Commun. Math. Phys. 249, 611–637 (2004)
Digital Object Identifier (DOI) 10.1007/s00220-004-1122-7

Communications in
Mathematical Physics

Boundary Maps for C^*-Crossed Products with $\mathbb{R}$ with an Application to the Quantum Hall Effect

J. Kellendonk[1], **H. Schulz-Baldes**[2]

[1] School of Mathematics, Cardiff University, Cardiff, CF24 4YH, Wales, U.K.
[2] Institut für Mathematik, TU Berlin, Strasse des 17. Juni 136, 10623 Berlin, Germany

Received: 24 July 2003 / Accepted: 23 January 2004
Published online: 18 June 2004 –

Abstract: The boundary map in K-theory arising from the Wiener-Hopf extension of a crossed product algebra with $\mathbb{R}$ is the Connes-Thom isomorphism. In this article the Wiener Hopf extension is combined with the Heisenberg group algebra to provide an elementary construction of a corresponding map on higher traces (and cyclic cohomology). It then follows directly from a non-commutative Stokes theorem that this map is dual w.r.t. Connes' pairing of cyclic cohomology with K-theory. As an application, we prove equality of quantized bulk and edge conductivities for the integer quantum Hall effect described by continuous magnetic Schrödinger operators.

1. Motivation and Main Result

In a commonly used approach to study aperiodic solids, particles in the bulk of the medium are described by covariant families of one-particle Schrödinger operators $\{H_\omega\}_{\omega\in\Omega}$, where Ω is the probability space of configurations furnished with an ergodic action of space translations. Crossed product algebras provide a natural framework for such families [Be86]. In particular their bounded functions are represented by elements of a C^*-crossed product, the so-called bulk algebra. The non-commutative topology of the C^*-algebra is a useful tool to construct topological invariants resulting from pairings between K-group elements and higher traces. Some of these invariants may be physically interpreted as topologically quantised quantities; the quantised Hall conductivity is such an example. The physics near a boundary of the solid can also be described by a C^*-algebra, the so-called edge algebra. The bulk algebra being essentially a crossed product of the edge algebra with $\mathbb{R}$ (or with $\mathbb{Z}$ in the tight binding approximation [KRS02]), both algebras are tied together in the Wiener-Hopf extension (or respectively the Toeplitz extension). This extension gives rise to boundary maps between the K-groups and the higher traces of the bulk and edge algebra which allow one to equate bulk and edge invariants. This topological relation and its physical interpretations is our main objective. We discuss one prominent physical example of this, the quantum Hall effect, where the

Hall conductivity may either be expressed as the Chern number of a spectral projection associated with a gap in the bulk spectrum [Be86, AS85, K87, ASS94, Be88, BES94] or as the non-commutative winding number of the unitary of time translation of the edge states corresponding to the gap by a characteristic time (the inverse of the gap width) [KRS02, KS03]. The mathematical background for this equality between bulk and edge invariants for continuous Schrödinger operators is the subject of the present article.

The mathematical framework is as follows. Consider an $\mathbb{R}$-action α on a C^*-algebra $\mathcal{B}$. Denote by τ the translation action of $\mathbb{R}$ on the half open space $\mathbb{R} \cup \{+\infty\}$ (with fixed point $+\infty$). This defines a crossed product C^*-algebra $\mathcal{B} \rtimes_\alpha \mathbb{R}$ and an extension of this C^*-algebra by another crossed product, $C_0(\mathbb{R} \cup \{+\infty\}, \mathcal{B}) \rtimes_{\tau\otimes\alpha} \mathbb{R}$, the so-called Wiener-Hopf extension. They form an exact sequence

$$0 \longrightarrow \mathcal{K} \otimes \mathcal{B} \longrightarrow C_0(\mathbb{R} \cup \{+\infty\}, \mathcal{B}) \rtimes_{\tau\otimes\alpha} \mathbb{R} \stackrel{\mathrm{ev}_\infty}{\longrightarrow} \mathcal{B} \rtimes_\alpha \mathbb{R} \longrightarrow 0, \qquad (1)$$

where ev_∞ is induced from the surjective homomorphism $C_0(\mathbb{R} \cup \{+\infty\}, \mathcal{B}) \to \mathcal{B}$ given by evaluating $f \in C_0(\mathbb{R} \cup \{+\infty\}, \mathcal{B})$ at $+\infty$ and $\mathcal{K}$ are the compact operators on $L^2(\mathbb{R})$. Rieffel has shown [R82] that the boundary maps $\partial_i : K_i(\mathcal{B} \rtimes_\alpha \mathbb{R}) \to K_{i+1}(\mathcal{B})$ in the corresponding six-term exact sequence are the inverses of the Connes-Thom isomorphism [C81]. In the physical context described above, the boundary maps relate the K-groups of the bulk algebra with the K-groups of the edge algebra.

