/ $CF_u \cong \mathbb{Z}\mathcal{A}_u$ as a $\mathbb{Z}[R_u]$ module (and by pull-back as a $\mathbb{Z}[\mathbb{Z}^N]$ module).

d/ F_u is locally a Cantor Set.

First we have a lemma also of independent interest in the next section.

Definition 9.5 Write $\overline{K}$ for the $\overline{D}'$-closure of the set $K' \cap NS \cap (Q+u)$ (recall K' from 9.1).

Lemma 9.6 $\overline{K}$ *is a compact open subset of F_u.*

Proof Closure is by definition so compactness follows immediately on observing that $K' \cap (Q+u) \cap NS$ is embedded $\overline{D}'$-isometrically in the space $M\widetilde{P}_u \times K'$ with metric $D' + \|.\|$ as the closed subset $\{(\widetilde{P}_v, v) \mid v \in K' \cap (Q+u) \cap NS\}$. But $M\widetilde{P}_u \times K'$ is compact.

For openness, we appeal to an argument similar to that of (6.1). Suppose, for a contradiction, that v_n is a $\overline{D}'$-convergent sequence in $(F \cap NS) \cap K'$ and that v'_n is a $\overline{D}'$ convergent sequence in $(F \cap NS) \setminus K'$ and that both sequences have the same limit $x \in \widetilde{\Pi}_u$. Therefore $v = \widetilde{\mu}(x)$ is the Euclidean limit of the v_n and v'_n and so $v \in \partial K$. But by construction $\widetilde{P}_{v_n}$ and $\widetilde{P}_{v'_n}$ have a different D' limit – a contradiction since the limit in each case must be $\widetilde{\eta}(x)$. □

Proof of Theorem 9.4 a/ The sets in $\mathcal{B}_u$ are clopen by Lemma 9.6 above. Therefore the metric topology on F_u we are considering, let us call it τ, is finer than the topology τ' generated by $\mathcal{B}_u$. Both topologies are invariant under the action of $\mathbb{Z}^N$ so that it suffices to show their equivalence on some closed r-ball X of F_u. Suppose that τ' were a Hausdorff topology. Then this equivalence would follow from the continuity of the identity map $(X, \tau) \to (X, \tau')$, because compactness of (X, τ) implies that of (X, τ') and so the map would automatically be a homeomorphism. On the other hand, we will proof below that $A \mapsto \overline{A}$ yields an isomorphism of Boolean algebras between $\mathcal{A}_u$ and $\mathcal{B}_u$. In particular, $\mathcal{B}_u$ is closed under symmetric differences so that τ' being T_1 already implies that it is Hausdorff.

It remains therefore to check that τ' is T_1, i.e. that the collection $\mathcal{B}_u$ contains a decreasing set of neighbourhoods around any point in F_u.

Certainly, if $a \neq b$ with $a, b \in F \cap (Q+u)$, then the assumption that $Int(K) \cap (Q+u) \neq \emptyset$ (2.6) (hence $Int(K') \cap (Q+u) \neq \emptyset$, interior taken in F) and the facts that K' is bounded and that $\pi'(\mathbb{Z}^N)$ is dense in $Q \cap F$, imply that there is some $v \in \mathbb{Z}^N$ such that $a \in (Int(K') \cap (Q+u)) + \pi'(v)$ and $b \notin \overline{(K' \cap (Q+u))} + \pi'(v)$ (Euclidean closure in $F \cap (Q+u)$). *i.e.* a and b are separated by the topology induced by $\widetilde{\mu}(\mathcal{B}_u)$.

In particular, if $x, y \in F_u$ and $y \in \cap\{B \in \mathcal{B}_u \mid x \in B\}$ then $\widetilde{\mu}(x) = \widetilde{\mu}(y)$.

But, if $x \neq y$ and $\widetilde{\mu}(x) = \widetilde{\mu}(y) = v$, then, by (3.7)e/, $\widetilde{\eta}(x) \neq \widetilde{\eta}(y)$ and we may suppose that there is a point $p \in \widetilde{\eta}(x) \setminus \widetilde{\eta}(y)$. We use the argument of Lemma 8.6 to show that then there are two sequences $v_n, v'_n \in \pi'(\mathbb{Z}^N)$ both converging to v in Euclidean topology and such that $p + (v_n - v) \in \Sigma$ and $p + (v'_n - v) \notin \Sigma$ for all n large enough. But then, for such a choice of n, $y \notin \overline{A}$ (closure of $A = (NS \cap (Q+u) \cap K') + \pi'(p + v_n)$ in $\overline{D}'$ metric) and $x \in \overline{A}$, a contradiction to the construction of y.

Therefore $x = y$ and so, by the local compactness (Lemma 9.6) of F_u we have the required basic property of the collection $\mathcal{B}_u$.

b/ This will follow from a/ and the equivalences in Lemma 9.2 if we can show that $\mathcal{A}_u$ is isomorphic to $\mathcal{B}_u$ as a Boolean algebra. To show this, it is enough to show that $A \mapsto \overline{A}$ (closure in $\overline{D}'$ metric) is 1-1 on $\mathcal{A}_u$; and for this it suffices to prove that if $A \in \mathcal{A}_u$ is non-empty, then its Euclidean closure has interior (in $(Q+u) \cap F$).

Note that $NS \cap K' = NS \cap Int(K')$, so that if $A \in \mathcal{A}_u$ then A is formed of the union and intersection of sets of the form $(NS \cap (Q + u) \cap Int(K')) + v$ $(v \in \pi'(\mathbb{Z}^N))$, and the subtraction of unions and intersections of sets of the form $(NS \cap (Q+u) \cap K') + v$. With this description and Lemma 2.5, A is equal to $NS \cap Int(\overline{A})$ (Euclidean closure, and interior in $(Q+u) \cap F$), and from this our conclusion follows.

c/ Elements of CF_u are finite sums of integer multiples of indicator functions of compact open sets. Such sets are finite unions of basic clopen sets from the collection in part a/. The isomorphism in part b/ completes the equation.

d/ Given the results of a/ and Lemma 9.6 it is sufficient to show that F_u has no isolated points. However, by the argument of part b/ and Lemma 2.5 we see that every clopen subset of F_u has $\widetilde{\mu}$ image with Euclidean interior (in $(Q+u) \cap F$) and so cannot be a single point. $\square$

10 A Cantor $\mathbb{Z}^d$ Dynamical System

In this section we describe a $\mathbb{Z}^d$ dynamical system whose mapping torus is equal to $M\mathcal{T}$. First, assuming projection data, (E, K, u) and $E \cap \mathbb{Z}^N = 0$, we find a suitable F to which to apply the construction of the previous section.

Definition 10.1 Suppose that G is a group intermediate to $\mathbb{Z}^N$ and R_u. The examples in our applications ahead are $G = \mathbb{Z}^N$, $G = R_u$ and $G = S_\mathcal{T}$, found in Theorem 7.4.

Fix a free generating set, $r_1, r_2, ..., r_N$, for G and suppose that the first $\dim \Delta$ of these generate the subgroup $G \cap E^\perp$ (this can be required by Lemma 2.9).

Let F be the real vector space spanned by $r_1, r_2, ..., r_n$, where $n = N - d$.

Note that, since $E \cap G = 0$ (by Lemma 2.9 and the assumption $E \cap \mathbb{Z}^N = 0$), F is complementary to E and, since $n \geq \dim \Delta$, F contains Δ.

Note that by Theorem 7.1, any two groups intermediate to R_u and $\mathbb{Z}^N$, G and G' say, differ only by some elements in $R_u \cap E^\perp$, a complemented subgroup of R_u. Thus we may fix their generating sets to differ only among those elements which generate $G \cap E^\perp$ or $G' \cap E^\perp$ respectively. With this convention therefore, the construction of F is independent of the choice of group intermediate to R_u and $\mathbb{Z}^N$ and F depends only on the data (E, K, u) chosen at the start.

Also, more directly, the elements $r_{\dim \Delta+1}, ..., r_N$ depend only on the data (E, K, u) chosen at the start.

Definition 10.2 Suppose G_0 is the subgroup of G generated by $r_1, r_2, ..., r_n$, and that G_1 is the complementary subgroup generated by the other d generators. Since $n \geq \dim \Delta$, G_1 depends only on the data (E, K, u) and not on the choice of G intermediate to $\mathbb{Z}^N$ and R_u.

Both groups, G_1 and G_0 act on F_u and E as subactions of R_u (Definition 9.1).

Let $X_G = F_u/G_0$, a space on which G_1 acts continuously.

Theorem 10.3 *Suppose that we have projection data (E, K, u) such that $E \cap \mathbb{Z}^N = 0$ and G, a group intermediate to $\mathbb{Z}^N$ and R_u. Then X_G is a Cantor set on which G_1 acts minimally and there is a commutative square of G_1 equivariant maps*

$$\begin{array}{ccc} F_u & \xrightarrow{q} & X_G \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu'} \\ F \cap (Q+u) & \longrightarrow & (F \cap (Q+u))/G_0. \end{array}$$

The set $\nu'^{-1}(v)$ is a singleton whenever $v \in (NS \cap F \cap (Q+u))/G_0$.

The space $(F \cap (Q+u))/G_0$ is homeomorphic to a finite union of tori each of dimension $(N - d - \dim \Delta)$. Indeed, this space can be considered as a topological group, in which case it is the product of a subgroup of $\widetilde{\Delta}/\Delta$ with the $(N - d - \dim \Delta)$-torus. The action of G_1 on this space is by rotation and is minimal.

Proof Assuming we have proved the fact that X_G is compact then the commuting square and its properties follow quickly. Therefore we look at X_G.

Since G_0 acts isometrically on F_u with uniformly discrete orbits (Theorem 6.8), q is open and locally a homeomorphism and so X_G inherits the metrizability of F_u, a base of clopen sets and the lack of isolated points (see Theorem 9.4 d/).

Now, let $Y_o = \{\sum_{1\leq j\leq n} \lambda_j r'_j \mid 0 \leq \lambda_j < 1\} \cap (Q+u)$, a subset of $F \cap (Q+u)$. Choose $J \subset \mathbb{Z}^N$ finite but large enough that $Y_1 = \cup_{v\in J}((K' \cap (Q+u)) + \pi'(v))$ contains $\overline{Y}_o$ (Euclidean closure). In particular $q(\cup_{v\in J}(\overline{K} + \pi'(v))) = X_G$, the image of a compact set (Lemma 9.6) under a continuous map. So X_G is also compact.

Therefore, we have checked all conditions that show X_G is a Cantor set.

Minimality follows from the minimality of the G action on F_u which in turn follows from the minimality of the $\mathbb{Z}^N + E$ action on $\widetilde{\Pi}_u$, proved analogously to (3.9).

The structure of the rotational factor system follows quickly from the first part of this lemma, the structure of $F \cap (Q+u)$, and Lemma 9.2. $\square$

Now we describe X_G as a fundamental domain of the action of G_0.

Definition 10.4 From the details of 10.3 we constuct a clopen fundamental domain for the action of G_0 on F_u. We let $G_0^+ = \{\sum_{1\leq j\leq n} \alpha_j r_j \mid \alpha_j \in \mathbb{N}\}$ and set $Y^+ = \cup_{r\in G_0^+}(Y_1 + r)$ and define $Y = Y^+ \setminus \cup_{r\in G_0^+, r\neq 0}(Y^+ + r)$.

Define $Y_G = \overline{\nu^{-1}(Y \cap NS)}$ (closure in the $\overline{D}'$ metric), a subset of F_u.

The following is immediate from this construction, using Lemma 9.6 and the equivariance of ν and ν' in Lemma 9.3 with respect to the R_u action.

Lemma 10.5 *With data (E, K, u), $E \cap \mathbb{Z}^N = 0$, and the definitions above, Y is a fundamental domain for the translation action by G_0 on $F \cap (Q+u)$. Moreover, Y_G is a compact open subset of F_u, and a fundamental domain for the action by G_0 on F_u.*

There is a natural homeomorphism $X_G \leftrightarrow Y_G$ which is G_1 equivariant. $\square$

Definition 10.6 Define CX_G to be the ring of continuous integer valued functions defined on X_G. Also define $C(F_u; \mathbb{Z})$ to be the ring of continuous integer valued functions defined on F_u without restriction on support, uniformity or magnitude (c.f. the definition of CF_u from 9.3). As $\mathbb{Z}[G_0]$ modules, the first is trivial and the second is defined as usual using the subaction of the R_u action on F_u. Both are non-trivial $\mathbb{Z}[G_1]$ modules.

The following combines Lemmas 10.3, 10.5 and Proposition 9.4 and will be of much importance in Chapter II.

Corollary 10.7 *With the data of Lemma 10.5,*

$$CF_u \cong CX_G \otimes \mathbb{Z}[G_0]$$

and

$$C(F_u;\mathbb{Z}) \cong Hom_{\mathbb{Z}}(\mathbb{Z}[G_0], CX_G)$$

as $\mathbb{Z}[G_0]$ *modules.* □

Definition 10.8 Let E' be the real span of $r_{n+1}, ..., r_N$ selected in Definition 10.1. This space contains G_1 (the integer span $r_{n+1}, ..., r_N$) as a cocompact subgroup.

Recall the definition of dynamical mapping torus **[PT]** which for the G_1 action on X_G may be equated with

$$MT(X_G, G_1) = (X_G \times E')/\langle (gx, v) - (x, v - g) \mid g \in G_1 \rangle.$$

Proposition 10.9 *Suppose that we have data* (E, K, u) *such that* $E \cap \mathbb{Z}^N = 0$*, and* G*, a group intermediate to* $\mathbb{Z}^N$ *and* R_u*. With the definitions above,* E' *is a d-dimensional subspace of* $\mathbb{R}^N$ *complementary to both* F *and* $E^\perp$*. Also* $MT(X_G, G_1) \equiv \widetilde{\Pi}_u/G$*.*

Proof The transformation $E' \longrightarrow E$ defined by $r_j \mapsto \pi(r_j)$, $n+1 \leq j \leq N$ is bijective since G_1 is complementary to G_0 and hence to the subset $G \cap E^\perp$ of G_0. From this we deduce the complementarity immediately.

From Lemma 10.3 we see that G_0 acts naturally on $MT(F_u, G_1)$ and that $MT(X_G, G_1) \equiv MT(F_u, G_1)/G_0$.

To form $MT(F_u, G_1)$ we take $F_u \times E'$ and quotient by the relation $(ga, v) \sim (a, v - g)$, $g \in G_1, a \in F_u, v \in \mathbb{R}^d$. Applying the inverse of the map of the first paragraph, we can re-express the mapping torus as $F_u \times E$ quotiented by the relation $(ga, w + \pi(g)) \sim (a, w)$, $g \in G_1, a \in F_u, w \in E$.

However, the action of G_1 on the F_u is that induced by translation on F_u^o by elements $\pi'(g) \mid g \in G_1$. So, working first on the space F_u^o, we have the equations

$$MT(F_u^o, G_1) = (F_u^o \times E)/\langle (a + \pi'(g), w + \pi(g)) - (a, w) \mid g \in G_1 \rangle$$

and, since $F_u^o \times E = NS$, we may write $(a, w) = v \in NS$ and so the quotient equals

$$NS/\langle v + g - v \mid g \in G_1 \rangle = NS/G_1$$

(recall that NS is G_1 invariant by Corollary 6.6). Then, by completing, we deduce the equation $MT(F_u, G_1) = \widetilde{\Pi}_u/G_1$ directly. A further quotient by G_0 completes the construction. □

Corollary 10.10 *Suppose that $\mathcal{T}$ is a projection method pattern with data (E, K, u) such that $E \cap \mathbb{Z}^N = 0$. Then $(X_\mathcal{T}, G_1)$ is a minimal Cantor $\mathbb{Z}^d$ dynamical system, whose mapping torus $MT(X_\mathcal{T}, G_1)$ is homeomorphic to $M\mathcal{T}$. The pattern dynamical system, $(M\mathcal{T}, E)$ is equal to the canonical $\mathbb{R}^d$ action on the mapping torus $(MT(X_\mathcal{T}, G_1), \mathbb{R}^d)$ up to a constant time change.*

Proof Choose $G = S_\mathcal{T}$ from Theorem 7.4 which gives $M\mathcal{T} \equiv \widetilde{\Pi}_u/G$. From this, all but the time change information follows quickly from 10.9 and 10.3, noting that $G_1 \cong \mathbb{Z}^d$.

To compare the two $\mathbb{R}^d$ actions, we apply the constant time change which takes the canonical $\mathbb{R}^d (\cong E')$ action on $MT(X_\mathcal{T}, \mathbb{Z}^d)$ to the canonical $\mathbb{R}^d (\cong E)$ action on $M\mathcal{T}$ by the isomorphism $\pi|_{E'}\colon E' \longrightarrow E$, mapping generators of the G_1 action $r_j \mapsto \pi(r_j)$ for $n+1 \le j \le N$. □

Examples 10.11 The dynamical system of 10.10 for the octagonal tiling is a $\mathbb{Z}^2$ action on a Cantor set, an almost 1-1 extension of a $\mathbb{Z}^2$ action by rotation on $\mathbb{T}^2$ (see **[BCL]**). For the Penrose tiling, it is also a Cantor almost 1-1 extension of a $\mathbb{Z}^2$ action by rotation on $\mathbb{T}^2$ (see **[R1]**), where we must check carefully that the torus factor has only one component (the only alternative of 5 components is excluded *ad hoc*).

The correspondence in 10.9 and 10.10 respects the structures found in Theorem 7.4 and so we deduce an analogue.

Definition 10.12 With data (E, K, u) and the selection $G = \mathbb{Z}^N$, perform the constructions of 10.1 and 10.2, writing the G_0 and X_G obtained there as G_u and X_u respectively.

Similarly, with $G = S_{\mathcal{T}}$, write G_0 as $G_{\mathcal{T}}$ and X_G as $X_{\mathcal{T}}$.

Also, with $G = R_u$, write G_0 as $\widetilde{G}_u$ and X_G as $\widetilde{X}_u$.

Note again that F and G_1 are the same for all three choices of G.

Corollary 10.13 *Suppose that we have data (E, K, u) such that $E \cap \mathbb{Z}^N = 0$. Then we can construct two Cantor dynamical systems, (X_u, G_1) and $(\widetilde{X}_u, G_1)$, the latter a finiteisometric extension of the former, together with a compact abelian group, M, which is a finite union of $(N - d - \dim \Delta)$-dimensional tori (independent of u) on which G_1 acts minimally by rotation, and a finite subgroup, Z_u, of M.*

These have the property that, if $\mathcal{T}$ is a projection method pattern with data (E, K, u), then there is a subgroup $Z_{\mathcal{T}}$ of Z_u and a commuting diagram of G_1-equivariant surjections

$$\begin{array}{ccccc}
\widetilde{X}_u & \longrightarrow & X_{\mathcal{T}} & \longrightarrow & X_u \\
\downarrow & & \downarrow & & \downarrow \\
M & \longrightarrow & M/Z_{\mathcal{T}} & \longrightarrow & M/Z_u
\end{array}$$

In this diagram, the top row consists of finite isometric extensions, the bottom row of group quotients and the vertical maps are almost 1-1.

Taking the G_1-mapping torus of this diagram produces the diagram of Theorem 7.4.

Proof With the groups $G_u, G_{\mathcal{T}}$ and $\widetilde{G}_u$ defined in (10.12). By the remark after 10.1, we know that $G_u < G_{\mathcal{T}} < \widetilde{G}_u$. Using the notation of sections 7, 9 and 10, set $Z_u = \widetilde{G}_u/G_u \cong R_u/\mathbb{Z}^N$ and $Z_{\mathcal{T}} = G_{\mathcal{T}}/G_u \cong S_{\mathcal{T}}/\mathbb{Z}^N$. Also set $M = ((Q + u) \cap F)/G_u$, and note that, since $\mathbb{Z}^N = G_1 + G_u$, a direct sum, G_u is in fact independent of the choice of non-singular u. Thus M is independent also of u (the uniform translation by u is irrelevant), so we attach no subscript.

With this notation and the equivalences above, together with the description of the systems, X_u and $\widetilde{X}_u$, using Theorem 10.3, we complete the result. $\square$

II Groupoids, C*-algebras, and their Invariants

1 Introduction

In this chapter we develop the connections between the pattern dynamical systems described in Chapter I and the pattern groupoid. As with the continuous hull, a pattern groupoid, which we write $\mathcal{G}\mathcal{T}^*$, can be defined abstractly for any pattern, $\mathcal{T}$, of Euclidean space and we refer to **[K1]** **[K3]** for the most general definitions. Here we give a special form for projection method patterns.

The (reduced) C^*-algebra, $C^*(\mathcal{G}\mathcal{T}^*)$, of this groupoid is a non-commuta-tive version of the mapping torus; this should be regarded as a more precise detector of physical properties of the quasicrystal and the discrete Schröding-er operators for the quasicrystal are naturally members of this algebra.

The initial purpose of this chapter is to compare the non-commutative structure, *i.e.* , the groupoid, of a pattern with the dynamical systems constructed before. In this regard, we cover similar ground to the work of Bellissard *et al.* **[BCL]** but, as noted in the introduction, applied to a groupoid sometimes different. Our Theorem 4.2 is the complete generalization of their connection between C^*-algebraic K-theory and dynamical cohomology, found for projection method tilings in 2 dimensions.

The dynamical system and C^*-algebras we associate to a pattern give rise to a range of ways of attaching an invariant to the pattern $\mathcal{T}$. These include C^* and topological K-theories, continuous groupoid cohomology, Čech cohomology and the dynamical or group (co)homology. The second main result of the chapter is to set up and demonstrate that all these invariants are isomorphic as groups, though the non-commutative invariants contain the richer structure of an ordered group. This structure appears likely to contain information relevant to subsequent investigations, only some of which is recoverable from the other invariants. On the other hand, as we shall see in later chapters, the group (co)homology invariants admit greater ease of computationand is often sufficient for other applications.

The final result of the chapter shows that all these common invariants provide an obstruction to the property of self similarity of a pattern. We shall use this obstruction in Chapter IV to show that almost all canonical projection method patterns fail to be substitution systems.

The organisation of this chapter is as follows. In Section 2 we define the various groupoids considered and their C^* algebras, and discuss their equivalences. In Section 3 we consider the notion of continuous similarity of topological groupoids. This is an important equivalence relation for us as continuously similar groupoids have the same groupoid cohomology. In Section 4 we define our invariants and prove them to be additively equivalent and in Section 5 we establish the role our invariants play in discussing self similarity.

2 Equivalence of Projection Method pattern groupoids

First we develop some general results about topological groupoids, appealing to the definitions in **[Ren]**. These will lead us to the notion of Equivalence of Groupoids which compares most naturally the groupoid C^*-algebras.

Also in this section, we define several groupoids which can be associated to a projection method pattern. We will show that many of these groupoids are related by equivalence.

Definition 2.1 We write the unit space of a groupoid $\mathcal{G}$ as $\mathcal{G}^o$, and write the range and source maps, $r, s\colon \mathcal{G} \longrightarrow \mathcal{G}^o$ respectively. Both these maps are continuous and, due to the existence of a Haar System in all our examples, we note that they are open maps as well.

Recall the *reduction* of a groupoid. Given a groupoid $\mathcal{G}$ with unit space, $\mathcal{G}^o$, and a subset, L, of $\mathcal{G}^o$, define the reduction of $\mathcal{G}$ to L as the subgroupoid ${}_L\mathcal{G}_L = \{g \in \mathcal{G} \mid r(g), s(g) \in L\}$ of $\mathcal{G}$, with unit space, L.

If L is closed then ${}_L\mathcal{G}_L$ is a closed subgroupoid of $\mathcal{G}$.

We also define $\mathcal{G}_L = \{g \in \mathcal{G} \mid s(g) \in L\}$ and note the maps $\rho\colon \mathcal{G}_L \longrightarrow \mathcal{G}^o$ and $\sigma\colon \mathcal{G}_L \longrightarrow L$ defined by r and s respectively.

We say $L \subset \mathcal{G}^o$ is *range-open* if, for all open $U \subset \mathcal{G}$, we have $r(\{x \in U : s(x) \in L\})$ open in $\mathcal{G}^o$.

Suppose a topological abelian group, H, acts by homeomorphisms on a topological space X, then we define a groupoid called the *transformation groupoid*, $\mathcal{G}(X, H)$, as the topological direct product, $X \times H$, with multiplication $(x, g)(y, h) = (x, g + h)$ whenever $y = gx$, and undefined otherwise. The unit space is $X \times \{0\}$.

This last construction is sometimes called the transformation group [**Ren**] or even the transformation group groupoid, but we prefer the usage to be found in [**Pa**].

We note that if H is locally compact, then $C^*(\mathcal{G}(X,H))$ can be naturally identified with $C_o(X)\times H$, the crossed product [**Ren**].

Lemma 2.2 *Suppose that H is an abelian metric topological group acting homeomorphically on X. Let $\mathcal{G} = \mathcal{G}(X,H)$ be the transformation groupoid and suppose that L is a closed subset of $X \equiv \mathcal{G}^o$.*

a/ If H is discrete and countable, then $\mathcal{G}^o$ is a clopen subset of $\mathcal{G}$, and L is range-open if and only if it is clopen in X.

b/ If there is an $\epsilon > 0$ such that for all neighbourhoods, $B \subset B(0,\epsilon)$, of 0 in H and all A open in L, we have BA open in X, then L is range-open.

Proof Only part b/ presents complications. Suppose that U is open in $X \times H$. We want to show that $r((L \times H) \cap U)$ is open. Pick $x = r(y,h) \in r(L\times H \cap U)$ and let $C \times (B+h)$ be a neighbourhood of (y,h) inside U, with B sufficiently small. Then $A = s(C) \cap L$ is open in L and $x \in (B+h)A = h(BA)$ an open subset of X by hypothesis. However, $(B+h)A \subset r((L\times H)\cap U)$ by construction, and so we have found an open neighbourhood of x in $r((L\times H)\cap U)$ as required. □

We use the constructions from [**MRW**] [**Rie**] without comment. In particular, we do not repeat the definition of (strong Morita) *equivalence* of groupoids or of C^*-algebras, which is quite complicated. For separable C^*-algebras strong Morita equivalence implies stable equivalence and equates the ordered K-theory (without attention to the scale). All our examples are separable.

Lemma 2.3 *Suppose that $\mathcal{G}$ is a locally compact groupoid and that $L \subset \mathcal{G}^o$ is a closed, range-open subset which intersects every orbit of $\mathcal{G}$. Then ${}_L\mathcal{G}_L$ is equivalent to $\mathcal{G}$ (in the sense of* [**MRW**]*) and the two C^* algebras, $C^*({}_L\mathcal{G}_L)$ and $C^*(\mathcal{G})$ are strong Morita equivalent.*

Proof It is sufficient to show that $\mathcal{G}_L \xrightarrow{\rho} \mathcal{G}^o$ is a left $(\mathcal{G} \xrightarrow{r,s} \mathcal{G}^o)$-module whose $\mathcal{G} \xrightarrow{r,s} \mathcal{G}^o$ action is free and proper, and that $\mathcal{G}_L \xrightarrow{\sigma} L$ is a right $({}_L\mathcal{G}_L \xrightarrow{r,s} L)$-module whose ${}_L\mathcal{G}_L \xrightarrow{r,s} L$ action is free and proper. In short, ${}_L\mathcal{G}_L$ is an abstract transversal of $\mathcal{G}$ and $\mathcal{G}_L$ a $(\mathcal{G}, {}_L\mathcal{G}_L)$-equivalence bimodule from which we can construct the $(C^*(\mathcal{G}), C^*({}_L\mathcal{G}_L))$-bimodule which shows strong Morita equivalence of the two algebras directly, *c.f.* [**MRW**] Thm 2.8.

