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Cohomology of Canonical Projection Tilings

A. H. Forrest[1], **J. R. Hunton**[2], **J. Kellendonk**[3]

[1] IMF, NTNU Lade, 7034 Trondheim, Norway. E-mail: alanf@matstat.unit.no
[2] The Department of Mathematics and Computer Science, University of Leicester, University Road, Leicester, LE1 7RH, England. E-mail: jrh7@mcs.le.ac.uk
[3] Fachbereich Mathematik, Sekr. MA 7-2, Technische Universität Berlin, 10623 Berlin, Germany. E-mail: kellen@math.tu-berlin.de

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Abstract: We define the cohomology of a tiling as the cocycle cohomology of its associated groupoid and consider this cohomology for the class of tilings which are obtained from a higher dimensional lattice by the canonical projection method in Schlottmann's formulation. We prove the cohomology to be equivalent to a certain cohomology of the lattice. We discuss one of its qualitative features, namely that it provides a topological obstruction for a generic tiling to be substitutional. We develop and demonstrate techniques for the computation of cohomology for tilings of codimension smaller than or equal to 2, presenting explicit formulae. These in turn give computations for the K-theory of certain associated non-commutative C^* algebras.

Introduction

Quasiperiodic tilings have become an active area of research in solid state physics due to their role in modeling quasicrystals [1–4], and the projection method in its various formulations [5–8] is one of the most common techniques to construct candidates for such tilings. This raises the question of characterization and even classification of such tilings. For that to be investigated one must first decide which properties of a tiling are essential for the physical properties of the solid. We take the point of view here that it is only the local structure of the tiling that matters, and even more, only its topological content, as captured, for example, by the continuous hull [22, 23] or the tiling groupoid [15, 10]. According to this point of view the tight binding model for particle motion in the tiling is not uniquely determined by the tiling but its form is constrained by the topology of the tiling, i.e. the Hamiltonian reflects the long range order of the tiling (though additional information is required to specify the interaction strengths, etc.). Our interest is thus in the topological invariants of tilings, in particular here with the cohomology and K-theory of the tiling groupoid.

Without additional mathematical structure of the tiling it is not clear how to obtain explicit results for its cohomology. Substitution tilings provide a class of tilings where such results can be obtained [9, 10] since they possess a symmetry which relates different scales. The present article is part of a programme to compute the tiling cohomology of projection tilings, those which may be obtained by projection from higher dimensional lattices. We consider here projection tilings defined by Laguerre complexes after Schlottmann [20]; see Definition 20 and the notation at the start of Sect. 3.1 for a precise description of the class of tilings considered. We present both qualitative and quantitative results.

Our qualitative results centre around giving sufficient conditions under which a rational version of the cohomology is infinitely generated. These conditions are in some sense almost always met and since the rational cohomology of substitution tilings is finitely generated we can conclude, Corollary 55, that canonical projection tilings are rarely substitutional. We cannot as yet offer an interpretation of the fact that some tilings produce finitely generated cohomology whereas others do not, but, if understood, it could well lead to a criterion to single out a subset of tilings relevant for quasicrystal physics from the vast set of tilings which may be obtained from the canonical projection method. In this context we point out that no canonical projection tiling is known to us which has infinitely generated cohomology but allows for local matching rules, cf. [11].

Our quantitative results are restricted to canonical projection tilings with small codimension (i.e. small difference between the rank of the projected lattice and the dimension of the tiling). We give closed formulæ, Theorems 63, 64 for the cohomology of such tilings in terms of the defining projection data. Formulæ for tilings of higher codimension can in principle be derived using more sophisticated tools from algebraic topology, along the lines of the methods employed at the end of [19]. As tilings obtained by the projection method belong to a large class of tilings whose cohomology is isomorphic to the (unordered) K-theory of the associated groupoid-C^* algebra [12], we also have explicit calculations for the K-theory of these algebras, Corollary 66. This (non-commutative) aspect of the topology of tilings has a direct interpretation in physics. The C^* algebra is the algebra of observables for particles moving in the tiling and its ordered K_0-group (or its image on a tracial state) may serve to “count” (or label) the possible gaps in the spectrum of the Hamilton operator which describes its motion [13–15]. In this context it is even more challenging to find an interpretation of the generators of the K_0-group when there are infinitely many. At first sight, all but finitely many of them appear to be infinitesimal.

This article has some parallels with the series [16–18] (see also [19]). Here however we study tilings as defined by Schlottmann’s variant of the projection method [20]; the calculations we present are consequently applicable to a wider class of tilings than those considered in [18] or at the end of [19].

The article is organized as follows. We describe the continuous dynamical system which can be assigned to any reasonable tiling in Sect. 1. Its associated transformation groupoid has orbits homeomorphic to the space in which the tiling is embedded. We derive the tiling groupoid as a reduction of this groupoid in Sect. 2; it is an r-discrete groupoid and we define tiling cohomology to be the cohomology of this groupoid. Again, this can be done for arbitrary tilings but one of the main features of the particular canonical projection tilings we consider, which make a computation of the cohomology feasible, is that one can find a $\mathbb{Z}^d$ Cantor dynamical system whose associated transformation groupoid is continuously similar to the tiling groupoid. This material is covered in Sect. 3 where we define precisely the class of tilings for which we obtain our results. This observation allows the tiling cohomology to be formulated in terms of group cohomology. In this part our work parallels that of Bellissard et al. [21] on the K-theoretic level. After two illustrative examples in Sect. 4 we discuss our qualitative results in Sect. 5 and the quantitative results in Sect. 6. In Sect. 7 we present the connection with K-theory and the non-commutative topological approach.

1. Continuous Tiling Dynamical Systems

In this section we set up some preliminary notions and definitions with the main aim being to introduce and begin to describe the *continuous hull* $M\mathcal{T}$, Definition 2, of a tiling $\mathcal{T}$.

In fact, this idea is not particular to the projection method tilings considered in the main work of this paper and in this section our definitions and results apply to a wide class of patterns. We specialise to the canonical projection tilings in Sect. 3.1 where we formally define this class.

In general, a d-dimensional tiling is a covering of $\mathbb{R}^d$ by closed subsets, called its tiles, which overlap at most at their boundaries and are usually subject to various other constraints, as for example being connected, uniformly bounded in size and the closures of their interiors; they may also be decorated. For this article though we shall assume that the tiles are (possibly decorated) polytopes with non-empty interiors and which touch face to face. Moreover, we require that the tilings are of *finite type*, see Definition 3.

Given a tiling $\mathcal{T}$ of $\mathbb{R}^d$, then $\mathbb{R}^d$ acts naturally on it by translation. Denote the tiling translated by x as $\mathcal{T} - x$. The closure of the orbit $\mathcal{T} - \mathbb{R}^d$ of $\mathcal{T}$ with respect to an appropriate metric gives rise to a dynamical system [22] whose underlying space is the continuous hull of $\mathcal{T}$. Thus our precise definition of the continuous hull will follow when we have chosen our metric.

There are several proposals for the metric used which are all based on comparing patches around the origin of $\mathbb{R}^d$. The basic idea is as follows. Represent a tiling $\mathcal{T}$ as a closed subset of $\mathbb{R}^d$ by the boundaries of its tiles and its decorations (if any) by small compact sets. Let B_r be the open ball of radius r around $0 \in \mathbb{R}^d$ and let $B_r(\mathcal{T}) := (B_r \cap \mathcal{T}) \cup \partial B_r$, a closed set. Two tilings, $\mathcal{T}$ and $\mathcal{T}'$, should be close to each other if $B_r(\mathcal{T})$ and $B_r(\mathcal{T}')$ coincide, possibly up to a small discrepancy, for large r. The different ways to quantify the allowed discrepancy lead to the different spaces which may be found in the literature.

Definition 1. *For tilings $\mathcal{T}$ and $\mathcal{T}'$ as above, define metrics D_0 and D by*

$$D_0(\mathcal{T}, \mathcal{T}') = \inf \left\{ \tfrac{1}{r+1} \mid B_r(\mathcal{T}) = B_r(\mathcal{T}') \right\},$$
$$D(\mathcal{T}, \mathcal{T}') = \inf \left\{ \tfrac{1}{r+1} \mid d_r(B_r(\mathcal{T}), B_r(\mathcal{T}')) < \tfrac{1}{r} \right\},$$

where d_r is the Hausdorff metric defined among closed subsets of the closed r-ball.

The first metric, D_0, allows no discrepancy; the completion of the $\mathbb{R}^d$ orbit of $\mathcal{T}$ under this metric would be non-compact. However, completion with respect to the metric D yields a compact space under very general conditions [22, 23]. Note also that D is not invariant under the action of $\mathbb{R}^d$ by translation, but this action is nevertheless uniformly continuous and can thus be extended to the completion.

Definition 2. *The* **continuous dynamical system associated to** $\mathcal{T}$ *is the pair $(M\mathcal{T}, \mathbb{R}^d)$, the closure $M\mathcal{T}$ of the orbit of $\mathcal{T}$ with respect to the metric D, and with the action of $\mathbb{R}^d$ induced by translation. Call $M\mathcal{T}$ the* **continuous hull** *of $\mathcal{T}$.*

Let $M_r(\mathcal{T})$ be the subset of (whole) tiles of $\mathcal{T}$ contained in B_r. As for $\mathcal{T}$, think of $M_r(\mathcal{T})$ as the closed subset defined by the boundaries and decorations of its tiles.

Definition 3. *A tiling $\mathcal{T}$ is called of* **finite type** *(or of* **finite pattern type**, *or of* **finite local complexity***) if for all r the set of translational congruence classes of sets $M_r(\mathcal{T} - x)$, $x \in \mathbb{R}^d$, is finite.*

The elements of the space $M\mathcal{T}$ may again be interpreted as tilings. While we continue to write $\mathcal{T}$ for the original tiling, we write T for a general element of $M\mathcal{T}$. If $\mathcal{T}$ is of finite type the elements $T \in M\mathcal{T}$ are those tilings in which each finite part can be identified with a finite part of a translate of $\mathcal{T}$. Thus, for each $T \in M\mathcal{T}$ and for each r, there exists an $x \in \mathbb{R}^d$ such that $B_r(T) = B_r(\mathcal{T} - x)$.

Definition 4. *Two tilings $\mathcal{T}, \mathcal{T}'$ are called* **locally isomorphic** *if for every r there exist $x, x' \in \mathbb{R}^d$ such that $B_r(\mathcal{T}) = B_r(\mathcal{T}' - x')$ and $B_r(\mathcal{T}') = B_r(\mathcal{T} - x)$. If every element of $M\mathcal{T}$ is locally isomorphic to $\mathcal{T}$ then $\mathcal{T}$ is called* **minimal**.

The tilings we are interested in here are all minimal. Note that a tiling being minimal directly implies that each orbit of the associated dynamical system is dense.

Finally, we have a third option for a metric on the orbit of $\mathcal{T}$, linking the spaces considered here with the work of [9]. The following metric defines the same topology as the metric considered there.

Definition 5. *Define the metric D_t by*

$$D_t(\mathcal{T}, \mathcal{T}') := \inf\{\tfrac{1}{r+1} \mid \exists x, x' \in B_{\frac{1}{2r}} : B_r(\mathcal{T} - x) = B_r(\mathcal{T}' - x')\}.$$

In this metric discrepancy is allowed only for small translations. As soon as two tilings differ by a rotation, however small, they will have a certain minimal non-zero distance. Thus closure with respect to D_t leads, for instance for the Pinwheel tilings [24], to a non-compact space, whereas closure with respect to D would still lead to a compact space.

Which kind of metric is to be used has, of course, to be adapted to the problem, but for our purposes the following result shows that the distinction between D and D_t is inessential.

Theorem 6. *Let $\mathcal{T}$ be a finite type tiling. Then $M\mathcal{T}$ is compact and equal to the completion of $\mathcal{T} - \mathbb{R}^d$ with respect to D_t.*

Proof. We start by showing that the two metrics D and D_t yield the same completion for finite type tilings. Clearly $D(T, T') \leq D_t(T, T')$ so we have to show that any D-Cauchy sequence is also a D_t-Cauchy sequence.

Suppose that $(T_i)_i$ is a D-Cauchy sequence converging to $T \in M\mathcal{T}$. Then for any r, $d_r(M_r(T_i), M_r(T)) \xrightarrow{i\to\infty} 0$. As $\mathcal{T}$ is a finite type tiling, we can find for all i which are larger than some i_0 an ϵ_i such that $M_r(T_i) = M_r(T) - \epsilon_i$ and $\epsilon_i \xrightarrow{i\to\infty} 0$. But then $B_{r-c}(T_i) = B_{r-c}(T - \epsilon_i)$, where c is an upper bound on the diameter of the tiles. Now choose i_r such that $\epsilon_{i_r} \leq 1/r$. Then, for any r, $D_t(T, T_{i_r}) \leq 1/(r+1)$. Thus a D-Cauchy sequence will also be a D_t-Cauchy sequence. In particular $M\mathcal{T}$ is equal to the completion of $\mathcal{T} - \mathbb{R}^d$ with respect to D_t. Its compactness for finite type tilings is well known, see, for example, [23]. $\square$

This result allows us to identify the open sets in $M\mathcal{T}$.

Definition 7. *Say that a finite subset P of tiles of a tiling T is a* **patch** *(or* **pattern**, *or* **cluster***) of it and write $P \subset T$. Define*

$$\mathcal{U}_P := \{T \in M\mathcal{T} | P \subset T\},$$

subsets of the continuous hull.

Theorem 8. *The collection of sets $\{B_\epsilon + x + \mathcal{U}_P\}$, $\epsilon > 0$, $x \in \mathbb{R}^d$, P a patch of $\mathcal{T}$, is a base for the topology of $M\mathcal{T}$.*

Proof. The previous result allows us to work with the metric D_t. Let $r(\epsilon) := \frac{1-\epsilon}{\epsilon}$ and $\mathcal{V}_r(T) = \{T' \in M\mathcal{T} | B_r(T) = B_r(T')\}$. Then we can describe the ϵ-neighbourhoods of T with respect to D_t as follows.

$$\begin{aligned} D_t(T, T') < \epsilon \text{ iff } \exists r > r(\epsilon)\ \exists x, x' \in B_{\frac{1}{2r}} : B_r(T - x) = B_r(T' - x') \\ \text{iff } T' \in \bigcup_{r > r(\epsilon)} \bigcup_{x \in B_{\frac{1}{2r}}} \left(B_{\frac{1}{2r}} + \mathcal{V}_r(T - x) \right). \end{aligned} \tag{1}$$

The tiling being of finite type implies that, for every $r > 0$ and every $T \in M\mathcal{T}$, there exists a finite set of pairs (x_i, P_i), $x_i \in \mathbb{R}^d$, P_i a patch of $\mathcal{T}$, such that $B_r(T') = B_r(T)$ whenever there is an i such that $P_i + x_i$ is a patch of T'. In other words, $\mathcal{V}_r(T) = \bigcup_i \mathcal{U}_{P_i + x_i}$. This shows that (1) is a union of sets of the above collection.

To show that $B_\epsilon + \mathcal{U}_P$ is open in the metric topology (which by continuity of the action implies that also $B_\epsilon + x + \mathcal{U}_P$ is open for $x \in \mathbb{R}^d$) we take a point T in it and show that a whole neighbourhood (with respect to D_t) of it lies in $B_\epsilon + \mathcal{U}_P$. Let R be large enough so that $\frac{1}{R} < \epsilon$ and P is a patch of $B_{R - \frac{1}{2R}}(T)$(we view here P as a closed subset much like a tiling). Then, for all $x \in B_{\frac{1}{2R}}$, $P \subset B_R(T - x) + x$ and hence $\mathcal{V}_R(T - x) \subset \mathcal{U}_P - x$. This implies that the $\frac{1}{R+1}$-neighbourhood of T lies in $B_\epsilon + \mathcal{U}_P$. □

The following observation will be useful in Sect. 3.2

Lemma 9. *Let P be a patch in a finite type tiling $\mathcal{T}$. Then $\mathcal{U}_P$ is compact.*

Proof. If $D(T, T')$ is small enough, and $T, T' \in \mathcal{U}_P$, then it is equal to $D_0(T, T')$. That $\mathcal{U}_P$ is complete and precompact with respect to the D_0-metric is proven in [15]. □

2. The Groupoid Approach to Tilings

To a given tiling one may associate an r-discrete groupoid called the *tiling groupoid*. This groupoid is special among other groupoids which may be assigned to the tiling in that its C^* algebra plays the role of the algebra of observables for particles moving in the tiling [15, 10]. It also determines the tiling up to topological equivalence [25]. The K-theory of the C^* algebra and the cohomology of the groupoid are – at least for canonical projection tilings – closely related, and may be considered as (non-commutative) invariants of the tiling. It is these invariants we discuss in this paper. We define the tiling groupoid in Sect. 2.2, but first we need to briefly recall some facts about groupoids.

2.1. Generalities. For a traditional definition of a topological groupoid, and as a general reference for most of the concepts introduced below like that of reduction, continuous similarity and continuous cocycle cohomology, we refer the reader to [26].

In a slightly different but equivalent way one may say that a groupoid $\mathcal{G}$ is a set with partially defined associative, cancellative multiplication and with unique inverses. *Partially defined* refers to the fact that multiplication is not defined for all elements, but only for a subset of $\mathcal{G} \times \mathcal{G}$, the *composable* elements. An *inverse* of x is a solution y of the equations $xyx = x$ and $yxy = y$, and for a groupoid this solution is required to be unique. Hence we may denote the inverse of x by x^{-1}. The inverse map $x \mapsto x^{-1}$ is an involution. Multiplication is *cancellative* if, provided it is defined, $xy = xz$ implies $y = z$, and this is the case whenever the composable elements are the pairs (x, y) for which $x^{-1}x = yy^{-1}$.