In the context of smooth crossed products, where $\mathcal{B}$ is a Fréchet algebra with smooth action α so that one obtains a smooth version of (1), Elliott, Natsume and Nest [ENN88] have given dual boundary maps for cyclic cohomology groups, namely isomorphisms $\#_\alpha : HC^n(\mathcal{B}) \to HC^{n+1}(\mathcal{B} \rtimes_\alpha \mathbb{R})$ which satisfy

$$\langle \#_\alpha \eta, x \rangle = -\frac{1}{2\pi} \langle \eta, \partial_i x \rangle , \qquad \eta \in HC^{i-1+2n}(\mathcal{B}) , \qquad x \in K_i(\mathcal{B} \rtimes_\alpha \mathbb{R}) , \quad (2)$$

where $\langle \cdot, \cdot \rangle$ denotes Connes' pairing between cyclic cocycles and K-group elements.

Our aim here is to obtain the same kind of result for α-invariant higher traces on C^*-algebras. One reason for doing this is that, whereas our estimates from [KS03] show that the operators relevant in the physical context described above lie in C^*-crossed products it is not clear whether they belong to the smooth sub-algebras used in [ENN88]. Another reason is to present a different proof with, as we believe, considerably simpler algebraic constructions so that it should henceforth be more easily accessible also to the non-expert. In fact, for the phsyical interpretation of Eq. (2) it is indispensible that all isomorphisms involved can be made explicit. In particular, it is essential that we can compute the boundary map ∂_0 on (classes of) spectral projections of the Schrödinger operator on gaps. Our proof establishes (2) directly for $i = 0$, the case needed for the application to the quantum Hall effect, whereas the equality is proven in [ENN88] first for $i = 1$ and then extended to $i = 0$ using the Takai duality and Connes' Thom isomorphism. Hence the non-expert reader can understand our result without prior knowledge, for instance, of Connes' Thom isomorphism. The proof we present uses continuous fields of C^*-algebras and is inspired by another article of Elliott, Natsume and Nest [ENN93].

More precisely, the result can be described as follows. An n-trace on a Banach algebra $\mathcal{B}$ is the character of an (unbounded) n-cycle $(\Omega, \int, d)$ over $\mathcal{B}$ having further continuity properties (*cf.* Def. 2). It is called α-invariant if α extends to an action of $\mathbb{R}$ on the graded differential algebra (Ω, d) by isomorphisms of degree 0 and $\int \circ \alpha = \int$ (and the above-mentioned continuity properties are α-invariant, *cf.* Def. 3). Let η be an n-trace which is the character of an α-invariant cycle $(\Omega, \int, d)$ over $\mathcal{B}$. We prove that

$$\#_\alpha \eta(f_0, \dots, f_{n+1}) = \sum_{k=1}^{n+1} (-1)^k \int (f_0 df_1 \cdots \nabla f_k \cdots df_{n+1})(0),$$
$$\nabla f(x) = \imath x f(x) \, ,$$

is an $n+1$-trace on the L^1-crossed product $L^1(\mathbb{R}, \mathcal{B}, \alpha)$. Furthermore, if $\mathcal{B}$ is a C^*-algebra then the pairing with $\#_\alpha \eta$ extends to the K-group of the C^*-crossed product $\mathcal{B} \rtimes_\alpha \mathbb{R}$ and satisfies the duality equation (2).

Sections 2 to 5 are devoted to explain the mathematical context and to prove the above result (Theorem 2 and Theorem 6). Theorem 6 follows from two main arguments, a homotopy argument (Theorem 5) and periodicity in cyclic cohomology. Although the latter is well-known we have added a detailed proof of its version adapted to our context (Theorem 4) in the appendix, hence making this work self-contained. In Sect. 6 we discuss the application of this result to the quantum Hall effect.

2. C^*-Algebraic Preliminaries

2.1. Crossed products by $\mathbb{R}$. Let $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{B})$ be an action of $\mathbb{R}$ on a C^*-algebra $\mathcal{B}$. It is required to be continuous in the sense that for all $A \in \mathcal{B}$, the function $x \in \mathbb{R} \mapsto \alpha_x(A)$ is continuous. The crossed product algebra $\mathcal{B} \rtimes_\alpha \mathbb{R}$ of $\mathcal{B}$ with respect to the action α of $\mathbb{R}$ is defined as follows [P79]. The linear space $C_c(\mathbb{R}, \mathcal{B})$ of compactly supported continuous functions with values in $\mathcal{B}$ is endowed with the $*$-algebra structure

$$(fg)(x) \;=\; \int_{\mathbb{R}} dy\, f(y)\, \alpha_y(g(x-y)) \, , \qquad f^*(x) \;=\; \alpha_x(f(-x))^* \, . \tag{3}$$

The L^1-completion of $C_c(\mathbb{R}, \mathcal{B})$, i.e. completion w.r.t. the norm $\|f\|_1 := \int_{\mathbb{R}} dx\, \|f(x)\|_{\mathcal{B}}$, is a Banach algebra, the L^1-crossed product denoted $L^1(\mathbb{R}, \mathcal{B}, \alpha)$. The crossed product algebra $\mathcal{B} \rtimes_\alpha \mathbb{R}$ is the completion of $L^1(\mathbb{R}, \mathcal{B}, \alpha)$ w.r.t. the C^*-norm $\|f\| := \sup_\rho \|\rho(f)\|$, where the supremum is taken over all bounded $*$-representations. It is not necessary to perform the middle step via the L^1-crossed product, but it is sometimes convenient to work with it when verifying that the integral kernel of a given operator belongs to $\mathcal{B} \rtimes_\alpha \mathbb{R}$. In this spirit, we can benefit in Sect. 6.2 from our results in [KS03]. By a continuity argument, one can simply work with functions $f : \mathbb{R} \to \mathcal{B}$ when performing calculations with elements of $\mathcal{B} \rtimes_\alpha \mathbb{R}$.