The definition of these actions is canonical and the freedom and properness of the actions is automatic from the fact that L intersects every orbit and from the properness and openness of the maps r, s. Indeed all the conditions follow quickly from these considerations except for the fact that $\mathcal{G}_L \xrightarrow{\rho} \mathcal{G}^o$ is a left $(\mathcal{G} \xrightarrow{r,s} \mathcal{G}^o)$-module; and the only trouble here is in showing that ρ is an open map. However, this is precisely the problem that range-openness is defined to solve. $\square$

Together with Lemma 2.2 above, this result gives a convenient corollary which unifies the r-discrete and non-r-discrete cases treated separately in [**AP**].

Corollary 2.4 *Suppose that (X, H) and $L \subset X$ obey either of the conditions of Lemma 2.2, then, writing $\mathcal{G} = \mathcal{G}(X, H)$, $C^*({}_L\mathcal{G}_L)$ and $C^*(\mathcal{G})$ are strong Morita equivalent.* $\square$

Before passing to more special examples, we remark that there is no obstruction to the generalisation of results 2.2, 2.3 and 2.4 to the case of non-abelian locally compact group actions, noting only that, for notational consistency with the definition of transformation groupoid, the group action on a space should then be written on the right.

We define a selection of groupoids associated with projection method patterns, all of them transformation groupoids.

Definition 2.5 Given a projection method pattern, $\mathcal{T}$, with data (E, K, u), fix $G = S_{\mathcal{T}}$ as the group obtained from $\mathcal{T}$ (Theorem I.7.4) so that $M\mathcal{T} = \widetilde{\Pi}_u/G$. Recall the definitions of $X_{\mathcal{T}} = X_{S_{\mathcal{T}}}$, $Y_{\mathcal{T}} = Y_{S_{\mathcal{T}}}$ and G_1 from I.10.2 and I.10.4.

We define in turn: $\mathcal{G}X_{\mathcal{T}} = \mathcal{G}(X_{\mathcal{T}}, G_1)$, from the G_1 action on $X_{\mathcal{T}}$, and $\mathcal{G}F_{\mathcal{T}} = \mathcal{G}(F_u, G)$, using the action of G on F_u, both defined in (I.8.1).

Also define $\mathcal{G}\widetilde{\Pi}_{\mathcal{T}} = \mathcal{G}(\widetilde{\Pi}_u, E + G)$.

All but the last of these groupoids are r-discrete (see [**Ren**]).

Lemma 2.6 *Suppose $\mathcal{T}$ is a projection method pattern with data (E, K, u) such that $E \cap \mathbb{Z}^N = 0$. The groupoids $\mathcal{G}F_{\mathcal{T}}$ and $\mathcal{G}X_{\mathcal{T}}$ are each a reduction of $\mathcal{G}\widetilde{\Pi}_{\mathcal{T}}$ to the closed range-open sets, F_u and $Y_{\mathcal{T}}$.*

Proof It is clear that $\mathcal{G}F_{\mathcal{T}}$ is a reduction of $\mathcal{G}\widetilde{\Pi}_{\mathcal{T}}$ to F_u. To prove that this a range-open set using Lemma 2.2, we take an open subset of F_u and examine the action of small elements of $E + G$ on it. Only the E action enters our consideration and then it is clear from (I.9.2) that if B is an open subset of E and A is an open subset of F_u, then as topological spaces, $BA \equiv A \times B$, which is clearly open in $\widetilde{\Pi}_u$.

Recall the notation of (I.10.12), and the homeomorphism $X_{\mathcal{T}} \leftrightarrow Y_{\mathcal{T}}$, found by Lemma I.10.5, putting $G = S_{\mathcal{T}}$. In effect, this homeomorphism equates $X_{\mathcal{T}}$ with a fundamental domain of the $G_{\mathcal{T}}$ action on F_u. This homeomorphism is G_1-equivariant if we equat e the G_1 action on $Y_{\mathcal{T}}$ with the induced action of G/G_0 on $F_u/G_0 = X_{\mathcal{T}} \equiv Y_{\mathcal{T}}$. But this is precisely the correspondence needed to equate $\mathcal{G}(X_{\mathcal{T}}, G_1)$ with the reduction of $\mathcal{G}F_{\mathcal{T}}$ to $Y_{\mathcal{T}}$ considered as a subset of the unit space of $\mathcal{G}F_{\mathcal{T}}$. Thus $\mathcal{G}X_{\mathcal{T}}$ is the reduction of $\mathcal{G}\widetilde{\Pi}_{\mathcal{T}}$ to $Y_{\mathcal{T}}$, and since $Y_{\mathcal{T}}$ is clopen in F_u the same argument as above shows that $Y_{\mathcal{T}}$ is closed and range-open in $\widetilde{\Pi}_u$. □

Now we define a groupoid connected more directly with the pattern, $\mathcal{T}$.

Definition 2.7 From Definition I.4.4, recall the two maps $M\widetilde{P}_u \longrightarrow M\mathcal{T} \xrightarrow{*} MP_u$ whose composition is π_*. Without confusion we name the second (starred) map π_* as well.

We also define a map $\eta_{\mathcal{T}}$ which is the composite $\widetilde{\Pi}_u \xrightarrow{\widetilde{\eta}} M\widetilde{P}_u \longrightarrow M\mathcal{T}$.

Note that $\eta(x) = \pi_*(\eta_{\mathcal{T}}(x))$ for all $x \in \widetilde{\Pi}_u$ and that, being a composition of open maps (I.3.9), $\eta_{\mathcal{T}}$ is an open map.

Define the *discrete hull* of $\mathcal{T}$ as $\Omega_{\mathcal{T}} = \{S \in M\mathcal{T} \mid 0 \in \pi_*(S)\}$.

The *pattern groupoid*, $\mathcal{G}\mathcal{T}$, is the space $\{(S, v) \in \Omega_{\mathcal{T}} \times E \mid v \in \pi_*(S)\}$ inheriting the subspace topology of $\Omega_{\mathcal{T}} \times E$. The restricted multiplication operation is $(S', v')(S, v) = (S', v + v')$, if $S = v'S'$, undefined otherwise. The unit space is $\mathcal{G}\mathcal{T}^o = \{(S, 0) \mid S \in \Omega_{\mathcal{T}}\}$, homeomorphic to $\Omega_{\mathcal{T}}$.

Also define $E_u^\perp = \widetilde{\mu}^{-1}(E^\perp)$, a space which is naturally homeomorphic to F_u; a correspondence made by extending the application of $\pi^\perp$, inverted by the extension of π'.

Lemma 2.8 *Suppose $\mathcal{T}$ is a projection method pattern with data (E, K, u) such that $E \cap \mathbb{Z}^N = 0$. The groupoid $\mathcal{G}\mathcal{T}$ is isomorphic to a reduction of $\mathcal{G}\widetilde{\Pi}_{\mathcal{T}}$ to a closed range-open set.*

Proof Let L be a compact open subset of $E_u^\perp$ so that $\eta_{\mathcal{T}}(L) = \Omega_{\mathcal{T}}$ and $\eta_{\mathcal{T}}$ is 1-1 on L. This can be constructed as follows. Define $L_o = \overline{NS \cap K \cap (Q + u)}$ where the closure is taken with respect to the

$\overline{D}'$ metric - a clopen subset of $E_u^\perp$ by (I.9.6). Let $L = L_o \backslash \cup \{gL_o \mid g \in G \cap E^\perp, g \neq 0\}$ (using the G action on $\widetilde{\Pi}_u$).

We claim that the reduction of $\mathcal{G}\widetilde{\Pi}_{\mathcal{T}}$ to L is isomorphic to the pattern groupoid defined above.

Suppose that $(x; g, v) \in \mathcal{G}\widetilde{\Pi}_{\mathcal{T}}$ and $x \in L$ and $(g+v)x \in L$, then $0 \in \eta(x) = \pi_*(\eta_{\mathcal{T}}(x))$ and $0 \in \eta((g+v)x)$. But note that the action by $v \in E$ on $x \in NS$ is $vx = x - v$ and so $\eta((g+v)x) = \eta(gx) - v = \pi_*(\eta_{\mathcal{T}}(gx)) - v = \pi_*(\eta_{\mathcal{T}}(x)) - v$. Thus $0, v \in \pi_*(\eta_{\mathcal{T}}(x))$ and the map $\psi\colon (x; g, v) \mapsto (\eta_{\mathcal{T}}(x), v)$ is well defined ${}_L\mathcal{G}\widetilde{\Pi}_{\mathcal{T}L} \longrightarrow \mathcal{G}\mathcal{T}$. The E and G equivariance of the maps used to define ψ show that the groupoid structure is preserved.

Conversely, if $0, -v \in \pi_*(\eta_{\mathcal{T}}(x))$, then there are, by construction of L, $g, g' \in G$ such that $gx, (g'+v)x \in L$. Thus $(gx; g'-g, v) \in {}_L\mathcal{G}\widetilde{\Pi}_{\mathcal{T}L}$ showing that ψ is onto. Also, the g, g' are unique by the construction of L above, and so ψ is 1-1. The continuity of ψ and its inverse is immediate, so we have a topological groupoid isomorphism, as required.

Thus we have shown that $\mathcal{G}\mathcal{T}$ is isomorphic to a reduction of $\mathcal{G}\widetilde{\Pi}_{\mathcal{T}}$ to the set L which is clearly closed.

Also, L is a subset of $E_u^\perp$, transverse to E, so that the same argument as 2.6 shows that L is range open.

It remains to show that L hits every orbit of $\mathcal{G}\widetilde{\Pi}_{\mathcal{T}}$ and for this it is sufficient to show that for any $x \in \widetilde{\Pi}_u$, $Gx \cap (L \times E) \neq 0$ (where we exploit the equivalence: $\widetilde{\Pi}_u \equiv E_u^\perp \times E$ (I.9.2). But this is immediate from the fact that $L \times E$ is a clopen subset of $\widetilde{\Pi}_u$ (I.9.2), and by minimality of the $G + E$ action on $\widetilde{\Pi}_u$ (as in I.3.9). $\square$

Combining the Lemmas above, we obtain the following.

Theorem 2.9 *Suppose that $\mathcal{T}$ is a projection method pattern with data (E, K, u) such that $E \cap \mathbb{Z}^N = 0$. The C^*-algebras $C^*(\mathcal{G}\mathcal{T})$, $C_o(F_u) \times G$ and $C(X_{\mathcal{T}}) \times G_1$ are strong Morita equivalent and thus their ordered K-theory (without attention to scale) is identical.* $\square$

Remark 2.10 We can compare the construction above with the "rope" dynamical system constructed by the third author [**K2**] exploiting the generalised grid method introduced by de Bruijn [**dB1**]. The rope construction actually shows that, in a wide class of tilings including the canonical projection tilings, there is a Cantor minimal system $(X, \mathbb{Z}^d)$ such that $\mathcal{G}(X, \mathbb{Z}^d)$ is a reduction of $\mathcal{G}\mathcal{T}$. By comparing the details of the proof above with [**K2**] it is possible to show directly that, in the case of non-degenerate canonical projection tilings, the rope dynamical system is conjugate to $(X_{\mathcal{T}}, G_1)$.

We note that the construction of Lemma 2.8 depends only on the data (E, K, u) and on G and from this we deduce the following.

Corollary 2.11 *Suppose we specify projection data (E, K, u), such that $E \cap \mathbb{Z}^N = 0$. Then, among projection method patterns, $\mathcal{T}$, with these data, the dynamical system $(M\mathcal{T}, E)$ determines $\mathcal{G}\mathcal{T}$ up to groupoid isomorphism.*

Thus among projection method patterns with fixed data, the dynamical invariants are at least as strong as the non-commutative invariants. □

Finally we reconnect the work of this section with the original construction of the tiling groupoid [**K1**].

Definition 2.12 Recall the notation $A[r] = (A \cap B(r)) \cup \partial B(r)$ *etc.* defined in I.3.1 and I.4.2, for $r \geq 0$ and $A \subset \mathbb{R}^N$ or E. Given two closed sets, A, A' define the distance $D_o(A, A') = \inf\{1/(r+1) \mid r > 0, A[r] = A'[r]\}$.

As a metric this can be used to compare point patterns in E or $\mathbb{R}^N$ (as in I.3.1 and I.4.2), or decorated tilings in E as described in I.4.1.

Given a tiling, $\mathcal{T}$, of E, bounded subsets which are the closure of their interior, the construction of the discrete hull in [**K1**] starts by placing a single puncture generically in the interior of each tile according to local information (usually just the shape, decoration and orientation of the tile itself–the position of the puncture would then depend only on the translational congruence class of the tile). So we form the collection of punctures, $\tau(\mathcal{T})$ of $\mathcal{T}$, a discrete subset of points in E.

We consider the set $\Omega^o_{\mathcal{T}} = \{\mathcal{T} + x \mid 0 \in \tau(\mathcal{T} + x) = \tau(\mathcal{T}) + x\}$, and define a modified hull, which we write $\Omega^*_{\mathcal{T}}$ in this section, as the D_o completion of this selected set of shifts of $\mathcal{T}$.

As in [**K1**] we consider only tilings $\mathcal{T}$ which are of *finite type* (or of finite pattern type as in [**AP**] or of finite local complexity) which means that for each r the set $\{\mathcal{T}'[r] | \mathcal{T}' \in \Omega^o_{\mathcal{T}}\}$ is finite. It is well known that canonical projection tilings are finite type.

From this hull, we define the groupoid, $\mathcal{G}\mathcal{T}^*$ exactly as for $\mathcal{G}\mathcal{T}$: $\mathcal{G}\mathcal{T}^* = \{(S, v) \in \Omega^*_{\mathcal{T}} \times E \mid v \in \tau(S)\}$ with the analogous rule for partial multiplication.

The assumption of local information dictates more precisely that up to isomorphism $\mathcal{GT}^*$ is independent of the punctures chosen and the map τ is continuous, E-equivariant, and 1-1 from $\Omega_{\mathcal{T}}^*$ with D_o metric to the space of Delone subsets of E also with D_o metric.

Remark 2.13 Although phrased in terms of tilings, this definition can in fact be applied to patterns as well, where the idea of puncture becomes now the association of a point with each unit of the pattern (I.4.1). In this case the finite type condition is equated with the condition that $\tau(\mathcal{T})$ is Meyer (see [**La1**]), and this is sufficient to prove the analogues of all the Lemmas below. However, we continue to use the language of tilings and, since every projection method pattern is pointed conjugate to a decorated finite type tiling (decorating the Voronoi tiles for P_u for example (I.7.5)), we lose no generality in doing so.

We note that when a projection method pattern $\mathcal{T}$ is in fact a tiling, the two definitions of hull (2.7 and above) given here seldom coincide nor do we obtain the same groupoids (but we note the important exception of the canonical projection tiling in 2.16 ahead). The remainder of this section shows that, never-the-less, the two groupoids, $\mathcal{GT}$ and $\mathcal{GT}^*$, are equivalent. We start by comparing D and D_o.

Lemma 2.14 *Suppose that $\mathcal{T}$ is a finite type tiling as above, then $\Omega_{\mathcal{T}}^o$ is precompact with respect to D_o. Further D and D_o generate the same topology on $\Omega_{\mathcal{T}}^o$.*

Proof The precompactness of $\Omega_{\mathcal{T}}^o$ is proved in [**K1**].

For any two tilings, we have $D(\mathcal{T},\mathcal{T}') \le D_o(\mathcal{T},\mathcal{T}')$ by definition, and so the topology of D_o is always finer than that of D.

Conversely, as a consequence of the finite type condition of $\mathcal{T}$ there is a number $\delta_o < 1$ such that if $0 < \epsilon < \delta_o$, then $\mathcal{T}+x, \mathcal{T}+x' \in \Omega_{\mathcal{T}}^o$ and $D(\mathcal{T}+x,\mathcal{T}+x') < \epsilon$ together imply that $\mathcal{T}+x$ and $\mathcal{T}+x'$ actually agree up to a large radius ($1/\epsilon - 1$ will do) and we conclude $D_o(\mathcal{T}+x,\mathcal{T}+x') < 2\epsilon$ as required. □

Consequently, $\Omega_{\mathcal{T}}^*$ is canonically a subspace of $M\mathcal{T}$ and we can consider its properties as such.

Lemma 2.15 *With respect to the E action on $M\mathcal{T}$, both $\Omega_{\mathcal{T}}$ and $\Omega_{\mathcal{T}}^*$ are range-open.*

Proof With the notation of the proof of Lemma 2.8, $\Omega_{\mathcal{T}} = \eta_{\mathcal{T}}(L)$, where L is a compact open subset of $E_u^{\perp}$. As in 2.8, L is range-open in $\widetilde{\Pi}_u$ and, since $\eta_{\mathcal{T}}$ is an open E-equivariant map, we deduce the same of $\eta_{\mathcal{T}}(L)$.

For $\Omega^*_{\mathcal{T}}$, we note that (as in Section 1, 3.4 before) by the finite type condition of $\mathcal{T}$, there is a number δ_o so that, if $x, x' \in E$, $\mathcal{T}' \in M\mathcal{T}$ and $0 < \|x - x'\| < \delta_o$, then $D(\mathcal{T}' - x, \mathcal{T}' - x') \geq \|x - x'\|/2$ and $D_o(\mathcal{T}' - x, \mathcal{T}' - x') = 1$. In particular, since $\Omega^*_{\mathcal{T}}$ is D_o-compact and hence a finite union of radius $1/2$ D_o-balls, we deduce that the map $\Omega^*_{\mathcal{T}} \times B(\delta_o/2) \longrightarrow M\mathcal{T}$, defined as $(\mathcal{T}', x) \mapsto \mathcal{T}' + x$, is locally injective and hence open. From here the range-openness of $\Omega^*_{\mathcal{T}}$ is immediate. $\square$

Theorem 2.16 *If $\mathcal{T}$ is at once a tiling and a projection method pattern with data (E, K, u), such that $E \cap \mathbb{Z}^N = 0$, then the tiling groupoid $\mathcal{G}\mathcal{T}^*$ as defined in Def. 2.12 is equivalent to $\mathcal{G}\mathcal{T}$ (Def. 2.7). Thus the respective C^*-algebras are strong Morita equivalent also.*

If $\mathcal{T}$ is a non-degenerate canonical projection tiling there is a puncturing procedure inducing an isomorphism between the two groupoids.

Proof Recall the definition of transformation groupoid and the action of E on $M\mathcal{T}$ and consider $\mathcal{G}(M\mathcal{T}, E)$. Using Lemmas 2.3 and 2.15, it suffices to show that the groupoids $\mathcal{G}\mathcal{T}$ and $\mathcal{G}\mathcal{T}^*$ are each a reduction of $\mathcal{G}(M\mathcal{T}, E)$ to the sets $\Omega_{\mathcal{T}}$ and $\Omega^*_{\mathcal{T}}$ respectively. But this is immediate from their definition.

To treat the case of canonical projection tilings, we note that its tiles are parallelepipeds, and that the point pattern consists of their vertices. Hence if we fix a small generic vector then adding this vector to each vertex gives exactly one puncture for each tile. Translating an element of $\Omega_{\mathcal{T}}$ by that vector therefore produces an element of $\Omega^*_{\mathcal{T}}$ if we use these punctures to define the latter. The map so defined is clearly a continuous bijection which is E-equivariant (with respect to the restricted E-action). Therefore it induces an isomorphism between $\mathcal{G}\mathcal{T}$ and $\mathcal{G}\mathcal{T}^*$. $\square$

3 Continuous similarity of Projection Method pattern groupoids

The aim of this section is to compare our groupoids in a second way: by *continuous similarity*. This gives most naturally an equivalence between groupoid cohomology.

We will show that many of the groupoids we associate with a projection pattern are related in this way also. Further background facts about groupoids and their cohomology and the idea of similarity may be found in [**Ren**].

Definition 3.1 Two homomorphisms, $\phi, \psi\colon \mathcal{G} \longrightarrow \mathcal{H}$ between topological groupoids are *continuously similar* if there is a function, $\Theta\colon \mathcal{G}^o \longrightarrow \mathcal{H}$ such that

$$\Theta(r(x))\phi(x) = \psi(x)\Theta(s(x)).$$

Two topological groupoids are *continuously similar* if there exist homomorphisms $\phi\colon \mathcal{G} \longrightarrow \mathcal{H}$, $\psi\colon \mathcal{H} \longrightarrow \mathcal{G}$ such that $\Phi_{\mathcal{G}} = \psi\phi$ is continuously similar to $id_{\mathcal{G}}$ and $\Phi_{\mathcal{H}} = \phi\psi$ is continuously similar to $id_{\mathcal{H}}$.

In all our examples, produced by Lemma 3.3 ahead, the function Θ is continuous but note that this is not required by Definition 3.1. Our interest in this relation lies in the following fact which we exploit in Section 4; see [**Ren**] for the definition of continuous cohomology $H^*(\mathcal{G};\mathbb{Z})$ of a topological groupoid $\mathcal{G}$.

Proposition 3.2 ([**Ren**], with necessary alterations for the continuous category) *If $\mathcal{G}$ and $\mathcal{H}$ are continuously similar then $H^*(\mathcal{G};\mathbb{Z}) = H^*(\mathcal{H};\mathbb{Z})$.* □

It turns out that the construction of continuous similarities follows closely the reduction arguments in the examples that interest us.

Lemma 3.3 *Suppose that (X, H) is a free topological dynamical system (i.e., $hx = x$ implies that h is the identity), with transformation groupoid $\mathcal{G} = \mathcal{G}(X, H)$, and that L, L' are two closed subsets of $\mathcal{G}^o$. Suppose there are continuous functions, $\gamma\colon L \longrightarrow H$, $\delta\colon L' \longrightarrow H$ which define continuous maps $\alpha\colon L \longrightarrow L'$ and $\beta\colon L' \longrightarrow L$ by $\alpha x = \gamma(x)x$ and $\beta x = \delta(x)x$. Then ${}_L\mathcal{G}_L$ and ${}_{L'}\mathcal{G}_{L'}$ are continuously similar.*

Proof Construct the two homomorphisms, $\phi\colon {}_L\mathcal{G}_L \rightarrow {}_{L'}\mathcal{G}_{L'}$ and $\psi\colon {}_{L'}\mathcal{G}_{L'} \rightarrow {}_L\mathcal{G}_L$ by putting $\phi(x, g) = (\alpha x, \gamma(gx) + g - \gamma(x))$ and $\psi(y, h) = (\beta y, \delta(hy) + g - \delta(y))$.

A quick check confirms that these are homomorphisms, and they are both clearly continuous. Moreover, $\phi\psi(y, h) = (\alpha\beta y, \gamma((\delta(hy) +$

$h-\delta(y))\beta(y))$ $+\delta(hy)+h-\delta(y)-\gamma(\beta y)$, a rather complicated expression which can be simplified if we note that $\gamma((\delta(hy)+h-\delta(y))\beta(y)) = \gamma((\delta(hy)+h)y) = \gamma(\beta hy)$, and define $\sigma(y)$ to be the element of H such that $\sigma(y)y = \alpha\beta y$. Then $\sigma(y) = \delta(y)+\gamma(\beta y)$, by definition, and so $\sigma(hy) = \gamma(h\beta(y)) + \delta(hy) = \gamma((\delta(hy)+h-\delta(y))\beta(y)) + \delta(hy)$. This gives $\phi\psi(y,h) = (\alpha\beta y, \sigma(hy)+h-\sigma(y))$.

It is now easy to see that $\phi\psi$ is continuously similar to the identity on ${}_{L'}\mathcal{G}_{L'}$ using the transfer function, $\Theta : L' \longrightarrow {}_{L'}\mathcal{G}_{L'}$ given by $\Theta(y) = (\alpha\beta y, -\sigma(y))$. This Θ happens also to be continuous.

Reciprocal expressions give the similarity between $\psi\phi$ and the identity on ${}_L\mathcal{G}_L$. $\square$

Remark 3.4 Note that if $L' = \mathcal{G}^o$, then Lemma 3.3 can be reexpressed in the following form. If L is a closed subset of $\mathcal{G}^o$ for which there is a continuous map $\gamma\colon \mathcal{G}^o \longrightarrow H$ such that $\gamma(x)x \in L$ for all $x \in \mathcal{G}^o$, then ${}_L\mathcal{G}_L$ is continuously similar to $\mathcal{G}$. (The condition on L implies that L intersects every H-orbit of $(\mathcal{G}^o, H)$, but the converse is not true.)

We apply this lemma and remark in two ways as we examine continuous similarities between the various groupoids of section 2.

Lemma 3.5 *Suppose that $\mathcal{T}$ is a projection method pattern and write $\mathcal{G}$ for $\mathcal{G}\widetilde{\Pi}_{\mathcal{T}}$. If L is a clopen subset of F_u then ${}_L\mathcal{G}_L$ is continuously similar to $\mathcal{G}$.*

Proof It suffices to find the function γ in the remark.

Pick an order $\succ$ on $G = S_{\mathcal{T}}$ in which every non-empty set has a minimal element. The set $EL = \{vy : v \in E, y \in L\}$ is naturally homeomorphic to $E \times L$ by Lem. I.9.2, and hence is clopen in $\widetilde{\Pi}_u$. By the minimality of the $E+G$ action on $\widetilde{\Pi}_u$, (which is proved analogouly to Lem. I.3.9), we have $\cup_{h\in G} hEL = \widetilde{\Pi}_u$, so that for each $x \in \widetilde{\Pi}_u$, there is a $\succ$-minimal $h \in G$ such that $hx \in EL$. Let $\gamma_0(x)$ be this g, and note that by the freedom and isometric action of G and the clopenness of EL, this function $x \mapsto \gamma_0(x)$ is continuous and maps $\widetilde{\Pi}_u$ to EL.

Now, given $\gamma_0(x)x \in EL$, then there is a unique $\gamma_1(x) \in E$ such that $\gamma_1(x)\gamma_0(x)x \in L$, and it is clear that $x \mapsto \gamma_1(x)$ is continuous as a map $\widetilde{\Pi}_u \longrightarrow E$. The desired map γ can now be taken as this composite. $\square$

Lemma 3.6 *Suppose that $\mathcal{T}$ is a finite type tiling for which we have chosen a puncturing and which is also a projection method pattern. Then $\mathcal{G}\mathcal{T}$ and $\mathcal{G}\mathcal{T}^*$ are continuously similar.*

Proof We construct the maps γ and δ as follows.

Recall the metric $D_o(A,B) = \inf\{1/(r+1) : A[r] = B[r]\}$ and the argument of Lemma 2.14 which shows that D and D_o are equivalent on each of the sets $\Omega_{\mathcal{T}}$ and $\Omega^*_{\mathcal{T}}$. (Actually the argument refers only to the second space, but the fact that $\pi_*\mathcal{T}$ is a Meyer pattern (see **[La1]** and Remark 2.13) allows it to be applied directly to the first space as well.)

We may assume without loss of generality that, for each $S \in M\mathcal{T}$, each point of $\pi_*(S)$ is in the interior of a tile of S (if not we shift all the tiles in $\mathcal{T}$ by a uniform short generic displacement and start again equivalently).

Suppose that $S \in \Omega_{\mathcal{T}}$. We know that $0 \in \pi_*(S)$ and that by assumption there is a unique tile in S which contains the origin in its interior. This tile has a puncture at a point v say, and so $S - v \in \Omega^*_{\mathcal{T}}$. So we have defined a map from $\Omega_{\mathcal{T}}$ to E, $\gamma\colon S \mapsto -v$ which is clearly continuous with respect to the D_o metric. Moreover the map, $\alpha\colon S \mapsto S - v$ has range $\Omega^*_{\mathcal{T}}$.

Conversely, let r be chosen so that every ball in E of radius r contains at least one point of $\pi_*(\mathcal{T}) = P_u$. Consider the sets $\pi_*(S) \cap B(r)$, as S runs over $\Omega^*_{\mathcal{T}}$ and note that there are only finitely many possibilities, i.e. the set $J = \{\pi_*(S) \cap B(r) : S \in \Omega^*_{\mathcal{T}}\}$ is a finite collection of non-empty finite subsets of $B(r)$. Furthermore, by the continuity of π_* on $\Omega^*_{\mathcal{T}}$ with respect to the D_o metric, for each $C \in J$, the set $\{S \in \Omega^*_{\mathcal{T}} : \pi_*(S) \cap B(r) = C\}$ is clopen.