The set $\mathcal{G}^0 = \{xx^{-1} | x \in \mathcal{G}\}$ is called the set of *units*; it is the image of the map $r : \mathcal{G} \to \mathcal{G}^0$ given by $r(x) = xx^{-1}$, which is called the *range* map. The map $s : \mathcal{G} \to \mathcal{G}^0$ given by $s(x) = x^{-1}x = r(x^{-1})$ is called the *source* map. Writing $u \sim v$ for $u, v \in \mathcal{G}^0$ whenever $r^{-1}(u) \cap s^{-1}(v) \neq \emptyset$ defines an equivalence relation; its equivalence classes are called the *orbits* of $\mathcal{G}$.

A *topological* groupoid is a groupoid with a topology with respect to which multiplication and inversion are continuous maps. Such a groupoid is called *r-discrete* if $\mathcal{G}^0$ is an open subset. This condition implies that $r^{-1}(u)$ is a discrete set for any unit u.

A groupoid is called *principal* if its elements are uniquely determined by their range and source, i.e. if the map $\mathcal{G} \to \mathcal{G}^0 \times \mathcal{G}^0$ given by $x \mapsto (r(x), s(x))$ is injective.

2.1.1. Transformation groupoids. Let M be a topological space with a right action of a topological group G by homeomorphisms, denoted here $(x, g) \mapsto x \cdot g$. The *transformation groupoid*[1] $\mathcal{G}(M, G)$ is the topological space $M \times G$ with product topology; two elements (x, g) and (x', g') are composable provided that $x' = x \cdot g$, and their product is then $(x, g)(x', g') = (x, gg')$. Inversion is then given by $(x, g)^{-1} = (x \cdot g, g^{-1})$. Hence, $r(x, g) = (x, 0)$ and we see that $\mathcal{G}(M, G)$ is r-discrete if G is discrete. Furthermore, $\mathcal{G}(M, G)$ is principal precisely when G acts fixed point freely. One of the examples we have in mind here is $\mathcal{G}(M\mathcal{T}, \mathbb{R}^d)$ which, however, is not r-discrete.

2.1.2. Reductions.

Definition 10. *Let $\mathcal{G}$ be a groupoid, $\mathcal{G}^0$ its unit space and L a closed subset of $\mathcal{G}^0$. Then ${}_L\mathcal{G}_L := s^{-1}(L) \cap r^{-1}(L)$ is a closed subgroupoid of $\mathcal{G}$ called the* **reduction of $\mathcal{G}$ to L**.

Two further conditions on L will play a major role here.

- A reduction is called *regular* if every orbit of $\mathcal{G}$ has a non-empty intersection with L.
- Say that L is *range-open* [16] if the set $r(s^{-1}(L) \cap U)$ is open whenever $U \subset \mathcal{G}$ is open.

A regular reduction of a groupoid $\mathcal{G}$ to a range-open subset L is for many purposes as good as the groupoid itself. Muhly et al. have established a notion of equivalence between groupoids which captures this phenomenon in greater generality [27]. We will not discuss this notion of equivalence here, we merely record its main consequence of interest to us: The K-groups of the C^* algebras associated to a groupoid $\mathcal{G}$ and its reduction ${}_L\mathcal{G}_L$ to a range open subset L which intersects each orbit are isomorphic as ordered groups.

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[1] or transformation group as in [26]

2.1.3. Continuous similarity. As just noted, the concept of reduction is particularly well adapted to yield an equivalence relation on groupoids which carries over to an equivalence relation on the C^* algebras they define. It turns out that for canonical projection tilings the K-groups of the C^* algebras are related to the cohomology of the groupoids, as discussed further in Sect. 7, but this relation is not clear on the level of arbitrary tiling-groupoids. On the other hand there is a natural equivalence relation on groupoids, that of *continuous similarity*, which immediately gives rise to an equality on cohomology as well as implying equivalence in the sense of Muhly et al. [28].

Definition 11. *Two homomorphisms ϕ and $\psi : \mathcal{G} \to \mathcal{R}$ between (topological) groupoids are (continuously) similar if there exists a function $\Theta : \mathcal{G}^0 \to \mathcal{R}$ such that*

$$\Theta(r(x))\phi(x) = \psi(x)\Theta(s(x)). \tag{2}$$

Two (topological) groupoids, $\mathcal{G}$ and $\mathcal{R}$, are called (continuously) similar if there exist homomorphisms $\phi : \mathcal{G} \to \mathcal{R}$, $\phi' : \mathcal{R} \to \mathcal{G}$ such that $\Phi_{\mathcal{G}} = \phi' \circ \phi$ is (continuously) similar to $\mathrm{id}_{\mathcal{G}}$ and $\Phi_{\mathcal{R}} = \phi \circ \phi'$ is (continuously) similar to $\mathrm{id}_{\mathcal{R}}$.

We are mainly interested in establishing continuous similarity of certain principal transformation groupoids. A useful lemma to test this is proved in [17, (3.3, 3.4)].

Proposition 12. *Let $\mathcal{G} = \mathcal{G}(X, G)$ be a principal transformation groupoid (so G acts freely on X) and L and L' closed subsets of $X \cong \mathcal{G}^0$. Suppose that $\gamma : L' \to G$ and $\gamma' : L \to G$ are two continuous functions which define continuous functions $L \to L'$: $x \mapsto x \cdot \gamma'(x)$ and $L' \to L$: $x \mapsto x \cdot \gamma(x)$. Then the reductions of $\mathcal{G}$ to L and to L' are continuously similar.*

Remark 13. If $L' = X$ then one can take $\gamma'(x)$ to be the identity in the group for all $x \in L$ and the condition comes down to finding a continuous function $\gamma : X \to G$ such that $x \cdot \gamma(x) \in L$ for all $x \in L$.

2.1.4. Continuous cocycle cohomology. Given a dynamical system (M, G) with discrete group G one standard topological invariant associated with it is the cohomology of G with coefficients in the G-module $C(M, \mathbb{Z})$ of integer-valued continuous functions with G action given by $(g \cdot f)(m) = f(m \cdot g)$. This cohomology may be interpreted as a groupoid cohomology of the groupoid $\mathcal{G}(M, G)$. This is the *continuous cocycle cohomology* for r-discrete groupoids and we will recall its definition here for constant coefficients following [26].

Let A be an abelian group and $\mathcal{G}$ be a groupoid. Then $\mathcal{G}$ acts on the trivial A-bundle $\mathcal{G}^0 \times A \xrightarrow{\rho} \mathcal{G}^0$ (with product topology) partially, namely $x \in \mathcal{G}$ can act on the element $(s(x), a)$ mapping it to $(r(x), a)$. We denote this action by Φ, writing the partial map given by $x \in \mathcal{G}$ as $\Phi(x)$. The action is continuous in the sense that when $f \in C(\mathcal{G}^0, A)$ is a continuous section of the bundle then the function $x \mapsto (r(x), f(s(x)))$ is continuous too.

Let $\mathcal{G}^{(0)} = \mathcal{G}^0$, and, for $n > 0$, let $\mathcal{G}^{(n)}$ be the subset of the n-fold Cartesian product of $\mathcal{G}$ (with relative topology) consisting of composable elements $(x_1, \ldots, x_n)$, that is, with $r(x_i) = s(x_{i-1})$. The n-cochains are the continuous functions $f : \mathcal{G}^{(n)} \to \mathcal{G}^0 \times A$ such that $\rho(f(x_1, \ldots, x_n)) = r(x_1)$ and, for $n > 0$, $f(x_1, \ldots, x_n) = (r(x_1), 0)$ provided one of the x_i is a unit. The n-cochains form an abelian group under pointwise addition. The coboundary operator δ^n is defined as

$$\delta^0(f)(x) = \Phi(x)f(s(x)) - f(r(x)),$$

and, for $n > 0$,

$$\begin{aligned}\delta^n(f)(x_0,\ldots,x_n) = {} & \Phi(x_0)(f(x_1,\ldots,x_n)) \\ & + \sum_{i=1}^{n}(-1)^i f(x_0,\ldots,x_{i-1}x_i,\cdots,x_n) \\ & +(-1)^{n+1} f(x_0,\ldots,x_{n-1}).\end{aligned}$$

Then $H^n(\mathcal{G}, A)$, the continuous cocycle cohomology of $\mathcal{G}$ in dimension n with (constant) coefficients A, is defined as $\ker \delta^n / \mathrm{im}\delta^{n-1}$. The following result is proved in [26].

Theorem 14. *Continuously similar groupoids have isomorphic cohomology with constant coefficients.*

Let us consider a transformation groupoid $\mathcal{G}(M, G)$ as an example (G discrete). In that case the n-cochains are maps $f\colon M \times G^n \to M \times A$ of the form

$$f(m, g_1,\ldots,g_n) = (m, \tilde{f}(g_1,\ldots,g_n)(m)),$$

where $\tilde{f}\colon G^n \to C(M, A)$ is a continuous map which, for $n > 0$, is the zero map when applied to $(g_1,\ldots,g_n)$ with any one $g_i = e$, the identity element in G. These are precisely the n-cochains of the group G with coefficients in $C(M, A)$, a G module with respect to the action $(g \cdot f)(m) = f(m \cdot g)$ [29]. Hence every n-cochain of the groupoid with coefficients in A determines an n-cochain of the group G with coefficients in $C(M, A)$, and vice versa. Moreover, under this identification δ^n becomes the usual coboundary operator in group cohomology, since the groupoid action is nothing other than the shift of base point given by the action of G.

Corollary 15. *There is a natural isomorphism between the continuous cocycle cohomology of the transformation groupoid $\mathcal{G}(M, G)$ with constant coefficients A and the group cohomology of G with coefficients in $C(M, A)$,*

$$H^n(\mathcal{G}(M, G), A) \cong H^n(G, C(M, A)).$$

In the main results of this paper we shall be interested in the cases $A = \mathbb{Z}$ and $A = \mathbb{Q}$.

2.2. The tiling groupoid. The *tiling groupoid* may be defined without referring to continuous tiling dynamical systems, as for example in [15,10], but for the purpose of the present work it is important to draw the connection [13,9]. Starting with the groupoid of the continuous tiling dynamical system $\mathcal{G}(M\mathcal{T}, \mathbb{R}^d)$ we construct the tiling groupoid as a reduction of it.

We first construct a closed, range-open subset $\Omega_{\mathcal{T}}$ of $M\mathcal{T}$. Choose a point in the interior of each tile of $\mathcal{T}$ – called its *puncture* – in such a way that translationally congruent tiles have their puncture at the same position. Let $\Omega_{\mathcal{T}}$ be the subset of tilings of $M\mathcal{T}$ for which a puncture of one of its tiles coincides with the origin $0 \in \mathbb{R}^d$. Note that $\Omega_{\mathcal{T}}$ intersects each orbit of $\mathbb{R}^d$.

Definition 16. *The* **tiling groupoid** *of $\mathcal{T}$, denoted by $\mathcal{G}_{\mathcal{T}}$, is the reduction of $\mathcal{G}(M\mathcal{T}, \mathbb{R}^d)$ to $\Omega_{\mathcal{T}}$. Note that, by construction, $\mathcal{G}_{\mathcal{T}}$ is r-discrete.*

Proposition 17. *Suppose $\Omega_{\mathcal{T}}$ contains only non-periodic, finite type tilings. Then $\Omega_{\mathcal{T}}$ is closed and range-open and $\mathcal{G}_{\mathcal{T}}$ coincides with the groupoid $\mathcal{R}$ defined in [15].*

Proof. We refer to [10] for the groupoid $\mathcal{R}$ and its properties. Under the hypothesis $\mathbb{R}^d$ acts fixed point freely on $M\mathcal{T}$ and hence $\mathcal{G}_\mathcal{T}$ is principal. Therefore the map between $\mathcal{G}_\mathcal{T}$ and $\mathcal{R}$ is given by $(T, x) \mapsto (T, T - x)$, which certainly preserves multiplication and inversion, is an isomorphism provided it preserves the topology. The tiling being of finite type implies that punctures of two different tiles have a minimal distance, δ say. Thus there exists an ϵ (which is roughly as large as δ) such that if $D(\mathcal{T}-x, \mathcal{T}-x') < \epsilon$ and $\mathcal{T} - x, \mathcal{T} - x' \in \Omega_\mathcal{T}$, then $D(\mathcal{T} - x, \mathcal{T} - x') = D_0(\mathcal{T} - x, \mathcal{T} - x')$. It follows that $\Omega_\mathcal{T}$ is the metric completion with respect to D_0 of the set of all $\mathcal{T}' \in \Omega_\mathcal{T}$ which are translates of $\mathcal{T}$. In particular, it is closed and the existence of a minimal distance δ between punctures directly implies range-openness, cf. [16]. Furthermore, the metric D_0 and the metric used in [15] to define the hull lead to the same completions. This shows that the above map $(T, x) \mapsto (T, T - x)$ restricts to a homeomorphism of the spaces of units of $\mathcal{G}_\mathcal{T}$ and of $\mathcal{R}$. As noted, $\mathcal{G}_\mathcal{T}$ is r-discrete and its topology is generated by the sets $U \times \{x\}$, U open in $\Omega_\mathcal{T}$. Images of those sets under the above map generate the topology of $\mathcal{R}$. □

We conclude this section with our basic definition of the *cohomology of a tiling*.

Definition 18. *The* **cohomology of the tiling** $\mathcal{T}$*, denoted by* $H^*(\mathcal{T})$*, is the continuous cocycle cohomology* $H^*(\mathcal{G}_\mathcal{T}, \mathbb{Z})$ *of* $\mathcal{G}_\mathcal{T}$ *with constant coefficients* $\mathbb{Z}$*.*

We shall see later on that for canonical projection tilings, $H(\mathcal{G}_\mathcal{T}, \mathbb{Z})$ is isomorphic to the Czech cohomology of $M\mathcal{T}$. It seems to be an interesting question whether this is true in general.

3. Quasiperiodic Tilings Obtained by Cut and Projection

The *projection method* (or *cut and projection method*) is a well known way of producing quasiperiodic point sets or tilings by projection of a certain subset of a periodic set in a higher dimensional space. In earlier versions, for example [5], the favorite set was the integer lattice $\mathbb{Z}^N$ but a price has to be paid for the simplicity of this choice if the kernel of the projection contains non-zero lattice points. An elegant way around this difficulty, which is applicable to almost all interesting examples, is to use root lattices instead of $\mathbb{Z}^N$ [30] and the construction we use here is related to that.

However, rather than looking at arbitrary point sets obtained by the projection method (for example with fractal acceptance domain) we want to focus in this article on tilings where the acceptance domain is canonical – after all these include the main candidates for the description of quasicrystals – and for these tilings there is another approach which is a bit more elaborate to start with but easier to handle when it comes to the later steps in the construction of the cohomology groups. The approach we are about to describe is based on polyhedral complexes and their dualization, it is therefore sometimes called the *dualization method*, but in the present context where we start with a higher-dimensional periodic set it can be simply considered as a variant of the projection method such as used in [16,17]. We follow its description as in the article by Schlottmann [20] and refer the reader also to the examples discussed in [31].

The organisation of this section is as follows. We formally define the construction considered in 3.1 and discuss some basic properties and examples. The remaining subsections form a sequence of descriptions of the associated hull for such tilings; the final description is the one which allows us to describe the tiling cohomology in the remainder of the paper.

3.1. Projection tilings after Schlottmann. We must first recall and set up some notation to discuss *Laguerre complexes*. Consider a point set W of a euclidean space $\mathcal{E}$ together with a weight function $w\colon W \to \mathbb{R}$ on it; write Θ for the pair (W, w). For $q \in W$, the set

$$L_\Theta(q) := \{x \in \mathcal{E} | \forall q' \in W : |x-q|^2 - w(q) \le |x-q'|^2 - w(q')\} \tag{3}$$

is called the *Laguerre domain* of q. It is convex and under rather weak conditions [20] on Θ all Laguerre domains are actually compact polytopes (of dimension smaller or equal to that of $\mathcal{E}$ or even empty sets) and the set of all Laguerre domains with non-empty interior provides the tiles of a tiling $T\Theta$ which is of finite type and face to face. Laguerre domains generalise the notion of Voronoi domains and specialise to them when the weight function is constant. The concept of Voronoi domains is a familiar one in solid state physics where they arise (under the name Brouillon zone or Wigner-Seitz cell) if one takes as W the dual of the crystal lattice. A non-constant weight function gives the means to enlarge certain Laguerre domains at the cost of others or even to surpress some altogether.

The faces of the Laguerre domains define a cell complex structure: this is the so-called *Laguerre complex*. We denote it by $\mathcal{L}_\Theta$ and the (closed) cells of dimension k by $\mathcal{L}_\Theta^{(k)}$. The data Θ specify another complex which is dual to $\mathcal{L}_\Theta$: the dual ξ^* of a k-cell ξ is the convex hull of the set of $q \in W$ whose corresponding Laguerre domains contain ξ as a face. Note that ξ^* depends on ξ and Θ and not only on ξ and $\mathcal{L}_\Theta$. It has codimension k. This dual complex is again a Laguerre complex, denoted $\mathcal{L}_{\Theta^*}$ for $\Theta^* = (W^*, w^*)$, where W^* is the set of vertices (0-cells) of $\mathcal{L}_\Theta$ and $w^* : W^* \to \mathbb{R}$ is given by $w^*(q^*) = |q^* - q|^2 - w(q)$ for some q such that q^* is a vertex of $L_\Theta(q)$. In particular, Θ^* also defines a tiling with the above properties.