Let $(\rho, \mathcal{H})$ be a representation of $\mathcal{B}$. It induces a representation $(\pi, L^2(\mathbb{R}, \mathcal{H}))$ of $\mathcal{B} \rtimes_\alpha \mathbb{R}$:

$$(\pi(f)\psi)(x) \;=\; \int_{\mathbb{R}} dy\, \rho(\alpha_{-x}(f(x-y)))\psi(y) \, . \tag{4}$$

2.2. C^*-fields. We follow the exposition of [L98] in defining a continuous field of C^*-algebras (or simply a C^*-field) $(\mathcal{C}, \{\mathcal{C}^\hbar, \varphi_\hbar\}_{\hbar \in I})$ over a locally compact Hausdorff space I. This consists of a C^*-algebra $\mathcal{C}$ (also called the total algebra of the field), a collection of C^*-algebras $\{\mathcal{C}^\hbar\}_{\hbar \in I}$, one for each point of the space I, with surjective algebra homomorphisms $\varphi_\hbar : \mathcal{C} \to \mathcal{C}^\hbar$ such that,

1. for $a \in \mathcal{C}$, $\|a\| = \sup_{\hbar \in I} \|\varphi_\hbar(a)\|$,
2. for all $a \in \mathcal{C}$, $\hbar \mapsto \|\varphi_\hbar(a)\|$ is a function in $C_0(I)$,
3. $\mathcal{C}$ is a left $C_0(I)$ module and, for $f \in C_0(I)$, $a \in \mathcal{C}$ we have $\varphi_\hbar(fa) = f(\hbar)\varphi_\hbar(a)$.

The construction is reminiscent of a fibre bundle, except there is no typical fibre, the algebras $\mathcal{C}^\hbar$ need not to be isomorphic even if I is connected, and so one cannot define how the $\mathcal{C}^\hbar$ are topologically glued together using local trivializations. This information is contained in the algebra $\mathcal{C}$, the total C^*-algebra of the field. In fact, continuous sections of the field are collections $\{a_\hbar\}_{\hbar\in I}$ for which $a \in \mathcal{C}$ exist such that $\varphi_\hbar(a) = a_\hbar$. $\mathcal{C}$ can therefore be seen as the algebra of continuous sections with pointwise (in $\hbar$) multiplication. A C^*-field is called trivial if $\mathcal{C} = C_0(I, \mathcal{B})$ for some C^*-algebra $\mathcal{B}$, $\mathcal{C}^\hbar = \mathcal{B}$ and $\varphi_\hbar$ the evaluation at $\hbar$.

All we are interested in here concerns the more special set up in which $I \subset \mathbb{R}$ and we have a collection of continuous $\mathbb{R}$-actions $\{\alpha^\hbar\}_{\hbar\in I}$ on a single C^*-algebra $\mathcal{B}$, $\alpha^\hbar : \mathbb{R} \to \mathrm{Aut}(\mathcal{B})$. Collecting these together we get an $\mathbb{R}$ action $\tilde{\alpha} : \mathbb{R} \to \mathrm{Aut}(C_0(I, \mathcal{B}))$ by

$$\tilde{\alpha}_t(f)(\hbar) = \alpha_t^\hbar(f(\hbar)) ,$$

which is continuous provided the above expression is continuous in $\hbar$ for all t and f which we hereby assume. Then $(C_0(I, \mathcal{B}) \rtimes_{\tilde{\alpha}} \mathbb{R}, \{\mathcal{B} \rtimes_{\alpha^\hbar} \mathbb{R}, \mathrm{ev}_\hbar\}_{\hbar\in I})$ is a continuous field of C^*-algebras [R89].

Example 1 (*Heisenberg group algebra*). The (polarized) Heisenberg group $\mathbb{H}_3$ is $\mathbb{R}^3$ as topological space, but with (non-abelian) multiplication

$$(a_1, a_2, a_3)(b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3 + a_1 b_2) .$$

It contains the subgroup $\mathbb{R}^2 = \{(a_1, a_2, a_3) \in \mathbb{H}_3 | a_1 = 0\}$ so that $\mathbb{H}_3$ can be identified with the semi-direct product $\mathbb{R}^2 \rtimes_{\tilde{\tau}} \mathbb{R}$, where $\tilde{\tau}_{a_1}(a_2, a_3) = (a_2, a_3 + a_1 a_2)$. The Heisenberg group algebra (i.e. the crossed product $\mathbb{C} \rtimes_{\mathrm{id}} \mathbb{H}_3$ defined in a similar way as for $\mathbb{R}$) can therefore be identified with the C^*-algebra $C_0(\mathbb{R}^2) \rtimes_{\tilde{\tau}} \mathbb{R}$ with $\tilde{\tau}_{a_1}(f)(a_2, a_3) = f(a_2, a_3 - a_1 a_2)$. Let $\varphi_{a_2} : C_0(\mathbb{R}^2) \rtimes_{\tilde{\tau}} \mathbb{R} \to C_0(\mathbb{R}) \rtimes_{\tau^{a_2}} \mathbb{R}$ be evaluation of the 2-component at a_2, i.e. $\varphi_{a_2}(f)(a_1)(a_3) = f(a_1)(a_2, a_3)$. Then $\mathrm{im}\,(\varphi_{a_2}) \cong C_0(\mathbb{R}) \rtimes_{\tau^{a_2}} \mathbb{R}$, where $\tau_{a_1}^{a_2}(g)(a_3) = g(a_3 - a_2 a_1)$ for $g : \mathbb{R} \to C_0(\mathbb{R})$. Furthermore $(C_0(\mathbb{R}^2) \rtimes_{\tilde{\tau}} \mathbb{R}, \{C_0(\mathbb{R}) \rtimes_{\tau^{a_2}} \mathbb{R}, \varphi_{a_2}\}_{a_2\in\mathbb{R}})$ is a C^*-field. Therefore a_2 plays the role of $\hbar$.