For each $C \in J$ choose an element $v = v(C) \in C$, and define $\delta(S) = -v(\pi_*(S) \cap B(r))$; this is continuous by construction. Then $S + \delta(S) \in \Omega_{\mathcal{T}}$. □

Now, appealing to 2.6 and 2.8, we can gather the results of this section into the following corollary.

Corollary 3.7 *Suppose that $\mathcal{T}$ is a projection method pattern. Then $\mathcal{GT}$, $\mathcal{G}\widetilde{\Pi}_{\mathcal{T}}$, $\mathcal{G}F_{\mathcal{T}}$ and $\mathcal{G}X_{\mathcal{T}}$ are all continuously similar. If $\mathcal{T}$ is also a finite type tiling, then these are all continuously similar to the tiling groupoid, $\mathcal{GT}^*$, of* **[K2]**.

Proof Lemmas (3.5) and (2.6) and (2.8) show that $\mathcal{GT}$, $\mathcal{G}F_{\mathcal{T}}$ and $\mathcal{G}X_{\mathcal{T}}$ are all continuously similar to $\mathcal{G}\widetilde{\Pi}_{\mathcal{T}}$. The second part is a restatement of (3.6). □

4 Pattern cohomology and K-theory

We are now in a position to define our topological invariants for projection method patterns and prove their isomorphism as groups.

Definition 4.1 For a projection method pattern, $\mathcal{T}$, in $\mathbb{R}^d$, we define for each $m \in \mathbb{Z}$ the following groups.

(a) $H^m(\mathcal{GT}, \mathbb{Z})$, the continuous groupoid cohomology of the pattern group-oid $\mathcal{GT}$;

(b) $H^m(M\mathcal{T})$, the Čech cohomology of the space $M\mathcal{T}$;

(c) $H_{d-m}(G_1, CX_{\mathcal{T}})$ and $H^m(G_1; CX_{\mathcal{T}})$, the group homology and cohomology of G_1 with coefficients $CX_{\mathcal{T}}$ (I.10.6);

(d) $H_{d-m}(S_{\mathcal{T}}; CF_u)$ and $H^m(S_{\mathcal{T}}; C(F_u; \mathbb{Z}))$, the group homology and cohomology of $S_{\mathcal{T}}$ with coefficients CF_u (I.9.3) or $C(F_u; \mathbb{Z})$ (I.10.6);

(e) $K_{d-m}(C^*(\mathcal{GT}))$, the C^* K-theory of $C^*(\mathcal{GT})$;

(f) $K_{d-m}(C(X_{\mathcal{T}}) \times G_1)$, the C^* K-theory of the crossed product $C(X_{\mathcal{T}}) \times G_1$;

and, for finite type tilings (2.12),

(g) the continuous groupoid cohomology $H^m(\mathcal{GT}^*, \mathbb{Z})$.

Theorem 4.2 *For a projection method pattern $\mathcal{T}$ and for each value of m, the invariants defined in (4.1)(a) to (d) are all isomorphic as groups. If $\mathcal{T}$ is also a finite type tiling, then these are also isomorphic to that defined in (4.1)(g).*

The invariants defined in (4.1)(e) and (f) are each isomorphic as ordered groups. Finally, all these invariants are related via isomorphisms of groups such as

$$K_m(C(X_{\mathcal{T}}) \times G_1) \cong \bigoplus_{j=-\infty}^{\infty} H^{m+d+2j}(G_1; CX_{\mathcal{T}}).$$

These invariants are, in all cases, torsion free, and those in parts (a) to (d) and (g) are non-zero only for integers m in the range $0 \leq m \leq d$.

Proof It is immediate from the definition **[Ren]** that if W is a locally compact space on which a discrete abelian group G acts freely by homeomorphisms then the continuous groupoid cohomology $H^*(\mathcal{G}(W, G); \mathbb{Z})$ is naturally isomorphic to the group cohomology $H^*(G, C(W; \mathbb{Z}))$ with coefficients the continuous compactly supported integer-valued functions on W, with $\mathbb{Z}[G]$-module structure dictated naturally by the G action on W. This proves the equality of (a) (and (g) where appropriate) with the cohomology versions of (c) and (d) from (3.7) and the fact that $\mathcal{G}F_{\mathcal{T}}$ and $\mathcal{G}X_{\mathcal{T}}$ are transformation groupoids.

By (I.10.10) the space $M\mathcal{T}$ is homeomorphic to the mapping torus $MT(X_{\mathcal{T}}, G_1)$ and as noted in **[FH]** the Čech cohomology $H^*(MT(X_{\mathcal{T}}, G_1))$ is isomorphic to the group cohomology $H^*(G_1, CX_{\mathcal{T}})$ (this is standard and follows, for example, by induction on the rank of G_1 with the induction step passing from $\mathbb{Z}^r$ to $\mathbb{Z}^{r+1}$ coming from the comparison of the Mayer-Vietoris decomposition of $MT(X_{\mathcal{T}}, G_1)$ along one coordinate with the long exact sequence in group cohomology coming from the extension $\mathbb{Z}^r \to \mathbb{Z}^{r+1} \to \mathbb{Z}$). This proves the isomorphism of (4.1)(b) with the cohomological invariant of (4.1)(c).

The isomorphism of $H^m(G_1, CX_{\mathcal{T}})$ with $H_{d-m}(G_1; CX_{\mathcal{T}})$ is simply
Poincaré duality for the group $G_1 \cong \mathbb{Z}^d$.

By Cor. I.10.7, a decomposition of CF_u as a $\mathbb{Z}[G_{\mathcal{T}}]$ module is given by $CX_{\mathcal{T}} \otimes \mathbb{Z}[G_{\mathcal{T}}]$ where $CX_{\mathcal{T}}$ is a trivial $\mathbb{Z}[G_{\mathcal{T}}]$ module. Standard homological algebra now tells us that

$$\begin{aligned} H_p(S_{\mathcal{T}}; CF_u) &\cong H_p(G_1 \oplus G_{\mathcal{T}}; CX_{\mathcal{T}} \otimes \mathbb{Z}[G_{\mathcal{T}}]) \\ &\cong H_p(G_1; CX_{\mathcal{T}}) \end{aligned}$$

establishing the isomorphism of (4.1)(c) and (d) in homology. A similar argument also works in cohomology for $C(F_u; \mathbb{Z})$.

The isomorphism of (4.1)(e) and (f) follows from the Morita equivalence of the underlying C^*-algebras in (2.7) and the isomorphism

$$K_m(C(X_{\mathcal{T}}) \times G_1) \cong \bigoplus_{j=-\infty}^{\infty} H^{m+d+2j}(G_1; CX_{\mathcal{T}})$$

is one of the main results of **[FH]**.

The torsion-freedom of these invariants also follows from the results of **[FH]**, while the vanishing of the (co)homological invariants outside the range of dimensions stated is immediate from their identification with the (co)homology of the group $G_1 \cong \mathbb{Z}^d$.

We make one further reduction of the complexity of the computation of these invariants. Recall first the construction of I.2.9 and I.10.1, in particular the equation $F \cap Q = V \oplus \widetilde{\Delta}$ splitting F into continuous and discrete directions, and in which $\pi'(\mathbb{Z}^N)$, the projection of $\mathbb{Z}^N$ onto F parallel to E, is dense. Recall also the map $\widetilde{\mu}\colon \Pi_u \longrightarrow Q + u$ defined in (I.4.3) for each $u \in NS$.

Definition 4.3 The restriction of $\widetilde{\mu}$ to F_u is written $\nu\colon F_u \longrightarrow F \cap (Q + u) = (F \cap Q) + \pi'(u)$; this map is $\pi'(\mathbb{Z}^N)$-equivariant and $|\nu^{-1}(v)| = 1$ precisely when $v \in NS \cap F \cap (Q + u)$ (see Lemma I.9.2).

Let $\Gamma_{\mathcal{T}} = \{v \in S_{\mathcal{T}} : \pi'(v) \in V\}$ and $CV_u = \{f \in CF_u : \nu(supp(f)) \subset V + \pi'(u)\}$, where *supp* refers to the support of the function. This is consistent with setting $V_u = \{x \in F_u : \nu(x) \in V + \pi'(u)\}$ and taking CV_u as the continuous integer valued functions on V_u with compact support. There is a natural decomposition $CF_u = CV_u \otimes_{\mathbb{Z}} \mathbb{Z}[\widetilde{\Delta}]$.

Lemma 4.4 *As a subgroup of $S_{\mathcal{T}} = G_{\mathcal{T}} + G_1$ (the decomposition of I.10.1), $\Gamma_{\mathcal{T}}$ satisfies $\Gamma_{\mathcal{T}} = (\Gamma_{\mathcal{T}} \cap G_{\mathcal{T}}) \oplus G_1$. Moreover, $\Gamma_{\mathcal{T}}$ is complemented in $S_{\mathcal{T}}$ by a group Γ_Δ, naturally isomorphic to $\widetilde{\Delta}$.*

With this splitting, the action of $\mathbb{Z}[S_{\mathcal{T}}] = \mathbb{Z}[\Gamma_{\mathcal{T}}] \otimes \mathbb{Z}[\Gamma_\Delta]$ on $CF_u = CV_u \otimes_{\mathbb{Z}} \mathbb{Z}[\widetilde{\Delta}]$ is the obvious one, and hence there is an isomorphism of homology groups $H_*(S_{\mathcal{T}}; CF_u) \cong H_*(\Gamma_{\mathcal{T}}; CV_u)$.

Proof The decompositions and restrictions on $G_{\mathcal{T}}$ and G_1 follow from the definition and the original construction of (I.10.1). The conclusion in homology is the same homological argument as used in the previous proof. □

We note that since $S_{\mathcal{T}} \supset \mathbb{Z}^N$ and $\pi^\perp(\mathbb{Z}^N)$ is dense in $Q \cap F$, the group $\Gamma_{\mathcal{T}}$ acts minimally on V and hence on V_u.

Corollary 4.5 *With the data above,*

$$K_n(C^*(\mathcal{G}\mathcal{T})) = \bigoplus_{j=-\infty}^{\infty} H_{n+2j}(\Gamma_{\mathcal{T}}; CV_u). \qquad \square$$

This is, in fact, the most computationally efficient route to these invariants and, with the exception of Chapter III, the one we shall use in the remainder of this memoir.

5 Homological conditions for self similarity

To motivate the direction we now move in, we give an immediate application of these invariants. In this section we show that the (co)homological invariants defined in Section 4 provide an obstruction to a pattern arising as a substitution system. This result will be used in Chapter IV to show that almost all canonical projection tilings fail to be self similar.

We adopt the construction of substitution tilings in [**AP**] considering only finite type tilings whose tiles are compact subsets of $\mathbb{R}^d$ homeomorphic to the closed ball. A substitution procedure as in [**AP**] is based on a map which assigns to each tile of a tiling $\mathcal{T}$ a patch of tiles (a tiling of a compact subset of $\mathbb{R}^d$) which covers the same set as the tile. Moreover, the map is $\mathbb{R}^d$-equivariant in the sense that translationally congruent tiles are mapped to translationally congruent patches. This is useful if there is a real constant $\lambda > 1$ such that the tiling which is obtained from $\mathcal{T}$, by first replacing all tiles with their corresponding patches and then stretching the resulting tiling by λ (keeping the origin fixed), belongs to $M\mathcal{T}$. This procedure of replacing and stretching can then be applied to all tilings of $M\mathcal{T}$ thus defining the substitution map $\sigma : M\mathcal{T} \to M\mathcal{T}$ which is assumed to be injective. As an aside, Anderson and Putnam show that under this condition σ is a hyperbolic homeomorphism. They establish the following property for substitution tilings, i.e. tilings which allow for a substitution map.

Theorem 5.1 [**AP**] *Suppose that $\mathcal{T}$ is a finite type substitution tiling of $\mathbb{R}^d$. Then $M\mathcal{T}$ is the inverse limit of a stationary sequence*

$$Y \xleftarrow{\gamma} Y \xleftarrow{\gamma} \cdots$$

of a compact Hausdorff space Y with a continuous map γ. □

Let us describe Y. Call a collared tile a tile of the tiling $\mathcal{T}$ decorated with the information of what are its adjacent tiles and a collared prototile the translational congruence class of a collared tile. As a topological space we identify the collared prototile with a tile it represents as a compact subset of $\mathbb{R}^d$, regardless of its decoration. By the finite type condition there are only finitely many collared prototiles. Let $\tilde{Y}$ be the disjoint union of all collared prototiles. Y is the quotient of $\tilde{Y}$ obtained upon identifying two boundary points x_i of collared prototiles t_i, $i = 1, 2$, if there are two collared tiles $\hat{t}_i$ in the tiling, $\hat{t}_i$ of class t_i, such that $\hat{x}_1 = \hat{x}_2$ where $\hat{x}_i$ is the point on the boundary of $\hat{t}_i$ whose position corresponds to that of x_i in t_i. If the tiles are polytopes which match face to face (i.e. two adjacent tiles touch at complete faces) then Y is a finite CW complex whose highest dimensional cells are the (interiors of) the tiles represented by the collared prototiles and whose lower dimensional cells are given by quotients of the set of their faces of appropriate dimension. The map γ above is induced by the substitution. Going through the details of its construction in [**AP**] one finds that in the case where tiles are polytopes which match face to face it maps faces of dimension l to faces of dimension l thus providing a cellular map. We are then in the situation assumed for the second part of [**AP**].

Corollary 5.2 *Suppose that $\mathcal{T}$ is a finite type tiling whose tiles are polytopes which match face to face. Then for each m, the rationalized Čech cohomology $H^m(M\mathcal{T}) \otimes \mathbb{Q}$ has finite $\mathbb{Q}$-dimension.*

Proof Like [**AP**] we obtain from (5.1) and the conclusion that Y is a finite CW complex and γ a cellular map that $H^m(M\mathcal{T}) = \varinjlim H^m(Y)$. So $H^*(M\mathcal{T}) \otimes \mathbb{Q} = \varinjlim (H^*(Y) \otimes \mathbb{Q})$. Thus the $\mathbb{Q}$ dimension of $H^*(M\mathcal{T}) \otimes \mathbb{Q}$ is bounded by that of $H^*(Y) \otimes \mathbb{Q}$ and this is finite since Y is a finite CW complex. □

The conclusion of (5.2) applies to much more general pattern constructions. Note that the only principle used is that the space $M\mathcal{T}$ of the tiling dynamical system is the inverse limit of a sequence of maps between uniformly finite CW complexes. We sketch a generalization whose details can be reconstructed by combining the ideas to be found in [**Pr**] and [**Fo**].

Finite type substitution tilings are analysed combinatorially by Priebe in her PhD Thesis [**Pr**] where the useful notion of *derivative tiling*, generalized from the 1-dimensional symbolic dynamical concept [**Du**], is developed. We do not pursue the details here except to note that the derivative of an almost periodic finite type tiling is almost periodic and finite type, and that the process of deriving can be iterated.

Suppose that $\mathcal{T}$ is an almost periodic finite type tiling. By means of repeated derivatives, and adapting the analysis of [**Fo**] for periodic lattices, we may build a Bratteli diagram, $\mathcal{B}$, for $\mathcal{T}$. Its set of vertices at level t is formally the set of translation classes of the tiles in the t^{th} derivative tiling, and the edges relating two consecutive levels, t and

$t+1$ say, are determined by the way in which the tiles of the $(t+1)^{\text{th}}$ derivative tiling are built out of the tiles of the t^{th} derivative tiling. The diagram $\mathcal{B}$ defines a canonical dimension group, $K_0(\mathcal{B})$. Adapting the argument of **[Fo]** we can define a surjection $K_0(\mathcal{B}) \longrightarrow H^d(M\mathcal{T})$ and hence a surjection $K_0(\mathcal{B}) \otimes \mathbb{Q} \longrightarrow H^d(M\mathcal{T}) \otimes \mathbb{Q}$.

In **[Pr]** it is shown that the repeated derivatives of a finite type aperiodic substitution tiling have a uniformly bounded number of translation classes of tiles and are themselves finite type. In particular, the number of vertices at each level of its Bratteli diagram $\mathcal{B}$ is bounded uniformly. Thus $K_0(\mathcal{B}) \otimes \mathbb{Q}$ is finite dimensional over $\mathbb{Q}$, being a direct limit of uniformly finite dimensional $\mathbb{Q}$ vector spaces, and so we reprove (5.2) for the case $m = d$.

It is worth extracting the full power of this argument since it applies to a wider class than the substitution tilings.

Theorem 5.3 *Suppose that $\mathcal{T}$ is a finite type tiling of $\mathbb{R}^d$ whose repeated derivatives have a uniformly bounded number of translation classes of tiles, then $H^d(\mathcal{G}\mathcal{T}) \otimes \mathbb{Q}$ is finite dimensional over $\mathbb{Q}$.* □

Therefore in Chapter IV, when we take a pattern, $\mathcal{T}$, compute its rationalized homology $H_0(\Gamma_{\mathcal{T}}; CV_u) \otimes \mathbb{Q}$ and find it is infinite dimensional, we know we are treating a pattern or tiling which is outwith the class specified in Theorem 5.3, and *a fortiori* outside the class of finite type substitution tilings.

III Approaches to Calculation I: Cohomology for Codimension One

1 Introduction

Our goal in this chapter is to demonstrate the computability of the invariants introduced in Chapter II and we do so by looking at the case where $N = d + 1$. In this case the lattice Δ is always trivial whenever $E \cap \mathbb{Z}^N = 0$.

Recall that when $\Delta = 0$ the projection pattern is determined by a small number of parameters – the dimensions d and N of the space E and the lattice $\mathbb{Z}^N \subset \mathbb{R}^N$, the slope of E in $\mathbb{R}^N$ and the shape of the acceptance domain K. We shall restrict ourselves to specific acceptance domains in later chapters, but the main result of this chapter (3.1) characterises the invariants of patterns on $\mathbb{R}^d$ arising as projection from $\mathbb{R}^{d+1}$ for more or less arbitrary acceptance domains K.

This chapter thus gives a complete answer to the so-called codimension 1 patterns (*i.e.*, $N - d = 1$). After restricting the shape of the acceptance domain we shall give in Chapter V an alternative analysis of this case together with formulæ for the ranks of the invariants in the codimension 2 and 3 cases, when these are finite.

When $N - d = 1$ one of the most important features of K is its number of path components. To facilitate our computations we examine in Section 2 a general technique which sometimes simplifies the computations of projection pattern cohomology when the acceptance domain is disconnected. We prove our main results in Section 3.

We note that the case $d = 1$ gives the classical Denjoy counterexample systems whose ordered cohomology is discussed in [**PSS**], an observation which has also been made by [**He**]. The result in this chapter, for $d = 1$ and H_* finitely generated, can be deduced from that paper [**PSS**] a task which has been carried out in [**He**].

2 Inverse limit acceptance domains

Suppose that K and K_i, $i = 1, 2, \ldots,$ are compact subsets of $E^\perp$ each of which is the closure of its interior, and suppose that $IntK =$

$\cup_i IntK_i$ is a disjoint union and that $\partial K = \cup \partial K_i$. Let $K_i^* = \underset{j \leq i}{\cup} K_j$, so that $IntK_i^* = \underset{j \leq i}{\cup} IntK_j$ is a disjoint union and $\partial K_i^* = \underset{j \leq i}{\cup} \partial K_j$.

We define $NS^i = \mathbb{R}^N \setminus (E + \mathbb{Z}^N + \partial K_i^*)$ and $\Sigma^i = K_i^* + E$. So for each $u \in NS^i$ we have $\widetilde{P}_u^i = \mathbb{Z}^N \cap \Sigma^i$ and $P_u^i = \pi(\widetilde{P}_u^i)$. From these we construct $\widetilde{\Pi}_u^i$, Π_u^i, MP_u^i and so on, as usual. In fact, in the following, we shall be interested only in the strip pattern $\widetilde{P}_u^i$.

Provided u is non-singular for K, and hence is non-singular for all K_i^*, we can take a space F complimentary to E in $\mathbb{R}^N$ and a corresponding group G_u (Def. I.10.2) which will play their usual roles for all sets of projection data $(E, \mathbb{R}^N, K_i^*)$ and $(E, \mathbb{R}^N, K)$. For each domain, K_i^*, we construct the corresponding F_u^i *etc.* The following lemma follows easily from the definitions.

Lemma 2.1 *Suppose $j < i$ is fixed throughout the statement of this lemma. Then NS^i is a dense subset of NS^j and $NS = \cap_k NS^k$. Moreover, for $u \in NS^i$, we have a natural continuous $E + \mathbb{Z}^N$ equivariant surjection $\widetilde{\Pi}_u^i \longrightarrow \widetilde{\Pi}_u^j$, and a natural continuous E equivariant surjection $M\widetilde{P}_u^i \longrightarrow M\widetilde{P}_u^j$; this latter is described equivalently by the formula $S \mapsto S \cap \Sigma^j$.*

We also have an $\mathbb{Z}^N$-equivariant map $F_u^i \longrightarrow F_u^j$, and a G_u-equivariant map $X_u^i \longrightarrow X_u^j$. All these maps respect the commutative diagrams of Chapter I and they map many-to-one only when the image is in (the appropriate embedding of) $NS^j \setminus NS^i$. □

Theorem 2.2 *With the notation and assumptions above, we have the following equivariant homeomorphisms.*

(a) $\widetilde{\Pi}_u \cong \varprojlim \widetilde{\Pi}_u^i$, $\quad E + \mathbb{Z}^N$ *equivariantly;*

(b) $M\widetilde{P}_u \cong \varprojlim M\widetilde{P}_u^i$, $\quad E$*-equivariantly;*

(c) $F_u \cong \varprojlim F_u^i$, $\quad \mathbb{Z}^N$*-equivariantly;*

(d) $X_u \cong \varprojlim X_u^i$, $\quad G_u$*-equivariantly.*

Proof Once again the results are straightforward from the definitions. The map $M\widetilde{P}_u \longrightarrow M\widetilde{P}_u^i$ is again equivalently written $S \mapsto S \cap \Sigma^i$. □

The following is now a direct consequence of (4.2), (2.2)(b) and the continuity of Čech cohomology on inverse limits.

Corollary 2.3 *There is a natural equivalence*

$$H^*(\mathcal{G}\widetilde{P}_u) = \varinjlim H^*(\mathcal{G}\widetilde{P}_u^i). \qquad \square$$

3 Cohomology in the case *d=N–1*

In this section we determine the cohomology for projection method patterns when $d = N - 1$. It is the only case for which we have such a complete answer.

Here E is a codimension 1 subspace of $\mathbb{R}^N$ and so $E \cap \mathbb{Z}^N = 0$ implies that $\Delta = 0$. Therefore, $MP_v = MP_u = M\widetilde{P}_u$ for all $u, v \in NS$ and so, given E and K, there is only one projection pattern torus MP to consider, no need to parametrise by u, and an equation $S_P = \mathbb{Z}^N$ (I.8.2). With this in mind, we shall avoid further explicit mention of any particular non-singular point u.

Write $e_1, ..., e_N$ for the usual unit vector basis of $\mathbb{R}^N$, which are also the generators of $\mathbb{Z}^N$. Choose the space F as that spanned by e_N, and so $G_{\mathcal{T}} = \langle e_N \rangle$ and $G_u = \langle e_1, e_2, ..., e_{N-1} \rangle$. Recall that we write $K' = \pi'(K) \subset F$, where π' is the skew projection onto F parallel to E and that π' maps K homeomorphically to K', preserving the boundary, $\partial K' = \pi'(\partial K) \cong \partial K$.

Now any compact subset of $E^\perp \equiv \mathbb{R}$ which is the closure of its interior is a countable union of closed disjoint intervals; and K is such a set. Thus ∂K and hence $\partial K'$ is countable. Pick $A = \{p_1, p_2, ...\}$, $p_j \in \partial K'$, to be a set of representatives of $\pi'(\mathbb{Z}^N)$ orbits of $\partial K'$, a countable and possibly finite set. Write $k \in \mathbb{Z}_+ \cup \infty$ for the cardinality of A.

Theorem 3.1 *If $\mathcal{T}$ is a projection method pattern with $d = N - 1$ and $E \cap \mathbb{Z}^N = 0$, then*

$$H^m(\mathcal{GT}) = H^m(\mathbb{T}^N \setminus k \text{ points }) = \begin{cases} \mathbb{Z}^{\binom{N}{m}} & \text{for } m \leq N-2, \\ \mathbb{Z}^{N+k-1} & \text{for } m = N-1, \\ 0 & \text{otherwise.} \end{cases}$$

An infinite superscript denotes the countably infinite direct sum of copies of $\mathbb{Z}$.

Proof We know that $IntK'$ is the union of a countable number of open intervals, whose closures, K_i, are disjoint. We use the notation and results of Section 2, setting $K_i^* = \bigcup_{j \leq i} K_j$ as the finite union of disjoint closed intervals, $\underset{j\leq i}{\cup}[s_je_N, r_je_N]$ say. As $M\mathcal{T} = M\widetilde{P}$, by (2.3) it is enough to compute the direct limit $\varinjlim H^*(M\widetilde{P}^i)$.

We consider the process of completion giving rise to the space $M\widetilde{P}^i$ which we consider as $\widetilde{\Pi}^i$ (the completion of the non-singular points NS^i) modulo the action of $\mathbb{Z}^N$. The limit points introduced in $\widetilde{\Pi}^i$ arise as the limit of patterns $P^i_{x_n}$ as x_n approaches a singular point, either from a positive e_N direction, or from a negative one. To be more precise, suppose that $x_n = x + t_ne_N \in NS$ is a sequence converging to $x \in \mathbb{R}^N$ in the Euclidean topology. If (t_n) is an increasing sequence, then $\lim_{n\to\infty} \widetilde{P}^i_{x_n}$ exists in the D metric and is the point pattern $(x + \mathbb{Z}^N) \cap (\underset{j\leq i}{\cup}(s_je_N, r_je_N] + E)$.

Likewise, if (t_n) is a decreasing sequence then $\lim_{n\to\infty} \widetilde{P}^i_{x_n}$ is the point pattern $(x + \mathbb{Z}^N) \cap (\underset{j\leq i}{\cup}[s_je_N, r_je_N) + E)$. These two patterns are the same if and only if $x \in NS^i$. If $x \notin NS^i$ then these two patterns define the two D limit points in $\widetilde{\Pi}^i$ over $x \in \mathbb{R}^N$. Thus the quotient $M\widetilde{P}^i \longrightarrow \mathbb{T}^N$ is 1-1 precisely when mapping to the set $NS^i/\mathbb{Z}^N$, and otherwise it is 2-1; we can picture the map intuitively as a process of "closing the gaps" made by cutting $\mathbb{T}^N$ along the finite set of hyperplanes $(\partial K_i^* + E)/\mathbb{Z}^N$, *c.f.* **[Le]**.

We examine the space $M\widetilde{P}^i$ in more detail. Given $r > 0$, consider the space $M^i_r = \{(S \cap B(r)) \cup \partial B(r) : S \in M\widetilde{P}^i_u\}$ endowed with the Hausdorff metric d_r among the set of all closed subsets of $B(r)$, the closed ball in $\mathbb{R}^N$ with centre 0 and radius r. By construction M^i_r is compact and, for $s \geq r$ and $i \geq j$, there are natural restriction maps $M^i_s \longrightarrow M^j_r$, whose inverse limit for fixed $i = j$ is $M\widetilde{P}^i$, and whose inverse limit over all i and r by (2.2) is $M\widetilde{P}$. The map $M^i_r \longrightarrow \mathbb{T}^N$ given by $\widetilde{P}_v \mapsto v \bmod \mathbb{Z}^N$, $v \in NS^i$, factors the canonical quotient $M\widetilde{P}^i \longrightarrow \mathbb{T}^N$.