We can now describe the projection method construction we shall study. Let $\Gamma \in \mathcal{E}$ be a lattice whose generators form a base for $\mathcal{E}$, let W be a finite union of Γ-orbits of points in $\mathcal{E}$, and let $w : W \to \mathbb{R}$ be a Γ-periodic function. Now let $E \subset \mathcal{E}$ be a linear affine subspace and let $\pi : \mathcal{E} \to E$ be the orthogonal projection. Write d for the dimension of E, $d^\perp$ for that of its orthocomplement $E^\perp$, and $\pi^\perp$ for $1-\pi$. We shall also write $x^\perp$ as shorthand for $\pi^\perp(x)$. An element $u \in \mathcal{E}$ is called *singular* if there is a $\beta \in \mathcal{L}_\Theta^{(d^\perp-1)}$ such that $\pi^\perp(u) \in \pi^\perp(\beta)$. Hence the set of singular points is $S = S^\perp + E$ where

$$S^\perp := \bigcup_{\beta \in \mathcal{L}_\Theta^{(d^\perp-1)}} \pi^\perp(\beta).$$

The set of non-singular points is denoted by NS. We can write it as $NS = \bigcap_{\beta\in\mathcal{L}_\Theta^{(d^\perp-1)}} E + E^\perp \backslash \beta^\perp$ which shows that it is a G_δ set (a countable intersection of open sets). Since $\beta^\perp$ has codimension 1 in $E^\perp$, NS is dense.

It is convenient to write $W_u = W + u$ and $w_u(q+u) = w(q)$ and define Θ_u as (W_u, w_u).

Definition 19. *For data W, w and E as above, each $u \in NS$ defines a tiling T_u whose tiles are the elements of the set*

$$\{\pi(\xi^*) | \xi \in \mathcal{L}_{\Theta_u}^{(d^\perp)}, \xi \cap E \neq \emptyset\}.$$

The dimension of $E^\perp$ is called the **codimension** *of T_u.*

That this is a tiling by Laguerre domains has been shown by Schlottmann [20]. In fact, T_u is the tiling $T(\tilde{W}_u^*, \tilde{w}_u^*)$ defined by the Laguerre-complex dual to $\mathcal{L}_{(\tilde{W}_u, \tilde{w}_u)}$, where $\tilde{W}_u = \pi(W_u)$ and $\tilde{w}_u(\pi(q+u)) = \max\{w(q') - |\pi^\perp(q'+u)|^2 | \pi(q') = \pi(q)\}$ (assuming it exists). Using this description one can see that one loses no generality in restricting to the cases in which $\pi^\perp(\Gamma)$ is dense in $E^\perp$ [20].

Definition 20. *A* **canonical projection tiling** *is a tiling T_u associated to data W, w, E and u as before that satisfies also the conditions*

(a) *that $\pi^\perp(\Gamma)$ lies dense in $E^\perp$;*
(b) *that $E \cap \Gamma = 0$;*
(c) *that up to translation, any $\xi^* \in \mathcal{L}_{\Theta^*}^{(d)}$ is uniquely determined by its projection $\pi(\xi^*)$;*
(d) *that for $\xi^*, \eta^* \in \mathcal{L}_{\Theta^*}^{(d)}$, $\xi^* = \eta^* + x$ implies $x \in \Gamma$.*
(e) *that for all $\beta \in \mathcal{L}_{\Theta}^{(d^\perp - 1)}$, the (affine) hyperplane H_β which is tangent to $\beta^\perp$ is a subset of $S^\perp$.*

Remark 21. Conditions (b),(c),(d) in this definition are not strictly necessary but will considerably simplify the exposition. (b) implies that the tilings are completely non-periodic. (c) and (d) can be made obsolete with the help of decorations, see Sect. 3.2.1. Condition (e) will not be relevant until Subsect. 3.3 and we shall ignore it for the remainder of this and the next subsection.

Example 22. Consider the example $W = \mathbb{Z}^N$, the integer lattice in $\mathbb{R}^N$, with standard basis $\{e_i, i = 1, \dots, N\}$ and vanishing weight function w. In this highly symmetric case, the dual complex to $\mathcal{L}_{\mathbb{Z}^N, w}$ differs only by a shift about $\delta = \frac{1}{2}\sum_i e_i$ from the original one. Writing $\gamma = \{\sum_{i=1}^N c_i e_i | 0 \le c_i \le 1\}$ for the unit cube, its translates by $\delta + z$, $z \in \mathbb{Z}^N$, are its Laguerre domains and it is not difficult to see that, when E is chosen such that $E \cap \mathbb{Z}^N = \{0\}$ the vertices of the tiling T_u defined in Definition 19 are the points

$$\{\pi(z) | z \in (\mathbb{Z}^N + u + \delta) \cap (E + \gamma)\}. \tag{4}$$

This set we referred to in [16] as the canonical projection pattern defined by the data $(\mathbb{Z}^N, E, u')$ with $u' = u + \delta$.

$\pi^\perp(\mathbb{Z}^N)$ lies dense in $E^\perp$ if and only if $E^\perp \cap \mathbb{Z}^N = \{0\}$. In this case one sees quickly that all further conditions of Definition 20 are met.

But $E^\perp \cap \mathbb{Z}^N$ is not always trivial, important examples where it is non-trivial are the Penrose tilings. This is the reason why we consider the apparently more elaborate construction with Laguerre complexes. It allows us to focus our attention to input data which satisfy (a) of Definition 20.

Let D be the real span of $E^\perp \cap \mathbb{Z}^N$ (assuming it is not trivial) and let V be the orthocomplement of D in $E^\perp$. Following [20] we factor the projection $\pi : \mathbb{R}^N \to E$ as $\pi = \pi_2 \circ \pi_1$, where $\pi_1 : \mathcal{E} \to E \oplus V$ is the orthogonal projection with kernel D and $\pi_2 : E \oplus V \to E$ has kernel V. We may then perform the construction of the projection method in two steps. First we produce the (periodic) tiling defined by the data $W = \mathbb{Z}^N$, $w = 0$, the subspace $E \oplus V$ and non-singular point u and using projection π_1. As noted, this tiling can be understood as a Laguerre complex, namely the one defined by the lattice $\pi_1(\mathbb{Z}^N)$ and weight function w given by $w(\pi_1(z)) = \max\{w(z') - |\pi_1^\perp(z' + u)|^2 | \pi_1(z') = \pi_1(z)\}$. In the second step we now use this new Laguerre complex and the projection π_2: to be precise, we use the data $\pi_1(\mathbb{Z}^N), w, E, \pi_1(u)$. Note that w remains zero after the first step if $\pi_1^\perp(u) \in \mathbb{Z}^N$, but, if $\pi_1^\perp(u) \notin \mathbb{Z}^N$, we have to expect that the maximal periodicity lattice of the Laguerre complex defined by $(\pi_1(\mathbb{Z}^N), w)$ is a sublattice of $\pi_1(\mathbb{Z}^N)$ containing the lattice $\mathbb{Z}^N \cap (E \oplus V)$. To summarize, even if $E^\perp \cap \mathbb{Z}^N \neq \{0\}$ we may construct tilings whose vertices are the points of (4) by Schlottmann's method from data which satisfy conditions (a) and (b) of Definition 20. The further conditions, in particular (e), have to be carefully verified.

The most famous class of tilings which may be constructed by the above method are the Penrose tilings. Here $N = 5$, E is a two dimensional invariant subspace of the rotation $e_i \mapsto e_{i+1}$ (i mod 5) and D is the span of δ. If $\pi_1^\perp(u) = -\delta$ then the new Laguerre complex $\mathcal{L}_{\pi_1(\mathbb{Z}^5),w}$ becomes the dual of the Voronoi complex (i.e. the Delaunay complex) of the root lattice A_4 [30]. The resulting tilings are the usual Penrose tilings. Other choices for $\pi_1^\perp(u)$ lead to the so-called generalized Penrose tilings.

We conclude this section by establishing some important properties of canonical projection tilings which will be of use later. First, for non-singular u and v, T_u is locally isomorphic to T_v and to any other element of its hull [20]; in fact, $MT_u = MT_v$ and the dynamical system (MT_u, E) is minimal (any orbit lies dense). We may therefore drop the index u to write $M\mathcal{T}$ for the continuous hull.

Given $u \in \mathcal{E}$ (not necessarily non-singular) we define

$$\tilde{P}_u := \{\xi \in \mathcal{L}^{(d^\perp)}_{\Theta_u} | 0 \in \pi^\perp(\xi)\}.$$

Lemma 23. *Let $\xi \in \tilde{P}_u$, $u \in NS$ and $P = \pi(\xi^*)$.*

1. *If $s \in -\xi^\perp + \Gamma$ such that $u + s \in NS$ then P is a tile of T_{u+s}.*
2. *If $s \in E + \Gamma$ then the converse holds: P being a tile of T_{u+s} implies $s \in -\xi^\perp + \Gamma$.*

Proof. First, let $s \in -\xi^\perp$ such that $u + s \in NS$ and $\xi \in \tilde{P}_u$. Then $\xi + s \in \mathcal{L}^{(d^\perp)}_{\Theta_{u+s}}$ and $0 \in \xi^\perp + s$. Hence $\xi + s \in \tilde{P}_{u+s}$ so that the dual of $\xi + s$ with respect to the data Θ_{u+s} projects (under π) onto a tile of T_{u+s}. This dual is $\xi^* + s$ (where ξ^* is the dual of ξ with respect to Θ_u) and hence projects onto P.

For the second statement split the given $s = s' + \gamma$ with $s' \in E, \gamma \in \Gamma$. Then $\pi(\xi^*) \in T_{u+s}$ whenever $\pi(\xi^*) - s' \in T_u$. Hence there is a $\eta \in \tilde{P}_u$ such that $\pi(\xi^*) - s' = \pi(\eta^*)$. By condition (c) this implies $\exists v \in E^\perp : \xi^* + v - s' = \eta^*$. By condition (d) we must have $v - s' \in \Gamma$. But then $\xi + v - s' = \eta \in \tilde{P}_u$. The latter implies $v \in -\xi^\perp$. The statement follows since $v + \Gamma = s' + \Gamma = s + \Gamma$. □

Lemma 24. *$u \in NS$ whenever $\forall \xi \in \tilde{P}_u : 0 \in Int\, \xi^\perp$.*

Proof. u is singular whenever there is a $\xi \in \mathcal{L}^{(d^\perp)}_{\Theta_u}$ such that $0 \in \partial\xi^\perp$. This ξ then belongs to $\tilde{P}_u$. □

For regular u and a patch P of T_u let

$$A_u(P) = \bigcap_{\xi \in \tilde{P}_u | \pi(\xi^*) \in P} -\xi^\perp.$$

For technical reasons we set $A_u(\emptyset) = E^\perp$. $A_u(P)$ is called the acceptance domain for P, for reasons which become clear in Corollary 26.

Lemma 25. *With the notation above*

1. *For all $u \in NS$ and all $r > 0$ there is a $\delta > 0$ such that $t \in E^\perp$, $u + t \in NS$, $|t| < \delta$ implies $M_r(T_u) = M_r(T_{u+t})$.*
2. *For all $u \in NS$ and all $\epsilon > 0$ there is a $\delta > 0$ such that $|u - v| < \delta$, $v \in NS$, implies $D(T_u, T_v) < \epsilon$.*

Proof. If r is large enough $A_u(M_r(T_u))$ is a finite intersection of convex polytopes. Since u is regular, 0 is an interior point of these polytopes and hence $A_u(M_r(T_u)))$ contains an open δ-neighbourhood of $0 \in E^\perp$. By Lemma 23.1 $|t| < \delta$ implies that $M_r(T_u) \subset T_{u+t}$. Hence $M_r(T_{u+t}) = M_r(T_u)$ which proves the first statement.

As for the second, given u and ϵ let $r > \frac{1}{\epsilon} - c$, where $c - 1$ is an upper bound for the diameter of the tiles. The first statement of the lemma insures the existance of a δ'_ϵ such that $t \in E^\perp$, $u + t \in NS$, $|t| < \delta'_\epsilon$ implies $D(T_u, T_{u+t}) < \epsilon$. Hence if $|u - v| < \delta'_\epsilon$, $v \in NS$, then $D(T_u, T_v) \leq D(T_u, T_{u+\pi^\perp(v-u)}) + D(T_{u+\pi^\perp(v-u)}, T_v) < \epsilon + \delta'_\epsilon$. Taking $\delta = \min\{\delta'_{\frac{\epsilon}{2}}, \frac{\epsilon}{2}\}$ then implies $D(T_v, T_w) < \epsilon$. □

Corollary 26. *Let P be a patch of T_u, $u \in NS$. Then $P \subset T_v$, for $v \in NS$ whenever $v - u \in A_u(P) + \Gamma$.*

Proof. First let $P = \pi(\xi^*)$, $\xi \in \tilde{P}_u$. Then we only have to improve the second part of Lemma 23. Let $r > 0$ such that $P \subset B_r$. Then we find from Lemma 25.1 a δ (depending on v) such that $t \in E^\perp$, $u + t \in NS$, $|t| < \delta$ implies $M_r(T_v) = M_r(T_{v+t})$. Since $E + \Gamma$ lies dense we can find arbitrarily small $t \in E^\perp$ so that $v + t - u \in E + \Gamma$. If $|t| < \delta$ we can combine the above with Lemma 23.2 to obtain that $P \subset T_v$ implies $v \in -\xi^\perp + \Gamma + B_{|t|}$. Since we can choose t arbitrarily small the statement of the corollary follows for $P = \pi(\xi^*)$.

Now the case of a general patch P is a simple consequence of the fact that $P \subset T_v$ whenever all tiles of P belong to T_v. □

Lemma 27. *Let $u \in NS$. Then $A(T_u) := \bigcap_r A_u(M_r(T_u)) = \{0\}$.*

Clearly $A(T_u)$ is convex and closed. If $0 \neq s \in A(T_u)$ then $A(T_u)$ must contain the interval $[0, s]$. Suppose that this is the case. Since the singular points are $\Gamma^\perp$ orbits of boundaries of compact polytopes and since $\Gamma^\perp$ is dense, $u + \mathrm{Int}[0, s]$ must contain a singular point. By convexity of the ξ, $u + [0, s] \in \mathrm{Int}\xi^\perp$ for all $\xi \in \tilde{P}_u$. In particular, $u + t$, $0 < t < s$, is an interior point of all $\xi^\perp$ for which $\xi \in \tilde{P}_{u+t}$. This shows by Lemma 24 that all points in $u + \mathrm{Int}[0, s]$ must be regular. This is a contradiction. □

Proposition 28. *Let $u, v \in NS$. Then $T_u = T_v$ whenever $u - v \in \Gamma$.*

Proof. If $T_u = T_v$ then $M_r(T_u) \subset T_v$ for all r. Hence, by Corollary 26 and Lemma 27 $v - u \in \left(\bigcap_r A_u(M_r(T_u))\right) + \Gamma = \Gamma$. □

3.2. The topology of $M\mathcal{T}$. For canonical projection tilings we have a much better description of the topology of the continuous hull; this is one of the crucial reasons why we can so successfully compute their cohomology.

First we use the tiling metric to define a metric on the space NS,

$$\bar{D}(v, w) := D(T_u, T_v) + |v - w|,$$

and let Π be the $\bar{D}$-completion of NS.

Lemma 29. *The action of $E+\Gamma$ on NS (by translation), the map $\eta_0\colon NS \to M\mathcal{T}$ by $x \mapsto T_x$, and the inclusion $\mu_0 : NS \hookrightarrow \mathcal{E}$ all extend to continuous maps on the completion Π. Furthermore, the extension of η_0, to $\eta\colon \Pi \to M\mathcal{T}$ is a local homeomorphism and the extension of μ_0 is a surjection $\mu : \Pi \to \mathcal{E}$ that is one to one on non-singular points.*

Proof. $\bar{D}$ is invariant under the Γ action and for small $s \in E$ we have that $\bar{D}(u+s, v+s)$ differs very little from $\bar{D}(u,v)$; this implies that the action of $E+\Gamma$ extends to one by homeomorphisms on Π. Uniform continuity of η_0 and μ_0 is clear, as one can bound the D-metric and the euclidian metric by the $\bar{D}$-metric. Hence both maps extend continuously.

To show that η is open recall from Proposition 28 that $\eta_0^{-1}(T_u) = u+\Gamma$. Hence, different preimages under η_0 of one single point have a minimal distance. In particular, any restriction of η_0 to some small open ball, smaller than that minimal distance, will be injective and we claim that a Cauchy-sequence in the image of such a restriction has a Cauchy sequence as preimage. This then shows that the restrictions extend to injective maps implying that η is a local homeomorphism. To prove our claim let $(T_{u_\nu})_\nu, u_\nu \in NS$, be a D-Cauchy sequence with $(u_\nu)_\nu$ belonging to a small ball (with respect to $\bar{D}$ in the relative topology). Observe that if $\bar{D}(u,v)$ is small ($u, v \in NS$) then $|\pi(u)-\pi(v)|$ is small as well and bounded by $2D(T_u, T_v)$. Hence we can choose the ball small enough so that convergence of T_{u_ν} implies that of $|\pi(u_\nu)|$ and hence also $T_{u_\nu^\perp}$ is a Cauchy sequence. But the latter is even a Cauchy sequence with respect to the metric D_0. Now $D_0(T_{u_\nu^\perp}, T_{u_{\nu+\mu}^\perp}) \to 0$ implies that $R_\nu = \sup\{R|\forall\mu : B_R(T_{u_\nu}) = B_R(T_{u_{\nu+\mu}})\}$ diverges and hence diameter of $A_{u_\nu}(M_{R_\nu}(T_{u_\nu}))$ shrinks to zero (Lemma 27) which implies, by Lemma 23, $|u_{\nu+\mu}^\perp - u_\nu^\perp| \to 0$. This shows that $(u_\nu)_\nu$ converges in the euclidian metric and therefore also in the $\bar{D}$-metric.