Example 2. If we have an $\mathbb{R}$-action α on a C^*-algebra $\mathcal{B}$ we can extend the above field of the Heisenberg group algebra in the following way: With the above $\mathbb{R}$-action $\tilde{\tau}$ on $C_0(\mathbb{R}^2)$ define $\tilde{\tau} \otimes \alpha : \mathbb{R} \to \mathrm{Aut}\, C_0(\mathbb{R}^2, \mathcal{B})$ by $(\tilde{\tau} \otimes \alpha)_{a_1}(f)(a_2, a_3) = \alpha_{a_1}(f(a_2, a_3 - a_2 a_1))$. Setting $a_2 = \hbar$ as above, this then yields a C^*-field $(C_0(\mathbb{R}^2, \mathcal{B}) \rtimes_{\tilde{\tau}\otimes\alpha} \mathbb{R}, \{C_0(\mathbb{R}, \mathcal{B}) \rtimes_{\tau^\hbar\otimes\alpha} \mathbb{R}, \varphi_\hbar\}_{\hbar\in\mathbb{R}})$ which will be of crucial importance later on. This C^*-field is trivial away from $\hbar = 0$, i.e., for $\hbar \neq 0$, $C_0(\mathbb{R}, \mathcal{B}) \rtimes_{\tau^1\otimes\alpha} \mathbb{R} \cong C_0(\mathbb{R}, \mathcal{B}) \rtimes_{\tau^\hbar\otimes\alpha} \mathbb{R}$ and $\ker(\varphi_0) \cong C_0(\mathbb{R}\backslash\{0\}, C_0(\mathbb{R}, \mathcal{B}) \rtimes_{\tau^1\otimes\alpha} \mathbb{R})$. However, $C_0(\mathbb{R}, \mathcal{B}) \rtimes_{\tau^0\otimes\alpha} \mathbb{R} \cong C_0(\mathbb{R}, \mathcal{B} \rtimes_\alpha \mathbb{R})$ is not isomorphic to $C_0(\mathbb{R}, \mathcal{B}) \rtimes_{\tau^1\otimes\alpha} \mathbb{R}$.

2.3. Extensions. Suppose that we have a surjective morphism between C^*-algebras $q : \mathcal{C} \to \mathcal{B}$. One then says that $\mathcal{C}$ is an extension of $\mathcal{B}$ by the ideal $\mathcal{J} := \ker(q)$.[1]

Example 3 (*Cone of an algebra*). The suspension of the algebra $\mathcal{B}$ is $S\mathcal{B} := C_0(\mathbb{R}, \mathcal{B})$. Its cone is given by $C\mathcal{B} := C_0(\mathbb{R} \cup \{+\infty\}, \mathcal{B})$. The cone is an extension of $\mathcal{B}$ by the ideal $S\mathcal{B}$, the morphism being $q = \mathrm{ev}_\infty$, the evaluation at $+\infty$.

[1] Some authors call $\mathcal{C}$ an extension of $\mathcal{J}$ by $\mathcal{B}$.

Example 4 (*Wiener-Hopf extension*). Let α be an $\mathbb{R}$-action on $\mathcal{B}$. We extend the $\mathbb{R}$-actions $\tau^\hbar\otimes\alpha$ on the suspension $S\mathcal{B}$ of Example 2 to the cone $C\mathcal{B}$ by setting $(\tau^\hbar\otimes\alpha)_t f(+\infty) = \alpha_t(f(+\infty))$. Hence evaluation at $+\infty$ yields a surjective algebra-homomorphism

$$\mathrm{ev}_\infty : C\mathcal{B} \rtimes_{\tau^\hbar\otimes\alpha} \mathbb{R} \longrightarrow \mathcal{B} \rtimes_\alpha \mathbb{R}\,. \tag{5}$$

For $\hbar = 1$ the corresponding extension is called the Wiener-Hopf extension for an $\mathbb{R}$-action α on $\mathcal{B}$ [R89]. The ideal is $\ker(\mathrm{ev}_\infty) = S\mathcal{B} \rtimes_{\tau^1\otimes\alpha} \mathbb{R}$ which appeared in Example 2, it is isomorphic to $\mathcal{K}\otimes\mathcal{B}$ [R82] (see also the Appendix). These form the ingredients of the exact sequence (1).