Define C^i_r as the set $\{v \in \mathbb{T}^N : (v + \mathbb{Z}^N) \cap (\partial\Sigma^i \cap Int B(r)) \neq \emptyset\}$. As before, $M^i_r \longrightarrow \mathbb{T}^N$ is 2-1 precisely on those points mapped to C^i_r and otherwise is 1-1.

The intersection $\partial\Sigma^i \cap B(r)$ is, for all r large enough compared with the diameter of K_i^*, equal to a finite union of codimension 1 discs, parallel to E, and of radius at least $r-1$, and at most r. Each of these discs has centre $\pi^\perp(a)$ for some $a \in \partial K_i^*$. Consider this collection of discs modulo $\mathbb{Z}^N$ and select two, say with centres $\pi^\perp(a)$ and $\pi^\perp(b)$, where $a, b \in \partial K_i^*$. Then, for r very large and $a - b \in \pi'(\mathbb{Z}^N)$, these discs will overlap modulo $\mathbb{Z}^N$. Since there are a finite number of such pairs in ∂K_i^* to consider, we have a universal r such that if $a, b \in \partial K_i^*$ and $a - b \in \pi'(\mathbb{Z}^N)$, then the disc with centre $\pi^\perp(a)$ overlaps, modulo $\mathbb{Z}^N$, the disc with centre $\pi^\perp(b)$. If $a - b \notin \pi'(\mathbb{Z}^N)$, then these discs will not overlap, modulo $\mathbb{Z}^N$. Hence for r sufficiently large, $\partial\Sigma^i \cap B(r)$ mod $\mathbb{Z}^N$ has precisely $|A_i|$ components.

For the same r, C_r^i mod $\mathbb{Z}^N$ is also a finite union of discs of radius at least $r-1$ and at most r; likewise C_r^i has exactly $|A_i|$ components, in direct correspondence with the elements of A_i.

The description above of the limiting points in $\widetilde{\Pi}^i$ as we approach C_r^i in a direction transverse to E, shows that M_r^i is homeomorphic to $\mathbb{T}^N$ with a small open neighbourhood of C_r^i removed. There is a natural homotopy equivalence with the space $\mathbb{T}^N \setminus C_r^i$.

We can now examine what happens as we let first r and then i tend to infinity in this construction. For the above sufficiently large r, the map $M_{r+1}^i \longrightarrow M_r^i$ is, up to homotopy, the injection from $\mathbb{T}^N \setminus C_{r+1}^i$ to $\mathbb{T}^N \setminus C_r^i$, and this is simply, up to homotopy, the identity from $\mathbb{T}^N \setminus |A_i|$ points to itself. Hence $H^*(M_r^i) = H^*(\mathbb{T}^N \setminus |A_i| \text{ points})$ and $H^*(M_r^i) \longrightarrow H^*(M_{r+1})$ is the identity showing that $H^*(M\widetilde{P}^i)$ is the cohomology of the torus with $|A_i|$ punctures.

Finally, for each i and for r sufficiently large (depending on i) the map $M_r^{i+1} \longrightarrow M_r^i$ is that induced by the inclusion of A_i in A_{i+1}, and this corresponds in the above description to the adding of a new puncture for each element of $A_{i+1} \setminus A_i$. In cohomology, the map $H^p(M_r^{i-1}) \longrightarrow H^p(M_r^i)$ is thus the identity for $p \neq N-1$, and in dimension d gives rise to the direct system of groups and injections $\cdots \longrightarrow \mathbb{Z}^{N-1+|A_i|} \longrightarrow \mathbb{Z}^{N-1+|A_{i+1}|} \longrightarrow \cdots$ which gives the required formula. □

We give an alternative proof of this theorem from a different perspective in Chapter V.

We note that the pattern dynamical system (X, G_1) is in fact a Denjoy example [**PSS**] generalized to a $\mathbb{Z}^{N-1}$ action and dislocation along k separate orbits.

Corollary 3.2 *Suppose that* Γ *is a dense countable subgroup of* $\mathbb{R}$ *finitely generated by* r *free generators. Suppose we Cantorize* $\mathbb{R}$ *by cutting and splitting along* k Γ*-orbits (as described e.g. in* [**PSS**]*) to form the locally Cantor space* R' *on which* Γ *acts continuously, freely and minimally. Consider the* Γ*-module* C *of compactly supported integer valued functions defined on* R'*. Then* $H^*(\Gamma, C) = H^*(\mathbb{T}^r \setminus k$ *points*$;\mathbb{Z})$. □

IV Approaches to Calculation II: Infinitely Generated Cohomology

1 Introduction

In this chapter we restrict ourselves to the classical projection tilings with canonical acceptance domains K (so K is the projection of the unit cube in $\mathbb{R}^N$). We examine the natural question of when such tilings arise also as substitution systems and show that the invariants of chapter 2 are effective and computable discriminators of such tiling properties.

Much of this chapter is devoted to giving a qualitative description of the cohomology of canonical projection method patterns. The main result is formulated in Theorem 2.9. It gives a purely geometric criterion for infinite generation of pattern cohomology and for infinite rank of its rationalisation. As a corollary of this, by the obstruction proved in II.5, we deduce that almost all canonical projection method patterns fail to be substitution systems; in fact for vast swathes of initial data *all* such patterns fail to be self similar.

The restriction to the canonical acceptance domain allows for a second, geometric description of the coefficient groups CF_u (I.9.3) and CV_u (II.4.3) in terms of indicator functions on particular polytopes, we call them *$\mathcal{C}_u$-topes.* In Section 2 we introduce this viewpoint, setting up the notation and definitions sufficient to state the main theorem. From here until the end of Section 5, our aim is to prove Theorem 2.9 by establishing criteria for the existence of infinite families of linearly independent $\mathcal{C}_u$-topes. In Sections 3 and 4 we construct such families in the *indecomposable* case, and complete the analysis for the general case in Section 5. The final Section 6 gives some general classes of patterns where the conditions of Theorem 2.9 are satisfied, so proving the generic failure of self-similarity for canonical projection method patterns.

2 The canonical projection tiling

For the first time in our studies, we narrow our attention to the classical projection method tilings of **[OKD]** **[dB1]**. This section outlines the simplifications to be found in this case, and describes the main result of the remainder of this paper; a sufficient condition for infinitely generated $H^d(\mathcal{GT})$. In Chapter V we will see that this condition is also neccessary.

Therefore we have data (K, E, u) where $K = \pi^\perp([0,1]^N)$ and $u \in NS$. From Section I.8 we see that, but for a few exceptional cases, we have $S_\mathcal{T} = \mathbb{Z}^N$, and if we elect either to exclude these exceptions (as most authors do) or to include them only in their most decorated form ($M\mathcal{T} = M\widetilde{P}_u$), we make $S_\mathcal{T} = \mathbb{Z}^N$ a standing assumption.

With Theorem I.9.4 we have a description of the topology of CF_u: it is generated by intersection and differences of shifts of a single compact open set $\overline{K}$, formed by completing $\pi' K \cap (Q+u) \cap NS$ (Lem. I.9.6). Topologically $CF_u \equiv CV_u \times \widetilde{\Delta}$ and CV_u inherits the subspace topology and a stabilizing subaction $\Gamma_\mathcal{T}$ of $S_\mathcal{T}$ (see II.4.3).

With the choice of K as a canonical acceptance domain above, we may follow more closely the work of Le [**Le**] and give other more geometrical descriptions of the elements of CF_u and CV_u.

Definition 2.1 For each $J \subset \{1, 2, ..., N\}$, we construct a subspace $e^J = \langle e_j : j \in J \rangle$ (the span) of $\mathbb{R}^N$, where $\{e_j\}$ is the standard unit basis of $\mathbb{R}^N$ or $\mathbb{Z}^N$.

Write $\dim F = n$.

Let $\mathcal{I} = \{J \subset \{1, 2, ..., N\} : \dim \pi'(e^J) = n - 1\}$ and define $\mathcal{I}^*$ to be the set of elements of $\mathcal{I}$ minimal with respect to containment.

Define $\mathbb{Z}^N_{n-1} = \cup \{e^J + v : |J| = n-1, v \in \mathbb{Z}^N\}$, i.e. the $n-1$-dimensional skeleton of the regular cubic CW decomposition of $\mathbb{R}^N$.

The following Lemma gives some combinatorial information about $\mathcal{I}^*$ and describes the singular points in F - they are formed by unions of affine subspaces of F of codimension 1.

Lemma 2.2 *With the construction above,*

i/ $\mathcal{I}^*$ is a sub-collection of the $n-1$-element subsets of $\{1, 2..., N\}$. Also every subspace of F of the form $\pi'(e^J)$, with $|J| = n-1$, is contained in $\pi'(e^{J'})$ for some $J' \in \mathcal{I}^*$.

ii/ $\mathbb{R}^N \setminus NS = \pi'^{-1}\pi'(\mathbb{Z}^N_{n-1})$ and $\pi'(\mathbb{Z}^N_{n-1}) = F \setminus NS$.

Proof i/ Straightforward.

ii/ With the data above, $\pi'(K)$ is a convex polytope in F, with interior, each of whose extreme points is of the form $\pi'(v)$ where $v \in \{0,1\}^N$. By i/, each of the faces of $\pi'(K)$ is contained in some $\pi'(e^J + v)$ where $v \in \{0,1\}^N$ and $J \in \mathcal{I}^*$. Also by i/, we have $\pi'(\mathbb{Z}^N_{n-1}) = \pi'(\cup\{e^J + v : v \in \mathbb{Z}^N, J \in \mathcal{I}^*\})$.

However, by definition, $F \setminus NS$ is the union of the faces of those polytopes of the form $\pi'(K+v)$, $v \in \mathbb{Z}^N$. Thus we deduce immediately that $\pi'(\mathbb{Z}^N_{n-1}) = \pi'(\cup\{e^J + v : v \in \mathbb{Z}^N, J \in \mathcal{I}^*\}) \supset F \setminus NS$.

Conversely, a simple geometric argument shows that for each $J \in \mathcal{I}^*$, there is a face of $\pi'(K)$ which is contained in $\pi'(e^J + v)$ for some $v \in \mathbb{Z}^N$.

Assuming the claim, we have, for each $J \in \mathcal{I}^*$ and each $v \in \mathbb{Z}^N$, some suitable shift of $\pi'(K)$, $\pi'(K + w)$ say, $w \in \mathbb{Z}^N$, one of whose faces, Φ say, contains the point $\pi'(v)$ as an extreme point, and $\Phi \subset \pi'(e^J + v)$. With this same Φ, therefore, we see that $\cup\{\Phi + \pi'(w) : w \in \mathbb{Z}^N\} \supset \cup\{\pi'(e^J + w) : w \in \mathbb{Z}^N\}$. Thus $\pi'(\mathbb{Z}^N_{n-1}) = \cup\{\pi'(e^J + v) : v \in \mathbb{Z}^N, J \in \mathcal{I}^*\} \supset F \setminus NS$. So we are done. $\square$

Definition 2.3 Recall the subspace V from (I.2) and write $\dim V = m$.

Now consider the set $\mathcal{I}^*(V) = \{J \in \mathcal{I}^* : \dim(\pi'(e^J) \cap V) = m - 1\}$.

The following Lemma shows that the sets $\mathcal{I}^*(V)$ can be found canonically from the sets $\mathcal{I}^*$ and describes the singular points in $V + \pi'(u)$ preparatory to a description of V_u (See II.4.3). Once again, the singular points in $V+\pi'(u)$ form a union of affine hyperplanes in $V+\pi'(u)$.

Lemma 2.4 *i/* $\mathcal{I}^*(V) = \{J \in \mathcal{I}^* : \pi'(e^J) \cap V \neq V\}$.

ii/ If $u \in NS$, *then* $(V + \pi'(u)) \setminus NS = (V + \pi'(u)) \cap \pi'(\mathbb{Z}^N_{n-1}) = (V + \pi'(u)) \cap (\cup\{\pi'(e^J + v) : v \in \mathbb{Z}^N, J \in \mathcal{I}^*(V)\})$.

Proof Follows directly from definition of V and Lemma 2.2 $\square$

With these notations in mind, we are well equipped to describe the topology of V_u. Although V_u is best described as a subspace of F_u formed from placing V as the affine subspace $V+\pi'(u)$ in F, we prefer to shift the whole construction back to V by applying a uniform shift by $-\pi'(u)$. There are some advantages later in having an origin and a vector space structure.

Definition 2.5 Given $u \in NS$, we use 2.4 i/ to define a set, $\mathcal{C}_u$, of $m-1$-dimensional affine subspaces of V whose elements are of the form $(\pi'(e^J + v) \cap (V + \pi'(u))) - \pi'(u)$ where $J \in \mathcal{I}^*(V)$ and $v \in \mathbb{Z}^N$. Such a space may also be written, $\pi'(e^J + v - u) \cap V$.

We say that a subset of V is a $\mathcal{C}_u$*-tope*, if

i/ it is compact and is the closure of its interior, and ii/ it is a polytope, each of whose faces is a subset of some element of $\mathcal{C}_u$.

We shall also say that a subset, B, of V_u is a $\mathcal{C}_u$*-tope* if B is clopen and $\nu(B) - \pi'(u)$ is a $\mathcal{C}_u$-tope subset of V in the sense above (recall ν from (I.9.2) above).

We shall show that the $\mathcal{C}_u$-topes generate the topology on V_u. To help this we describe a third possible topology as it is found in [**Le**].

For each element, α, of $\mathcal{C}_u$, a hyperplane in V, consider the half-spaces $H_\alpha^\pm$ defined by it. The sets $H_\alpha^\pm \cap NS$ can be completed in the $\overline{D}$ metric to a closed and open subset of V_u which we write $\overline{H}_\alpha^\pm$. We also call these subsets of V_u *half-spaces.*

Proposition 2.6 *The following three collections of subsets of* V_u *are the same:*

i/ The collection of $\mathcal{C}_u$*-topes.*

ii/ The collection of compact open subsets of V_u*.*

iii/ The collection of those finite unions and intersections of half-spaces in V_u *which are compact.*

Proof Every $\mathcal{C}_u$-tope in V_u is clearly an element of collection iii/ since every $\mathcal{C}_u$-tope in V is a finite intersection and union of half spaces in V.

Conversely, every open half-space in V is a countable, locally finite, union of the interiors of $\mathcal{C}_u$-topes (in V). Therefore any pre-compact intersection of half spaces can be formed equivalently from some union and intersection of a finite collection of $\mathcal{C}_u$-topes, i.e. is a $\mathcal{C}_u$-tope, and each $\mathcal{C}_u$-tope in V_u is an intersection of clopen half-spaces.

Now compare collection ii/ with iii/ and i/.

Suppose that $\overline{H}$ is a half-space in V_u defined by the hyperplane $W \in \mathcal{C}_u$. As noted in 2.5, W has the form $\pi'(e^J + v - u) \cap V$ for some $J \in \mathcal{I}^*(V)$ and $v \in \mathbb{Z}^N$, and our choice of half-space gives a corresponding choice of half-space in V on one side or other of W. Since $J \in \mathcal{I}^*(V) \subset \mathcal{I}^*$ (2.4 i/), this in turn defines a choice of half-space in F one side or other of $\pi'(e^J + v - u)$. Write H for the closed half-space of F chosen this way.

Consider the collection of N-dimensional cubes in the $\mathbb{Z}^N$ lattice whose image under the projection π' is contained entirely in $H + \pi'(u)$. By Lemma 2.4 ii/, the boundary of $H + \pi'(u)$ is an $(n-1)$-dimensional hyperplane in F, the image under π' of an $(n-1)$-dimensional subspace of $\mathbb{R}^N$ contained in $\mathbb{Z}^N_{n-1}$. Therefore the cubes collected above line up against this boundary exactly and, projected under π', they cover $H + \pi'(u)$ exactly. In particular, $H + \pi'(u)$ is covered by $\Gamma_{\mathcal{T}}$-translates of $\pi'([0,1]^N)$. Restricting this construction to the space $(V + \pi'(u)) \cap NS$, we find that $\overline{H}$ is a locally finite countable union of shifts of $\overline{K}$. In short, collection iii/ is contained in collection ii/.

Conversely, we see that $\pi'([0,1]^N) \cap NS = \pi'((0,1)^N) \cap NS$ so we are sure that the intersection $(V + \pi'(u)) \cap ((\pi'([0,1]^N)) + \pi'(v))$ (for any choice of $v \in \mathbb{Z}^N$) is a polytope subset of $V + \pi'(u)$ with interior. The faces of this polytope are clearly subsets of NS, i.e. contained in elements of $\mathcal{C}_u$ shifted along with V by $\pi'(u)$. Thus every $\Gamma_{\mathcal{T}}$ translate of $\overline{K}$ intersects V_u either non-emptily or as a $\mathcal{C}_u$-tope. Collection ii/ is contained in collection i/ therefore. $\square$

Since V_u is locally compact, we can describe the topology on V_u, defined in II.4.3, equivalently as that generated by $\mathcal{C}_u$-topes, or as that generated by half-spaces. This gives an alternative description of CV_u.

Corollary 2.7 *Let $V' = (V + \pi'(u)) \cap NS) - \pi'(u)$. The group CV_u is naturally isomorphic to the group of integer-valued functions, $V' \longrightarrow \mathbb{Z}$, generated by indicator functions of sets of the form $C \cap V'$ where C is a $\mathcal{C}_u$-tope. This group isormophism is a ring and $\mathbb{Z}[\Gamma_{\mathcal{T}}]$-module isomorphism as well.* $\square$

Now we describe an important set of points.

Definition 2.8 Write $\mathcal{P}$ for the set of points in V which can found as the 0-dimensional intersection of m elements of $\mathcal{C}_u$. Note that, under the assumptions on $\mathcal{C}_u$, $\mathcal{P}$ is a non-empty countable set, invariant under shifts by $\Gamma_{\mathcal{T}}$.

Say that $\mathcal{P}$ *is finitely generated* if $\mathcal{P}$ is the disjoint union of a finite number of $\Gamma_{\mathcal{T}}$ orbits, *infinitely generated* otherwise.

We may now express the main theorem of this Chapter.

Theorem 2.9 *Given a canonical projection method pattern, $\mathcal{T}$ and the constructions above. If $\mathcal{P}$ is infinitely generated, then $H^d(\mathcal{GT})$ is infinitely generated and $H^d(\mathcal{GT}) \otimes \mathbb{Q}$ is infinite dimensional.*

In Chapter V (Theorem V.2.4) we prove the converse i.e. that $H^d(\mathcal{GT})$ infinitely generated is equivalent to $\mathcal{P}$ infinitely generated.

We complete the proof of Theorem 2.9 in Section 5, but the final step and basic idea, of the proof can be presented already.

Proposition 2.10 *Suppose that G is a torsion-free abelian group and that H is an abelian group, with $H/2 := H/2H$ its reduction* mod 2. *Suppose that there is a group homomorphism, $\phi : G \longrightarrow H/2$, such that $Im\phi$ is infinitely generated as a subgroup of $H/2$ (equivalently infinite dimensional as a $\mathbb{Z}/2$ vector space); then G is infinitely generated as an abelian group, and $G \otimes \mathbb{Q}$ is infinitely generated as a $\mathbb{Q}$ vector space.*

Proof It is sufficient to prove the statement concerning $G \otimes \mathbb{Q}$.

Suppose that $\phi(s_n)$ is a sequence of independent generators for $Im\phi$ and suppose that there is some relation

$$\sum_{n=1}^{m} q_n s_n = 0$$

for $q_n \in \mathbb{Q}$. Since G is torsion-free, we can assume the q_n are integers and have no common factor; in particular, they are not all even. Applying the map ϕ then gives a non-trivial relation among the $\phi(s_n)$, a direct contradiction, as required. □

Therefore, to prove 2.9, we shall find a homomorphism from $H^d(\mathcal{GT})$, equivalently $H_0(\Gamma_{\mathcal{T}}, CV_u)$, to an infinite sum of $\mathbb{Z}/2$ whose image is infinitely generated. This is completed in full generality in Theorem 5.4. In order to construct independent elements of $H_0(\Gamma_{\mathcal{T}}, CV_u)$ and its image, we must consider the geometry of the $\mathcal{C}_u$-topes defined in 2.5.

3 Constructing $\mathcal{C}$-topes

To prove that a group or $\mathbb{Q}$-vector space is infinitely generated, we must produce independent generators. In the case of $H_0(\Gamma_{\mathcal{T}}, CV_u)$, we must find elements of CV_u which remain independent modulo Γ-boundaries. In any case, we must at least produce some elements of CV_u, and in this section we start with constructions of the simplest objects in this space: indicator functions of $\mathcal{C}_u$-topes.

Rather than refer constantly to the original tiling notation, we abstract the construction conveniently, basing our development on a general collection of affine hyperplanes, $\mathcal{C}$, of a vector space, V, with group action, Γ. Always, the example in mind is $\mathcal{C} = \mathcal{C}_u$ (2.5), V (I.2) and $\Gamma = \Gamma_{\mathcal{T}}$ (II.4.3), but the construction is potentially more general. However, the first few definitions and constructions are the slightest generalization of those of section 2.

Definition 3.1 Suppose that V is a vector space of dimension m and that Γ is a finitely generated free abelian group acting minimally by translation on V. Thus we write $w \mapsto w + \gamma$ for the group action by $\gamma \in \Gamma$, and we may think of Γ as a dense subgroup of V without confusion.

Suppose that $\mathcal{C}$ is a countable collection of affine subspaces of V such that each $W \in \mathcal{C}$ has dimension $m-1$, and such that, if $W \in \mathcal{C}$ and $\gamma \in \Gamma$, then $W + \gamma \in \mathcal{C}$.

We suppose that the number of Γ orbits in $\mathcal{C}$ is finite.

If $W \in \mathcal{C}$, then we define a unit normal vector, $\lambda(W)$ (with respect to some inner-product). The set $\mathcal{N}(\mathcal{C}) = \{\lambda(W) : W \in \mathcal{C}\}$ is finite and we suppose that we have chosen the $\lambda(W)$ consistently so that $-\lambda(W) \notin \mathcal{N}(\mathcal{C})$.

We suppose that $\mathcal{N}(\mathcal{C})$ generates V as a vector space.

We can consider the intersections of elements of $\mathcal{C}$.

Definition 3.2 Given $0 \leq k \leq m-1$, define $\mathcal{C}^{(k)}$ to be the collection of k-dimensional affine subspaces of V formed by intersection of elements of $\mathcal{C}$. Thus $\mathcal{C}$ can be written $\mathcal{C}^{(m-1)}$, and, to be consistent with the notation of section 2, $\mathcal{C}^{(0)}$ can be written $\mathcal{P}$.

Define a *singular flag*, $\mathcal{F}$, to be a sequence of affine subspaces, $(\theta_0, \theta_1, ..., \theta_{m-1})$ of V such that $\theta_j \in \mathcal{C}^{(j)}$ for all $0 \leq j \leq m-1$, and $\theta_j \subset \theta_{j+1}$ for all $0 \leq j \leq m-1$. The set of singular flags is written $\mathcal{J}_o$.

It is clear that each $\mathcal{C}^{(k)}$ and $\mathcal{J}_o$ supports a canonical Γ action. We write $\mathcal{J} = \mathcal{J}_o/\Gamma$, the set of Γ orbits in $\mathcal{J}_o$.

Definition 3.3 We say that a subset, C, of V is a $\mathcal{C}$*-tope*, if

i/ C is compact and is the closure of its interior, and

ii/ C is a polytope, each of whose $(m-1)$-dimensional faces is a subset of some element of $\mathcal{C}$.

Definition 3.3 Convex $\mathcal{C}$-topes are finite objects whose geometry and combinatorics are immediately and intuitively related. Therefore we think of the definition of face, edge and vertex, and more generally

k-dimensional face, in this case, as the intuitive one. Likewise the idea of incidence of edge on vertex, face on edge, etc is intuitive.

Suppose that $C \subset V$ is a convex $\mathcal{C}$-tope and that $\mathcal{F} = (\theta_j)$ is a singular flag. We say that *$\mathcal{F}$ is incident on C* if each θ_k contains a k-dimensional face of C.

We write $[\mathcal{F}]$ for the Γ-orbit class of $\mathcal{F}$. We say that *$\mathcal{F}$ is uniquely incident on C* if $\mathcal{F}$ is incident on C, but no other $\mathcal{F}' \in [\mathcal{F}]$ is incident on C.

The main aim of this section is to build convex $\mathcal{C}$ topes on which certain singular flags are uniquely incident. This will not happen in every possible circumstance however, and we develop an idea of decomposability which will break up the space V into a direct sum of spaces on which such constructions can be made. In section 5 we shall show how to recombine these pieces.

Construction 3.5 Suppose that A is a finite set of non-zero vectors in V which spans V, and no pair of which is parallel. The example we have in mind is $\mathcal{N}(\mathcal{C})$, the set of normals.

A *decomposition* of A is a partition $A = A_1 \cup A_2$ such that $V_1 \cap V_2 = 0$ where each V_j is the space spanned by A_j, $j = 1, 2$.

A is *indecomposable* if no such decomposition is possible. It is not hard to show that every set A has a unique partition into indecomposable subsets.

Suppose that $B \subset A$ is a basis for V. Then, by stipulating that B is an orthonormal basis, we define an inner product which we write as square brackets: $[.,.]_B$.

Then we define a finite graph $G(B; A)$ with vertices B and an edge from x to y whenever there is a $z \in A \setminus B$ such that $[x, z]_B \neq 0$ and $[y, z]_B \neq 0$. We do not allow loops.

The following is elementary.

Lemma 3.6 *Suppose that A spans V, then the following are equivalent:*

i/ A is indecomposable

ii/ for all bases $B \subset A$, $G(B; A)$ is connected

iii/ for some basis $B \subset A$, $G(B; A)$ is connected. □

Remark 3.7 Note that if $\phi : V \longrightarrow V$ is a linear bijection, then A is indecomposable if and only if $\phi(A)$ is indecomposable. Therefore, the condition $\mathcal{N}(\mathcal{C})$ indecomposable can be stipulated without reference to a particular inner product, although an inner product must be used to define the normals. We use this freedom in Theorem 3.10 ahead.

Construction 3.8 Suppose that A and B are as above, giving an inner product $[.,.]_B$ to V and defining a graph $G(B;A)$. Choose $b \in B$ and let W_b be the hyperplane in V orthogonal to b and let π_b be the orthogonal projection of V onto W_b.

Consider the sets $\pi_b(A)$ and $\pi_b(B)$. Apart from $0 = \pi_b(b)$ the latter equals $B \setminus b$ precisely, an othonormal basis for W_b. The former contains $B \setminus b$ and other vectors which may be of various lengths and pairs of which may be parallel. From this set, we form A_b a new set of vectors in W_b by taking, for each class of parallel elements of $\pi_b(A)$ a single unit length representative (ignoring 0). If the class in question is one which contains some $b' \in B \setminus b$, then we let b' be the representative chosen. Write $B_b = B \setminus b$, so that $B_b \subset A_b$ is a basis.

We consider the sets A_b, B_b in W_b and form the graph $G(B_b; A_b)$ with respect to the inner product $[.,.]_{B_b}$, the restriction of the inner product $[.,.]_B$ to W_b.

If $\mathcal{C}$ is a collection of affine hyperplanes in V, then we define $\mathcal{C}_b$ to be collection of affine hyperplanes in W_b of the form $W_b \cap W$ where $W \in \mathcal{C}$ is chosen so that W is not parallel to W_b.

The following useful lemma comes straight from the definition.

Lemma 3.9 *i/ The graph $G(B_b; A_b)$ is formed from $G(B;A)$ by removing the vertex b and all its incident edges from $G(B;A)$.*

ii/ If $A = \mathcal{N}(\mathcal{C})$, then $A_b = \mathcal{N}(\mathcal{C}_b)$.