To show that μ is almost one to one on non-singular points observe that μ can also be viewed as the extension of the identity map id : $(NS, \bar{D}) \to (NS, \|\cdot\|)$ to the completions (here $(NS, \bar{D})$ and $(NS, \|\cdot\|)$ is the standard notation for the incomplete metric spaces, $\|\cdot\|$ standing for the euclidean metric). Above we showed that id is uniformly continuous and Lemma 25.2 shows that its inverse is pointwise continuous. So if $u \in NS$ and $(x_\nu)_\nu$ is a $\bar{D}$-Cauchy sequence in NS converging to $x \in \Pi$, then $\mu(u) = \mu(x)$ implies that $(x_\nu)_\nu$ must be a $\|\cdot\|$-Cauchy sequence converging to $u \in NS$. The pointwise continuity of the identity map $(NS, \|\cdot\|) \to (NS, \bar{D})$ implies therefore that $x = u$. □

Corollary 30. *The map η induces an E-equivariant homeomorphism between the orbit space Π/Γ and $M\mathcal{T}$.*

Proof. Proposition 28 and Lemma 29 imply that η maps Γ-orbits onto single tilings. To show that $\eta(x) = \eta(y)$ implies $y \in x+\Gamma$ (we denote the extended action of $\gamma \in \Gamma$ on Π also simply additively) we first recall from the last lemma that NS as a subset of Π is the preimage of $NS \subset \mathcal{E}$ under μ, a continuous map. Therefore NS is also a dense G_δ subset of Π.

Let $\eta(x_1) = \eta(x_2)$ but $x_1 \neq x_2$. Fix $\delta > 0$, by the Hausdorff property we may find $\bar{D}$-open U_i such that $x_i \in U_i$, U_i is contained in the δ-neighbourhood (with respect to $\bar{D}$) of x_i, and $\eta(U_1) = \eta(U_2)$. Since η is continuous and open, $\eta(U_i \cap NS)$ is a G_δ-dense subset of $\eta(U)$. Therefore $\eta(U_1 \cap NS) \cap \eta(U_2 \cap NS)$ is not empty. So take $u_i \in U_i \cap NS$ such that $\eta(u_1) = \eta(u_2)$. By Proposition 28 we find a $\gamma \in \Gamma$ such that $u_1 - u_2 = \gamma$.

Therefore $\bar{D}(x_1, x_2 + \gamma) \le \bar{D}(x_1, u_1) + \bar{D}(u_2 + \gamma, x_2 + \gamma)$ which tends to 0 if $\delta \to 0$. Hence $x_1 = x_2 + \gamma$. E-equivariance is clear. □

We have thus another dynamical system $(\Pi, E + \Gamma)$ which plays the role of a "universal covering" (not in its strict sense) of the continuous tiling dynamical system.

Remark 31. We can compare this construction with the so-called torus parametrisation of projection tilings [32]; this also parallels a discussion which was carried out for tilings related to $\mathbb{Z}^N$ (not necessarily canonical) in [16]. There is a surjection $\mu' \colon M\mathcal{T} \to \mathcal{E}/\Gamma$ which makes commutative the diagram

$$\begin{array}{ccc} \Pi & \stackrel{\mu}{\to} & \mathcal{E} \\ \eta \downarrow & & \downarrow \\ M\mathcal{T} & \stackrel{\mu'}{\to} & \mathcal{E}/\Gamma \end{array} . \tag{5}$$

All maps are E-equivariant and μ is $E+\Gamma$ equivariant; μ' is one to one on (classes of) nonsingular points. The dense set NS/Γ of the torus $\mathcal{E}/\Gamma$ therefore yields a parametrization of a dense set of tilings. In fact it can be shown that $\mathcal{E}/\Gamma$ parametrizes the remaining set of tilings up to changes on sets of tiles having zero density in the tiling. This torus parametrization is very useful for analyzing symmetry properties of the tilings [32].

We need now to describe the topology of Π. Recall from Sect. 1 that a base of the topology of $M\mathcal{T}$ is generated by sets $B_\epsilon + x + \mathcal{U}_P$, $\epsilon > 0$, $x \in E$, P a patch in T. For $u \in E^\perp \cap NS$ Lemma 23 can be reformulated to say that $P \subset T_x$ for $x \in u + E + \Gamma$ whenever $x \in A_u(P) + u + \Gamma$. For $u \in E^\perp \cap NS$ we let

$$\mathcal{A}_u = \{(A_u(P) \cap \Gamma^\perp) + u + y^\perp | P \subset T_u, y \in \Gamma\} \cup \{\emptyset\}.$$

Then, by the interpretation of $A_u(P)$ we see that $\mathcal{A}_u$ is closed under intersection. In fact, if $y \in \Gamma$ then $A_u(P) \cap (A_u(P') + y^\perp) = A_u(P \cup (P' + \pi(y)))$ provided $P \cup (P' + \pi(y)) \subset T_u$ and $\emptyset$ otherwise.

It is also useful to have another description of $\mathcal{A}_u$ which shows that the collection $\mathcal{B} := \{\overline{A} | A \in \mathcal{A}_u\}$ of closed subsets in Π does not depend on u. For $X \subset \mathcal{L}_\Theta^{(d^\perp)}$, let $A(X) := \bigcap_{\xi \in X} -\xi^\perp$ and

$$\mathcal{A}'_u := \{A(X) \cap (\Gamma^\perp + u) | X \subset \mathcal{L}_\Theta^{(d^\perp)} \text{ finite}\} \cup \{\emptyset\}.$$

Then $A_u(P) + u = A(X)$, where $X = \{\xi \in \tilde{P}_u | \pi(\xi^*) \in P\} + u$ which shows that $\mathcal{A}_u \subset \mathcal{A}'_u$. On the other hand let $v \in A(X) \cap (\Gamma^\perp + u)$. Then $\forall \xi \in X\colon \pi(\xi^*) \in T_v$ and $v - u = \gamma^\perp$ for some $\gamma \in \Gamma$. It follows that $\{\pi(\xi^*) | \xi \in X\} + \pi(\gamma)$ is a patch in T_u. Hence $\mathcal{A}_u = \mathcal{A}'_u$. But from the form of $\mathcal{A}'_u$ it is clear that $\mathcal{B}$ does not depend on u.

Theorem 32. *The collection $\{B_\epsilon + x + U | U \in \mathcal{B}, \epsilon > 0, x \in E\}$ is a base of the topology of Π. In particular, Π is homeomorphic to $E_c^\perp \times E$ (with the product topology) where $E_c^\perp = \overline{E^\perp \cap NS}$ (the $\bar{D}$-closure of $E^\perp \cap NS$ in Π).*

Proof. Let P be a patch of T_u, $u \in \mathcal{E}^\perp \cap NS$. From Lemma 23 follows that for $x \in u + E + \Gamma$, $P \subset T_x$ whenever $x \in A_u(P) + u + \Gamma$. Let $X(P) = (A_u(P) + u) \cap NS$. Since $\mathcal{U}_P$ is closed $\eta^{-1}(\mathcal{U}_P) = \overline{X(P)} + \Gamma$. Furthermore, if $\gamma \in \Gamma$ is not trivial then $\bar{D}(X(P), X(P) + \gamma) > \delta$, for some $\delta > 0$ (here we mean the obvious extension of $\bar{D}$

to subsets). Hence, for all $x \in E + \Gamma$, $B_\epsilon + x + \overline{X(P)}$ is an open set. We conclude that the above collection consists indeed of open sets and its image under η is a collection of sets of which forms a base of the topology of $M\mathcal{T}$.

Now let $V \subset \Pi$ be open and of diameter smaller than $\frac{\delta}{2}$. Then $\eta(V)$ is open and hence of the form $\eta(V) = \bigcup_{(\epsilon,x,P)\in I} B_\epsilon + x + \mathcal{U}_P$, where I is an index set containing triples with $\epsilon > 0$, $x \in E + \Gamma$, $P \subset T_u$. If we choose ϵ small enough and the patches P large enough we can make sure that $B_\epsilon + x + \overline{X(P)}$ has $\bar{D}$-distance at least δ to $B_\epsilon + x + \gamma + \overline{X(P)}$ provided $\gamma \in \Gamma$ is non-trivial. Then V is the union of those $B_\epsilon + x + \overline{X(P)}$, $(\epsilon, x, P) \in I$ which contain one of its points.

That Π has the above form of a product space is now clear. $\square$

Corollary 33. *The collection $\mathcal{B}$ is a base of compact open neighbourhoods for $E_c^\perp$. In particular, $E_c^\perp$ is a totally disconnected set without isolated points.*

Proof. That $\mathcal{B}$ is a base of the topology follows directly from the last theorem. That its sets are compact follows from compactness, Lemma 9, of the sets $\mathcal{U}_P$, $P \subset T_u$. $\square$

3.2.1. Decorated tilings. Sometimes it is useful to decorate the tiles of a tiling, usually with small compact sets like arrows. One reason for introducing decorations in the present framework is to get around the hypotheses (c) and (d) made in Definition 20. If it happens that two translationally non-congruent faces of $\mathcal{L}^{(d)}_{\Theta^*}$ project onto the same tile we can distinguish them by means of a decoration: the projection images of faces are decorated by arrows which have equal shape for equal translational congruence class but different shape for different classes. Decorating has to be taken into account in the general framework in the way that tiles, patches, and tilings are decorated objects. This means for Lemma 23, for instance, that the tile P is no longer just the set $\pi(\xi^*)$ but this set together with the decoration. Likewise we have to understand patches in Corollary 26 as subsets of decorated tiles. The description of the hull and notably Theorem 32 remain as stated if one takes into account that the tiling is the decorated one. It is important to note that we need only finitely many different decorations for that so that the decorated tiling remains finite type. In the same way we can handle the case in which the translation subgroup of $\mathcal{L}^{(d)}_{\Theta^*}$ is larger than Γ or a fundamental domain for it contains several translationally congruent faces. We can distinguish them again by decorations of which we need only finitely many.

A different reason for introducing decorations is to introduce matching conditions or break the symmetry of the tiles. For instance, the octagonal and decagonal tilings are canonical projection tilings which have matching rules only after (a symmetry breaking) decoration. We now indicate how certain (quasiperiodic) decorations can be incorporated in the projection method. This situation is in so far different from the above in that we suppose to start with a canonical projection tiling which we want to decorate and ask how this modifies the topology of the hull.

We saw that the sets of $\mathcal{B}$ have the interpretation of acceptance domains. If a nonsingular point u belongs to such a set then this can be interpreted by saying that a certain patch occurs at T_u. If we introduce by hand additional faces in the Laguerre-complex $\mathcal{L}_{\Theta_u}$ we started with we divide a $d^\perp$-cell $\xi^\perp$ into several components. Each component may serve as acceptance domain for a decorated tile, the bare tile is $\pi(\xi^*)$ and for its decoration we can take a label or a small compact set like an arrow. We need to make sure that there are as many different decorations as there are new components and we need to require that the additional faces form Γ-orbits so that the new Laguerre complex remains

Γ-invariant. This also insures that the decorated tiling remains minimal. If we now take the new faces into account by taking as a base for the topology the sets corresponding to the above components then we end up with a similar description of the continuous hull in the decorated case as in the undecorated one. Certainly arbitrary decorations could not be handled like this, but those which define matching rules for the (then decorated) octagonal and decagonal tilings do.

3.3. A description of the topology by singular planes. We now bring into play the final hypothesis of the main Definition 20 of canonical projection tilings,

(e) For all $\beta \in \mathcal{L}_\Theta^{(d^\perp-1)}$, the (affine) hyperplane H_β which is tangent to $\beta^\perp$ is a subset of $S^\perp$.

What we require here is that for all β, the stabilizer of H_β with respect to the action of Γ given by $\lambda \mapsto \lambda + \gamma^\perp$ has rank at least $d^\perp$ and that its lattice spacing is small enough compared with the inner diameter of $\beta^\perp$ to insure that $\beta^\perp$ intersects each of its orbits. This is certainly the case for $W = \mathbb{Z}^N$, $w = 0$, but holds in many other interesting cases.

We call the hyperplanes H_β *singular planes*. Using hypothesis (e) we get a further description of the topology of $E_c^\perp$. It allows us to write the singular points in $E^\perp$ as $S^\perp = \bigcup_{\beta\in\mathcal{L}_\Theta^{(d^\perp-1)}} H_\beta$ which is clearly invariant under the action of Γ given by $\lambda \mapsto \lambda + \gamma^\perp$. The set $\mathcal{C}$ of all singular planes is invariant under Γ as well and, since $\mathcal{L}_\Theta^{(d^\perp-1)}$ contains only a finitely many Γ-orbits, $\mathcal{C}$ consists of a finite number of Γ-orbits, too.

Definition 34. *A compact polytope in $E^\perp$ is called a $\mathcal{C}$-**tope** if it is the closure of its interior and if all its boundary faces are subsets of singular planes. A subset of $E_c^\perp$ is called a $\mathcal{C}$-tope if it is the $\bar{D}$-closure of the set of non-singular points of a $\mathcal{C}$-tope in $E^\perp$.*

Theorem 35. *The characteristic functions on $\mathcal{C}$-topes generate $C_c(E_c^\perp, \mathbb{Z})$, the compactly supported continuous, integer valued functions on $E_c^\perp$.*

Proof. $\mathcal{C}$-topes form the set of finite unions of sets of $\mathcal{B}$. The latter being clopen and forming a base of the topology, their corresponding characteristic functions generate $C_c(E_c^\perp, \mathbb{Z})$. Since $1_{U\cup V} + 1_{U\cap V} = 1_U + 1_V$ the statement follows. □

Remark 36. For $\Gamma = \mathbb{Z}^{d+d^\perp}$ Le [11] gave a description of the topology of $E_c^\perp$ which we relate to the above. For $x \in E^\perp$ let c_x be a connected component of $E^\perp \setminus \bigcup_{x\in H\in\mathcal{C}} H$, an open subset of $E^\perp$ called a corner. Note that $c_x = E^\perp$ if $x \in NS$. Let $E_L^\perp = \{(x, c_x) | x \in E^\perp\}$ with topology generated by the sets

$$\mathcal{U}_{(x,c_x)} = \{(y, c_y) | y \in \overline{c_x}, c_x \cap c_y \neq \emptyset\}.$$

Clearly, the projection onto the first factor is a continuous surjective map $E_L^\perp \to E^\perp$. This is Le's description of a transversal for the continuous hull. Let U be a $\mathcal{C}$-tope in $E^\perp$. Then

$$U_L := \{(x, c_x) | x \in U, c_x \cap \mathrm{Int} U \neq \emptyset\}$$

is a preimage of U in $E_L^\perp$ which is a finite union of $\mathcal{U}_L$'s and hence open. Let $\mathcal{B}_L$ be the collection of all sets obtained in this way. Then the topology of $E_L^\perp$ is generated by $\mathcal{B}_L$

since we can realize the sets $\mathcal{U}_{(x,c_x)}$ as (infinite) unions. We leave it to the reader to verify that the map $\mathcal{B} \to \mathcal{B}_L$ given by $U \mapsto \mu(U)_L$ is a bijection preserving the operations intersection, union, and symmetric difference. Then $C_0(E_c^\perp)$ is isomorphic to $C_0(E_L^\perp)$ and $E_c^\perp$ is homeomorphic to $E_L^\perp$.

3.4. A variant of the tiling groupoid for canonical projection tilings. For canonical projection tilings it is convenient to use a slightly different groupoid which is isomorphic to a reduction of the tiling groupoid. It is also continuously similar to it. In [17] it is called the *pattern groupoid*.

Let ϵ be a small vector in E which is not parallel to any of the faces of tiles. To a vertex v, associate the tile which contains in its interior $v + \epsilon$; this defines an injection between the vertices of a projection tiling and its tiles. We assume that ϵ is small enough so that the associated tile contains this vertex. Let $\Omega\mathcal{T}$ be the subset of $M\mathcal{T}$ given by those tilings which have a vertex on $0 \in E$. As for $\Omega_\mathcal{T}$ one shows that $\Omega\mathcal{T}$ is a closed range-open subset which intersects each orbit of $\mathcal{G}(M\mathcal{T}, E))$. Thus we define the reduction

$$\mathcal{G}\mathcal{T} := {}_{\Omega\mathcal{T}}\mathcal{G}(M\mathcal{T}, E))_{\Omega\mathcal{T}}$$

of $\mathcal{G}(M\mathcal{T}, E))$. Now consider a new set of punctures for $\mathcal{T}$, a subset of the old one, namely give only those tiles a puncture which are associated to vertices as described above. This choice can be made locally since we only have to test the vertices of the tile itself to decide whether we select its puncture to become a new one. Call $\Omega'_\mathcal{T}$ the subset of tilings of $M\mathcal{T}$ for which a new puncture lies on 0. By letting the new punctures tend to the corresponding vertices one immediately sees that the reduction ${}_{\Omega'_\mathcal{T}}\mathcal{G}(M\mathcal{T}, E))_{\Omega'_\mathcal{T}}$ is isomorphic to $\mathcal{G}\mathcal{T}$. Furthermore, ${}_{\Omega'_\mathcal{T}}\mathcal{G}(M\mathcal{T}, E))_{\Omega'_\mathcal{T}}$ is the reduction to $\Omega'_\mathcal{T}$ of $\mathcal{G}_\mathcal{T}$ which, as noted in [10] is continuously similar to it. A similar argument can also be found in [17].