Example 5 (*Extension of Heisenberg group algebra*). By repeating the constructions of Example 2 but with $\mathbb{R}^2$ replaced by $\mathbb{R}\times(\mathbb{R}\cup\{+\infty\})$ and actions extended as above, one obtains the C^*-field $(C_0(\mathbb{R}, C\mathcal{B}) \rtimes_{\tilde\tau\otimes\alpha} \mathbb{R}, \{C\mathcal{B} \rtimes_{\tau^\hbar\otimes\alpha} \mathbb{R}, \varphi_\hbar\}_{\hbar\in\mathbb{R}})$. The map in (5) now extends to a surjection which we also denote by ev_∞,

$$\mathrm{ev}_\infty : C_0(\mathbb{R}, C\mathcal{B}) \rtimes_{\tilde\tau\otimes\alpha} \mathbb{R} \to C_0(\mathbb{R}, \mathcal{B}) \rtimes_\alpha \mathbb{R}\,, \tag{6}$$

whose kernel is $C_0(\mathbb{R}, S\mathcal{B}) \rtimes_{\tilde\tau\otimes\alpha} \mathbb{R}$. Each algebra is the total algebra of a C^*-field so that one actually has a field of extensions, the fibre at $\hbar = 1$ being the Wiener-Hopf extension.

3. *K*-Theoretic Preliminaries

This introduction is mainly meant to fix notations. For a complete definition of the K-groups for a Banach algebra $\mathcal{B}$, *cf.* [Bl86]. We denote by $[\mathcal{B}]_0$ the homotopy classes of projections of $\mathcal{B}$ and by $\mathcal{B}^+$ the unitalisation, $\mathcal{B}^+ = \mathcal{B}\times\mathbb{C}$ with $(A,\lambda)(A',\lambda') = (AA' + \lambda A' + A\lambda', \lambda\lambda')$. The C^*-inductive limit of the matrix algebras $M_n(\mathcal{B}^+)$ is denoted by $M_\infty(\mathcal{B}^+)$. $[M_\infty(\mathcal{B}^+)]_0$ is a monoid under addition of homotopy classes of projections, $[p]_0+[q]_0 = [\mathrm{diag}(p,q)]_0$. The K_0-group $K_0(\mathcal{B})$ of $\mathcal{B}$ is obtained from the monoid $[M_\infty(\mathcal{B}^+)]_0$ by Grothendieck's construction and then factorizing out the added unit.

Let $U(\mathcal{B})$ be the group of unitaries $u\in\mathcal{B}^+$ such that $u-1\in\mathcal{B}$ (the 1 is here the unit in $\mathcal{B}^+$). We denote by $[\mathcal{B}]_1$ the homotopy classes of $U(\mathcal{B})$. The algebraic limit of the groups $U(M_n(\mathcal{B}))$ is denoted by $U(M_\infty(\mathcal{B}))$ and then $K_1(\mathcal{B}) = [M_\infty(\mathcal{B})]_1$. The (non-abelian) product in $U(M_n(\mathcal{B}))$ induces a product in $[M_\infty(\mathcal{B})]_1$ which is abelian and therefore denoted additively.

3.1. Elliott-Natsume-Nest map. Suppose we have a continuous field of C^*-algebras $(\mathcal{C}, \{\mathcal{C}^\hbar, \varphi_\hbar\}_{\hbar\in I})$ over $I = [0,1]$ which is trivial away from $\hbar = 0$. This means that there are isomorphisms $\phi_\hbar : \mathcal{C}^1 \to \mathcal{C}^\hbar$ for $\hbar > 0$ such that $\phi : C_0((0,1], \mathcal{C}^1) \to \ker(\varphi_0)$: $\varphi_\hbar(\phi(f)) = \phi_\hbar(f(\hbar))$ is an isomorphism. The following theorem shows that in this situation one obtains maps $[\mathcal{C}^0]_i \to [\mathcal{C}^1]_i$ which induce homomorphisms $K_i(\mathcal{C}^0) \to K_i(\mathcal{C}^1)$. We call these maps ENN-maps.

Theorem 1 [ENN93]. *Consider a continuous field of C^*-algebras $(\mathcal{C}, \{\mathcal{C}^\hbar, \varphi_\hbar\}_{\hbar\in I})$ over $I = [0,1]$ which is trivial away from $\hbar = 0$. For any projection $p\in\mathcal{C}^0$ there is a projection valued section $\tilde p\in\mathcal{C}$ such that $\varphi_0(\tilde p) = p$. For any $u\in U(\mathcal{C}^0)$ there is a section $\tilde u\in U(\mathcal{C})$ such that $\varphi_0(\tilde u) = u$. The maps $\mu_i : [\mathcal{C}^0]_i \to [\mathcal{C}^1]_i$: $\mu_0([p]_0) = [\varphi_1(\tilde p)]_0$, $\mu_1([u]_1) = [\varphi_1(\tilde u)]_1$ are well-defined and induce homomorphisms $\mu_i : K_i(\mathcal{C}^0) \to K_i(\mathcal{C}^1)$.*