Theorem 3.10 *Suppose that $m = \dim V > 1$ and that $0 \notin A \subset V$ spans V, and A has no parallel elements. Suppose that $B \subset A$ is a basis for V and that $G(B;A)$ is connected, then there is a closed convex polytope, C, of V, with interior, such that*

i/ The normal of each face of C is an element of A.

ii/ v is a vertex of C which is at the intersection of exactly m faces of C and each of these faces is normal to some element of B.

iii/ The vertex v is uniquely defined by property ii/.

(All normals are taken with respect to the inner product $[.,.]_B$.)

Proof We suppose, without loss of generality, that $v = 0$ and proceed by induction on $m = |B| \geq 2$.

Suppose that $|B| = 2$. By graph connectedness, we find $a \in A \setminus B$ with non-zero components in each B coordinate direction. Thus we can construct easily a triangle, C, in V with the required properties.

For larger values of $|B|$, we proceed as follows.

As in Construction 3.8, define, for each $b \in B$, W_b to be the hyperplane normal to b and let $\pi_b : V \longrightarrow W_b$ be the orthogonal projection.

A simple argument shows that we can find $b_o \in B$ so that $G(B; A) \setminus b_o$ (i.e. removing the vertex b_o and all incident edges) is connected. Thus, appealing to the description of Lemma 3.9 and by induction on $|B|$, we can find a convex compact polytope subset, C_1, of W_{b_o} which obeys conditions i/ to iii/ above with respect to the basis $B \setminus b$ and hyperplanes with normals parallel to some $\pi_{b_o}(a) : a \in A \setminus b$.

Suppose that W' is a hyperplane in W_{b_o} which contains a face of C_1 and has normal $\pi_{b_o}(a)$ for some $a \in A$. Then W' is in fact of the form $W \cap W_{b_o}$, where W is the unique hyperplane in V, normal to a, containing the space W'. Indexing the faces of C_1 with j say, each face is contained in W'_j with normal $\pi_{b_o}(a_j)$. So construct a hyperplane W_j in V as above, with normal a_j. In the case of the faces of C_1, incident on 0 and normal to $b \in B \setminus b_o$ say, we make sure that we choose $a_j = b$, so that in this case $W_j = W_b$ as defined above.

Now we build C_2, a convex closed subset of V, as follows. For each face j of C_1, let H_j be the closed half-space defined by W_j, containing the set C_1. Let $C_2 = \cap_j H_j$, where the intersection is indexed over all faces of C_1, so that $C_1 = C_2 \cap W_{b_o}$.

C_2 is almost certainly unbounded, but it is important to note that none of its faces is normal to b_o. Furthermore, by construction of C_1, there is only one edge in C_2 which sits at the intersection of $m - 1$ hyperplanes normal to some element of $B \setminus b_o$, and this edge contains the point 0 in its interior and is parallel to b_o.

To produce a bounded set, we intersect C_2 with a compact convex polytope C_4 whose faces are mostly normal to elements of B. We construct C_4 as follows.

For each $b \in B \setminus b_o$, let H_b be the half space of V defined by the hyperplane W_b and containing C_1. Now let W_b^+ an affine translation of W_b beyond the other side of C_1. Precisely, we argue as follows: since W_b contains a face of the convex set C_1, either $[b, c]_B \geq 0$ for all $c \in C_1$, or $[b, c]_B \leq 0$ for all $c \in C_1$; we construct W_b^+ in the first case leaving the second case to symmetry. Since C_1 is compact, there is an upper bound $r > [b, c]_B$ for all $c \in C_1$. Let $W_b^+ = rb + W_b$. Now let H_b^+ be the half space defined by W_b^+ and containing C_1.

Let H_{b_o} be a half space of V defined by W_{b_o}; it doesn't matter which one. The intersection,

$$C_3 = \bigcap_{b \in B} H_b \cap \bigcap_{b \in B \setminus b_o} H_b^+$$

is therefore a semi-infinite rectangular prism whose semi-infinite axis runs parallel to b_o and whose base contains C_1.

We now form an oblique face to truncate this prism. Let $a_o \in A \setminus b_o$ be chosen so that $[a_o, b_o]_B \neq 0$, as can indeed be done by the assumed connectedness of $G(B; A)$. Let W_{a_o} be an affine hyperplane of V with normal a_o; we shall detail its placement soon. Note that, by constuction, W_{a_o} intersects every line parallel to b_o, in particular we may place W_{a_o} so that it intersects (the interior of) all the edges of C_3 parallel to b_o. Let H_{a_o} be the half-space defined by W_{a_o} and containing C_1. Let $C_4 = H_{a_o} \cap C_3$.

The properties of C_4 are summarized: C_4 is compact and convex: all its faces, except exactly one, are normal to some element of B; exactly one of its faces is normal to b_o and that face contains C_1: 0 is a vertex of C_4 and it is at the intersection of exactly m faces each normal to some element of B: the other vertices of C_4 for which this can be said are in W_{b_o} but are all outside C_1.

Let $C = C_2 \cap C_4$. This is clearly a compact convex polytope in V and C_1 is a face of C. By construction, the point 0 is a vertex of C and obeys property ii/. Any other vertex of C with property ii/ must be found in C_1 since there are no other faces of C_2 or C_4 normal to b_o. However, no other vertex of C_1 can have property ii/ by its original definition. $\square$

Although the theorem above makes no reference to group actions, we can adapt it for use in section 4.

Theorem 3.11 *Suppose $\mathcal{C}$ is a collection of hyperplanes in V, $\dim V > 1$, with Γ action as in Def. 3.1, and suppose that $\mathcal{N}(\mathcal{C})$ is indecomposable. Suppose that $\mathcal{F}$ is a singular flag (3.2). Then there is a convex $\mathcal{C}$-tope on which $\mathcal{F}$ is uniquely incident.*

Proof A singular flag is a descending sequence of singular spaces and so we can also express it as the sequence of spaces,

$$W_1,\ W_1 \cap W_2,\ \cap_{1 \leq i \leq 3} W_i,\ \ldots,\ \cap_{1 \leq i \leq k} W_i,\ \ldots,\ \cap_{1 \leq i \leq m} W_i,$$

where $W_i : 1 \leq i \leq m$ are transverse elements of $\mathcal{C}$. Let $\{v\} = \cap_{1 \leq i \leq m} W_i$.

Fix some inner product $[.,.]$ in V so that the W_i are orthogonally transverse and let $B = \{\lambda(W_i) : 1 \leq i \leq m\}$, where the normal λ is taken with respect to this inner product. B is an orthonormal basis for V. We define $\mathcal{N}(\mathcal{C})$ with respect to this inner product (recall Remark 3.7 above) and, to fit it into past notation, let $A = \mathcal{N}(\mathcal{C})$.

By hypothesis, A is indecomposable. Thus the graph $G(B; A)$ is connected, by 3.6, and so we may form by 3.10 a convex polytope, C_o, in V with the properties outlined in 3.10.

i/ The normal of each face of C_o is an element of A.

ii/ v is a vertex of C_o which is at the intersection of exactly m faces of C_o and each of these faces is normal to some element of B.

iii/ The vertex v is uniquely defined by property ii/.

However, we know that the orbit of an element, W, of $\mathcal{C}$ is dense in V in the sense that for every affine hyperplane, W', of V, parallel to W, and every $\epsilon > 0$, there is an W'', in the Γ orbit class of W, such that W' and W'' are separated by a vector of length at most ϵ. Therefore we may adjust C_o slightly without disturbing the combinatorial properties of its faces to form a $\mathcal{C}$-tope, C, with the same properties. The vertex v need not be disturbed atall.

However, then it is clear that $\mathcal{F}$ is uniquely incident on $\mathcal{C}$. $\square$

4 The indecomposable case

We continue to consider the abstracted situation of section 3 and reintroduce the analysis of 2.6 and 2.7 as a definition.

Definition 4.1 In the space, $V' = V \setminus \cup\{W : W \in \mathcal{C}\}$, define $\mathcal{A}_\mathcal{C}$ to be the collection of subsets of V' of the form $C \cap V'$, where C is a $\mathcal{C}$-tope, and with the empty set thrown in as well.

We write $CV_\mathcal{C}$ for the ring of integer-valued functions generated by indicator functions of elements of $\mathcal{A}_\mathcal{C}$.

Compare this definition with the construction of the topology of V_u in section 2.

Recall the set of singular flags, $\mathcal{J}_o$, for V and $\mathcal{C}$ as above, the Γ action on $\mathcal{J}_o$, and the set $\mathcal{J}$ of Γ orbits in $\mathcal{J}_o$. This transfers by a coordinatewise action to a Γ action on groups such as $\oplus_{\mathcal{J}_o}\mathbb{Z}/2$, the $\mathcal{J}_o$-indexed direct sum of $\mathbb{Z}/2$. In this case $(\oplus_{\mathcal{J}_o}\mathbb{Z}/2)/\Gamma = \oplus_{\mathcal{J}}\mathbb{Z}/2$ canonically.

The main results of this section (4.2) and (4.6) are two similar technical results. Here is the first.

Proposition 4.2 *With the constructions and notation of section 3 we suppose that $m = \dim V > 1$ and $\mathcal{N}(\mathcal{C})$ is indecomposable. Then, there is a Γ-equivariant homomorphism*

$$\xi_o : CV_{\mathcal{C}} \to \oplus_{\mathcal{J}_o}\mathbb{Z}/2$$

with the following property: for each singular flag $\mathcal{F}$ there is an element $e \in CV_{\mathcal{C}}$ so that $\xi_o(e)$ has value 1 at coordinate $\mathcal{F}$ and value 0 at all other coordinates $\mathcal{F}' \in [\mathcal{F}]$.

The proof follows directly from section 3. Our aim is to build a Γ-equivariant homomorphsism from $CV_{\mathcal{C}}$ to $\oplus_{\mathcal{J}_o}\mathbb{Z}/2$ with certain further properties. We use the following lemma to make the construction.

Lemma 4.3 *i/ From 4.1 above, $\mathcal{A}_{\mathcal{C}}$ is an algebra generated by sets of the form, $C \cap V'$, where C is a convex $\mathcal{C}$-tope.*

ii/ There is a one-to-one correspondance between a group homomorphisms $\xi : CV_{\mathcal{C}} \longrightarrow G$ to an abelian group G and maps ξ' from convex $\mathcal{C}$-topes to G with the property that if C_1 and C_2 are interior disjoint convex $\mathcal{C}$-topes and if $C_1 \cup C_2$ is convex, then $\xi'(C_1 \cup C_2) = \xi'(C_1) + \xi'(C_2)$.

iii/ If Γ acts homomorphically on G then ξ is Γ-equivariant whenever the corresponding ξ' is Γ-equivariant.

Proof i/ Every $\mathcal{C}$-tope can be decomposed into a finite interior-disjoint union of convex $\mathcal{C}$-topes.

ii/ Using the decomposition of part i/ we can define a homomorphism $\xi : \mathcal{A}_{\mathcal{C}} \longrightarrow G$ as follows: Take a $\mathcal{C}$-tope, C, break it into interior-disjoint cover by convex $\mathcal{C}$-topes, C_j. Let $\xi(C \cap V') = \sum \xi'(C_j)$.

To see that this is well-defined, consider a second such decomposition $C = \cup_i C_i'$. Let $D_{i,j}$ be the closure in V of $V' \cap C_i' \cap C_j$; this is either empty or a convex $\mathcal{C}$-tope. With the convention $\xi'(\emptyset) = 0$, the condition on ξ' extends inductively to show that $\xi'(C_j) = \sum_i \xi'(D_{i,j})$ and $\xi'(C_i') = \sum_j \xi'(D_{i,j})$. Therefore $\sum_j \xi'(C_j) = \sum_i \xi'(C_i')$ since both are equal to $\sum_{i,j} \xi'(D_{i,j})$.

It is clear that ξ is additive for interior-disjoint unions in $\mathcal{A}_{\mathcal{C}}$, and so extends to $CV_{\mathcal{C}}$. It is straightforward to construct a map ξ' from ξ.

iii/ Follows quickly from the construction of ii/. $\square$

Proof of 4.2 It is easy to define a map ξ' on convex $\mathcal{C}$-topes which reflects the geometric idea of incidence of a singular flag. Suppose C is a convex $\mathcal{C}$-tope, then define $\xi'(C)$ to be an element of $\oplus_{\mathcal{J}_o}\mathbb{Z}/2$ whose entry at the $\mathcal{F}$ coordinate ($\mathcal{F} \in \mathcal{J}_o$ of course) is 1 if and only if $\mathcal{F}$ is incident on C.

It is clear that ξ' is Γ-equivariant.

To show that that ξ' is additive, in the sense of Lemma 4.3 ii/, we consider two interior disjoint convex $\mathcal{C}$-topes C_1 and C_2, whose union is a convex $\mathcal{C}$-tope, and fix a particular singular flag, $\mathcal{F}$. Three cases should be checked, the third breaking down into two subcases.

Note first the general principle that a singular subspace (of any dimension, t say) containing a face (of dimension t) of $C_1 \cup C_2$ must therefore contain a face (of dimension t) of C_1 or a face (of dimension t) of C_2. The converse, of course, is not true as faces between C_1 and C_2 meet and fall into the interior of a higher dimensional faces of $C_1 \cup C_2$; and this is generally the only way that faces can be removed from consideration. So we have a second general principle that a singular subspace contains a face of $C_1 \cup C_2$ when it contains a face of C_1 and no face of C_2 (matching dimensions always).

Case a/: $\mathcal{F}$ is incident neither on C_1 nor on C_2. By our first general principle, it is immediate that $\mathcal{F}$ is not incident on $C_1 \cup C_2$.

Case b/: $\mathcal{F}$ is incident on precisely one of C_1 or C_2. The second general principle above shows then that $\mathcal{F}$ is incident on $C_1 \cup C_2$.

Case c/: $\mathcal{F} = (\theta_j)_{0 \le j < N}$ is incident on both C_1 and C_2. Suppose that θ_t is the singular subspace (of dimension t) containing a face F_1 (of dimension t) of C_1 and a face F_2 (of dimension t) of C_2. We analyse two possibilities: i/ $F_1 = F_2$: ii/ F_1 and F_2 are interior disjoint (as subsets of θ_t). Note that we have used the convexity of $C_1 \cup C_2$ and other assumed properties to make this dichotomy.

Case ci/: Here F_1 is no longer a face of $C_1 \cup C_2$ and θ_t does not contain a face of $C_1 \cup C_2$. Thus $\mathcal{F}$ is not incident on $C_1 \cup C_2$.

Case cii/: Here θ_t contains the face $F_1 \cup F_2$ of $C_1 \cup C_2$. However, consider θ_{t-1} (note that $t \ge 1$ automatically in case ii/) which contains a face of both C_1 and C_2 of dimension $t-1$. In fact, by convexity, this face must be $F_1 \cap F_2$. However, this set is not a $t-1$ dimensional face of $C_1 \cup C_2$, having been absorbed into the interior of the t-dimensional face $F_1 \cup F_2$.

Either way, in case c/ $\mathcal{F}$ is not incident on $C_1 \cup C_2$.

Combining all these three cases gives the additivity mod 2 required of ξ'. Therefore, we define a Γ-equivariant homomorphism $\xi_o : CV_{\mathcal{C}} \longrightarrow \oplus_{\mathcal{J}_o}\mathbb{Z}/2$ as required.

The element e is provided by the construction of 3.11. Consider a convex $\mathcal{C}$-tope, C produced by Theorem 3.11 from the singular flag $\mathcal{F}$. The indicator function of C is an element of $CV_{\mathcal{C}}$ which we will write e. The properties claimed of e in the theorem follow automatically from unique incidence of $\mathcal{F}$ on C. □

Γ-equivariance of the homomorphism from Proposition 4.2 allows us to build the commuting diagram

$$\begin{array}{ccc} CV_{\mathcal{C}} & \xrightarrow{\xi_o} & \oplus_{\mathcal{J}_o}\mathbb{Z}/2 \\ \downarrow^{q} & & \downarrow^{q} \\ H_0(\Gamma; CV_{\mathcal{C}}) & \xrightarrow{\xi_*} & \oplus_{\mathcal{J}}\mathbb{Z}/2 \end{array}$$

to define ξ_*, where the maps q quotient by the action of Γ. In the indecomposable case therefore, Theorem 5.4 below is a direct consequence of this observation and the reader who is only interested in the indecomposable case can jump directly to that theorem replacing the first words "Given the data above" by the first sentence of Proposition 4.2. In the decomposable case, however, we need a lot more work to establish such a diagram.

We prove an analogous technical result for the case $\dim V = 1$. It seems necessary and slightly surprising that its proof is not as directly geometric.

For the remainder of the section, therefore, we assume that $\dim V = 1$, but, for technical reasons, we also relax the assumption that Γ is a subgroup of V. Rather Γ is a free abelian group acting minimally by translation, but some of these translations may be 0. In this case we decompose $\Gamma = \Gamma_0 \oplus \Gamma_1$ where Γ_1 acts minimally and freely and Γ_0 fixes every point in V.

We note that in the case $\dim V = 1$, the flags are simply single points from $\mathcal{P}$ (sections 2 and 3). Thus there is a canonical correspondence between $\mathcal{J}_o$ and $\mathcal{P}$, which is clearly Γ-equivariant.

Construction 4.4 Consider the direct sum group $L = \oplus_{\mathcal{J}}\mathbb{Z}$ whose elements can be considered as collections of integers indexed by elements of $\mathcal{J}$, or equivalently by Γ-orbit classes in $\mathcal{P}$. Let $h_j : j = 1, 2, .., k$ (with the obvious adaption for $k = \infty$) be the canonical free generating set for L.

Let $\mathcal{P}_1, ...\mathcal{P}_k$ be the Γ-orbit classes of $\mathcal{P}$ and choose $x_0 \in \mathcal{P}_1$ and, for each $1 \leq j \leq k$, also choose $x_j \in \mathcal{P}_j$ such that $x_j > x_0$ (recall that $\mathcal{P} \subset V = \mathbb{R}$ in the case $\dim V = 1$).

According to definition 4.1 above, $V' = V \setminus \mathcal{P}$, and the sets $I_j = [x_0, x_j] \cap V'$ are elements of $\mathcal{A}_\mathcal{C}$. Thus the functions f_j which indicate respective I_j, are elements of $CV_\mathcal{C}$.

This allows us to define a homomorphism $\beta : L \longrightarrow CV_\mathcal{C}$, defined $\beta(h_j) = f_j$ for each $1 \leq j \leq k$. Consider $\oplus_{\Gamma_1} L$, the Γ_1 indexed direct sum of copies of L, whose elements we shall consider as Γ_1-indexed elements of L, $(g_\gamma)_{\gamma \in \Gamma_1}$, all but a finite number of which are 0. Note that, by correponding the coordinate indices Γ-equivariantly, $\oplus_{\Gamma_1} L$ is Γ-equivariantly isomorphic to $\oplus_{\mathcal{J}_o} \mathbb{Z}$. (The Γ_0 component of the action acts trivially).

Now consider the homomorphism $\beta^* : \oplus_{\Gamma_1} L \longrightarrow CV_\mathcal{C}$, defined as

$$\beta^*((g_\gamma)_{\gamma \in \Gamma_1}) = \sum_{\gamma \in \Gamma_1} \gamma \beta(g_\gamma)$$

Lemma 4.5 *With the construction above, β^* is Γ-equivariant.*

i/ $\beta^* : \oplus_{\Gamma_1} L \longrightarrow CV_\mathcal{C}$ is injective.

ii/ The image of β^* is complemented in $CV_\mathcal{C}$.

Proof The proof relies on a construction whose proper generalization is made in Chapter 5. In the case $\dim V = 1$, however, it is easy enough to describe directly.

Consider those subsets, $I_{a,b}$, of $\mathcal{A}_\mathcal{C}$ formed from a $\mathcal{C}$-tope interval $[a, b] \cap V'$ (much as the sets I_j were formed above). In this case $a, b \in \mathcal{P}$ necessarily. Therefore to this set, we associate an element of $\oplus_\mathcal{P} \mathbb{Z}$ namely one which is 0 at every coordinate except the a coordinate, where it is 1, and at the b coordinate, where it is -1. We write this element $\delta(I_{a,b}) = 1_a - 1_b$ in an obvious notation. This function is additive in the sense of Lemma 4.3 and so extends to a homomorphism $\delta : CV_\mathcal{C} \longrightarrow \oplus_\mathcal{P} \mathbb{Z}$. Moreover, δ is clearly Γ-equivariant on such intervals, and hence it is equivariant when extended to a group homomorphism.

Recall the elements, f_j, of $CV_\mathcal{C}$ defined before. Consider an equation of the form $\sum_{1 \leq j \leq k, \gamma \in \Gamma_1} t_{j,\gamma} \gamma f_j = 0$ (where $t_{j,\gamma} \in \mathbb{Z}$ equal 0 for all but a finite set of indices). If we apply δ to this, we find $\sum_{1 \leq j \leq k, \gamma \in \Gamma_1} t_{j,\gamma} \gamma \delta(f_j) = 0$, a sum in $\oplus_\mathcal{P} \mathbb{Z}$, and we can now use the equation $\delta(f_j) = 1_{x_0} - 1_{x_j}$.

Decompose $\oplus_{\mathcal{P}}\mathbb{Z} = \oplus_j \oplus_{\mathcal{P}_j} \mathbb{Z}$ and examine what happens on each $\oplus_{\mathcal{P}_j}\mathbb{Z}$.

Consider first an index $j \geq 2$. In the component $\oplus_{\mathcal{P}_j}\mathbb{Z}$ for such a j, the sum reduces to $\sum_{\gamma\in\Gamma_1} t_{j,\gamma}\gamma 1_{x_j} = 0$ whence $t_{j,\gamma} = 0$ for all $\gamma \in \Gamma_1$ since Γ_1 acts freely on $\mathcal{P}_j$.

The case remaining is $j = 1$ for which the sum reduces to

$$\Big(\sum_{1\leq j\leq k,\gamma\in\Gamma_1} t_{j,\gamma}\gamma 1_{x_0}\Big) - \Big(\sum_{\gamma\in\Gamma_1} t_{1,\gamma}\gamma 1_{x_1}\Big) = 0$$

however, by the last paragraph, this reduces to $\sum_{\gamma\in\Gamma_1} t_{1,\gamma}\gamma(1_{x_0} - 1_{x_1}) = 0$. But $x_1 = \gamma_1(x_0)$ for some $\gamma_1 \in \Gamma_1$ and so we find $\gamma_1 f = f$ where $f = \sum_{\gamma\in\Gamma_1} t_{1,\gamma}\gamma 1_{x_0}$. This implies that $f = 0$ (Γ_1 acts freely on $\oplus_{\mathcal{P}}\mathbb{Z}$) and so we have $t_{1,\gamma} = 0$ for all $\gamma \in \Gamma_1$ as well.

In conclusion, the equation $\sum_{1\leq j\leq k,\gamma\in\Gamma_1} t_{j,\gamma}\gamma f_j = 0$ implies $t_{j,\gamma} = 0$ for all j and γ, and so i/ follows.

To show ii/ we note first that $CV_{\mathcal{C}}$ is isomorphic to a countable direct sum of $C(X;\mathbb{Z}) \equiv \oplus_\infty\mathbb{Z}$ (X Cantor) and so is itself free abelian. Therefore to prove complimentarity of the image of β^* it's enough to show that $CV_{\mathcal{C}}/Im\beta^*$ is torsion free.

Therefore, we argue to the contrary and suppose that we have an element $g = (g_\gamma)_{\gamma\in\Gamma_1} \in \oplus_{\Gamma_1}L$ for which $\beta^*(g) = tf$ for some $t \in \mathbb{Z}$, $t \geq 2$, and $f \in CV_{\mathcal{C}}$. This equation may be written $\sum_{1\leq j\leq k,\gamma\in\Gamma_1} t_{j,\gamma}\gamma f_j = tf$, where $\beta(g_\gamma) = \sum_{1\leq j\leq k} t_{j,\gamma} f_j$.

The analysis we have just completed in part i/ can be performed equally well modulo t. Therefore each $t_{j,\gamma} = 0 \bmod t$ and we deduce that each $\beta(g_\gamma) = tf_\gamma$ for some $f_\gamma \in \beta(L)$. In particular $f = \sum_{\gamma\in\Gamma_1} \gamma f_\gamma \in \beta^*(\oplus_{\Gamma_1}L)$. Thus every element of $\beta^*(\oplus_{\Gamma_1}L)$ which can be divided in $CV_{\mathcal{C}}$ can also be divided in $\beta^*(\oplus_{\Gamma_1}L)$, i.e. the quotient is torsion-free as required. $\square$

This gives the key to the second and final technical result of this section.

Proposition 4.6 *Assume the constructions and notation of 4.4 and section 3: in particular we suppose that* $\dim V = 1$*. Then there is a surjective* Γ*-equivariant homomorphism* $\xi_o : CV_{\mathcal{C}} \to \oplus_{\mathcal{J}_o}\mathbb{Z}/2$.

Proof Lemma 4.5 builds an injective Γ-equivariant homomorphism $\beta^* : \oplus_{\mathcal{J}_o}\mathbb{Z} \longrightarrow CV_{\mathcal{C}}$ whose image is complemented in $CV_{\mathcal{C}}$. Therefore we have automatically a surjective Γ-equivariant homomorphism $\xi : CV_{\mathcal{C}} \longrightarrow \oplus_{\mathcal{J}_o}\mathbb{Z}$ reversing this (i.e. $\xi\beta^* =$ identity). Extending ξ by the reduction mod 2, $\oplus_{\mathcal{J}_o}\mathbb{Z} \longrightarrow \oplus_{\mathcal{J}_o}\mathbb{Z}/2$, gives us the homomorphism $\xi_o : CV_{\mathcal{C}} \longrightarrow \oplus_{\mathcal{J}_o}\mathbb{Z}/2$. □

5 The decomposable case

Proposition 4.2 is very close to a proof of Theorem 2.9 in the case that $\mathcal{N}(\mathcal{C})$ is indecomposable. Rather than finish the argument for that case, we clear up the decomposable case and so complete the most general proof. Once again we exploit the generalities of section 3.

Construction 5.1 Suppose that $\mathcal{N}(\mathcal{C})$ is decomposable and that $\mathcal{N}_1, \ldots, \mathcal{N}_k$ are its components. Let V_j be the vector space spanned by $\mathcal{N}_j$, defined for each $1 \leq j \leq k$. Without loss of generality (see Remark 3.7) we can impose an inner product on V which will make the V_j mutually orthogonal and we do this from the start.

Therefore $V = \sum_j V_j = \oplus_j V_j$ is an orthogonal direct sum. Let π_j be the orthogonal projection onto V_j with kernel $\sum\{V_i : i \neq j\}$.

Let $\mathcal{C}_j$ be the set of those elements of $\mathcal{C}$ whose normal is contained in V_j. Thus, for each j, $\mathcal{N}_j = \mathcal{N}(\mathcal{C}_j)$, an indecomposable subset of V_j. The Γ action on $\mathcal{C}$ leaves each of the sets $\mathcal{C}_j$ invariant.

The natural Γ action on V_j is more complicated than mere restriction. We consider $\pi_j(\Gamma)$ as a subgroup of V_j and let Γ act by translation: $\gamma(v) = v + \pi_j(\gamma)$. We call this the *projected action*.

On each V_j we construct the sets of singular flags, $\mathcal{J}_{oj}$ and $\mathcal{J}_j = \mathcal{J}_{oj}/\Gamma$, and the spaces $CV_{j\,\mathcal{C}_j^*}$ according to section 3, using the projected action of Γ on V_j and the singular hyperplanes $\mathcal{C}_j^* = \{W \cap V_j : W \in \mathcal{C}_j\}$. Note again that, for each j, $\mathcal{N}(\mathcal{C}_j^*) = \mathcal{N}(\mathcal{C}_j)$ is indecomposable as a spanning subset of V_j. Moreover, $\mathcal{P}_j$, the intersection point set defined using $\mathcal{C}_j^*$ in V_j, is equal to $\pi_j(\mathcal{P})$ which in turn is equal to $\mathcal{P} \cap V_j$.