Without loss of generality we may assume that $0 \in W$, our Γ-invariant set we start with, and that the Laguerre domain of 0 has interior and therefore 0 is a vertex of the dual complex. Let $u \in E^\perp \cap NS$ be such that 0 is a vertex of T_u. All vertices of T_u are contained in $\pi(W_u)$ which can be written in the form $\pi(W_u) = \bigcup_{x \in X} x + \pi(\Gamma)$ for a finite subset $X \in E$ of points which are all in different $\pi(\Gamma)$ orbits, 0 being one of them. Therefore, if $s \in E$ and 0 is a vertex of T_{u-s} then $s \in x + \pi(\Gamma)$ for some $x \in X$. Using Proposition 28 we find that $\eta^{-1}(T_{u-s}) \cap E_c^\perp \times \{x\}$ is not empty provided 0 is a vertex of T_{u-s}. By continuity and closedness of $E_c^\perp$ this extends to arbitrary $T \in \Omega\mathcal{T}$. So if we let $L\mathcal{T} := \eta^{-1}(\Omega\mathcal{T}) \cap E_c^\perp \times X$ then $\eta^{-1}(\Omega\mathcal{T}) = L\mathcal{T} + \Gamma$.

Lemma 37. *$\mathcal{G}\mathcal{T}$ is isomorphic to the reduction ${}_{L\mathcal{T}}\mathcal{G}(\Pi, E + \Gamma)_{L\mathcal{T}}$, where $L\mathcal{T}$ is as above.*

Proof. The map ${}_{L\mathcal{T}}\mathcal{G}(\Pi, E + \Gamma)_{L\mathcal{T}} \to {}_{\Omega\mathcal{T}}\mathcal{G}(M\mathcal{T}, E)_{\Omega\mathcal{T}}$ given by $(y, s + \gamma) \mapsto (\eta(y), s)$ is a groupoid homomorphism. It is injective, because no two points of X belong to the same $\pi(\Gamma)$ orbit, and surjective, because $\eta(L\mathcal{T}) = \Omega\mathcal{T}$. Continuity follows from the continuity properties of η. □

3.5. Discrete tiling dynamical systems for canonical projection tilings. We now bring to fruition the work of the preceding subsections and prove that the groupoids constructed so far from a canonical projection tiling are continuously similar to that arising from a minimal action of $\mathbb{Z}^d$ on a Cantor-set. This gives us the key, in Sects. 4, 5 and 6, to qualitatively and quantitatively describing the cohomology of these tilings.

Let F be a subspace which is complementary to E, thus $F \cap E = 0$ and $F + E = \mathcal{E}$. We denote by π' the projection onto F with kernel E (so it is not orthogonal except if $F = E^\perp$). The restriction of π' to $u + \Gamma^\perp$ ($u \in E^\perp \cap NS$) extends to a homeomorphism between $E_c^\perp$ and $F_c = \overline{F \cap NS}$ (its closure in Π) and we can write $\Pi = F_c \times E$ with the product topology. Since $E \cap \Gamma = \{0\}$, $\pi'(\Gamma)$ is isomorphic to Γ so that we have a natural minimal action of Γ on F, $x \cdot \gamma = x - \pi'(\gamma)$, without fixed points. The extension of this action to F_c defines a minimal dynamical system (F_c, Γ) also without fixed points.

Proposition 38. *$\mathcal{G}(F_c, \Gamma)$ is continuously similar to $\mathcal{G}(\Pi, E + \Gamma)$.*

Proof. We apply Proposition 12 taking $L = F_c$ (which is closed) and $\gamma\colon \Pi \to E + \Gamma$ to be the extension of $\pi : \mathcal{E} \to E$. □

Now we decompose $\Gamma \cong \mathbb{Z}^{d+d^\perp}$ into complementary subgroups, $\Gamma = G_0 \oplus G_1$, where $G_0 \cong \mathbb{Z}^{d^\perp}$ and $G_0' := \pi'(G_0)$ spans F. Define

$$X := F_c / G_0$$

so that we obtain (X, G_1), a minimal dynamical system without fixed points.

Proposition 39. *$\mathcal{G}(F_c, \Gamma)$ is continuously similar to $\mathcal{G}(X, G_1)$.*

Proof. We claim that F_c has a clopen fundamental domain Y for G_0. The proposition follows then from Proposition 12 upon using $L = Y$ and $\gamma\colon F_c \to \Gamma$, $\gamma(x)$ being the unique element of G_0 such that $x \cdot \gamma(x) \in Y$. The latter is indeed continuous since the preimage of a lattice point is a translate of the fundamental domain and therefore open.

To prove the claim pick any $\xi \in \mathcal{L}_\Theta^{(d^\perp)}$ such that $\xi^\perp$ has interior. Since G_0' spans F it has a compact fundamental domain Y^0. By density of $\Gamma^\perp$ there is a finite subset $0 \in J \subset \Gamma$ such that $Y^1 = \bigcup_{\gamma \in J} (-\xi^\perp + \gamma^\perp)$ covers Y^0. It follows that

$$Y_c^1 := \bigcup_{\gamma \in J} (\overline{(-\xi^\perp \cap NS)} + \gamma^\perp)$$

is a compact open subset of F_c and $Y_c^1 + G_0' = F_c$. Now let G_0^+ be a positive cone of G_0' which satisfies $G_0' = G_0^+ \cup (-G_0^+)$ thus implying a total order. We claim that $Y := Y_c^1 \backslash (Y_c^1 + G_0^+ \backslash \{0\}) \cap Y_c^1$ is a clopen fundamental domain. Clopenness is easy to see. So let $x \in F_c$. Clearly, the set of all $g \in G_0'$ such that $x + g \in Y_c^1$ is non-empty and finite. The unique minimal element g_0 of this set is the only one satisfying $x + g_0 \in Y$. □

Proposition 40. *$\mathcal{G}\mathcal{T}$ is continuously similar to $\mathcal{G}(E_c^\perp, \Gamma)$.*

Proof. From Lemma 37 we know that $\mathcal{G}\mathcal{T}$ is isomorphic to the reduction ${}_{L\mathcal{T}}\mathcal{G}(\Pi, E + \Gamma)_{L\mathcal{T}}$. Let $(L\mathcal{T})_x := L\mathcal{T} \cap E_c^\perp \times \{x\}$, $x \in X$. If $u \in E^\perp \cap NS$ such that 0 is a vertex of T_u and $v \in u + E \cap (L\mathcal{T})_x$ then $v = u - s$ with $s \in x + \pi(\Gamma)$. Hence there is a unique $g \in \Gamma$ such that $v + x - g \in E^\perp$. Now $\eta(v + x - g) = \eta(u)$ contains 0 as a vertex and hence $v + x - g \in (L\mathcal{T})_0$. We define a map $\gamma' : (L\mathcal{T})_x \to E + \Gamma$ first on the dense set $u + E \cap (L\mathcal{T})_x$ by $\gamma'(v) = x - g$, with g as above, and then extend it by continuity. Applying Proposition 12 with $L = L\mathcal{T}$, $L' = (L\mathcal{T})_0$, $\gamma : L' \to E + \Gamma$, $\gamma(x) = 0$, and

$\gamma' : L\mathcal{T} \to E + \Gamma$ as above, we find that ${}_{L\mathcal{T}}\mathcal{G}(\Pi, E + \Gamma)_{L\mathcal{T}}$ is continuously similar to ${}_{L'}\mathcal{G}(\Pi, E + \Gamma)_{L'}$. The latter is equal to the reduction of $\mathcal{G}(E_c^\perp, \Gamma)$ to L'. L' is clopen (in the topology of $E_c^\perp$) and hence $\mu(L')$ contains an open set.

We claim that there exists a choice of decomposition $\Gamma = G_0 + G_1$ with the properties stated before Proposition 39 and such that L' contains a clopen fundamental domain Y for G_0. It then follows again from Proposition 12 upon using the same map γ as in Proposition 39 (Y is a subset of L') that ${}_{L'}\mathcal{G}(\Pi, E + \Gamma)_{L'}$ is continuously similar to $\mathcal{G}(E_c^\perp, \Gamma)$. This then proves the proposition.

It remains to prove the claim. Since $\Gamma^\perp$ is dense in $E^\perp$ we can choose $d^\perp$ elements of Γ which generate a group H isomorphic to $\mathbb{Z}^{d^\perp}$, such that $H^\perp$ spans $E^\perp$, and has a fundamental domain Y' in $E^\perp$ contained in $\mu(L')$. Let G_0 be the group generated by H and representatives for the torsion elements of Γ/H. It is a free abelian group of rank $d^\perp$ which contains H and $G_0^\perp$ cannot be dense in $E^\perp$. By the same construction as in the proof of the last proposition we obtain from Y' a fundamental domain Y for G_0 in $E_c^\perp$ which is contained in L since $\mu(Y) \subset Y'$. $\square$

Corollary 41. $H^*(\mathcal{T}) \cong H^*(\Gamma, C(F_c, \mathbb{Z})) \cong H^*(G_1, C(X, \mathbb{Z}))$.

A direct consequence of the above corollary is that $H^k(\mathcal{T})$ is trivial if k exceeds the rank of G_1, which is d, the dimension of the tiling. Furthermore, using that $H^0(G_1, C(X, \mathbb{Z})) = \{f \in C(X, \mathbb{Z}) | \forall g \in G_1 : g \cdot f = f\}$ [29], minimality of the G_1 action implies that $H^0(\mathcal{T}) = \mathbb{Z}$. Finally, if M is a G_1-module then $H^d(G_1, M) = \mathrm{Coinv}(G_1, M)$ is the group of coinvariants [29]

$$\mathrm{Coinv}(G_1, M) := M/\langle\{m - g \cdot m | m \in M, g \in G_1\}\rangle.$$

By the corollary $H^d(\mathcal{T})$ is thus equal to $C(X, \mathbb{Z})/E(G_1)$ where $E(G_1)$ is subgroup of $C(X, \mathbb{Z})$ generated by the elements $f - g \cdot f$ for $g \in G_1$ and $(g \cdot f)(x) = f(x \cdot g)$.

Remark 42. The dynamical systems of the form (X, G_1) defined above *a priori* depend on the position of F and on the choice of G_0. However, in a certain sense they are all equivalent, namely their groupoids are all continuously similar and they are all reductions of one big groupoid. They are not all isomorphic, as an investigation of the order unit of the K_0-group of the C^* algebra they define shows.

The dependence on F is inessential. The map π' induces a Γ equivariant homeomorphism between $E_u^\perp$ and F_c. Different F's therefore lead to isomorphic dynamical systems (F_c, Γ). Taking F as the span of G_0 one verifies directly that $M\mathcal{T}$ is the mapping torus of (X, G_1) [16]. One consequence of this (though not one we make use of below) is the following.

Corollary 43. *The tiling cohomology of non-periodic canonical projection tilings is isomorphic to the Czech cohomology of their continuous hull.*

We do not know whether this result is true for general tilings.

Remark 44. Consider the case $\Gamma = \mathbb{Z}^{d^\perp + d}$, $F = E^\perp$ and G_0 generated by, say, the first $d^\perp$ basis elements e_i. Then the dynamical system is the rope dynamical system of [10].

Remark 45. We conclude Sect. 3 by summarizing the structure of (X, G_1) in a commutative diagram which is the discrete analogue of (5); see [16] for the neccessary proofs.

$$\begin{array}{ccc} F_c & \xrightarrow{\mu} & F \\ \eta \downarrow & & \downarrow \\ X & \xrightarrow{\mu'} & F/G_0\,. \end{array}$$

The maps are Γ (respectively G_1) equivariant where the G_1-action on the $d^\perp$-torus F/G_0 is by rotations (constant shifts). X is a Cantor set and the surjection $\mu'\colon X \to F/G_0$ is one to one for nonsingular points of X which form a dense G_δ subset. Thus (X, G_1) is an almost one to one extension of a relatively simple system, that of rotations on a torus. The crucial topological information is encoded in the set on which μ' is not injective.

4. Examples

Before we proceed to give a qualitative picture of tiling cohomology and to describe methods for calculation, we discuss the two simplest examples which we believe show typical features. Both are one-dimensional tilings obtained from an integer lattice, so by Corollary 41 only $H^0(\mathcal{T})$ and $H^1(\mathcal{T})$ are non-zero. As noted, by minimality $H^0(\mathcal{T}) = \mathbb{Z}$ and $H^1(\mathcal{T})$ is identified in the last section as a group of coinvariants.

Example 46. In our first example we take $W = \mathbb{Z}^2$, $w = 0$ and $d = 1$. Here E is specified by a vector $(1, \nu)$ and ν has to be irrational to meet the requirement $E \cap \mathbb{Z}^2 = \{0\}$. Clearly, $E^\perp$ is generated by $(-\nu, 1)$ and the singular planes are simply points, namely the points of $\pi^\perp(\mathbb{Z}^2)$ (we ignore the shift by δ). Identifying $E^\perp$ with $\mathbb{R}$ we have $\pi^\perp(\mathbb{Z}^2) = \mathbb{Z} + \nu\mathbb{Z}$ (after a suitable rescaling). Hence $C_c(E_c^\perp, \mathbb{Z})$ is generated by indicator functions $1_{[a,b]}$ (on the $\bar{D}$-closure of $[a, b] \cap NS$) with $a, b \in \mathbb{Z} + \nu\mathbb{Z}$, $a < b$. How many of them are cohomologous? Clearly, $1_{[a,b]} \sim 1_{[0,b-a]}$ and there are unique $n, m \in \mathbb{Z}$ such that $b - a = n + \nu m$. Defining $1_{[a,b]} = -1_{[b,a]}$ in the case of $a > b$, we get

$$1_{[0,b-a]} = 1_{[0,n]} + 1_{[n,n+\nu m]} \sim n 1_{[0,1]} + m 1_{[0,\nu]}$$

which shows that the coinvariants are $\mathbb{Z}^2$ provided the two generators given by the classes of $1_{[0,1]}$ and of $1_{[0,\nu]}$ are independent. This will be shown in Sect. 7. Let us mention in this context that the above tilings are very close to being substitutional [33] (they are strictly substitutional only for ν a quadratic irrationality).

The above result shows that whatever the irrational ν is $H^1(\mathbb{Z}^2, C_c(E_c^\perp, \mathbb{Z})) = \mathbb{Z}^2$. This demonstrates that cohomology is not a very fine invariant to distinguish tilings, at least in these low dimensions. We shall see in Sect. 7 how further structure can be added.

Example 47. In our second example we take $W = \mathbb{Z}^3$, $w = 0$ and $d = 1$. Here we consider only the case where $E^\perp \cap \mathbb{Z}^3 = \{0\}$ because the other leads essentially to the previous example. In this case, the singular planes are lines which are $\pi^\perp(\mathbb{Z}^3)$-translates of $H_\alpha = \langle e_\alpha^\perp \rangle$, $\alpha = 1, 2, 3$ (again up to the shift by δ). Any two H_α span $E^\perp$.

We claim that the result for the cohomology differs drastically from the previous example in that the coinvariants are infinitely generated. Fix $g_1, g_2 \in \pi^\perp(\mathbb{Z}^3)$ and let U be the rhombus (we assume it has interior) whose boundary faces lie in $H_1 \cup (H_1 + g_1) \cup H_2 \cup (H_2 + g_2)$. Clearly, 1_U, the indicator function on the $\bar{D}$-closure of $U \cap NS$, belongs to $C_c(E_c^\perp, \mathbb{Z})$. Let, for $\alpha = 1, 2$, π_1 (π_2) be the projection onto H_1 (H_2) which has kernel H_2 (H_1) and let $\Gamma_\alpha = \pi_\alpha(\pi^\perp(\mathbb{Z}^3))$. Then for all $\lambda_\alpha \in \Gamma_\alpha$ also $1_{U+\lambda_1+\lambda_2} \in C_c(E_c^\perp, \mathbb{Z})$. How many of them are cohomologous? Let us try to repeat the construction of the first example. Clearly

$$1_{U+\lambda_1+\lambda_2} \sim 1_{U+\lambda_1'+\lambda_2'} \quad \text{if } \lambda_1 + \lambda_2 - \lambda_1' - \lambda_2' \in \pi^\perp(\mathbb{Z}^3).$$

But since the rank of Γ_α is at least 2 (because it is dense in H_α) we see that the number of $\pi^\perp(\mathbb{Z}^3)$ orbits of points in $\Gamma_1 + \Gamma_2$ (which is the number of elements in $(\Gamma_1 +$

$\Gamma_2)/\pi^\perp(\mathbb{Z}^3))$ is infinite. Therefore the construction used in the first example cannot be used here to reduce the generators to a finite set. This does not prove our claim but it does indicate a crucial point, namely that there are infinitely many orbits of points which are intersections of singular planes. From this we will conclude in the next section that the tilings of the second example cannot be substitutional.