Proof. (We only recall how these maps are constructed, for the rest, see [ENN93].) Let p be a projection in $\mathcal{C}^0$. Since φ_0 is surjective, there exists a selfadjoint section $x \in \mathcal{C}$ with $\varphi_0(x) = p$. By Property 2 of C^*-fields, we find for any $\delta > 0$ an ϵ such that $\|\varphi_\hbar(x^2 - x)\| < \delta$ for $\hbar < \epsilon$. For small δ the spectrum of $\varphi_\hbar(x)$ is close to $\{0, 1\}$ and we can find a continuous function $f : \mathbb{R} \to \mathbb{R}$, vanishing for $t < a$ and being 1 for $t > b$ where $0 < a < b < 1$ and (a, b) does not intersect the spectra of $\varphi_\hbar(x)$, $\hbar \leq \epsilon$. Then $f(x)$ is another section with $\varphi_0(f(x)) = p$, but such that $\varphi_\hbar(f(x))$ are projections for $\hbar \leq \epsilon$. Now the section can be extended by the constant section since the field is trivial away from $\hbar = 0$. The resulting section is $\tilde{p}$ where $\varphi_\hbar(\tilde{p}) = \varphi_\epsilon(f(x))$ if $\hbar \geq \epsilon$.

The choice of x is not canonical, but it is not difficult to see that the homotopy class of $\tilde{p}$ is uniquely determined since any other choice $\tilde{p}'$ needs to be close to $\tilde{p}$ at small $\hbar$. The case of unitaries works in a similar way.

With canonically extended $\varphi_\hbar$, the field $(M_n(\mathcal{C}^+), \{M_n(\mathcal{C}^{\hbar^+}), \varphi_\hbar\}_{\hbar \in I})$ is a continuous field of C^*-algebras which is trivial away from 0. The above construction applies therefore also to elements in $[M_n(\mathcal{C}^{0^+})]_0$ and $[M_n(\mathcal{C}^0)]_1$ and induces homomorphisms between the corresponding K-groups. □

3.2. Boundary maps in K-theory. Suppose given an extension $\mathcal{C} \xrightarrow{q} \mathcal{B}$ by $\mathcal{J} := \ker(q)$. What interests us here are two maps, the boundary maps in K-theory, which measure the extent to which the map induced by q on homotopy classes is not surjective. The first of these maps is the exponential map

$$\exp \,:\, K_0(\mathcal{B}) \to K_1(\mathcal{J})$$

which is induced from the map $\exp : [\mathcal{B}]_0 \to [\mathcal{J}]_1$ defined as follows: Let p be a projection in $\mathcal{B}$. Since q is surjective, there exists an $x \in \mathcal{C}$ such that $q(x) = p$. Since p is selfadjoint we can choose x selfadjoint and define

$$\exp[p]_0 \;:=\; [u]_1 \;, \qquad u \;=\; e^{2\pi\imath x} \;.$$

If we apply the above to the cone (Example 3) given by $C\mathcal{B} \xrightarrow{\mathrm{ev}_\infty} \mathcal{B}$, then $\ker(\mathrm{ev}_\infty)$ is the suspension of $\mathcal{B}$ and the exponential map is the so-called Bott map $\exp = \beta : K_0(\mathcal{B}) \to K_1(S\mathcal{B})$,

$$\beta[p]_0 = [e^{2\pi\imath\chi p}]_1,$$

where $\chi : \mathbb{R} \to [0, 1]$ is a continuous function with $\lim_{t\to-\infty} \chi(t) = 0$ and $\lim_{t\to\infty} \chi(t) = 1$.

The second map of interest is the index map $\mathrm{ind} : K_1(\mathcal{B}) \to K_0(\mathcal{J})$ defined as follows: Given $V \in U(M_n(\mathcal{B}))$ defining a class in $K_1(\mathcal{B})$, let $W \in U(M_{2n}(\mathcal{C}))$ be a lift of $\begin{pmatrix} V & 0 \\ 0 & V^* \end{pmatrix}$. Then

$$\mathrm{ind}([V]_1) \;=\; \left[W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W^* \right]_0 - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right]_0 \,.$$

The index map of the extension defined by $C\mathcal{B} \xrightarrow{\mathrm{ev}_\infty} \mathcal{B}$ is denoted by Θ. The fact [Bl86] that the compositions $\Theta\beta : K_0(\mathcal{B}) \to K_0(SS\mathcal{B})$ and $\beta\Theta : K_1(\mathcal{B}) \to K_1(SS\mathcal{B})$ are isomorphisms is called Bott periodicity.

3.3. Boundary maps of Wiener-Hopf extension. We want to express the exponential and the index map of the Wiener-Hopf extension $\mathrm{ev}_\infty : C\mathcal{B} \rtimes_{\tau^1 \otimes \alpha} \mathbb{R} \longrightarrow \mathcal{B} \rtimes_\alpha \mathbb{R}$ (discussed in Example 4) using an ENN-map. Herefore we use the C^*-fields of Example 5 restricted to $[0, 1] \in \mathbb{R}$. They form the extension

$$\mathrm{ev}_\infty : C([0,1], C\mathcal{B}) \rtimes_{\tilde{\tau} \otimes \alpha} \mathbb{R} \to C([0,1], \mathcal{B}) \rtimes_\alpha \mathbb{R} \,. \tag{7}$$