We abbreviate the $\mathbb{Z}[\Gamma]$ module $CV_{j\,\mathcal{C}_j^*}$ as CV_j in what follows.

Lemma 5.2 *Given the constructions above, we have the following relations:*

i/ There is a canonical bijection $\mathcal{C} \leftrightarrow \cup_j \mathcal{C}_j^*$.

ii/ There is a canonical Γ-equivariant bijection $\Pi_j \mathcal{P}_j \leftrightarrow \mathcal{P}$ (direct product of sets), where Γ acts on the direct product diagonally by its projected actions.

iii/ $V \equiv \oplus_j V_j$ is a canonical orthogonal direct sum on which the usual Γ action on V is retrieved as the diagonal action of the projected actions of Γ.

iv/ $CV \equiv \otimes_j CV_j$ *canonically as a* $\mathbb{Z}[\Gamma]$*-module, where the module action is given by the diagonal action of the projected actions.*

v/ There is a canonical Γ*-equivariant surjection* $\sigma : \mathcal{J}_o \longrightarrow \Pi_j \mathcal{J}_{oj}$ *(direct product of sets) and a natural* Γ*-equivariant injection* $\tau : \Pi_j \mathcal{J}_{oj} \longrightarrow \mathcal{J}_o$ *whose composition* $\sigma\tau$ *is the identity.*

Proof The first four parts all follow from the fact that, for each j, the singular spaces in V break into two classes. Those spaces which are parallel to V_j do not impinge on its singular geometry at all. And those singular spaces, W, which are not parallel, intersect V_j in exactly the same way as they project to V_j, i.e. $\pi_j(W) = W \cap V_j$.

Part v/ is more involved. Consider a singular flag, $\mathcal{F}$, in V with the singular plane set $\mathcal{C}$. The bijection of part i/ can be detailed: each element of $\mathcal{C}$ is of the form $V_1 + V_2 + ... + V_{j-1} + W_j + V_{j+1} + .. + V_k$, an orthogonal sum, where $W_j \in \mathcal{C}_j^*$. Thus any intersection of elements of $\mathcal{C}$ can be written as an orthogonal sum $W_1 + W_2 + ... + W_k$, where, for each j, W_j is an intersection of elements from $\mathcal{C}_j^*$.

Now consider the sequence of singular spaces listed in $\mathcal{F}$ and their orthogonal decomposition as above. As we read from point to hyperplane, their dimension rises by exactly one and so the orthogonal summands rise in dimension, but only in one of the summands and only by one dimension. Suppose in the orthogonal decomposition of this increasing sequence, we ignore all directions but the jth say. Then we find a nested sequence of singular spaces in V_j whose dimension rises by at most 1 at each step, either going all the way up to V_j or (for precisely one value of j) stopping one dimension short. Extract the sequence as a strictly increasing subsequences, neglecting the last term if it happens to be V_j, and call the result $\mathcal{F}_j$. For each j, $\mathcal{F}_j$ will be a singular flag in V_j with respect to the singular planes $\mathcal{C}_j^*$. The map $\mathcal{F} \mapsto (\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_k)$ is σ.

To show it's onto, we construct τ. Given $\mathcal{F}_j = (W_{j,0}, ..., W_{j,m_j-1})$ (where dimension of V_j is m_j) consider the sequence: $W_{1,0} + W_{2,0} + ... + W_{k,0}$, $W_{1,1} + W_{2,0} + ... + W_{k,0}$, $W_{1,2} + W_{2,0} + ... + W_{k,0}$,..., $W_{1,m_1-1} + W_{2,0} + ... + W_{k,0}$, $V_1 + W_{2,0} + ... + W_{k,0}$, $V_1 + W_{2,1} + ... + W_{k,0}$, ..., $V_1 + W_{2,m_2-1} + ... + W_{k,0}$, $V_1 + V_2 + W_{3,0} + ... + W_{k,0}$,..., $V_1 + V_2 + ... V_{k-1} + W_{k,m_k-2}$, $V_1 + V_2 + ... V_{k-1} + W_{k,m_k-1}$, where all sums are orthogonal sums in V. This is a singular flag, in V with respect to $\mathcal{C}$, which we shall call $\tau(\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_k)$.

It is immediate from definition that σ and τ are Γ-equivariant and that $\sigma\tau$ is the identity. $\square$

Note that although $\mathcal{P}$ may have an infinite number of Γ-orbits, each of the $\mathcal{P}_j$ may have only finitely many Γ orbits under the projected action. This subtle problem cuts out the possibility of an easy argument by induction on the number of indecomposable components of $\mathcal{N}(\mathcal{C})$.

Applying τ coordinate-wise, we find

Corollary 5.3 *There is a* Γ*-equivariant homomorphism* $\tau^+ : \otimes_j \oplus_{\mathcal{J}_{oj}} \mathbb{Z}/2 \longrightarrow \oplus_{\mathcal{J}_o}\mathbb{Z}/2$. □

Now we are ready for the main result of this section.

Theorem 5.4 *With the data above, there is a homomorphism* $\xi_*: H_0(\Gamma; CV_\mathcal{C}) \longrightarrow \oplus_\mathcal{J}\mathbb{Z}/2$ *such that for all* $v \in \mathcal{P}$*, there is a singular flag* $\mathcal{F} = (\theta_j)_{1\leq j<m} \in \mathcal{J}_o$ *so that* $\theta_0 = \{v\}$*, and an element* $e_v \in H_0(\Gamma, CV_\mathcal{C})$ *such that* $\xi_*(e_v)$ *has value* 1 *at coordinate* $[\mathcal{F}]$ *(i.e. the* Γ*-orbit class of* $\mathcal{F}$*, an element of* $\mathcal{J}$ *(3.3)).*

Proof Suppose that $\mathcal{N}(\mathcal{C}) = \cup_j \mathcal{N}(\mathcal{C}_j)$ is an indecomposable partition, forming the spaces V_j, of dimension $m_j = \dim V_j$, etc as above.

Note that on each V_j, Γ acts by translation by elements of $\pi_j(\Gamma)$ and that action may or may not be free. Proposition 4.6 applies in this case by hypothesis. Also the proof of Proposition 4.2 does not depend on the freedom of the Γ action. No complication arises therefore if we stick with the projected Γ action on each V_j even though the action may not be free.

For each j, $\pi_j(\Gamma)$ is the projection of a dense subset of V and so itself is dense in V_j. Therefore the Γ action on each V_j is minimal and $rank(\pi_j(\Gamma)) > 1$.

Given $v \in \mathcal{P}$, we find $\pi_j(v) = v_j \in \mathcal{P}_j$. By Propositions 4.2 and 4.6 we find for each j a homomorphism

$$\xi_{oj} : CV_j \longrightarrow \oplus_{\mathcal{J}_{oj}}\mathbb{Z}/2,$$

a singular flag $\mathcal{F}_j$ in V_j such that $\mathcal{F}_{j0} = \{v_j\}$, and an element $e_{v,j} \in CV_j$ so that $\xi_o(e_{v,j})$ has value 1 in the coordinate $\mathcal{F}_j$ and 0 in the coordinates of all other flags from the Γ-orbit $[\mathcal{F}_j]$ of $\mathcal{F}$ (if $\dim V_j = 1$ flags are just points and the latter properties follow from surjectivity). The homomorphisms ξ_{oj} are independent of the choice of v.

The equation $CV_\mathcal{C} = \otimes_j CV_j$ allows us to build the homomorphism

$$\otimes\xi_{oj} : CV_\mathcal{C} \longrightarrow \otimes_j \oplus_{\mathcal{J}_{oj}} \mathbb{Z}/2$$

and using τ^+ of Cor 5.3, we can continue this homomorphism to $\oplus_{\mathcal{J}_o}\mathbb{Z}/2$. This homomorphism is clearly Γ-equivariant and so we deduce a quotiented homomorphism: $\xi_* : H_0(\Gamma; CV_{\mathcal{C}}) \longrightarrow \oplus_{\mathcal{J}}\mathbb{Z}/2$ which completes a square as required

$$\begin{array}{ccc} CV_{\mathcal{C}} & \xrightarrow{\tau^+(\otimes_j \xi_{oj})} & \oplus_{\mathcal{J}_o}\mathbb{Z}/2 \\ \downarrow^q & & \downarrow^q \\ H_0(\Gamma; CV_{\mathcal{C}}) & \xrightarrow{\xi_*} & \oplus_{\mathcal{J}}\mathbb{Z}/2 \end{array} .$$

Given v, the element $e_v = \otimes_j e_{v,j} \in CV_{\mathcal{C}}$ is mapped by $\tau^+(\otimes_j \xi_{oj})$ to an element with value 1 at the coordinate $\mathcal{F} = \tau(\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_k)$ and with value 0 at coordinates $\mathcal{F}' = \tau(\mathcal{F}'_1, \mathcal{F}'_2, ..., \mathcal{F}'_k)$ where $\mathcal{F}'_j \in [\mathcal{F}_j]$ but $\mathcal{F} \neq \mathcal{F}'$. In particular, $\tau^+(\otimes_j \xi_{oj})(e_v)$ has value 0 at all coordinates of the Γ-orbit $[\mathcal{F}]$ of $\mathcal{F}$ except at $\mathcal{F}$ itself. Hence $q\tau^+(\otimes_j \xi_{oj}) : CV_{\mathcal{C}} \longrightarrow \oplus_{\mathcal{J}}\mathbb{Z}/2$ maps e_v to an element with value 1 at the coordinate $[\mathcal{F}]$. The zero dimensional element of $\mathcal{F}$ is $\{v\}$ by construction. □

Proof of Theorem 2.9 Suppose that $\mathcal{P}$ is infinitely generated and that $v_1, v_2, ...$ are representatives of distinct Γ-orbit classes. By Theorem 5.4, we find a homomorphism $\xi_* : H_0(\Gamma; CV_{\mathcal{C}}) \longrightarrow \oplus_{\mathcal{J}}\mathbb{Z}/2$, singular flags $\mathcal{F}_j \in \mathcal{J}_o$ so that the zero dimensional element of $\mathcal{F}_j$ is $\{v_j\}$, and elements $e_j \in H_0(\Gamma, CV_{\mathcal{C}})$ such that $\xi_*(e_j)$ has value 1 at coordinate $[\mathcal{F}_j]$.

By taking a subsequence if necessary we may assume that for $i < j$, all the values $\xi_*(e_i)$ have value 0 at the $[\mathcal{F}_j]$ coordinate. In particular, we have ensured that the set $\{\xi_*(e_j) : j \geq 1\}$ is $\mathbb{Z}/2$ independent in $\oplus_{\mathcal{J}}\mathbb{Z}/2$. By Proposition 2.10 we have $H_0(\Gamma, CV_{\mathcal{C}}) \otimes \mathbb{Q}$ infinitely generated.

Now to prove Theorem 2.9, we apply this analysis to the construction of section 2, using the same V, and setting $\mathcal{C} = \mathcal{C}_u$ and $\Gamma = \Gamma_{\mathcal{T}}$. □

6 Conditions for infinitely generated cohomology

To apply Theorem 2.9 we must be able to count the orbits in $\mathcal{P}$. This is a geometric exercise, and each case will have its own peculiarities. We present in this section elementary general conditions which are sufficient to give infinite orbits in $\mathcal{P}$.

Recall the general set-up from IV.3, and the construction of $\mathcal{P}$ as points which are the proper intersection of $m = \dim V$ hyperplanes picked from $\mathcal{C}$.

Definition 6.1 Suppose that $W_1, .., W_m$ is a set of hyperplanes chosen from $\mathcal{C}$, intersecting in a single point p. For each subset A of $\{1, \cdots, m\}$ the intersection

$$W_A := \bigcap_{i \in A} W_i$$

has dimension $m - |A|$. We write $A^c = \{1, \cdots, m\} \setminus A$ and define $\Gamma_A = \{x \in V | \exists \gamma \in \Gamma : \{x + p\} = W_A \cap (W_{A^c} + \gamma)\}$. Finally let $\Gamma^A \subset \Gamma$ be the stabilizer of W_A.

We think of Γ_A as the projection of Γ onto W_A along W_{A^c}. The following is straightforward from the definitions.

Lemma 6.2 *With the notation above:*
i/ Γ_A is a group with Γ^A as a subgroup.
ii/ $\Gamma_A + p = \mathcal{P} \cap W_A$.
iii/ If $q \in \mathcal{P} \cap W_A$, then $(\Gamma + q) \cap W_A = \Gamma^A + q$.
iv/ If $A_1, A_2 \subset \{1, \cdots, m\}$ are disjoint, then $\Gamma^{A_1 \cup A_2} = \Gamma^{A_1} \cap \Gamma^{A_2}$.

This gives immediately an easy way to determine whether we have an infinite number of orbits in $\mathcal{P}$.

Proposition 6.3 *If, for some choice of $W_1, ..., W_m$ and $A \subset \{1, \cdots, m\}$, the stabilizer Γ^A has infinite index in Γ_A (equivalently, if $\operatorname{rk} \Gamma^A < \operatorname{rk} \Gamma_A$), then $\mathcal{P}$ is infinitely generated.*

Proof By Lemma 6.2 ii/ and iii/ the orbits in $\mathcal{P}$ which intersect W_A are enumerated precisely by the cosets of Γ^A in Γ_A. This is infinite by assumption. □

We can pursue the construction above a little further to get even a sharper condition for infinitely generated $\mathcal{P}$. Given $A_1, A_2 \subset \{1, \cdots, m\}$, note that $A_1 \cup A_2 = \{1, 2, .., m\}$ implies $W_{A_1} \cap W_{A_2} = \{p\}$ and therefore $\Gamma_{A_1} \cap \Gamma_{A_2} = \{0\}$. This together with Lemma 6.2 gives the following result.

Lemma 6.4 *For every choice of $W_1, ..., W_m$ as above and for every pair of sets, $A_1, A_2 \subset \{1, \cdots, m\}$ we have,* *i/ if* $A_1 \cap A_2 = \emptyset$, *then*

$$\operatorname{rk}\Gamma^{A_1} + \operatorname{rk}\Gamma^{A_2} - \operatorname{rk}\Gamma^{A_1\cup A_2} \leq \operatorname{rk}\Gamma \leq \operatorname{rk}\Gamma_{A_1} + \operatorname{rk}\Gamma_{A_2} - \operatorname{rk}(\Gamma_{A_1}\cap\Gamma_{A_2}),$$

ii/ if $A_1 \cup A_2 = \{1, 2, .., m\}$, *then*

$$\Gamma^{A_1} + \Gamma^{A_2} \subset \Gamma^{A_1\cap A_2} \subset \Gamma_{A_1\cap A_2} \subset \Gamma_{A_1} + \Gamma_{A_2}$$

both sums being direct. □

Corollary 6.5 *If* $\mathcal{P}$ *is finitely generated, then for every choice of* $W_1, ..., W_m$ *and for every pair of sets,* $A_1, A_2 \subset \{1, \cdots, m\}$ *we have,*
i/ if $A_1 \cap A_2 = \emptyset$, *then*

$$\operatorname{rk}\Gamma^{A_1} + \operatorname{rk}\Gamma^{A_2} - \operatorname{rk}\Gamma^{A_1\cup A_2} = \operatorname{rk}\Gamma,$$

ii/ if $A_1 \cup A_2 = \{1, 2, .., m\}$, *then*

$$\operatorname{rk}\Gamma^{A_1} + \operatorname{rk}\Gamma^{A_2} = \operatorname{rk}\Gamma^{A_1\cap A_2}.$$

Proof Suppose $A_1 \cap A_2 = \emptyset$. Then $\Gamma^{A_1\cup A_2} = \Gamma^{A_1} \cap \Gamma^{A_2} \subset \Gamma_{A_1} \cap \Gamma_{A_2}$ which, by Proposition 6.3, implies $\operatorname{rk}\Gamma_{A_1\cup A_2} \leq \operatorname{rk}(\Gamma_{A_1} \cap \Gamma_{A_2})$. Hence, by Lemma 6.4 $\operatorname{rk}\Gamma \leq \operatorname{rk}\Gamma_{A_1} + \operatorname{rk}\Gamma_{A_2} - \operatorname{rk}\Gamma_{A_1\cup A_2}$ and now Proposition 6.3 and Lemma 6.4 i/ allow to conclude i/. ii/ follows directly from Proposition 6.3 and Lemma 6.4 ii/. □

Definition 6.6 Let us now look at a situation in which we pick more hyperplanes, $\mathcal{W} = \{W_1, \cdots, W_f\}$, $f > m = \dim V$, from $\mathcal{C}$ demanding that the set $\mathcal{N}(\mathcal{W})$ of normals of the planes (defined as in Definition 3.1 but for the subset $\mathcal{W} \subset \mathcal{C}$) is indecomposable in the sense of 3.5. This requires that $\mathcal{N}(\mathcal{W})$ spans V and by (3.6) is equivalent to the fact that all graphs $G(B; \mathcal{N}(\mathcal{W}))$ with $B \subset \mathcal{N}(\mathcal{W})$ a basis for V are connected. Let us denote by $\mathcal{I}_l$ the collection of subsets $A \subset \{1, \cdots, f\}$ of $m - l$ elements such that W_A has dimension l (compare with 2.1). Note that any $A \in \mathcal{I}_0$ defines a basis B_A by the normals to all W_i, $i \in A$.

Theorem 6.7 *With* $\mathcal{W}$ *as above suppose that* $\mathcal{P}$ *is finitely generated. Then, for all* $A \in \mathcal{I}_l$

$$\operatorname{rk}\Gamma^A = l\frac{\operatorname{rk}\Gamma}{\dim V}.$$

In particular $\dim V$ *divides* $\mathrm{rk}\,\Gamma$.

Proof Fix i and choose an $A \in \mathcal{I}_0$ which contains i (this is clearly possible). Applying Corollary 6.5 i/ iteratively we obtain

$$\sum_{i \in A} \mathrm{rk}\,\Gamma^{\{i\}} = (m-1)\mathrm{rk}\,\Gamma. \tag{6.1}$$

By hypothesis there exists a $j \in A$ such that the vertices of $G(B_A, \mathcal{N}(\mathcal{W}))$ corresponding to the normals of W_i and W_j are linked. Hence there is a $k \notin A$ such that the normal of W_k has nonvanishing scalar product $[\cdot,\cdot]$ with the normals of W_i and W_j. This implies that, both, $(A \backslash \{i\}) \cup \{k\}$ and $(A \backslash \{j\}) \cup \{k\}$ belong to $\mathcal{I}_0$. Hence we can apply (6.1) to both sets to conclude $\nu_i = \nu_j$. By assumption $G(B_A; \mathcal{N}(\mathcal{W}))$ is connected so that a repetition of the argument shows that ν_i does not depend on the choice of $i \in A$. This proves the theorem for $l = m-1$ ($l = m$ is clear). The statements for $l < m-1$ follow now inductively from Corollary 6.5 i/. $\square$

Theorem 6.7 contains as a special case a condition for canonical projection method patterns as to whether they have infinitely generated cohomology which can quickly be checked.

Corollary 6.8 *Suppose that* $\mathcal{T}$ *is a tiling in* $\mathbb{R}^d$*, topologically conjugate (I.4.5) to a canonical projection method pattern with data* (E, u)*, and suppose that* $E \cap \mathbb{Z}^N = 0$*. If* $N - \mathrm{rk}\,\Delta$ *is not divisible by* $N - \mathrm{rk}\,\Delta - d$ *then* $H_0(\mathcal{G}\mathcal{T}_u) \otimes \mathbb{Q}$ *is infinite dimensional, and so* $\mathcal{T}_u$ *is not a substitution tiling.*

Proof In the tiling case $\mathrm{rk}\,\Gamma_{\mathcal{T}_u} = N - \mathrm{rk}\,\Delta$ and $\dim V = N - d - \mathrm{rk}\,\Delta$. $\square$

Examples 6.9 (Cf. Examples I.2.7) The Octagonal tiling, a canonical projection tiling with $N = 4$, $\Delta = 0$ and $d = 2$, and the Penrose tiling, a canonical projection tiling with $N = 5$, $\Delta \cong \mathbb{Z}$ and $d = 2$, are both substitutional and hence have finitely generated cohomology. Theorem 6.7 therefore tells us that the stabilizers of the hyperplanes (here lines) have rank two.

It is clear that in the generic placement of planes the ranks of the stabilizers of the intersections of hyperplanes will have smaller rank than compatible with the last theorem. Thus we deduce:

Theorem 6.10 *Suppose that $\mathcal{T}$ is a tiling in $\mathbb{R}^d$, topologically conjugate to a canonical projection method pattern with data (E, u) and $N > d+1$, and suppose that E is in generic position. Then $H_0(\mathcal{G}\mathcal{T}) \otimes \mathbb{Q}$ is infinite dimensional, and $\mathcal{T}$ is not a substitution tiling.* $\square$

V Approaches to Calculation III: Cohomology for Small Codimension

1 Introduction

The last chapter was devoted to the case in which the cohomology groups of a canonical projection tiling were infinitely generated. Now we turn to the opposite case. In particular we shall assume that $\mathcal{P}$ has only finitely many Γ orbits, i.e. is finitely generated (IV.2.8). In that case we find that the cohomology groups of canonical projection tilings are finitely generated free abelian groups and we can provide explicit formulae for their ranks if the codimension is smaller or equal to 3. We obtain a formula for their Euler characteristic even in any codimension. Although we saw that finitely generated $\mathcal{P}$ is non-generic this seems to be the case of interest for quasicrystal physics. In particular we will present calculations for the Ammann-Kramer tiling as an example. It is a three dimensional analog of the Penrose tilings and often used to model icosahedral quasicrystals.

The material presented here extends **[FHK]** to the codimension 3 case. This is important, because all known tilings which are used to model icosahedral quasicrystals are obtained from projection out of a 6-dimensional periodic structure with codimension 3. Unlike in **[FHK]** we use here a spectral sequence derived from a double complex to prove our formulae.

Perhaps the first use of spectral sequences in calculation of tiling cohomology or K-theory is found in **[BCL]**. However, we note that the spectral sequence we use here differs significantly from the spectral sequence found in **[BCL]**. While the sequence of **[BCL]** produces an isomorphism of the K-theory of the groupoid C^*-algebra of the tiling with its cohomology in 2 dimensions (as is generalized in Chapter II), our sequence represents a geometric decomposition of the tiling cohomology itself which is a powerful calculating tool.

In Section 2 we recall the set-up and state the main results (Thms. 2.4, 2.5, 2.8 for arbitrary codimension and Thm. 2.7 for codimension 3). In Section 3 we explain one part of the double complex, namely the one which is related to the structure of the set of singular points, and in Section 4 we recall the other half which is simply group homology. We put both together in Section 5 where we employ the machinery of spectral sequences to proof our results. In principle there is no obstruction to pushing the calculation further to higher codimension, it is rather the complexity which becomes overwhelming to do this in practice. In the final section we sketch how our formulae apply to the Ammann-Kramer tiling.

2 Set up and statement of the results

In the second chapter we defined the cohomology of a tiling as the cohomology of one of its groupoids (it turned out that it does not matter which one we take) and interpreted it in various forms, in particular as a standard dynamical invariant of one of the systems. This dynamical system is of the following type: Consider a dense lattice Γ of rank N in a euclidian space V as it arises generically if one takes $N > \dim V$ vectors of V and considers the lattice they generate. Let K be a compact set which is the closure of its interior and consider the orbit $S = \partial K + \Gamma \subset V$ of ∂K under Γ. We showed in Chapters I and II how such a situation arises for projection method patterns and how in this case the rather simple dynamical system (V, Γ) (Γ acting by translation) extends to a dynamical system $(\overline{V}, \Gamma)$ which coincides with the old one on the dense G_δ-set $V \backslash S$. $\overline{V}$ is locally a Cantor set and obtained from V upon disconnecting it along the points of S. We are interested in calculating the homology groups $H_p(\Gamma, C\overline{V})$ of the group Γ with coefficients in the compactly supported integer valued continuous functions over $\overline{V}$. Already for the results of Chapter IV we specialized to the situation in which S is a union of a collection $\mathcal{C}$ of hyperplanes, the collection consisting of finitely many $\Gamma_{\mathcal{T}}$-orbits. We denoted there $\overline{V}$ by $V_{\mathcal{C}}$. We will restrict our attention to this case here too. In the context of projection method patterns (Def. I.4.4), where $\Gamma = \Gamma_{\mathcal{T}}$ and $V = V + \pi'(u)$ (II.4.3), this means that we consider only polytopal acceptance domain K and such that the orbit of a face under the action of Γ contains the hyperplane it spans (this is hypothesis H3 in [**FHK**]). The $\Gamma_{\mathcal{T}}$-orbits of these hyperplanes constitute the set $\mathcal{C}$. In particular, we rule out fractal acceptance domain although this case might be important for quasicrystal physics.

Set-up 2.1 All our results below depend only on the context described by the data $(V, \Gamma, \mathcal{W})$, a dense lattice Γ of finite rank in a Euclidean space V with a finite family $\mathcal{W} = \{W_i\}_{i=1,\cdots,f}$ of (affine) hyperplanes whose normals span V. Thus they are not specific for the tilings considered here but can be applied e.g. also to the situation of [**FHK**]. Our aim is to analyse the homology groups $H_p(\Gamma, CV_{\mathcal{C}})$ where $\mathcal{C} := \{W + \gamma : W \in \mathcal{W}, \gamma \in \Gamma\}$.

We extend the notation of IV.6.6.

Definitions 2.2 Given a finite family $\mathcal{W} = \{W_i\}_{i=1,\cdots,f}$ of hyperplanes, recall that J_l is the collection of subsets $A \subset \{1,\cdots,f\}$ of $\dim V - l$ elements such that W_A has dimension l. On W_A we have an action of $\Gamma^{|A|}$:

$$\vec{x}\cdot W_A := \bigcap_{i\in A}(W_i - x_i).$$

We call $\vec{x}\cdot W_A$ a *singular space* or singular l-space if we want to specify its dimension l. Let $\mathcal{P}_l$ be the quotient $\Gamma^{|A|}\times J_l/\sim$ with $(\vec{x},A)\sim(\vec{x}',A')$ if $\vec{x}\cdot W_A = \vec{x}'\cdot W_{A'}$. It is in one to one correspondence to the set of singular l-spaces. We denote equivalence classes by $[\vec{x},A]$ and the space $\vec{x}\cdot W_A$ also by $W_{[\vec{x},A]}$. On $\mathcal{P}_l$ we have an action of Γ: $y\cdot[\vec{x},A] = [y\cdot\vec{x},A]$ where $(y\cdot\vec{x})_i = y + x_i$. We denote the orbit space $\mathcal{P}_l/\Gamma$ by I_l. Since this action coincides with the geometric action of Γ (by translation) on the singular l-spaces we can use the elements of I_l to label the Γ-orbits of singular l-spaces. Note that the map $\{1,\cdots,f\}\to I_{\dim V-1}$ which assigns to i the orbit of $[0,\{i\}]$ is surjective but not necessarily injective. The stabilizer of a singular l-space depends only on its orbit class, if the label of its orbit is $\Theta\in I_l$ we denote the stabilizer by Γ^Θ.

Fix $\hat\Theta\in\mathcal{P}_{l+k}$, $l+k<\dim V$ and let $\mathcal{P}_l^{\hat\Theta} := \{\hat\Psi\in\mathcal{P}_l | W_{\hat\Psi}\subset W_{\hat\Theta}\}$. Then Γ^Θ (Θ the orbit class of $\hat\Theta$) acts on $\mathcal{P}_l^{\hat\Theta}$ (diagonally, by the same formula as above) and we let $I_l^{\hat\Theta} = \mathcal{P}_l^{\hat\Theta}/\Gamma^\Theta$, the orbit space. It labels the Γ^Θ-orbits of singular l-spaces in the $l+k$-dimensional space $W_{\hat\Theta}$. We can naturally identify $I_l^{\hat\Theta}$ with $I_l^{\hat\Theta'}$ if $\hat\Theta$ and $\hat\Theta'$ belong to the same Γ-orbit and so we define I_l^Θ, for the class $\Theta\in I_{l+k}$. I_l^Θ is the subset of I_l labelling those orbits of singular l-spaces which have a representative that lies in singular space whose label is Θ. Finally we denote

$$L_l = |I_l|,\quad L_l^\Theta = |I_l^\Theta|$$

where $\Theta\in I_{l+k}$, $l+k<\dim V$.