5. Conditions for Infinitely Generated Cohomology

The cohomology groups of a canonical projection tiling, as defined in Sect. 2.2, contain rich information about the tiling. With the analysis of Sect. 3 we shall see in Sect. 6 that they are completely computable, at least for projections of small codimension. In this section we examine instead the qualitative behaviour for generic projection tilings of the rationalisations of these cohomology groups. Although rational cohomology, $H^*(\mathcal{G}_\mathcal{T}, \mathbb{Q})$, is a somewhat cruder invariant, it still proves useful. In the following subsection it will allow us to comment on the relationship between canonical projection tilings and tilings defined by a substitution system.

Recall the set of singular points $S^\perp$ in $E^\perp$, defined in Sect. 3.1, and the assumption (e) of our Definition 20 of a canonical projection tiling.

Definition 48. *We call a point $x \in S^\perp$ an* **intersection point** *if there are $d^\perp$ singular planes which intersect uniquely at x.*

Let $\mathcal{P}$ be the set of intersection points. Clearly, $\mathcal{P}$ is invariant under the action of Γ. Let $\Omega(\mathcal{P}) = \mathcal{P}/\Gamma$ be the orbit space. One of the main results of [19] is the following theorem (see also [17]).

Theorem 49 (*[17,19]***).** *$\Omega(\mathcal{P})$ is an infinite set if and only if $H^*(\mathcal{G}\mathcal{T}, \mathbb{Q})$ is infinitely generated.*

We do not repeat its proof here, but rather explain how to obtain criteria under which $\Omega(\mathcal{P})$ is infinite.

Choose $d^\perp$ singular planes H_β, indexed now simply by $\alpha = 1, \ldots, d^\perp$, which intersect in exactly one point. Let $S' := \bigcup_\alpha (H_\alpha + \Gamma^\perp)$ and let $\mathcal{P}' = \mathcal{P} \cap S'$, a subset which is clearly Γ-invariant. Write L_α for $\bigcap_{\alpha' \neq \alpha} H_{\alpha'}$, a line, and let $\pi_\alpha : E^\perp \to L_\alpha$ be the (not necessarily orthogonal) projection with kernel H_α. Then $\Gamma^\alpha := \{\gamma \in \Gamma | L_\alpha + \gamma = L_\alpha\}$, the stabilizer of L_α, can be naturally identified with a subgroup of $\Gamma_\alpha = \pi_\alpha(\Gamma^\perp)$.

Lemma 50. *If* $\mathrm{rank}\Gamma^\alpha < \mathrm{rank}\Gamma_\alpha$ *then $\Omega(\mathcal{P})$ is an infinite set.*

Proof. Let $x \in L_\alpha \cap \mathcal{P}'$. Then, by construction, $x + \Gamma_\alpha \in \mathcal{P}'$, too. The latter set may be decomposed in its Γ^α-orbits and if $\mathrm{rank}\Gamma^\alpha < \mathrm{rank}\Gamma_\alpha$ there are infinitely many. On the other hand, intersection points of $L_\alpha \cap \mathcal{P}'$ which lie in different Γ^α-orbits lie also in different Γ-orbits. □

This gives the following easily checked criterion; it also shows that $\Omega(\mathcal{P})$ being an infinite set is a generic feature.

Corollary 51. *If* $\mathrm{rank}\Gamma^\alpha < 2$ *then $\Omega(\mathcal{P})$ is an infinite set.*

Proof. Density of $\Gamma^\perp$ implies that of Γ_α. Hence $\mathrm{rank}\Gamma_\alpha \geq 2$. □

Corollary 52. *If $d^\perp > d$ then $\Omega(\mathcal{P})$ is an infinite set.*

Proof. We showed above $\mathrm{rank}\Gamma_\alpha \geq 2$. In particular, $\sum_\alpha \mathrm{rank}\Gamma_\alpha \geq 2d^\perp$. The statement of the lemma follows therefore from the observation that $\Omega(\mathcal{P})$ is an infinite set if $(\bigoplus_\alpha \Gamma_\alpha)/\Gamma^\perp$ is infinite and the latter is the case whenever $\sum_\alpha \mathrm{rank}\Gamma_\alpha > d + d^\perp$. □

The claim of our second example in Sect. 4 follows from this last result and the discussion of the next subsection.

With a little more thorough analysis [17] one can show that if $\Omega(\mathcal{P})$ is a finite set then $\frac{d}{d^\perp}$ must be an integer. A further result, accessible with the algebraic-topological methods of [19], is the following.

Theorem 53. *[19] If $\Omega(\mathcal{P})$ is a finite set then each $H^r(\mathcal{GT}, \mathbb{Z})$ is a finitely generated free abelian group for $r = 0, \ldots, d$ and is zero for other r.*

5.1. Comparison with substitution tilings. In addition to those tilings which arise from the canonical projection method there is another very important class for which cohomology can be computed. These are the finite type tilings which allow for a locally invertible (primitive) substitution. We briefly discuss these tilings and show, with the aid of the results of the previous section, that tiling cohomology gives effective criteria for distinguishing whether a tiling can come from one or the other of these two classes. In particular, we shall see that generically canonical projection tilings do not allow for a locally invertible substitution.

A substitution of a tiling T (the terms inflation and deflation are also used in this context) is roughly speaking a rule according to which each tile of T gets substituted by a collection of tiles (a patch) such that these patches fit together to form a new tiling which is locally isomorphic to T. Furthermore, the translational congruence class of the patch which substitutes a tile depends only on the translational congruence class of that tile and the relative position between two patches only on the relative position between the two tiles which they substitute. Therefore, the rule is specified when it is given for any translational congruence class of tiles (of which there are only finitely many) and for all possible relative positions two neighbouring tiles can have (of which there are also only finitely many). One of the major examples is the *octagonal tiling* whose substitution rule is shown in Fig. 1. The octagonal tiling is also an example of a tiling that can be obtained as a canonical projection tiling and the question naturally arises of obtaining criteria for deciding the possible origins, whether as substitutions, projections or both, of any given tiling.

There are additional conditions which turn out to be useful to assume a substitution satisfies, such as local invertibility; we refer the reader to [9] and [10] for details.

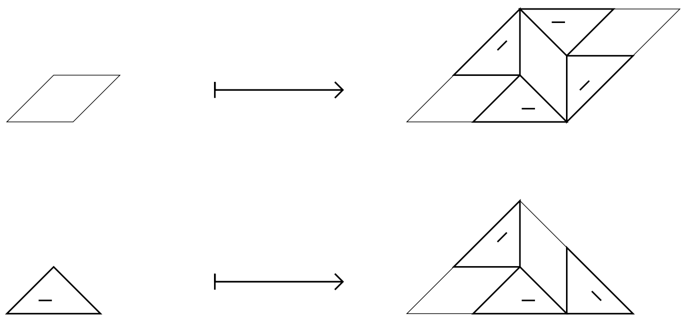

Fig. 1. Substitution of the octagonal tiling (triangle version)

Under such suitable conditions, [9] and [10] develop methods for the computation of substitution tilings.

Of the two approaches to compute the cohomology of substitution tilings that of [9] is based on the continuous dynamical system $(M\mathcal{T}, \mathbb{R}^d)$ whereas that of [10] is based on the tiling groupoid $\mathcal{G}_{\mathcal{T}}$. We sketch here the latter. The essential observation of this approach is that a primitive invertible substitution gives rise to a homeomorphism Θ (the Robinson map) between $\Omega_{\mathcal{T}}$ and the space of paths $\mathcal{P}_\Sigma$ on a certain oriented graph Σ. In the case where the substitution forces its border (see [15]) the connectivity matrix σ of Σ is a power of the substitution matrix. A natural principal topological groupoid $\mathcal{G}_\Sigma$ is associated with the path space, namely the one given by tail equivalence: two paths are tail equivalent if they agree up to finitely many edges. The tiling groupoid $\mathcal{G}_{\mathcal{T}}$, which is always principal for such substitution tilings, is identified via Θ with a subset of $\mathcal{P}_\Sigma \times \mathcal{P}_\Sigma$ and hence can be compared with $\mathcal{G}_\Sigma$. In fact, $\mathcal{G}_\Sigma$ is a subset of $G_{\mathcal{T}}$ (but not a closed one). This construction allows for a description of $\mathrm{Coinv}(\mathcal{G}_{\mathcal{T}}; \mathbb{Z})$, the group of coinvariants of $\mathcal{G}_{\mathcal{T}}$ with integer coefficients, a group which coincides with the cohomology group $H^d(\mathcal{T})$ of Sect. 2.2 when $\mathcal{T}$ arises also from the projection method (or, in the language of [10], when the tiling reduces to a $\mathbb{Z}^d$-decoration).

Theorem 54 ([10]**).** *For substitution tilings as discussed, the group of coinvariants* $\mathrm{Coinv}(\mathcal{G}_{\mathcal{T}}; \mathbb{Z})$ *is a quotient of the group of coinvariants of* $\mathcal{G}_\Sigma$*. Moreover,* $\mathrm{Coinv}(\mathcal{G}_\Sigma; \mathbb{Z})$ *is the direct limit of the system*

$$\mathbb{Z}^N \stackrel{\sigma}{\to} \mathbb{Z}^N \stackrel{\sigma}{\to} \cdots ,$$

where N is the number of vertices of Σ (which in the border forcing case coincides with the translational tile-classes).

Corollary 55. *A necessary condition for a canonical projection tiling to be substitutional is that $\Omega(\mathcal{P})$ is a finite set. Consequently, canonical projection tilings are generically non-substitutional and in particular no canonical projection tiling with $d^\perp$ not dividing d is substitutional.*

Proof. Suppose a canonical projection tiling $\mathcal{T}$ is substitutional. Then Theorem 54 tells us that $H^d(\mathcal{T})$ can be expressed as a direct limit of finitely generated free abelian groups. Such a limit need not to be finitely generated itself but when rational coefficients are considered instead of integer ones then the direct limit becomes that of the system

$$\mathbb{Q}^N \stackrel{\sigma}{\to} \mathbb{Q}^N \stackrel{\sigma}{\to} \cdots ,$$

namely $\mathbb{Q}^R$ where R is the rank of σ^n for large n. The first part of the corollary now follows from Theorem 49; the remainder follows from the results and comments of the preceding section. □

Remark 56. It is worth comparing the above result with a similar one due to Pleasants who uses the theory of algebraic number fields [34]. In the context of tilings obtained by the projection method there is an approach to the construction of substitutions which is based on the torus-parametrization. It is most powerful not when tilings are considered but when projection point patterns are looked at (though these are closely related to tilings, see [16]). For a lattice $\Gamma \subset \mathcal{E}$, a subspace E, and an acceptance domain $A \subset E^\perp$ (satisfying certain rather weak conditions) the projection point pattern given by the triple (Γ, E, A) is the point set $P_A := \pi((E+A)\cap\Gamma)$. The canonical choice for A corresponds to one where $P_A = \{\pi(\xi) | \xi \in \tilde{P}^0\}$ with $\tilde{P}^0$ the set of vertices (0-cells) of the lift of a canonical projection tiling T (constructed from the same data with constant weight function). In that case, A is a polytope, but in [34] A is allowed to be more general.

For the more general acceptance domains, the notion of substitution generalises to that of an *inflation*, a linear map λ [34] (or even affine linear [32]) which has E as one of its eigenspaces, with eigenvalue of modulus greater than 1, preserves Γ, and is contracting in a space F complementary to E. For λ to be a local inflation, i.e. an inflation which can be defined as a map on translational congruence classes, leads to a criterion on the acceptance domain A.

The method of Pleasants [34] is designed to construct projection point patterns with a given (finite) symmetry group of isometries. It is based on the result that every representation of a finite isometry group acting on $\mathbb{R}^d$ can be written as a matrix representation where the matrices take their entries in a real algebraic number field $\mathcal{K}$ of (finite) degree p. This number field $\mathcal{K}$ is then used to construct a decomposition $\mathbb{R}^{dp} = E \oplus E^\perp$, where $\dim E = d$, and a lattice Γ so that the point pattern with the desired symmetry is the projection point pattern constructed from data (Γ, E) and a general acceptance domain in $E^\perp$. In [34] Pleasants comes to the conclusion that local inflations always exist but, for $p > 2$, never for polytopal acceptance domains (so in particular not for the canonical one) whereas this obstruction is absent for $p = 2$. Note that $\dim E^\perp \geq \dim E$ in his construction, with equality holding only for $p = 2$, a result in agreement and comparable to our Corollary 52.

The direct limit of rational vector spaces in the proof of Corollary 55 is finitely generated, but the corresponding limit of underlying free abelian groups need not be finitely generated; indeed limits with divisibility can easily occur. Corollary 55 and Theorem 53 now imply the following.

Corollary 57. *A necessary condition for a substitution tiling $\mathcal{T}$ to arise also as a canonical projection tiling is that $Coinv(\mathcal{G}_\mathcal{T}; \mathbb{Z})$ is a finitely generated free abelian group.*

6. Explicit Formulae for Codimension $d^\perp \leq 2$

We turn now to methods of computation and present quantitative results for the cohomology of canonical projection tilings of codimension smaller than or equal to 2. The restriction to small codimension is a matter of simplification: in principle, the calculations can be carried out for any codimension, but in practice become quite complicated. Algebraic topology provides sophisticated tools to organize such calculations, namely spectral sequences, and we exploit their full power elsewhere [18, 19]. However, they are not really necessary for codimensions strictly less than 3 and we present here alternative, elementary methods of computation for these codimensions.

Throughout this section we assume that $\Omega(\mathcal{P})$ is finite, which we saw in Theorems 49 and 53 was necessary and sufficient to ensure that the cohomology is finitely generated and free abelian. In fact, the results below are independent of these theorems and show directly that if $d^\perp \leq 2$ then $H^*(\mathcal{T})$ is finitely generated and free abelian.

The calculations rely on the description of the topology of $E_c^\perp$ by singular planes developed in Sect. 3. Recall that $\mathcal{C}$ is a countable collection of singular planes with only finitely many Γ-orbits; we index the orbits by I. We know that the normals of the singular planes span $E^\perp$ and that $\Gamma^\perp$ lies dense in it. We now simplify the notation in writing Γ in place of $\Gamma^\perp$.

By Corollary 41 the task is to compute the cohomology of the group Γ with values in $C(E_c^\perp, \mathbb{Z})$ and the strategy is as follows. We recognize $C_c(E_c^\perp, \mathbb{Z})$, the compactly supported functions, as an Γ-module in a (finite) exact sequence of Γ-modules and use the functorial properties of cohomology, in particular that it turns short exact sequences into long exact ones. As the other modules in the exact sequence are effectively lower dimensional we can proceed recursively.

In practice it turns out to be more convenient to use homology in place of cohomology. This makes no essential difference: the fact that $E^\perp$ has $d^\perp$ non-compact independent directions together with Poincaré duality [29] gives an isomorphism [17]

Lemma 58. $H^k(\Gamma, C(E_c^\perp, \mathbb{Z})) \cong H_{d-k}(\Gamma, C_c(E_c^\perp, \mathbb{Z}))$.

6.1. Group homology. As a general reference to group homology we refer to [29]. Homology of a group Γ is defined using any projective resolution of $\mathbb{Z}$ by $\mathbb{Z}\Gamma$ modules of the group; here $\mathbb{Z}\Gamma$ denotes the free $\mathbb{Z}$ module on the basis elements of Γ; we write $[\gamma]$ for the basis element corresponding to $\gamma \in \Gamma$.

We choose here the following free resolution. Let $\{e_1, \ldots, e_N\}$ be a basis of $\Gamma \cong \mathbb{Z}^N$. Then $\Lambda\Gamma$, the exterior algebra over Γ, is the free graded $\mathbb{Z}$-module $\Lambda\Gamma = \bigoplus_{k=0}^N \Lambda_k\Gamma$, where $\Lambda_k\Gamma$ has basis $\{e_{i_1} \wedge \cdots \wedge e_{i_k} | 1 \leq i_j < i_{j+1} \leq N\}$ with antisymmetric multiplication (denoted by $\wedge$), i.e. the only relations are $e_i \wedge e_j = -e_j \wedge e_i$. Our resolution is

$$0 \to \Lambda_N\Gamma \otimes \mathbb{Z}\Gamma \xrightarrow{\partial} \Lambda_{N-1}\Gamma \otimes \mathbb{Z}\Gamma \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Lambda_0\Gamma \otimes \mathbb{Z}\Gamma \xrightarrow{\Sigma} \mathbb{Z} \to 0,$$

where tensor products are over $\mathbb{Z}$ and the $\mathbb{Z}\Gamma$ action on $\Lambda_r\Gamma \otimes \mathbb{Z}\Gamma$ is trivial on $\Lambda_r\Gamma$ and is the permutation representation on $\mathbb{Z}\Gamma$. The maps ∂ are defined as follows. We may regard $\mathbb{Z}\Gamma$ as Laurent polynomials in N variables $\{t_1, \ldots, t_N\}$ with integer coefficients. Addition in $\mathbb{Z}\Gamma$ then corresponds to multiplication of Laurent-polynomials. Then ∂ is the unique $\mathbb{Z}\Gamma$-linear derivation of degree 1 determined by $\partial(e_i) = (t_i - 1)$, and $\Sigma(t_i) = 1$.