The C^*-field corresponding to $C([0,1], \mathcal{B}) \rtimes_\alpha \mathbb{R}$ is trivial and that corresponding to the ideal $C([0,1], S\mathcal{B}) \rtimes_{\tilde{\tau} \otimes \alpha} \mathbb{R}$ satisfies the conditions of Theorem 1 to give rise to ENN-maps

$$\mu_i : [S\mathcal{B} \rtimes_{\mathrm{id} \otimes \alpha} \mathbb{R}]_i \to [S\mathcal{B} \rtimes_{\tau \otimes \alpha} \mathbb{R}]_i \,. \tag{8}$$

Proposition 1. *Let* exp *and* ind *be exponential and index maps of the Wiener-Hopf extension* (5)*. Then* $\mu_1 \beta = \exp$ *and* $\mu_0 \Theta = \mathrm{ind}$*. Here we have used the identification* $C\mathcal{B} \rtimes_{\mathrm{id} \otimes \alpha} \mathbb{R} \cong C(\mathcal{B} \rtimes_\alpha \mathbb{R})$.

Proof. A projection $p \in \mathcal{B} \rtimes_\alpha \mathbb{R}$ defines a constant section in $C([0,1], \mathcal{B}) \rtimes_\alpha \mathbb{R}$. If $x \in C([0,1], C\mathcal{B}) \rtimes_{\tilde{\tau} \otimes \alpha} \mathbb{R}$ is a selfadjoint lift of the constant section under (7) then, by definition, $\mu_1 [e^{2\pi i \varphi_0(x)}]_1 = [e^{2\pi \imath \varphi_1(x)}]_1$. Furthermore $\exp[p]_0 = [e^{2\pi \imath \varphi_1(x)}]_1$ since $\varphi_1(x)$ is a lift of p in (5). The claim follows since $\varphi_0(x)$ is a lift of p in the extension $C\mathcal{B} \rtimes_{\mathrm{id} \otimes \alpha} \mathbb{R} \overset{\varphi_\infty}{\to} \mathcal{B} \rtimes_\alpha \mathbb{R}$, and χp a lift of p in the extension $C(\mathcal{B} \rtimes_\alpha \mathbb{R}) \overset{\varphi_\infty}{\to} \mathcal{B} \rtimes_\alpha \mathbb{R}$. Under the identification stated in the lemma, $e^{2\pi \imath \chi p}$ is therefore homotopic to $e^{2\pi \imath \varphi_0(x)}$. The argument involving the index map is similar. □

4. Higher Traces on Banach Algebras

For background information on cyclic cohomology and higher traces (or n-traces) see [C94]. Given an associative algebra $\mathcal{B}$ let $C^n_\lambda(\mathcal{B})$ be the set of $n+1$-linear functionals on $\mathcal{B}$ which are cyclic in the sense that $\eta(A_1, \cdots, A_n, A_0) = (-1)^n \eta(A_0, \cdots, A_n)$. Define the boundary operator $b : C^n_\lambda(\mathcal{B}) \to C^{n+1}_\lambda(\mathcal{B})$:

$$\begin{aligned} b\eta(A_0, \cdots, A_{n+1}) = &\sum_{j=0}^{n} (-1)^j \eta(A_0, \cdots, A_j A_{j+1}, \cdots, A_{n+1}) \\ &+ (-1)^{n+1} \eta(A_{n+1} A_0, \cdots, A_n) \,. \end{aligned}$$

An element $\eta \in C^n_\lambda(\mathcal{B})$ satisfying $b\eta = 0$ is called a cyclic n-cocycle and the cyclic cohomology $HC(\mathcal{B})$ of $\mathcal{B}$ is the cohomology of the complex $0 \to C^0_\lambda(\mathcal{B}) \to \cdots \to C^n_\lambda(\mathcal{B}) \overset{b}{\to} C^{n+1}_\lambda(\mathcal{B}) \to \cdots$.

4.1. Cycles. A very convenient way of looking at cyclic cocycles is in terms of characters of graded differential algebras with graded closed traces $(\Omega, d, \int)$ over $\mathcal{B}$. Here $\Omega = \bigoplus_{n \in \mathbb{N}_0} \Omega^n$ is a graded algebra (we denote by $\deg(a)$ the degree of a homogeneous element a) and d is a graded differential on Ω of degree 1. A graded trace on the subspace Ω^n is a linear functional $\int : \Omega^n \to \mathbb{C}$ which is cyclic in the sense that $\int w_1 w_2 = (-1)^{\deg(w_1)\deg(w_2)} \int w_2 w_1$. It is closed if it vanishes on $d(\Omega^{n-1})$. In the situation below there is a largest number n for which Ω^n is non-trivial. This n is called the top degree of Ω. The graded trace will be a graded trace on the sub-space of top degree.

Definition 1. *An n-dimensional cycle is a graded differential algebra* (Ω, d) *of top degree n together with a closed graded trace* $\int$ *on* Ω^n*. A cycle* $(\Omega, d, \int)$ *is called a cycle over* $\mathcal{B}$ *if there is an algebra homomorphism* $\mathcal{B} \to \Omega^0$*.*

We will assume here that the homomorphism $\mathcal{B} \to \Omega^0$ is injective and hence identify $\mathcal{B}$ with a sub-algebra of Ω^0. The connection with cyclic cocycles is given by the following proposition [C94].