Note that $\mathcal{P}_0$ can be identified with $\mathcal{P}$ (Def. IV.2.8). So our assumption of this chapter is that L_0 is finite. The proof of the following lemma is straightforward.

Lemma 2.3 *If L_0 is finite then L_l^Θ is finite for all Θ and l.* □

The proof of the following five theorems will be given in Section 5. The first one is the converse of Theorem IV.2.9.

Theorem 2.4 *Suppose given data $(V, \Gamma, \mathcal{W})$ as in (2.1) with L_0 finite. Then $H_p(\Gamma, CV_{\mathcal{C}}) \otimes \mathbb{Q}$ has finite rank over the rational numbers $\mathbb{Q}$.*

We denote

$$D_p = \operatorname{rk} H_p(\Gamma, CV_{\mathcal{C}}) \otimes \mathbb{Q}.$$

Theorem 2.5 *Suppose given data $(V, \Gamma, \mathcal{W})$ as in (2.1) with L_0 finite. Then $H_p(\Gamma, CV_{\mathcal{C}})$ is free abelian. In particular it is uniquely determined by its rank which is D_p.*

Note that Theorems 2.4 and 2.5 exclude the possibility that $H(\Gamma, CV_{\mathcal{C}})$ would contain e.g. the dyadic numbers as a summand, a case which occurs frequently in cohomology groups of substitution tilings.

Recall that, if the normals of the hyperplanes $\mathcal{W}$ form an indecomposable set in the sense of IV.3.4 and L_0 is finite then Theorem IV.6.7 implies that the rank of the stabilizer Γ^Θ depends only on the dimension of the plane it stabilizes, i.e.

$$\operatorname{rk} \Gamma^\Theta = \nu \dim \Theta$$

where $\nu = \frac{\operatorname{rk} \Gamma}{\dim V}$ and $\dim \Theta = l$ provided $\Theta \in I_l$.

Recall Theorem III.3.1 applied to the present situation where we have data $(V, \Gamma, \mathcal{W})$ with L_0 finite and $\dim V = 1$. Then $H_p(\Gamma, CV_{\mathcal{C}}) = \mathbb{Z}^{D_p}$ with

$$D_p = \binom{\nu}{p+1}, \quad p > 0, \tag{2.1}$$

$$D_0 = (\nu - 1) + L_0. \tag{2.2}$$

If $\{M_i : i \in I\}$ is a family of submodules of some bigger module we denote by $\langle M_i : i \in I \rangle$ their span. For a finitely generated lattice G we let ΛG be the exterior ring (which is a $\mathbb{Z}$-module) generated by it. In **[FHK]** we obtained the following theorem.

Theorem 2.6 *Given data $(V, \Gamma, \mathcal{W})$ as in (2.1) with* $\dim V = 2$. *Suppose that L_0 is finite and that the normals of the hyperplanes $\mathcal{W}$ form an indecomposable set in the sense of IV.3.4. Then*

$$D_p = \binom{2\nu}{p+2} + L_1 \binom{\nu}{p+1} - r_{p+1} - r_p, \quad p > 0,$$

$$D_0 = \binom{2\nu}{2} - 2\nu + 1 + L_1(\nu - 1) + e - r_1$$

where

$$r_p = \mathrm{rk}\,\langle \Lambda_{p+1}\Gamma^\alpha : \alpha \in I_1\rangle,$$

and the Euler characteristic is

$$e := \sum_p (-1)^p D_p = -L_0 + \sum_{\alpha\in I_1} L_0^\alpha.$$

The main result of this chapter is an extension of this result to codimension 3.

Theorem 2.7 *Given data* $(V, \Gamma, \mathcal{W})$ *as in (2.1) with* $\dim V = 3$. *Suppose that* L_0 *is finite and that the normals of the hyperplanes* $\mathcal{W}$ *form an indecomposable set in the sense of IV.3.4. Then, for* $p > 0$,

$$D_p = \binom{3\nu}{p+3} + L_2\binom{2\nu}{p+2} + \tilde{L}_1\binom{\nu}{p+1} - R_p - R_{p+1},$$

$$D_0 = \sum_{j=0}^{3}(-1)^j\binom{3\nu}{3-j} + L_2\sum_{j=0}^{2}(-1)^j\binom{2\nu}{2-j}$$

$$+\tilde{L}_1\sum_{j=0}^{1}(-1)^j\binom{\nu}{1-j} + e - R_1$$

where $\tilde{L}_1 = -L_1 + \sum_{\alpha\in I_2} L_1^\alpha$,

$$R_p = \mathrm{rk}\,\langle \Lambda_{p+2}\Gamma^\alpha\colon \alpha \in I_2\rangle - \mathrm{rk}\,\langle \Lambda_{p+1}\Gamma^\Theta\colon \Theta \in I_1\rangle$$
$$+ \sum_{\alpha\in I_2}\mathrm{rk}\,\langle \Lambda_{p+1}\Gamma^\Theta\colon \Theta \in I_1^\alpha\rangle,$$

and the Euler characteristic is

$$e := \sum_p (-1)^p D_p = L_0 - \sum_{\alpha\in I_2} L_0^\alpha + \sum_{\alpha\in I_2}\sum_{\Theta\in I_1^\alpha} L_0^\Theta - \sum_{\Theta\in I_1} L_0^\Theta.$$

The last two theorems make combinatorial patterns for higher codimensional tilings apparent. We are able to present one such for the

Euler characteristic of the general case. To set this up define a *singular sequence* to be a (finite) sequence $c = \Theta_1, \Theta_2, ..., \Theta_k$ of Γ-orbits of singular spaces strictly ascending in the sense that $\Theta_j \in I^{\Theta_{j+1}}_{\dim \Theta_j}$, $\dim \Theta_j < \dim \Theta_{j+1}$, and $\dim \Theta_1 = 0$. An example of such a sequence is the Γ-orbit of a singular flag, but the dimension in the sequence can also jump by more than one. The length of the chain c is k, written $|c| = k$.

Theorem 2.8 *Given data $(V, \Gamma, \mathcal{W})$ with L_0 finite. Then the Euler Characteristic equals*

$$e := \sum_p (-1)^p D_p = \sum (-1)^{|c|+\dim V}$$

where the sum is over all singular chains c.

3 Complexes defined by the singular spaces

Let $\mathcal{C}'$ be an arbitrary countable collection of affine hyperplanes of V', a linear space, and define $\mathcal{C}'$-topes as in IV.3.2: compact polytopes which are the closure of their interior and whose boundary faces belong to hyperplanes from $\mathcal{C}'$.

Definition 3.1 For n at most the dimension of V' let $C^n_{\mathcal{C}'}$ be the $\mathbb{Z}$-module generated by the n-dimensional faces of convex $\mathcal{C}'$-topes satisfying the relations

$$[U_1] + [U_2] = [U_1 \cup U_2]$$

for any two n-dimensional faces U_1, U_2, for which $U_1 \cup U_2$ is as well a face and $U_1 \cap U_2$ has no interior (i.e. nonzero codimension in U_1). (The above relations then imply $[U_1] + [U_2] = [U_1 \cup U_2] + [U_1 \cap U_2]$ if $U_1 \cap U_2$ has interior.)

If we take $\mathcal{C}' = \mathcal{C} := \{W + x : W \in \mathcal{W}, x \in \Gamma\}$, our collection of singular planes, then $C^n := C^n_{\mathcal{C}}$ carries an obvious Γ-action, namely $x \cdot [U] = [U + x]$. It is therefore an Γ-module. Recall Definition IV.3.2 in which we defined $V_{\mathcal{C}}$ by identifying $CV_{\mathcal{C}}$ with $C^{\dim V}$ as Γ-module. An isomorphism between $C^{\dim V}$ and $CV_{\mathcal{C}}$ is given by assigning to $[U]$ the indicator function on the connected component containing $U \backslash S$ (which is a clopen set). Moreover, C^0 is a free $\mathbb{Z}$-module, its generators are in one to one correspondence to the elements of $\mathcal{P}$. The following proposition is from **[FHK]**.

Proposition 3.2 *There exist* Γ*-equivariant module maps* δ *and* ϵ *such that*

$$0 \to C^{\dim V} \xrightarrow{\delta} C^{\dim V - 1} \xrightarrow{\delta} \cdots C^0 \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

is an exact sequence of Γ*-modules and* $\epsilon[U] = 1$ *for all vertices* U *of* $\mathcal{C}$*-topes.*

Proof For a subset R of Γ let $\mathcal{C}_R := \{W + r : W \in \mathcal{W}, r \in R\}$ and $S_R = \{x \in W : W \in \mathcal{C}_R\}$. Let $\mathcal{R}$ be the set of subsets $R \subset \Gamma$ such that all connected components of $V \backslash S_R$ are bounded and have interior. $\mathcal{R}$ is closed under union and hence forms an upper directed system under inclusion. Let V_R be the disjoint union of the closures of the connected components of $V \backslash S_R$. $R \subset R'$ gives rise to a natural surjection $V_{R'} \to V_R$ which is the continuous extension of the inclusion $V \backslash S_{R'} \subset V \backslash S_R$ and $V_{\mathcal{C}}$ is the projective limit of the V_R. For any $R \in \mathcal{R}$, the $\mathcal{C}_R$-topes define a regular polytopal CW-complex

$$0 \to C^{\dim V}_{\mathcal{C}_R} \xrightarrow{\delta_R} C^{\dim V - 1}_{\mathcal{C}_R} \xrightarrow{\delta_R} \cdots C^0_{\mathcal{C}_R} \to 0, \tag{3.1}$$

with boundary operators δ_R depending on the choices of orientations for the n-cells ($n > 0$) **[Mas]**. Moreover, this complex is acyclic (V is contractible), i.e. upon replacing $C^0_{\mathcal{C}_R} \to 0$ by $C^0_{\mathcal{C}_R} \xrightarrow{\epsilon_R} \mathbb{Z} \to 0$ where $\epsilon_R[U] = 1$, (3.1) becomes an exact sequence. Let us constrain the orientation of the n-cells in the following way: Each n-cell belongs to a unique singular n-space $W_{\hat{\Theta}}$, $\hat{\Theta} \in \mathcal{P}_n$. We choose its orientation such that it depends only on the class of $\hat{\Theta}$ in I_l (i.e. we choose an orientation for all parallel $W_{\hat{\Theta}}$ and then the cell inherits it as a subset). By the same principle, all cells of maximal dimensions are supposed to have the same orientation. Then the cochains and boundary operators δ_R share two crucial properties: first, if $R \subset R'$ for $R, R' \in \mathcal{R}$, then we may identify $C^n_{\mathcal{C}_R}$ with a submodule of $C^n_{\mathcal{C}_{R'}}$ and under this identification $\delta_R(x) = \delta_{R'}(x)$ for all $x \in C^n_{\mathcal{C}_R}$, and second, if U and $U + x$ are $\mathcal{C}_R$-topes then $\delta_R[U + x] = \delta_R[U] + x$. The first property implies that the directed system $\mathcal{R}$ gives rise to a directed system of acyclic cochain complexes, and hence its direct limit is an acyclic complex, and the second implies, together with the fact that for all $x \in \Gamma$ and $R \in \mathcal{R}$ also $R + x \in \mathcal{R}$, that this complex becomes a complex of Γ-modules. The statement now follows since $C^n_{\mathcal{C}}$ is the direct limit of the $C^n_{\mathcal{C}_R}$, $R \in \mathcal{R}$. $\square$

Let $C^l_{[\vec{x},A]}$ be the restriction of C^l to polytopes which belong to $W_{[\vec{x},A]}$. We can naturally identify $C^l_{[\vec{x},A]}$ with $C^l_{[\vec{x}',A']}$ if $y \cdot [\vec{x},A] = [\vec{x}',A']$ for some $y \in \Gamma$. Moreover, in this case the complexes obtained by restriction of δ,

$$0 \to C^l_{[\vec{x},A]} \stackrel{\delta^{[\vec{x},A]}}{\longrightarrow} C^{l-1}_{[\vec{x},A]} \cdots C^0_{[\vec{x},A]} \to 0,$$

are isomorphic for all $[\vec{x},A]$ of the same orbit class in I_l. With this in mind we define $((C^l_\Theta)_{l\in\mathbb{Z}}, \delta^\Theta)$ as a complex isomorphic to the above complex where $\Theta \in I_l$ is the class of $[\vec{x},A]$. Thus every $\Theta \in I_l$ defines a new acyclic complex.

Lemma 3.3

$$C^l \cong \bigoplus_{\Theta\in I_l} C^l_\Theta \otimes \mathbb{Z}[\Gamma/\Gamma^\Theta]$$

and if $\Theta \in I_{l+k}$, $l+k < \dim V$, *then*

$$C^l_\Theta \cong \bigoplus_{\Psi\in I^\Theta_l} C^l_\Psi \otimes \mathbb{Z}[\Gamma^\Theta/\Gamma^\Psi].$$

Proof First observe that

$$C^l \cong \bigoplus_{[\vec{x},A]\in\mathcal{P}_l} C^l_{[\vec{x},A]},$$

because an l-face of a $\mathcal{C}$-tope belongs a unique singular l-space. From the definition of C_Θ and the observation that $y \cdot [\vec{x},A] = [\vec{x},A]$ whenever y leaves W_A invariant the first statement of the lemma follows. The proof for the second is similar. □

4 Group homology

Recall that the homology of a discrete group Γ' with coefficients in a Γ'-module M is defined as the homology of a complex which is obtained from a resolution of $\mathbb{Z}$ by (projective) Γ'-modules upon application of the functor $\otimes_{\Gamma'} M$. We use here a free and finite resolution of $\Gamma' = \Gamma \cong \mathbb{Z}^N$ by its exterior module so that the complex looks like

$$0 \to \Lambda_N\Gamma \otimes M \stackrel{\partial}{\longrightarrow} \cdots \Lambda_0\Gamma \otimes M \to 0$$

with boundary operator ∂ given by

$$\partial(e_{i_1}\cdots e_{i_k}\otimes m)=\sum_{j=1}^{k} e_{i_1}\cdots \hat{e}_{i_j}\cdots e_{i_k}(e_{i_j}\cdot m-m)$$

denoting by $\hat{e}_{i_j}$ that it has to be left out and by $g\cdot$ the action of $g\in\Gamma$ on M. In particular, $H_k(\Gamma,\mathbb{Z}\Gamma)$ is trivial for all $k>0$ and isomorphic to $\mathbb{Z}$ for $k=0$.

Suppose that we can split $\Gamma=\Gamma'\oplus\Gamma''$ and let us compute $H_k(\Gamma,\mathbb{Z}\Gamma'')$ where $\mathbb{Z}\Gamma''$ is the free $\mathbb{Z}$-module generated by Γ'' which becomes an Γ-module under the action of Γ given by $(g\oplus h)\cdot h'=h+h'$. Then we can identify

$$\Lambda_k\Gamma\otimes\mathbb{Z}\Gamma''\cong\bigoplus_{i+j=k}\Lambda_i\Gamma'\otimes\Lambda_j\Gamma''\otimes\mathbb{Z}\Gamma'' \tag{4.1}$$

and under this identification $\partial\otimes 1$ becomes $(-1)^{deg}\otimes\partial'$ where ∂' is the boundary operator for the homology of Γ''. It follows that

$$H_k(\Gamma,\mathbb{Z}\Gamma'')\cong\bigoplus_{i+j=k}\Lambda_i\Gamma'\otimes H_j(\Gamma'',\mathbb{Z}\Gamma'')\cong\Lambda_k\Gamma'.$$

As a special case, $H_k(\Gamma,\mathbb{Z})\cong\Lambda_k\Gamma\cong\mathbb{Z}^{\binom{N}{k}}$. Now let $\epsilon:\mathbb{Z}\Gamma''\to\mathbb{Z}$ be the sum of the coefficients, i.e. $\epsilon[h]=1$ for all $h\in\Gamma''$. We shall later need the following lemma:

Lemma 4.1 *With the identifications $H(\Gamma,\mathbb{Z}\Gamma'')\cong\Lambda\Gamma'$ and $H(\Gamma,\mathbb{Z})\cong\Lambda\Gamma$ the induced map $\epsilon_k:H_k(\Gamma,\mathbb{Z}\Gamma'')\to H_k(\Gamma,\mathbb{Z})$ becomes the inclusion $\Lambda_k\Gamma'\hookrightarrow\Lambda_k\Gamma$.*

Proof Using the decomposition (4.1) it is not difficult to see that the induced map $\epsilon_k:\bigoplus_{i+j=k}\Lambda_i\Gamma'\otimes H_j(\Gamma'',\mathbb{Z}\Gamma'')\to\bigoplus_{i+j=k}\Lambda_i\Gamma'\otimes H_j(\Gamma'',\mathbb{Z})$ preserves the bidegree and must be the identity on the first factors of the tensor product. Since $H_k(\Gamma'',\mathbb{Z}\Gamma'')$ is trivial whenever $k\neq 0$ and one dimensional for $k=0$, ϵ_k can be determined by evaluating ϵ_0 on the generator of $H_0(\Gamma'',\mathbb{Z}\Gamma'')$ and one readily checks that this gives a generator of $H_0(\Gamma'',\mathbb{Z})$ as well. □

Finally, we note two immediate corollaries of Lemma 3.3.

Corollary 4.2

$$H(\Gamma,C^l)\cong\bigoplus_{\Theta\in I_l}H(\Gamma^\Theta,C^l_\Theta).$$ □

Corollary 4.3 *If* $\Theta \in I_{l+k}$, $l + k < \dim V$, *then*

$$H(\Gamma^\Theta, C^l_\Theta) \cong \bigoplus_{\Psi \in I^\Theta_l} H(\Gamma^\Psi, C^l_\Psi). \qquad \square$$

5 The spectral sequences

In the last two sections we described two complexes. They can be combined to yield a double complex and then spectral sequence techniques can be applied to obtain the information we want. We do not explain these techniques in detail but only set up the notation, see, for example, [**Br**] for an introduction. Consider a double complex (E^0, ∂, δ) which is a bigraded module $E^0 = (E^0_{pq})_{p,q\in\mathbb{Z}}$ with two commuting differential operators ∂ and δ of bidegree $(-1, 0)$ and $(0, -1)$, respectively. The associated total complex is $TE^0 = ((TE^0)_k)_{k\in\mathbb{Z}}$ with $(TE^0)_k = \bigoplus_{p+q=k} E^0_{pq}$ and differential $\delta + \partial$. Furthermore one can form two spectral sequences of bigraded modules [**Br**]. The first one, $(E^k)_{k\in\mathbb{N}_0}$, is equipped with differential operators d^k of bidegree $(k - 1, -k)$, i.e.

$$\mathrm{d}^k_{pq} : E^k_{pq} \longrightarrow E^k_{p+k-1\, q-k}, \tag{5.1}$$

such that $E^{k+1} = H_{\mathrm{d}^k}(E^k)$. (Here and below we use also the notation $Z_d(C^k)$ and $H_d(C^k)$ for the degree k cycles and homology resp. of a complex (C, d).) E^0 is the original module, $\mathrm{d}^0 = \partial$, $\mathrm{d}^1 = \delta_*$ and the higher differentials become more and more subtle. The second sequence, $(\tilde{E}^k)_{k\in\mathbb{N}_0}$, is obtained in the same way except that one interchanges the role of the two differentials, i.e. $\tilde{E}^0 = E^0$, $\tilde{\mathrm{d}}^0 = \delta$, $\tilde{\mathrm{d}}^1 = \partial_*$. The two spectral sequences may look rather different but their "limits" are related to the homology of the total complex. This relation may be quite subtle but in our case we need only the following. Suppose that E^0_{pq} is non-trivial only for finitely many p, q. Then the higher differentials vanish for both sequences from some k on so that the modules stabilize and we have well defined limit modules E^∞, $\tilde{E}^\infty$. If the modules are moreover vector spaces then

$$\bigoplus_{p+q=k} E^\infty_{pq} \cong \bigoplus_{p+q=k} \tilde{E}^\infty_{pq} \cong H_{\partial+\delta}((TE^0)_k).$$

If the modules are only over some ring such as the integer numbers we may at least conclude that finitely generated E^∞ implies that $\tilde{E}^\infty$ and $H_{\partial+\delta}(TE^0)$ are finitely generated as well.

Inserting the complex from Proposition 3.1 for M in the last section we get the double complex $(\Lambda_p\Gamma \otimes C^q, \partial_{pq}, \delta_{pq})$ where $\partial_{pq} = \partial_p \otimes 1$ and $\delta_{pq} = (-1)^p \otimes \delta_q$.

Proposition 5.1 *Consider the spectral sequence $\tilde{E}^k$ derived from the double complex $(E_0^0 = \Lambda_p\Gamma \otimes C^q, \partial_{pq}, \delta_{pq})$ when starting with homology in δ. It satisfies*

$$\bigoplus_{p+q=k} \tilde{E}^\infty_{pq} \cong \Lambda_k\Gamma.$$

In particular, $\mathrm{rk}\left(H_{\delta+\partial}(\bigoplus_{p+q=k} \Lambda_p\Gamma \otimes C^q) \otimes \mathbb{Q}\right) = \binom{\mathrm{rk}\,\Gamma}{k}$.

Proof The first page of that spectral sequence is

$$\tilde{E}^1_{pq} = H_\delta(\Lambda_p\Gamma \otimes C^q) \cong \begin{cases} \Lambda_p\Gamma & \text{for} \quad q = 0, \\ 0 & \text{for} \quad q > 0. \end{cases}$$

Under the isomorphism $\tilde{E}^1_{p0} \cong \Lambda_p\Gamma$, $\tilde{\mathrm{d}}^1$ becomes trivial and all other differentials $\tilde{\mathrm{d}}^k$ vanish because they have bidegree $(-k, k-1)$. Hence $\tilde{E}^\infty = \tilde{E}^1$ from which the first statement follows. The second statement is then clear. □

The more difficult spectral sequence will occupy us for the rest of this chapter. Its first page is obtained when starting with homology in ∂, i.e.

$$E^1_{pq} = H_\partial(E^0_{pq}) = H_p(\Gamma, C^q),$$

and the first differential is $\mathrm{d}^1 = \delta_*$. We realize that $E^1_{p\,\dim V}$ is what we want to compute. For that to carry out we have to determine the ranks of the higher differentials which is quite involved. On the other hand the proofs of Theorems 2.4, 2.5, and 2.8 involve only the general fact that the higher differentials d^k have bidegree $(k-1, -k)$ and begin with that. All three proofs are by induction on the dimension of V the one-dimensional case following from Eqs. (2.1),(2.2).

Proof of Theorem 2.4 Theorem 2.4 is a rational result. So we work with a rationalized version of the above. Corollary 4.2 yields

$$E^1_{pq} \cong \bigoplus_{\Theta \in I_q} H_p(\Gamma^\Theta, C^q_\Theta).$$

Hence by Lemma 2 and induction on the dimension of V we find that $E^1_{pq} \otimes \mathbb{Q}$ is a finite dimensional vector space, provided $q < \dim V$. This implies that $E^k_{pq} \otimes \mathbb{Q}$ is finite dimensional, provided $q < \dim V$, for arbitrary k, and is trivial for $q > \dim V$ or $p > \operatorname{rk} \Gamma$. On the other hand by Proposition 5.1 $E^\infty_{pq} \otimes \mathbb{Q}$ has to be finite dimensional for all p, q. By construction $E^\infty_{p \dim V}$ is obtained upon taking successively the kernels of the higher differentials. So if $E^1_{p \dim V} \otimes \mathbb{Q}$ was infinite, one of the ranks of the higher differentials would have to be infinite. But this would contradict the finite dimensionality of $E^k_{pq} \otimes \mathbb{Q}$ for $q < \dim V$. □

Proof of Theorem 2.5 It is a result of **[FH]** that $H_p(\Gamma, CV_{\mathcal{C}})$ is torsion free. Therefore Theorem 2.5 follows if we can show that $H_p(\Gamma, CV_{\mathcal{C}})$ is finitely generated. Now we use Lemma 2.2 and Corollary 4.2 to see inductively that E^1_{pq} is finitely generated, provided $q < \dim V$. This implies that E^k_{pq} is finitely generated, provided $q < \dim V$, for arbitrary k. By Eq. (5.1) we get sequences

$$E^{k+1}_{p \dim V} \hookrightarrow E^k_{p \dim V} \xrightarrow{\mathrm{d}^k} E^k_{p+k-1 \dim V-k}$$

which are exact at $E^k_{p \dim V}$ and whose right hand side modules are finitely generated if $k \geq 1$. Hence $E^k_{p \dim V}$ is finitely generated if $E^{k+1}_{p \dim V}$ is finitely generated. But $E^\infty_{p \dim V}$ is finitely generated by Proposition 5.1. Again inductively we conclude therefore that $E^1_{p \dim V}$ is finitely generated. □

Proof of Theorem 2.8 By Theorem 2.4 the dimensions of the rational vector spaces $E^k_{pq} \otimes \mathbb{Q}$ are finite. From Eq. (5.1) it follows therefore that $\sum_{p,q} (-1)^{p+q} \dim E^k_{pq} \otimes \mathbb{Q}$ is independent of k. Choosing $k = \infty$ we see from Proposition 5.1 that this sum vanishes. With $e_q = \sum_p (-1)^p \dim H_p(\Gamma, C^q) \otimes \mathbb{Q}$ we thus get

$$e = e_{\dim V} = -\sum_{q=0}^{\dim V - 1} (-1)^{q + \dim V} e_q.$$

But $\dim H_p(\Gamma, C^q) \otimes \mathbb{Q} = \sum_{\Theta \in I_q} \dim H_p(\Gamma^\Theta, C^q_\Theta) \otimes \mathbb{Q}$ by Corollary 4.2 and we can use induction on $\dim V$ to see that $e_q = \sum_{\Theta \in I_q} \sum (-1)^{q+|c|-1}$ where the second sum is over all singular sequences c whose last element is Θ. Inserting this into the above formula for e gives directly the result. □

The strategy to prove the remaining Theorems 2.6 and 2.7 is as follows. Suppose that we know $d_{pq} = \dim E^1_{pq} \otimes \mathbb{Q}$ for $q < \dim V$. Then we can determine $d_{p\dim V}$ from these data, the dimensions of the rationalized total homology groups (Proposition 5.1), and the ranks of the differentials. To carry that out we have to consider only modules over $\mathbb{Q}$. We will therefore simplify our notation by suppressing $\otimes\mathbb{Q}$ and e.g. write E^k_{pq} instead of $E^k_{pq} \otimes \mathbb{Q}$. We consider first a more abstract situation.

Definition 5.2 Consider a double complex $(E^0_{pq}, \delta_{pq}, \partial_{pq})$ of finitely generated $\mathbb{Q}$-modules with its spectral sequence $(E^k)_{k\in\mathbb{N}_0}$, i.e. $E^1_{pq} := H_\partial(E^0_{pq})$ and $\mathrm{d}^1 = \delta_*$. We write $d_{pq} = \dim E^1_{pq}$. Assume that

- E1 There exist finite M, N such that E^0_{pq} is non-trivial only for $0 \leq q \leq M$ and $0 \leq p \leq N$,
- E2 $d_{p0} = 0$ for $p \geq 1$.

This is sufficient to ensure that the spectral sequence converges and we can define $N_k := \sum_{p+q=k} \dim E^\infty_{pq}$.

For $M = 1, 2, 3$, we now determine the ranks d_{pM} in terms of d_{pq}, $q < M$, and N_k and the ranks of the higher differentials d^k.