Given a Γ-module M, then $H_*(\Gamma, M)$, the homology of the group Γ with coefficients in M, is defined as the homology of the complex

$$0 \to \Lambda_N\Gamma \otimes \mathbb{Z}\Gamma \otimes_\Gamma M \xrightarrow{\partial\otimes 1} \cdots \xrightarrow{\partial\otimes 1} \Lambda_0\Gamma \otimes \mathbb{Z}\Gamma \otimes_\Gamma M \to 0$$

where, for two Γ-modules M_1, M_2, $M_1 \otimes_\Gamma M_2$ is the quotient of the algebraic tensor product (over $\mathbb{Z}$) $M_1 \otimes M_2$ by the relations $\gamma \cdot m_1 \otimes m_2 = m_1 \otimes \gamma \cdot m_2$.

Remark 59. An easy exercise in the definitions shows that $H_k(\Gamma, \mathbb{Z}\Gamma)$ is trivial for all $k > 0$ and is equal to $\mathbb{Z}$ for $k = 0$. More generally, suppose that $\Gamma = G \oplus H$ and let us compute $H_*(\Gamma, \mathbb{Z}H)$, where $\mathbb{Z}H$ is the free $\mathbb{Z}$-module generated by H made into a Γ-module by the action $(g \oplus h) \cdot h' = h + h'$. Then we can identify

$$\Lambda_k\Gamma \otimes \mathbb{Z}\Gamma \otimes_\Gamma \mathbb{Z}H \cong \bigoplus_{i+j=k} \Lambda_i G \otimes \Lambda_j H \otimes \mathbb{Z}H \tag{6}$$

and under this identification $\partial \otimes 1$ becomes $(-1)^{deg} \otimes \partial'$, where ∂' is the boundary operator for the homology of H. It follows that

$$H_k(\Gamma, \mathbb{Z}H) \cong \bigoplus_{i+j=k} \Lambda_i G \otimes H_j(H, \mathbb{Z}H) = \Lambda_k G.$$

As a special case, $H_k(\Gamma, \mathbb{Z}) = \Lambda_k\Gamma \cong \mathbb{Z}^{\binom{N}{k}}$.

Now let $\Sigma\colon \mathbb{Z}H \to \mathbb{Z}$ be the $\mathbb{Z}\Gamma$ module homomorphism given by the sum of the coefficients, i.e. $\Sigma[h] = 1$ for all $h \in H$. We shall need the following lemma later.

Lemma 60. *Under the identifications $H_*(\Gamma, \mathbb{Z}H) \cong \Lambda_* G$ and $H_*(\Gamma, \mathbb{Z}) \cong \Lambda_*\Gamma$ the induced map $\Sigma_k\colon H_k(\Gamma, \mathbb{Z}H) \to H_k(\Gamma, \mathbb{Z})$ becomes the embedding $\Lambda_k G \hookrightarrow \Lambda_k\Gamma$.*

Proof. Using the decomposition (6) the induced map

$$\Sigma_k\colon \bigoplus_{i+j=k} \Lambda_i G \otimes H_j(H, \mathbb{Z}H) \to \bigoplus_{i+j=k} \Lambda_i G \otimes H_j(H, \mathbb{Z})$$

preserves the bidegree and must be the identity on the first factors in the tensor products. Since $H_k(H, \mathbb{Z}H)$ is trivial whenever $k \neq 0$ and one dimensional for $k = 0$, Σ_k can be determined by evaluating Σ_0 on the generator of $H_0(H, \mathbb{Z}H)$; the result follows. □

The basic tool in the calculations below is the following. Whenever we have a short exact sequence of $\mathbb{Z}\Gamma$-modules $0 \to A \stackrel{\varphi}{\to} B \stackrel{\psi}{\to} C \to 0$ we get a long exact sequence of homology groups

$$\cdots \stackrel{\psi_{k+1}}{\to} H_{k+1}(\Gamma, C) \stackrel{\gamma_{k+1}}{\to} H_k(\Gamma, A) \stackrel{\varphi_k}{\to} H_k(\Gamma, B) \stackrel{\psi_k}{\to} H_k(\Gamma, C) \cdots .$$

The maps φ_k and ψ_k are the induced homomorphisms and the γ_k are the connecting homomorphisms. For details see [29].

6.2. A CW-like complex. Let $\mathcal{C}'$ be an arbitrary countable collection of affine hyperplanes of F', a linear space, and define $\mathcal{C}'$-topes as before: compact polytopes which are the closures of their interiors and whose boundary faces belong to hyperplanes from $\mathcal{C}'$. For n at most the dimension of F' let $C^n_{\mathcal{C}'}$ be the $\mathbb{Z}$-module generated by the n-dimensional faces of convex $\mathcal{C}'$-topes satisfying the relations

$$[U_1] + [U_2] = [U_1 \cup U_2]$$

for any two faces U_1, U_2, for which $U_1 \cup U_2$ is as well a convex face and $U_1 \cap U_2$ has no interior (i.e. nonzero codimension in U_1). These relations then imply $[U_1] + [U_2] = [U_1 \cup U_2] + [U_1 \cap U_2]$ if $U_1 \cap U_2$ has interior. If we take $\mathcal{C}' = \mathcal{C}$, our collection of singular planes from Sect. 3, then $C^n := C^n_{\mathcal{C}}$ is a $\mathbb{Z}\Gamma$ module under the action $\gamma \cdot [U] = [U + \gamma]$. As $\mathbb{Z}\Gamma$-modules, $C^{d^\perp} \cong C_c(E^\perp_c, \mathbb{Z})$, the isomorphism being given by assigning to $[U]$ the indicator function on the closure of $U \cap NS$ (which is clopen). Moreover, C^0 is a free $\mathbb{Z}\Gamma$-module with basis in one to one correspondence with the intersection points $\mathcal{P}$.

Proposition 61. *There exist Γ-equivariant module maps δ and Σ such that*

$$0 \to C^{d^\perp} \stackrel{\delta}{\to} C^{d^\perp - 1} \stackrel{\delta}{\to} \cdots C^0 \stackrel{\Sigma}{\to} \mathbb{Z} \to 0, \tag{7}$$

is an exact sequence of Γ-modules and $\Sigma[U] = 1$ for all vertices U of $\mathcal{C}$-topes.

Proof. Let I be the indexing set for Γ orbit classes of singular planes. For a subset R of Γ (which we identified with $\Gamma^\perp \subset E^\perp$) let $\mathcal{C}_R := \{H_i + r | r \in R, i \in I\}$ and $S_R = \{x \in H | H \in \mathcal{C}_R\}$. Let $\mathcal{R}$ be the set of subsets $R \subset \Gamma$ such that all connected components of $E^\perp \backslash S_R$ are bounded and have interior. $\mathcal{R}$ is closed under union and hence forms an upper directed system under inclusion. For any $R \in \mathcal{R}$, the $\mathcal{C}_R$-topes define a regular polytopal CW-complex

$$0 \to C^{d^\perp}_{\mathcal{C}_R} \stackrel{\delta_R}{\to} C^{d^\perp-1}_{\mathcal{C}_R} \stackrel{\delta_R}{\to} \cdots C^0_{\mathcal{C}_R} \to 0, \tag{8}$$

with boundary operators δ_R depending on the choices of orientations for the n-cells ($n > 0$) [35]. Moreover, this complex is acyclic ($E^\perp$ is contractible), i.e. upon replacing $C^0_{\mathcal{C}_R} \to 0$ by $C^0_{\mathcal{C}_R} \stackrel{\Sigma_R}{\to} \mathbb{Z} \to 0$, where $\Sigma_R[U] = 1$, (8) becomes an exact sequence. Let us constrain the orientation of the n-cells in the following way: for each $n < d^\perp$ there are finitely many subsets $J \subset I$ such that $\dim \bigcap_{i\in J} H_i = n$ and J is maximal. Each n-cell belongs to a subspace parallel to one of the $\bigcap_{i\in J} H_i$ and we choose its orientation such that it depends only on the corresponding J (i.e. we choose an orientation for $\bigcap_{i\in J} H_i$ and then the cell inherits it as a subset). By the same principle, all $d^\perp$-cells are supposed to have the same orientation. Then the cochains and boundary operators δ_R share two crucial properties: first, if $R \subset R'$ for $R, R' \in \mathcal{R}$, then we may identify $C^n_{\mathcal{C}_R}$ with a submodule of $C^n_{\mathcal{C}_{R'}}$ and under this identification $\delta_R(x) = \delta_{R'}(x)$ for all $x \in C^n_{\mathcal{C}_R}$, and second, if U and $U + x$ are $\mathcal{C}_R$-topes then $\delta_R[U + x] = \delta_R[U] + x$. The first property implies that the directed system $\mathcal{R}$ gives rise to a directed system of acyclic cochain complexes, and hence its direct limit is an acyclic complex, and the second implies, together with the fact that for all $\gamma \in \Gamma$ and $R \in \mathcal{R}$ also $R + \gamma \in \mathcal{R}$, that this complex becomes a complex of Γ-modules. The statement now follows since $C^n_{\mathcal{C}}$ is the direct limit of $C^n_{\mathcal{C}_R}$ for all n. □

6.3. Solutions for $d^\perp = 1, 2$. Based on the results of the last two sections we now calculate the homology groups $H_k(\Gamma, C^{d^\perp})$ for $d^\perp = 1, 2$.

Lemma 62. *Given a CW-like complex as in Sect. 6.2,*

$$H_k(\Gamma, C^0) = \begin{cases} 0 & \textit{for} \quad k > 0, \\ \mathbb{Z}^L & \textit{for} \quad k = 0, \end{cases} \tag{9}$$

where L is the number of Γ-orbits of vertices of $\mathcal{C}$-topes, i.e. $L = |\Omega(\mathcal{P})|$.

Proof. Since Γ acts fixpoint-freely we have $\Lambda\Gamma \otimes \mathbb{Z}\Gamma \otimes_\Gamma C^0 \cong \Lambda\Gamma \otimes \mathbb{Z}\Gamma \otimes \mathbb{Z}^L$ which directly implies the result. □

Theorem 63. *Let $\mathcal{T}$ be a d-dimensional canonical projection tiling of codimension* 1.

$$H^{d-k}(\mathcal{T}) \cong \begin{cases} \mathbb{Z}^{\binom{d+1}{k+1}} & \textit{for} \quad k > 0, \\ \mathbb{Z}^{d+L} & \textit{for} \quad k = 0. \end{cases}$$

Proof. In the case $d^\perp = 1$, (7) is the short exact sequence

$$0 \to C^1 \stackrel{\delta}{\to} C^0 \stackrel{\Sigma}{\to} \mathbb{Z} \to 0 \tag{10}$$

and we use the resulting long exact sequence of homology groups for the computation. By the last lemma, apart from the lowest degree every third homology group in that sequence is trivial so that $H_k(\Gamma, C^1) \cong H_{k+1}(\Gamma, \mathbb{Z})$ for $k > 0$. The remaining part of the sequence has the form $0 \to \mathbb{Z}^{d+1} \to H_0(\Gamma, C^1) \to \mathbb{Z}^L \to \mathbb{Z} \to 0$ and hence $H_0(\Gamma, C^1) = \mathbb{Z}^{d+L}$ as claimed. □

Note that at this stage (for very low codimension) we did not need to know explicitly the morphisms involved.

Recall the description of the topology of $E^\perp$ for canonical projection tilings by singular planes. These planes were organized in Γ-orbits, indexed by a finite set I, and we choose representatives H_α, for each $\alpha \in I$.

Theorem 64. *Let $\mathcal{T}$ be a d-dimensional canonical projection tiling of codimension* 2,

$$H^{d-k}(\mathcal{T}) \cong \begin{cases} \mathbb{Z}^{\binom{d+2}{k+2} - r_k - r_{k+1} + \sum_{\alpha \in I} \binom{\nu_\alpha}{k+1}} & \textit{for} \quad k > 0, \\ \mathbb{Z}^{\binom{d+2}{2} - d - L - 1 - r_1 + \sum_{\alpha \in I} (\nu_\alpha + l_\alpha - 1)} & \textit{for} \quad k = 0, \end{cases} \tag{11}$$

where ν_α is the rank of Γ^α (the stabilizer of H_α), l_α the number of Γ^α-orbits of intersection points in H_α, and r_k the rank of the module generated by the submodules $\Lambda_{k+1}\Gamma^\alpha \subset \Lambda_{k+1}\Gamma$ for all $\alpha \in I$.

Proof. Inserting $C_0^0 := \delta(C^1)$ we break the exact sequence (7) into two short exact ones

$$\begin{tikzcd}
0 \arrow[r] & C^2 \arrow[r, "\delta"] & C^1 \arrow[rr, "\delta"] \arrow[dr] & & C^0 \arrow[r, "\Sigma"] & \mathbb{Z} \arrow[r] & 0. \\
& & & C_0^0 \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

$0 \to C_0^0 \to C^0 \to \mathbb{Z} \to 0$ can be treated as in the codimension 1 case. Taking into account that the rank of Γ is $d + 2$ one gets

$$H_k(\Gamma, C_0^0) \cong \begin{cases} \mathbb{Z}^{\binom{d+2}{k+1}} & \text{for} \quad k > 0, \\ \mathbb{Z}^{d+L+1} & \text{for} \quad k = 0. \end{cases} \tag{12}$$

Let us have a closer look at C^1. For n at most 1 let C_α^n be the sub-module of C^n generated by the n-dimensional faces which belong to H_α, $\alpha \in I$. As before we denote by Γ^α the stabilizer of H_α and we let $\hat{\Gamma}^\alpha$ be a complementary subgroup, i.e. $\Gamma = \Gamma^\alpha \oplus \hat{\Gamma}^\alpha$ (recall that Γ / Γ^α has no torsion). Then

$$C^1 \cong \bigoplus_{\alpha \in I} C_\alpha^1 \otimes \mathbb{Z}\hat{\Gamma}^\alpha, \tag{13}$$

because any 1-dimensional face belongs to a translate of some H_α. Moreover the action of $\Gamma^\alpha \oplus \hat{\Gamma}^\alpha$ on C^1 is such that the first summand acts non-trivially only on the first factors, C_α^1, and the second only on the second factors, $\mathbb{Z}\hat{\Gamma}^\alpha$. In particular, $\mathbb{Z}\Gamma \otimes_\Gamma C^1 \cong \bigoplus_{\alpha \in I} \mathbb{Z}\Gamma^\alpha \otimes_{\Gamma^\alpha} C_\alpha^1 \otimes \mathbb{Z}\hat{\Gamma}^\alpha$ as Γ-modules which implies

$$H_*(\Gamma, C^1) \cong \bigoplus_{\alpha \in I} H_*(\Gamma^\alpha, C_\alpha^1). \tag{14}$$

Restricting the boundary maps δ and Σ to C^n_α we get a short exact sequence

$$0 \to C^1_\alpha \overset{\delta_\alpha}{\to} C^0_\alpha \overset{\Sigma_\alpha}{\to} \mathbb{Z} \to 0. \tag{15}$$

As in Theorem 63 and combined with Eq. (14) we obtain

$$H_k(\Gamma, C^1) \cong \begin{cases} \mathbb{Z}^{\sum_{\alpha\in I}\binom{\nu_\alpha}{k+1}} & \text{for} \quad k > 0, \\ \mathbb{Z}^{\sum_{\alpha\in I}(\nu_\alpha + l_\alpha - 1)} & \text{for} \quad k = 0, \end{cases} \tag{16}$$

where ν_α and l_α are as defined in the statement of the theorem. Note that the l_α are all finite since we required L to be finite. Equations (12, 16) give us part of the information needed to determine $H_*(\Gamma, C^2)$ from the exact sequence

$$0 \to C^2 \overset{\delta}{\to} C^1 \overset{\delta}{\to} C^0_0 \to 0, \tag{17}$$

but we have to determine explicitly one further morphism since we have no longer enough trivial groups in the resulting long exact sequence. We shall determine the induced morphism

$$\beta_* := \delta_* : H_*(\Gamma, C^1) \to H_*(\Gamma, C^0_0). \tag{18}$$

Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 \to & C^1_\alpha \otimes \mathbb{Z}\hat{\Gamma}^\alpha & \overset{\delta_\alpha\otimes 1}{\to} & C^0_\alpha \otimes \mathbb{Z}\hat{\Gamma}^\alpha & \overset{\Sigma_\alpha\otimes 1}{\to} & \mathbb{Z}\hat{\Gamma}^\alpha & \to 0 \\ & \downarrow \delta_\alpha \otimes 1 & & \downarrow & & \downarrow \Sigma^\alpha & \\ 0 \to & C^0_0 & \hookrightarrow & C^0 & \overset{\Sigma}{\to} & \mathbb{Z} & \to 0 \end{array}$$

where the middle vertical arrow is the inclusion, the right vertical arrow the sum of the coefficients, $\Sigma^\alpha[\gamma] = 1$, and the left vertical arrow the map of interest. In fact, β_k is the direct sum over all α of $(\delta_\alpha \otimes 1)_k : H_k(\Gamma, C^1_\alpha \otimes \mathbb{Z}\hat{\Gamma}^\alpha) \to H_k(\Gamma, C^0_0)$. This diagram gives rise to two long exact sequences of homology groups together with vertical maps, all commuting, $(\delta_\alpha \otimes 1)_*$ being one of them. Now use that for $k > 0$, $H_k(\Gamma, C^0_\alpha \otimes \mathbb{Z}\hat{\Gamma}^\alpha) = H_k(\Gamma, C^0) = 0$ so that we can express $(\delta_\alpha \otimes 1)_*$ through Σ^α_*. In fact, the triviality of these groups imply that $H_k(\Gamma, C^0_\alpha \otimes \mathbb{Z}\hat{\Gamma}^\alpha) \cong H_{k+1}(\Gamma, \mathbb{Z}\hat{\Gamma}^\alpha)$ and $H_k(\Gamma, C^0_0) \cong H_{k+1}(\Gamma, \mathbb{Z})$, for $k > 0$, and with these identifications

$$(\delta_\alpha \otimes 1)_k = \Sigma^\alpha_{k+1}.$$

By Lemma 60 the map Σ^α_k becomes the embedding $\Lambda_k\Gamma^\alpha \hookrightarrow \Lambda_k\Gamma$ under the above identifications. For $k > 0$ therefore, the rank of β_k is equal to the rank of the span of the submodules $\Lambda_{k+1}\Gamma^\alpha$, $\alpha \in I$, in $\Lambda_{k+1}\Gamma$, the number defined as r_k in the statement of the theorem. The long exact sequence corresponding to (17) implies

$$H_k(\Gamma, C^2) \cong H_{k+1}(\Gamma, C^0_0)/\mathrm{im}\,\beta_{k+1} \oplus H_k(\Gamma, C^1) \cap \ker \beta_k.$$

Since, for $k > 0$, $\dim H_k(\Gamma, C^1) \cap \ker \beta_k = \dim H_k(\Gamma, C^1) - r_k$ we get the desired result (the case $k = 0$ is similar), provided the homology groups are torsion free. That this is the case we know from [12]. $\square$

6.4. Example: octagonal tilings. We provide here one example, the octagonal tilings. A whole list of results for codimension 2 tilings could be obtained by evaluating (11) with a computer [36].