Lemma 5.3 *If $M = 1$ then, with the notation of Definition 5.2,*

$$d_{p1} = N_{p+1} + \begin{cases} d_{00} - N_0 & \text{for} \quad p = 0, \\ 0 & \text{for} \quad p > 0. \end{cases}$$

Proof If $M = 1$ then (5.1) implies that $\mathrm{d}^k = 0$ for $k \geq 2$ and hence $E^\infty = E^2$. With the notation

$$a_{pq} := \mathrm{rk}\, \mathrm{d}^1_{p\,q+1} : E^1_{p\,q+1} \to E^1_{pq} \tag{5.2}$$

we get

$$\dim E^\infty_{pq} = \begin{cases} d_{00} - a_{00} & \text{for} \quad p = q = 0, \\ d_{01} - a_{00} & \text{for} \quad p = 0, q = 1, \\ d_{p1} & \text{for} \quad N \geq p \geq 1, q = 1, \\ 0 & \text{otherwise}, \end{cases}$$

which when a_{00} is eliminated for N_0 yields the statement. $\square$

Let

$$b_{pq} := \operatorname{rk} \tilde{\delta}_{p\,q+1\,*} : E^1_{p\,q+1} \to H_\partial(Z_\delta(E^0_{pq})), \tag{5.3}$$

where $\tilde{\delta}_{p\,q+1\,*}$ is the map induced from $\tilde{\delta}_{p\,q+1}$ which in turn is obtained from $\delta_{p\,q+1}$ by restricting its target space, namely it forms the exact sequence $0 \to E^0_{p\,q+2} \xrightarrow{\delta_{p\,q+2}} E^0_{p\,q+1} \xrightarrow{\tilde{\delta}_{p\,q+1}} Z_\delta(E^0_{pq}) \to 0$.

Lemma 5.4 *If $M = 2$ then, with the notation of Definition 5.2,*

$$d_{p2} = N_{p+2} + d_{p1} - b_{p+1\,0} - b_{p0}, \quad p > 0$$

$$d_{02} = N_2 + d_{01} - b_{10} - N_1 + N_0 - d_{00}.$$

Proof If $M = 2$ then (5.1) still implies that $\mathrm{d}^k = 0$ for $k \geq 2$, because of E2. But now, with the notation (5.2),

$$\dim E^\infty_{pq} = \begin{cases} d_{00} - a_{00} & \text{for} \quad p = q = 0, \\ d_{01} - a_{00} - a_{01} & \text{for} \quad p = 0, q = 1, \\ d_{p1} - a_{p1} & \text{for} \quad N \geq p \geq 1, q = 1, \\ d_{p2} - a_{p1} & \text{for} \quad N \geq p \geq 0, q = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $\mathrm{d}^1_{p\,q+1} : E^1_{p\,q+1} \to E^1_{pq}$ is the map induced on the homology groups from $\delta_{p\,q+1}$. From the exact sequence $0 \to E^0_{p2} \xrightarrow{\delta_{p2}} E^0_{p1} \xrightarrow{\tilde{\delta}_{p1}} Z_\delta(E^0_{p0}) \to 0$ we conclude

$$a_{p1} = \dim\ker \tilde{\delta}_{p1\,*} = d_{p1} - b_{p0}$$

which now implies the statement. □

Lemma 5.5 *If $M = 3$ then, with the notation of Definition 5.2 and (5.3)*

$$d_{p3} = N_{p+3} + d_{p2} + d_{p+1\,1} - b_{p1} - b_{p+1\,1} - b_{p+1\,0} - b_{p+2\,0}, \quad p > 0,$$

$$d_{03} = N_3 - N_2 + N_1 - N_0 + d_{02} + d_{11} - d_{01} + d_{00} - b_{11} - b_{20}.$$

Proof Now (5.1) implies that $\mathrm{d}^k = 0$ only for $k > 2$ so that $E^3 = E^\infty$. With the notation

$$\mu_{pq} := \mathrm{rk}\, \mathrm{d}^2_{p-1\, q+2} : E^2_{p-1\, q+2} \to E^2_{pq}$$

and (5.2) we get

$$\dim E^\infty_{pq} = \begin{cases} d_{00} - a_{00} & \text{for} \quad p = q = 0, \\ d_{01} - a_{00} - a_{01} & \text{for} \quad p = 0, q = 1, \\ d_{p1} - a_{p1} - \mu_{p1} & \text{for} \quad N \geq p \geq 1, q = 1, \\ d_{p2} - a_{p1} - a_{p2} & \text{for} \quad N \geq p \geq 0, q = 2, \\ d_{p3} - a_{p2} - \mu_{p+1\, 1} & \text{for} \quad N > p \geq 0, q = 3, \\ d_{N3} - a_{N2} & \text{for} \quad p = N, q = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Let us recall the definition of d^2. Look at the commuting diagram

$$\begin{array}{ccccccccc} 0 & \to & E^0_{p3} & \xrightarrow{\delta_{p3}} & E^0_{p2} & \xrightarrow{\delta_{p2}} & E^0_{p1} & \xrightarrow{\delta_{p1}} & \cdots \\ & & & & \downarrow\tilde{\delta}_{p2} & \overset{\imath_{p0}}{\nearrow} & \downarrow\tilde{\delta}_{p1} & \overset{\imath_{p0}}{\nearrow} & \\ & & & & Z_\delta(E^0_{p1}) & & Z_\delta(E^0_{p0}) & & \end{array}.$$

Let $\theta : H_\partial(Z_\delta(E^0_{p1})) \to H_\partial(E^0_{p-1\,3}) = E^1_{p-1\,3}$ be the connecting map of the exact sequence $0 \to E^0_{p3} \xrightarrow{\delta_{p3}} E^0_{p2} \xrightarrow{\tilde{\delta}_{p2}} Z_\delta(E^0_{p1}) \to 0$. Then $\mathrm{d}^2_{p-1\,3} : E^2_{p-1\,3} \to E^2_{p1}$ is the map induced by $\imath_{p1\,*} \circ \theta^{-1} : E^1_{p-1\,3} \to E^1_{p1}$. Since θ^{-1} identifies $E^2_{p-1\,3} = Z_{d^1}(E^1_{p-1\,3}) \overset{\theta^{-1}}{\cong} H_\partial(Z_\delta(E^0_{p1}))/\mathrm{Im}\tilde{\delta}_{p2\,*}$ we get

$$\mu_{p1} = \mathrm{rk}\, \imath_{p1\,*} - \mathrm{rk}\, \delta_{p2\,*} = \dim\ker \tilde{\delta}_{p1\,*} - \mathrm{rk}\, \delta_{p2\,*} = (d_{p1} - b_{p0}) - a_{p1}.$$

Furthermore, using $a_{p2} = \dim\ker \tilde{\delta}_{p2\,*} = d_{p2} - b_{p1}$ one obtains the statement. □

Definition 5.6 We now specify to double complexes of the form $E^0_{pq} = \Lambda_p\Gamma \otimes C^q$, $\partial_{pq} = \partial_p \otimes 1$ and $\delta_{pq} = (-1)^p \otimes \delta_q$ as they arise in our application however tacitly changing the ring to be $\mathbb{Q}$. Below, d_{pq} shall always denote the dimension of $E^1_{pq} := H_\delta(\Lambda_p\Gamma \otimes C^q)$.

Clearly, E^0_{pq} is non-trivial only for $0 \leq p \leq \mathrm{rk}\,\Gamma$ and $0 \leq q \leq \dim V$. Furthermore, $E^1_{p0} = H_\delta(\Lambda_p\Gamma \otimes C^0) = H_p(\Gamma, C^0)$. Since C^0 is always a free Γ-module one has $H_p(\Gamma, C^0) = 0$ for $p > 0$ and the dimension of $H_0(\Gamma, C^0)$ is simply $|I_0|$, the number of orbits of in $\mathcal{P}$:

$$d_{00} = L_0.$$

In particular E1 and E2 are satisfied. Furthermore, $N_k = \binom{\operatorname{rk}\Gamma}{k}$.

The case $\dim V = 1$ (2.1,2.2) can now be immediately obtained using Lemma 5.3 and Proposition 5.1.

Proof of Theorem 2.5 A proof can also be found in [**FHK**] but we repeat it here using the framework of spectral sequences. So let $\dim V = 2$. We seek to apply Lemma 5.4. For that we need to determine d_{p1} and b_{p0}.

To determine d_{p1} we use Corollary 4.2. $H(\Gamma^\alpha, C^1_\alpha)$ can be computed using the double complex $(\Lambda_p\Gamma^\alpha \otimes C^q_\alpha, \partial^\alpha_{pq}, \delta^\alpha_{pq})$. So let us consider the spectral sequence which arises when starting with homology in ∂^α. As for the case $\dim V = 1$ one concludes that E^1_{p0} is non-trivial only for $p = 0$ and the dimension of E^1_{p0} equal to L^α_0, the number of orbits of points in $\mathcal{P} \cap W_\alpha$. Since $\operatorname{rk}\Gamma^\alpha = \nu$ we get

$$d_{p1} = L_1 \binom{\nu}{p+1}, \quad p > 0,$$

$$d_{01} = L_1(\nu - 1) + \sum_{\alpha \in I_1} L^\alpha_0.$$

It remains to compute $b_{p0} = \operatorname{rk}\tilde{\delta}_{p1\,*} : H_p(\Gamma, C^1) \to H_p(\Gamma, Z_\delta(C^0))$. For that look at the following commuting diagram of exact sequences

$$\begin{array}{ccccccc}
0 \to & C^1_\alpha \otimes \mathbb{Z}[\Gamma/\Gamma^\alpha] & \stackrel{\delta^\alpha_1 \otimes 1}{\longrightarrow} & C^0_\alpha \otimes \mathbb{Z}[\Gamma/\Gamma^\alpha] & \to & \mathbb{Z}[\Gamma/\Gamma^\alpha] & \to 0 \\
& \downarrow \tilde{\delta}^\alpha_1 \otimes 1 & & \downarrow & & \downarrow \epsilon^\alpha & \\
0 \to & Z_\delta(C^0) & \hookrightarrow & C^0 & \stackrel{\epsilon}{\longrightarrow} & \mathbb{Z} & \to 0
\end{array}$$

where the middle verticle arrow is the inclusion, the right vertical arrow the sum of the coefficients, $\epsilon^\alpha[\gamma] = 1$, and the direct sum over all $\alpha \in I_1$ of the left vertical arrows is $\tilde{\delta}_{p1}$. The above commutative diagram gives rise to two long exact sequences of homology groups together with vertical maps, all commuting, $(\tilde{\delta}^\alpha_{p1} \otimes 1)_* : H_p(\Gamma, C^1_\alpha \otimes \mathbb{Z}[\Gamma/\Gamma^\alpha]) \to H_p(\Gamma, Z_\delta(C^0))$ being one of them. Now we use that for $p > 0$, $H_p(\Gamma, C^0_\alpha \otimes \mathbb{Z}[\Gamma/\Gamma^\alpha]) = 0$ and $H_p(\Gamma, C^0) = 0$ which implies $H_p(\Gamma, C^1_\alpha \otimes \mathbb{Z}[\Gamma/\Gamma^\alpha]) \cong H_{p+1}(\Gamma, \mathbb{Z}[\Gamma/\Gamma^\alpha]) \cong H_{p+1}(\Gamma^\alpha, \mathbb{Z})$ and $H_p(\Gamma, Z_\delta(C^0)) \cong H_{p+1}(\Gamma, \mathbb{Z})$ and

$$(\tilde{\delta}^\alpha_{p1} \otimes 1)_* = \epsilon^\alpha_{p+1}.$$

By Lemma 4.1 the map ϵ_p^α can be identified with the inclusion $\Lambda_p\Gamma^\alpha \hookrightarrow \Lambda_p\Gamma$. For $p>0$ therefore,

$$b_{p0} = \mathrm{rk}\,\langle \mathrm{Im}(\tilde{\delta}_{p1}^\alpha \otimes 1)_* : \alpha \in I_1\rangle = \mathrm{rk}\,\langle \Lambda_{p+1}\Gamma^\alpha : \alpha \in I_1\rangle. \tag{5.4}$$

Direct application of Lemma 5.4 yields now the formulas for the dimensions stated in Theorem 2.2. □

Proof of Theorem 2.6 Let $\dim V = 3$. To apply Lemma 5.5 we need to determine $d_{p1}, d_{p2}, b_{p0}, b_{p1}$. To calculate d_{p1} we use Corollary 4.2 and Lemma 5.3 to obtain, for $\Theta \in I_1$,

$$\dim H_p(\Gamma^\Theta, C^1_\Theta) = \binom{\nu}{p+1} + \begin{cases} L_0^\Theta - 1 & \text{for} \quad p=0, \\ 0 & \text{for} \quad p>0. \end{cases}$$

Hence

$$d_{p1} = L_1 \binom{\nu}{p+1} + \begin{cases} -L_1 + \sum_{\Theta\in I_1} L_0^\Theta & \text{for} \quad p=0, \\ 0 & \text{for} \quad p>0. \end{cases}$$

To calculate d_{p2} we consider the complexes, $\alpha \in I_2$,

$$0 \to C_\alpha^2 \xrightarrow{\delta^\alpha} C_\alpha^1 \xrightarrow{\delta^\alpha} C_\alpha^0 \to 0.$$

The appropriate double complex to consider is therefore $(\Lambda_p\Gamma^\alpha \otimes C_\alpha^q, \partial_{pq}^\alpha, \delta_{pq}^\alpha)$. Repeating the arguments of the proof of Theorem 2.2 (but now using Corollary 4.3 in place of 4.2) we get

$$\dim H_p(\Gamma^\alpha, C^2_\alpha) = \binom{2\nu}{p+2} + L_1^\alpha \binom{\nu}{p+1} - r_p^\alpha - r_{p+1}^\alpha \quad p>0,$$

$$\dim H_0(\Gamma^\alpha, C^2_\alpha) = \binom{2\nu}{2} - 2\nu + 1 + L_1^\alpha(\nu-1) - r_1^\alpha - L_0^\alpha + \sum_{\Theta\in I_1^\alpha} L_0^\Theta$$

where

$$r_p^\alpha = \mathrm{rk}\,\langle \Lambda_{p+1}\Gamma^\Theta : \Theta \in I_1^\alpha\rangle.$$

Writing $\mathcal{F} = \sum_{\alpha\in I_2} L_1^\alpha$ we thus get

$$d_{p2} = L_2 \binom{2\nu}{p+2} + \mathcal{F}\binom{\nu}{p+1} - \sum_{\alpha\in I_2} (r_p^\alpha + r_{p+1}^\alpha), \quad p>0,$$

$$d_{02} = L_2\left(\binom{2\nu}{p+2} - 2\nu + 1\right) + \mathcal{F}(\nu - 1) \\ + \sum_{\alpha\in I_2}\left(\sum_{\Theta\in I_1^\alpha} L_0^\Theta - L_0^\alpha - r_1^\alpha\right).$$

We now compute b_{p0} and b_{p1}. For the first we can again repeat the arguments which yield (5.4) to obtain

$$b_{p0} = \operatorname{rk}\langle \Lambda_{p+1}\Gamma^\Theta : \Theta \in I_1\rangle.$$

b_{p1} is the rank of $\tilde{\delta}_{p2\,*}$, i.e. the rank of the map induced from $\tilde{\delta}_2 : C^2 \to Z_\delta(C^1)$. To determine it we consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & C^2_\alpha \otimes \mathbb{Z}[\Gamma/\Gamma^\alpha] \arrow[r, "\delta_2^\alpha\otimes 1"] \arrow[d, "\tilde{\delta}_2^\alpha \otimes 1"] & C^1_\alpha \otimes \mathbb{Z}[\Gamma/\Gamma^\alpha] \arrow[r, "\tilde{\delta}_1^\alpha\otimes 1"] \arrow[d] & Z_\delta(C^0_\alpha) \otimes \mathbb{Z}[\Gamma/\Gamma^\alpha] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z_\delta(C^1) \arrow[r, hook] & C^1 \arrow[r, "\delta_1"] & Z_\delta(C^0) \arrow[r] & 0.
\end{tikzcd}$$

The middle and right vertical maps are the obvious inclusions. By Corollary 4.2 the direct sum of the left vertical arrows of the diagrams which arise if $\alpha \in I_2$ is $\tilde{\delta}_2$. The diagram gives rise to a commutative diagram of which the following is one degree

$$\begin{tikzcd}
H_p(\Gamma, C^2_\alpha \otimes \mathbb{Z}[\Gamma/\Gamma^\alpha]) \arrow[r] \arrow[d, "(\tilde{\delta}^\alpha_{p2}\otimes 1)_*"] & H_p(\Gamma, C^1_\alpha \otimes \mathbb{Z}[\Gamma/\Gamma^\alpha]) \arrow[r, "(\tilde{\delta}^\alpha_{p1}\otimes 1)_*"] \arrow[d] & H_p(\Gamma, Z_\delta(C^0_\alpha) \otimes \mathbb{Z}[\Gamma/\Gamma^\alpha]) \arrow[d] \\
H_p(\Gamma, Z_\delta(C^1)) \arrow[r] & H_p(\Gamma, C^1) \arrow[r, "\delta_{p1\,*}"] & H_p(\Gamma, Z_\delta(C^0))
\end{tikzcd} \tag{5.5}$$

also with exact sequences. In particular,

$$b_{p1} = \operatorname{rk}\langle \operatorname{Im}(\tilde{\delta}^\alpha_{p2}\otimes 1)_* : \alpha \in I_2\rangle.$$

Now we can follow the same analysis as in the proof of Theorem 2.2 to see that, for $p > 0$, (5.5) is isomorphic to

$$\begin{array}{ccccccc}
\to & H_p(\Gamma, C^2_\alpha \otimes \mathbb{Z}[\Gamma/\Gamma^\alpha]) & \to & \bigoplus_{\Theta\in I_1^\alpha} \Lambda_{p+1}\Gamma^\Theta & \xrightarrow{\beta_p^\alpha} & \Lambda_{p+1}\Gamma^\alpha & \to \\
& \Big\downarrow (\tilde\delta^\alpha_{p2}\otimes 1)_* & & \Big\downarrow j_p^\alpha & & \Big\downarrow \imath_p^\alpha & \\
\to & H_p(\Gamma, Z_\delta(C^1)) & \to & \bigoplus_{\Theta\in I_1} \Lambda_{p+1}\Gamma^\Theta & \xrightarrow{\gamma_p} & \Lambda_{p+1}\Gamma & \to
\end{array} \tag{5.6}$$

where β_p^α is the direct sum over $\Theta \in I_1^\alpha$ of the inclusions $\Lambda_{p+1}\Gamma^\Theta \hookrightarrow \Lambda_{p+1}\Gamma^\alpha$, γ_p is the direct sum over $\Theta \in I_1$ of the inclusions $\Lambda_{p+1}\Gamma^\Theta \hookrightarrow \Lambda_{p+1}\Gamma$, and j_p^α and $\imath_p^\alpha$ are the obvious inclusions. As we are working with finitely generated $\mathbb{Q}$-modules we obtain from (5.6)

$$\mathrm{Im}(\tilde\delta^\alpha_{p2}\otimes 1)_* \cong \mathrm{Im}\imath^\alpha_{p+1}/(\mathrm{Im}\imath^\alpha_{p+1}\cap \mathrm{Im}\gamma_{p+1}) \oplus j_p^\alpha(\ker\beta_p^\alpha).$$

Since $I_1 = \bigcup_{\alpha\in I_2} I_1^\alpha$ the direct sum over $\alpha \in I_1^\alpha$ of j_p^α is surjective. Hence $\mathrm{Im}\gamma_p \subset \langle \mathrm{Im}\imath_p^\alpha : \alpha \in I_2\rangle$ and therefore

$$\langle \mathrm{Im}\imath_p^\alpha/(\mathrm{Im}\imath_p^\alpha \cap \mathrm{Im}\gamma_p) : \alpha\in I_2\rangle = \langle \mathrm{Im}\imath_p^\alpha : \alpha \in I_2\rangle/\mathrm{Im}\gamma_p.$$

Furthermore, using also that $\imath_p^\alpha$ is injective, we get $j_p^\alpha(\ker\beta_p^\alpha) = \mathrm{Im}j_p^\alpha \cap \ker\gamma_p$ and

$$\langle \mathrm{Im}j_p^\alpha \cap \ker\gamma_p : \alpha\in I_2\rangle = \ker\gamma_p.$$

Thus we have

$$\langle \mathrm{Im}(\tilde\delta^\alpha_{p2}\otimes 1)_* : \alpha \in I_2\rangle \cong \langle \mathrm{Im}\imath^\alpha_{p+1} : \alpha\in I_2\rangle/\mathrm{Im}\gamma_{p+1} \oplus \ker\gamma_p$$

which, since $\mathrm{rk}\,\gamma_p = b_{p0}$, implies

$$b_{p1} = \mathrm{rk}\,\langle \Lambda_{p+2}\Gamma^\alpha : \alpha\in I_2\rangle - b_{p+1\,0} + L_1\binom{\nu}{p+1} - b_{p0}.$$

With these results we obtain from Lemma 5.5 the ranks stated in Theorem 2.3. □

6 Example: Ammann-Kramer tilings

In Chapter IV we showed that generically, canonical projection tilings have infinitely generated cohomology. Nevertheless, all tilings known to us which are used to describe quasicrystals have finitely generated cohomology. In **[GK]** a list of results for the known 2-dimensional tilings used by quasicrystallographers was presented. Here we present the first result for a 3-dimensional projection tiling with finitely generated cohomology. It is the Ammann-Kramer tiling. The tiling was invented before the discovery of quasicrystals **[KrNe]** and rediscovered in **[DK]** **[Els]**. It has been used to describe icosahedral quasicrystals in **[EH]** **[HE]**. It is sometimes also called 3-dimensional Penrose tiling because it generalizes in a way Penrose's 2-dimensional tilings. A short description of this tiling and further 3-dimensional tilings related icosahedral symmetry is given in **[KrPa]**. The tiling is obtained by the canonical projection method from the data $(\mathbb{Z}^6, E)$, $\mathbb{Z}^6$ being the lattice generated by an orthonormal basis of $\mathbb{R}^6$, and E being obtained from symmetry considerations involving the representation theory of the icosahedral group. Projecting $\mathbb{Z}^6$ orthogonally onto the orthocomplement V of E one obtaines the (dense) lattice Γ generated by the six vectors

$$\begin{pmatrix}\tau\\1\\0\end{pmatrix}, \begin{pmatrix}-\tau\\1\\0\end{pmatrix}, \begin{pmatrix}0\\\tau\\-1\end{pmatrix}, \begin{pmatrix}0\\\tau\\1\end{pmatrix}, \begin{pmatrix}1\\0\\-\tau\end{pmatrix}, \begin{pmatrix}1\\0\\\tau\end{pmatrix}.$$

with respect to an orthonormal basis of V. Here $\tau = \frac{\sqrt{5}-1}{2}$ and we have the usual relation $\tau^2 + \tau - 1 = 0$. The acceptance domain is the orthogonal projection of the 6-dimensional unit cube into V and forms the triacontrahedron whose vertices are the linear combinations with coefficients $0, 1$ of the above vectors. It has 30 faces which are all triangles and the affine hyperplanes $\mathcal{W}$ are the planes spanned by these triangles. Although we have an action of the icosahedral group (which even permutes the faces) the geometry of their intersections is quite complex, but can be summarized as follows.

Singular subspaces in V: The two triangles which are in opposite position of the triacontrahedron belong to the same orbit (under the action of Γ) and representatives of the 15 orbits of singular planes are given by the spans of all possible pairs of vectors from our list of 6 vectors above. Thus I_2 has $L_2 = 15$ elements. The 15 planes can be gathered into five orthogonal triples. Pairs of these planes intersect in singular lines and triples intersect in singular points. Singular lines occur in three classes: Intersections of 5 planes simultaneously at a line (class I - 6 possible orientations): Intersections of 3 planes only (class II - 15 possible orientations): Intersections of 2 planes only (class III - 10 possible orientations). Each of the class I and III orientations is a single orbit, but each of the class II orientations splits into two orbits. Thus if we write $I_1 = I_1^{\mathrm{I}} \cup I_1^{\mathrm{II}} \cup I_1^{\mathrm{III}}$, a disjoint union according to the class, we have $|I_1^{\mathrm{I}}| = 6$, $|I_1^{\mathrm{II}}| = 2 \times 15$, and $|I_1^{\mathrm{III}}| = 10$. In particular, we have $L_1 = 46$ orbits of singular lines, arranged in 31 possible orientations. A similar calculation finds $L_0 = 32$ orbits of singular points formed at the intersection of three or more singular planes: Briefly, every singular point lies at the intersection of some orthogonal triple of planes. Given a particular othogonal triple, those points which are at the intersection of three planes in this orientation split into 8 orbits (because intersections of orthogonal planes give class II lines). However, two of those orbits are found at intersections of planes parallel to every other orientation (e.g. the orbit of the origin), and the remaining 6 are unique to the particular orthogonal triple chosen. This adds up to $6 \times 5 + 2 = 32$ orbits therefore.

Singular subspaces in W_i, $i \in \mathcal{P}_2$: Since all W_i lie in one orbit of the semidirect product of $\pi^{\perp}(\mathbb{Z}^6)$ with the icosahedral group it doesn't matter which i to take. Within a singular plane W_i there are 2 directions of each class of lines, but each of the class II lines is in two orbits. Thus if we denote the class of i in I_2 by α and decompose $I_1^{\alpha} = I_1^{\alpha\,\mathrm{I}} \cup I_1^{\alpha\,\mathrm{II}} \cup I_1^{\alpha\,\mathrm{III}}$ disjointly according to the classes of its singular lines we have $|I_1^{\alpha\,\mathrm{I}}| = 2$, $|I_1^{\alpha\,\mathrm{II}}| = 2 \times 2$, and $|I_1^{\alpha\,\mathrm{III}}| = 2$, giving a total of $L_1^{\alpha} = 8$ orbits of singular lines in 6 possible directions. In particular $\tilde{L}_1 = 15 \times 8 - 46 = 74$. A careful calculation finds the singular points arranged into $L_0^{\alpha} = 8$ orbits.

Singular subspaces in $W_{\hat{\Theta}}$, $\hat{\Theta} \in \mathcal{P}_1$: In each class I and class III line we have exactly two orbits of singular points. In each class II line we have exactly 4 orbits. Hence $L_0^{\Theta} = 2$ if Θ, the class of $\hat{\Theta}$ in I_1, is a class I or class III line whereas otherwise $L_0^{\Theta} = 4$. This gives $\sum_{\Theta \in I_1} L_0^{\Theta} = 2 \times 6 + 4 \times 30 + 2 \times 10 = 152$ and $\sum_{\Theta \in I_1^{\alpha}} L_0^{\Theta} = 2 \times 2 + 4 \times 4 + 2 \times 2 = 24$. Altogether $e = 120$.

To find the ranks R_1, R_2, a more detailed analysis of the generators of the stabilizers and their inner products must be made. Since this involves finding the rank of matrices of up to 15×15 in size, it is best checked by computer: $R_1 = 69$ and $R_2 = 9$.

Specializing Theorem 2.3 to $\dim V = 3$ and $\operatorname{rk} \Gamma = 6$ we obtain as the only nonzero ranks

$$D_3 = 1$$

$$D_2 = 6 + L_2 - R_2$$

$$D_1 = 15 + 4L_2 + \tilde{L}_1 - R_1 - R_2$$

$$D_0 = 10 + 3L_2 + \tilde{L}_1 + e - R_1.$$

This yields the following numbers for the ranks of the homology groups:

$$D_0 = 180, \quad D_1 = 71, \quad D_2 = 12, \quad D_3 = 1.$$

So to conclude we have:

$$K_0(C^*(\mathcal{G}\mathcal{T})) \cong \mathbb{Z}^{192} \qquad K_1(C^*(\mathcal{G}\mathcal{T})) \cong \mathbb{Z}^{72}$$

for the Ammann-Kramer tiling $\mathcal{T}$. We are grateful to Franz Gähler for assistence in completing the last step on the computer and for verifying these results generally.

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