The (undecorated) octagonal tilings are two dimensional tilings which may be constructed from the data $(\mathbb{Z}^4, 0, E)$, the four dimensional integer lattice $\mathbb{Z}^4$ (with standard basis $\{e_i\}_{i=1,\ldots,4}$) and the two dimensional invariant subspace of the eightfold symmetry C_8: $e_i \mapsto e_{i+1}$ for $i = 1, 2, 3$ and $e_4 \mapsto -e_1$ (the group C_8 acts as rotation by $\frac{\pi}{4}$) [37, 38]. It consists of squares and 45^0-rhombi all edges having equal length. $E^\perp$ is, of course, also an invariant subspace of the eightfold symmetry and the singular planes (which are lines) are well known, they are the tangents to the boundary faces of the projection of the unit cube into $E^\perp$ which is a regular octagon. They are translates under $\pi^\perp(\mathbb{Z}^4)$ of the four lines spanned by $e_i^\perp$ which form an orbit under C_8 (we may ignore the shift by δ). From these lines we get all our information, the numbers L, ν_i, l_i, $I = \{1, \ldots, 4\}$, and r_1, r_2, r_3 (higher r_k are unecessary since $d = 2$). Usually it is not so easy to determine L but in our case it is easy to see that apart from the orbit of the intersection point at 0 there are only two other ones: the orbit of $\frac{1}{\sqrt{2}}(e_1^\perp + e_3^\perp)$ and that of $\frac{1}{\sqrt{2}}(e_2^\perp + e_4^\perp)$. Hence $L = 3$. Clearly, Γ^1 is spanned by $e_1^\perp$ and $e_2^\perp - e_4^\perp$ and hence $\nu_1 = 2$ and $l_1 = 2$ which carries over to all i by symmetry. Finally, $r_1 = 3$ and $r_k = 0$ for $k \geq 2$ as $\nu_i = 2$. Inserting the numbers yields

$$H^0(\mathcal{T}) = \mathbb{Z}, \quad H^1(\mathcal{T}) = \mathbb{Z}^5, \quad H^2(\mathcal{T}) = \mathbb{Z}^9.$$

This result is in agreement with a calculation we made using Anderson and Putnam's method [9] for substitution tilings: the octagonal tiling is also substitutional, its substitution is given in Fig. 1 of Sect. 5.1.

7. The Non-Commutative Approach

We conclude by connecting the cohomology of a tiling, as we have been discussing, with its non-commutative topological invariants. The starting point of the non-commutative approach is the observation that the orbit spaces of the dynamical systems arising from the tiling are non-Hausdorff. In fact, for a (completely) non-periodic tiling $\mathcal{T}$, no two points in $M\mathcal{T}/\mathbb{R}^d$ can be separated by open neighbourhoods. Connes' non-commutative geometry was motivated by the desire to analyse such spaces. In the non-commutative topological approach [39] one studies the properties of the (non-commutative) C^* algebra associated with the dynamical system $(M\mathcal{T}, \mathbb{R}^d)$. This algebra is the crossed product algebra of $C(M\mathcal{T})$, the algebra of continuous functions over $M\mathcal{T}$, with the group $\mathbb{R}^d$. We denote it by $C(M\mathcal{T}) \times \mathbb{R}^d$. Topologically, this algebra may be described by its K-theory [40, 41]. It turns out that the K-groups are closely related to the Czech-cohomology of $M\mathcal{T}$. The K-groups, however, contain additional information in the form of a natural order structure on the K_0-group and this is the advantage of the non-commutative approach. We have seen in Example 46 that cohomology without extra structure is not a very fine invariant.

Equally well mathematically, but from a more physically motivated point of view, we can work with the formulation of the quotient $M\mathcal{T}/\mathbb{R}^d$ as the space of orbits of the tiling groupoid $\mathcal{G}_\mathcal{T}$ (or of $\mathcal{GT}$). The C^* algebra whose K-theory provides the non-commutative topological invariant is then the corresponding groupoid-C^* algebra [26, 15]. The importance of this groupoid C^* algebra for physical systems lies in the fact that it provides an abstract definition of the algebra of observables [15,10] for particles moving in the tiling; the scaled ordered K_0-group and its image under a tracial state governs the gap labelling.

If $\mathcal{T}$ is a canonical projection tiling $\mathcal{G}\mathcal{T}$ and $\mathcal{G}_{\mathcal{T}}$ are equivalent in the sense of Muhly et al. to the transformation groupoid $\mathcal{G}(X, G_1)$. This is proven directly in [16] but it also follows from our analysis of Sect. 3.5 where similarity of the two groupoids has been shown. By application of the theory of Muhly et al. [27] we obtain

Theorem 65. *The K-groups of $C(M\mathcal{T}) \times \mathbb{R}^d$ and of the groupoid-C^* algebras of $\mathcal{G}_{\mathcal{T}}$ and of $\mathcal{G}(X, G_1)$ are isomorphic, the isomorphism preserving the order on the K_0-group.*

The isomorphism between the first two K-groups was already observed in [9]. Of particular importance for the present case is the following relationship between K-theory and cohomology proved in [12]: if $(X, \mathbb{Z}^d)$ is a minimal $\mathbb{Z}^d$-dynamical system where X is homeomorphic to the Cantor set then

$$K_i(C(X) \times \mathbb{Z}^d) \cong \bigoplus_j H^{d-i+2j}(\mathbb{Z}^d, C(X, \mathbb{Z}))$$

as unordered groups. Thus, in view of Corollary 41,

Corollary 66. *For a canonical projection tiling $\mathcal{T}$,*

$$K_i(C^*(\mathcal{G}\mathcal{T})) \cong \bigoplus_j H^{d-i+2j}(\mathcal{T})$$

as unordered groups.

It is an interesting question whether this result is true for finite type tilings in general. As already mentioned, the isomorphism of the corollary neglects the information contained in the order structure on the K_0-group. One can cure for this at least partly by looking at the order on $H^d(\mathcal{T})$, the group of coinvariants, which is induced by the unique invariant probability measure on $\Omega\mathcal{T}$ (the dynamical system $(M\mathcal{T}, \mathbb{R}^d)$ is uniquely ergodic). That measure defines a group homomorphism $C_c(E_c^{\perp}, \mathbb{Z}) \to \mathbb{R}$ which by invariance induces a homomorphism $\tau : H^d(\mathcal{T}) \to \mathbb{R}$. The subset $\tau^{-1}(\mathbb{R}^{>0})$ is closed under addition and defines a positive cone of $H^d(\mathcal{T})$ which sits inside the positive cone of $K_0(C^*(\mathcal{G}\mathcal{T}))$ and contains already a good portion of the information, including that needed for the standard gap-labelling. In fact, for $d = 1$, where $H^1(\mathcal{T}) = K_0(C^*(\mathcal{G}\mathcal{T}))$, this order is precisely the order defined on the K_0-group in the standard way [40].

With this information at hand let us come back to Example 46, the canonical projection tiling with data $W = \mathbb{Z}^2$, $w = 0$, $d = 1$, and E specified by an irrational number ν. To keep track of this dependence we write $\mathcal{T}^{(\nu)}$ for a canonical projection tiling obtained from such data. The unique invariant probabibity measure on $\Omega\mathcal{T}^{(\nu)}$ is the pull back under μ of the Lebesgue measure on $E^{\perp}$ normalized in such a way that $\pi^{\perp}(\gamma)$ (the projection of the unit cell) has measure 1. From this we see that with $[1_{[a,b]}]$ denoting the coinvariant class of $1_{[a,b]}$,

$$\tau([1_{[a,b]}]) = \frac{b-a}{1+\nu}.$$

In particular, the rank of $\tau(H^1(\mathcal{T}^{(\nu)}))$ is 2 and hence $H^1(\mathcal{T}^{(\nu)}) \cong \mathbb{Z}^2$. Now, $\tau(n[1_{[0,1]}] + m[1_{[0,\nu]}]) > 0$, for $n, m \in \mathbb{Z}$, whenever (n, m) has positive scalar product with $(1, \nu)$ and hence belongs to the upper right half space defined by $E^{\perp}$ in $\mathbb{R}^2$. It follows that $K_0(\mathcal{G}\mathcal{T}^{(\nu)})$

is order isomorphic to $K_0(\mathcal{GT}^{(\nu')})$ whenever there exists a matrix $M \in GL(2,\mathbb{Z})$ such that $\nu' = \frac{M_{11}\nu+M_{12}}{M_{21}\nu+M_{22}}$. Note that in the above cases τ is injective. We remark without further explanation that the order unit improves the invariant even more. $K_0(\mathcal{GT}^{(\nu)})$ and $K_0(\mathcal{GT}^{(\nu')})$ are order isomorphic with isomorphism preserving the order unit if and only if $\nu' = \pm\nu$.

Returning to Example 47, the canonical projection tiling with data $W = \mathbb{Z}^3$, $w = 0$, $d = 1$, the unique invariant probability measure on $\Omega\mathcal{T}$ is again the pull back under μ of the Lebesgue measure on $E^\perp$ normalized in such a way that $\pi^\perp(\gamma)$ has measure 1. Thus all the elements $[1_{U+\lambda_1+\lambda_2}] - [1_U]$ are mapped to 0 by τ. In fact, one can show that the image of τ is finitely generated so that in this case all but finitely many generators of the K_0-group are neither positive nor negative, i.e. that almost all are infinitesimal.

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References

1. Steinhardt, P.J. and Ostlund, S.: *The Physics of Quasicrystals*. Singapore: World Scientific, 1987
2. Janot, C. and Mosseri, R.: *Proc. 5th Int. Conf. on Quasicrystals*. Singapore: World Scientific, 1995
3. Axel, F. and Gratias, D.: *Behond Quasicrystals*. Berlin–Heidelber–New York: Springer, 1995
4. Moody, R.V.: *The Mathematics of Long Range Aperiodic Order*. Dordrecht: Kluwer, 1997
5. Duneau, M. and Katz, A.: Quasiperiodic patterns and icosahedral symmetry. J. Physique **47**, 181–196 (1986)
6. Oguey, C., Katz, A. and Duneau, M.: A geometrical approach to quasiperiodic tilings. Commun. Math. Phys. **118**, 99–118 (1988)
7. de Bruijn, N.G.: Algebraic theory of Penrose's nonperiodic tilings of the plane. Kon. Nederl. Akad. Wetensch. Proc. Ser. **A84**, 38–66 (1981)
8. Kramer, P. and Schlottmann, M.: Dualisation of Voronoi domains and Klotz construction: a general method for the generation of quasiperiodic tilings. J. Phys. **A 22**, L1097 (1989)
9. Anderson, J.E. and Putnam, I.F.: Topological invariants for substitution tilings and their associated C^*-algebras. Ergod. Th. and Dynam. Sys. **18**, 509–537 (1998)
10. Kellendonk, J.: The local structure of tilings and their integer group of coinvariants. Commun. Math. Phys. **187**(1), 115–157 (1997)
11. Le, T.T.Q.: Local rules for quasiperiodic tilings. In: R.V. Moody (ed.) *The Mathematics of Long Range Aperiodic Order*. Dordrecht: Kluwer, 1997, pp. 331–366
12. Forrest, A.H. and Hunton, J.: The cohomology and K-theory of commuting homeomorphisms of the Cantor set. Ergod. Th. and Dynam. Sys. **19**, 611–625 (1999)
13. Bellissard, J.: Gap labelling theorems for Schrödinger's operators. In: Waldschmidt, M., Moussa, P., Luck, J.M. and Itzykson, C. (eds.) *From Number Theory to Physics*. Berlin–Heidelberg–New York: Springer-Verlag, 1992, pp. 538–630
14. Bellissard, J., Bovier, A. and Ghez, J.M.: Gap labelling theorems for one dimensional discrete Schrödinger operators. Rev. Math. Phys. **4**, 1–38 (1992)
15. Kellendonk, J.: Non commutative geometry of tilings and gap labelling. Rev. Math. Phys. **7**, 1133–1180 (1995)
16. Forrest, A.H., Hunton, J. and Kellendonk, J.: Projection quasicrystals I: Toral rotations. SFB-preprint No. 340, 1998
17. Forrest, A.H., Hunton, J. and Kellendonk, J.: Projection quasicrystals II: Versus substitutions. SFB-preprint No. 396, 1999
18. Forrest, A.H., Hunton, J.R. and Kellendonk, J.: Projection quasicrystals III: Cohomology. SFB-preprint No. 459, 2000

19. Forrest, A.H., Hunton, J.R. and Kellendonk, J.: Topological invariants for projection method patterns. To appear in Mem. Amer. Math. Soc.
20. Schlottmann, M.: Periodic and quasi-periodic Laguerre tilings. Int. J. Mod. Phys. B **7**, 1351–1363 (1993)
21. Bellissard, J.,Contensou, E. and Legrand, A.L.: K-théorie des quasi-cristeaux, image par la trace: le cas du réseau octogonal. C. R. Acad. Sci. Paris, Série I **326**, 197–200 (1998)
22. Rudolph, D.J.: Markov tilings of $\mathbb{R}^n$ and representations of $\mathbb{R}^n$ actions. Contemporary Mathematics **94**, 271–290 (1989)
23. Radin, C. and Wolff, M.: Space tilings and local isomorphism. Geom. Ded. **42**, 355–360 (1992)
24. Radin, C.: The Pinwheel tilings of the plane. Annals of Math. **139**, 661–702 (1994)
25. Kellendonk, J.: Topological equivalence of tilings. J. Math. Phys. **38** (4), 1823–1842 (1997)
26. Renault, J.: *A Groupoid approach to C^*-Algebras*. Lecture Notes in Math. **793**. Berlin–Heidelberg–New York: Springer-Verlag, 1980
27. Muhly, P.S., Renault, J.N. and Williams, D.P.: Equivalence and isomorphism for groupoid C^*-algebras. J. Operator Theory **17**, 3–22 (1987)
28. Renault, J.: Private communication.
29. Brown, K.S.: *Cohomology of Groups*. Berlin–Heidelberg–New York: Springer-Verlag, 1982
30. Baake, M., Joseph, D., Kramer, P. and Schlottmann, M.: Root lattices and quasicrystals. J. Phys. A: Math. Gen. **23**, L1037–L1041 (1990)
31. Gähler, F. and Stampfli, P.: The dualisation method revisited: dualisation of product Laguerre complexes as a unifying framework. Int. J. Mod. Phys. B **7**, 1333–1349 (1993)
32. Baake, M., Hermisson, J. and Pleasants, P.: The torus parametrization of quasiperiodic LI-classes. J. Phys. **A 30**, 3029–3056 (1997)
33. Mingo, J.A.: C^*-algebras associated with one-dimensional almost periodic tilings. Commun. Math. Phys. **183**, 307–337 (1997)
34. Pleasants, P.A.B.: The construction of quasicrystals with arbitrary symmetry group. In: Janot, C. and Mosseri, R. (eds): Proc. 5th Int. Conf. on Quasicrystals. Singapore: World Scientific, 1995. pp. 22–30
35. Massey, W.S: *A Basic Course in Algebraic Topology*. Berlin–Heidelberg–New York: Springer-Verlag, 1991
36. Gähler, F. and Kellendonk, J.: Cohomology groups for projection tilings of codimension 2. Material Science and Engineering **294–296**, 438–440 (2000)
37. Beenker, F.P.M.: Algebraic theory of non-periodic tilings of the plane by two simple building blocks: A square and a rhombus. Thesis, Techn. Univ. Eindhoven, TH-report 82-WSK-04, 1982
38. Socolar, J.E.S. Simple octagonal and dodecagonal quasicrystals. Phys. Rev. B **39**, (15), 10519–10551 (1989)
39. Connes, A.: *Non Commutative Geometry*. London–New York: Academic Press, 1994
40. Blackadar, B.: *K-Theory for Operator Algebras*. MSRI Publications 5. Berlin–Heidelberg–New York: Springer-Verlag, 1986
41. Wegge-Olson, N.E.: *K-theory of C^*-algebras. A friendly approach.* Oxford: Oxford University Press, 1993

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