Proposition 2. *Any cycle of dimension n over* $\mathcal{B}$ *defines a cyclic n-cocycle through what is called its character:*

$$\eta(A_0, \dots, A_n) = \int A_0 dA_1 \cdots dA_n \, .$$

Conversely, any cyclic n-cocycle arises as the character of an n-cycle.

A (bounded) trace over $\mathcal{B}$ is an example of a cyclic 0-cocycle. Taking $\Omega = \mathcal{B}, d = 0$, $\int$ to be that trace, we have a realization of the trace as character of a 0-cycle.

For our purposes, the cyclic cohomology of C^*-algebras is too small, because we need multilinear functionals which are unbounded. A particular class of unbounded cyclic cocycles suitable for our purposes is given by the higher traces [C94, C86]. These are characters of cycles over dense sub-algebras $\mathcal{B}'$ of $\mathcal{B}$ satisfying a continuity condition. It will be useful to relax the requirement of $\mathcal{B}$ being a C^*-algebra and rather consider Banach algebras.

Definition 2. *An n-trace on a Banach algebra* $\mathcal{B}$ *is the character of an n-cycle* $(\Omega', d, \int)$ *over a dense sub-algebra* $\mathcal{B}'$ *of* $\mathcal{B}$ *such that for all* $A_1, \dots, A_n \in \mathcal{B}'$ *there exists a constant* $C = C(A_1, \dots, A_n)$ *such that*

$$\left| \int (X_1 dA_1) \cdots (X_n dA_n) \right| \leq C \|X_1\| \cdots \|X_n\| \, , \tag{9}$$

for all $X_j \in \mathcal{B}'^+$*.*

Condition (9) may be rephrased by saying that for all $A_1, \dots, A_n \in \mathcal{B}'$ the apriori densely defined multi-linear functional

$$\mathcal{B}^{\times n} \to \mathbb{C} : \quad (X_1, \dots, X_n) \mapsto \int (X_1 dA_1) \cdots (X_n dA_n)$$

extends to a bounded multi-linear functional. Denoting by $p(A_1, \dots, A_n)$ the norm of that functional, i.e. the best possible constant C in (9), we have a family of maps $\mathcal{B}^{\times n} \to \mathbb{R}$ satisfying

$$\begin{aligned} p(A_1, \dots, \lambda A_j + \lambda' A_j', \dots, A_n) \leq |\lambda| \, & p(A_1, \dots, A_j, \dots, A_n) \\ & + |\lambda'| \, p(A_1, \dots, A_j', \dots, A_n) \, . \end{aligned}$$

But since d is a derivation, it also satisfies

$$\begin{aligned} p(A_1, \dots, A_j A_j', \dots, A_n) \leq \|A_j'\| \, & p(A_1, \dots, A_j, \dots, A_n) \\ & + \|A_j\| \, p(A_1, \dots, A_j', \dots, A_n) \, . \end{aligned}$$

For simplicity, rather than considering cycles $(\Omega', d, \int)$ over a dense sub-algebra $\mathcal{B}'$, we shall consider triples $(\Omega, d, \int)$ as in Definition 1 with Ω being a Banach algebra, $\mathcal{B} \subset \Omega^0$, but allowing for the possibility that d and $\int$ are only densely defined. If the character is densely defined and satisfies (9), we call the triple $(\Omega, d, \int)$ an unbounded n-cycle. The role of (9) is to insure the existence of a third algebra $\mathcal{B}''$, $\mathcal{B}' \subset \mathcal{B}'' \subset \mathcal{B}$, to which the character can be extended (by continuity) and such that the inclusion $i : \mathcal{B}'' \hookrightarrow \mathcal{B}$ induces an isomorphism between $K(\mathcal{B}'')$ and $K(\mathcal{B})$ [C86].

An example of a cycle for the commutative algebra $\mathcal{B} = C(M)$ of continuous functions over a compact manifold without boundary is given by the algebra of exterior forms with its usual differential $(\Omega(M), d)$ and graded trace equal to integration of n-forms, $n = \dim(M)$. This is an unbounded cycle. One may take $\mathcal{B}' = C^\infty(M)$ and $p(A_1, \dots, A_n) = \int |dA_1 \cdots dA_n|$, where (locally) $|dA_1 \cdots dA_n| = |f| d\,\mathrm{vol}$ if $dA_1 \cdots dA_n = f d\,\mathrm{vol}$. Note that $p(A_1, \dots, A_n)$ is not continuous in A_j w.r.t. the supremum norm, which is the C^*-norm of $\mathcal{B}$.

A 0-trace is a (possibly unbounded) linear functional tr which is cyclic and satisfies (9). A positive trace is a positive linear functional tr which is cyclic. It might be unbounded (with dense domain), but it always satisfies $|\mathrm{tr}(AX)| \le \mathrm{tr}(|A|)\|X\|$ if A is trace class and hence (9) holds with $\mathcal{B}'$ being the ideal of trace class elements.

Here we need to construct higher traces on a Banach algebra $\mathcal{B}$ on which is given a differentiable action of $\mathbb{R}^n$ leaving a (possibly unbounded) trace invariant. This is essentially Ex. 12, p. 254 of [C94].

Proposition 3. *Let $\mathcal{B}$ be a Banach algebra with a differentiable action of $\mathbb{R}^n$ and $\mathcal{T}$ be an invariant positive trace on $\mathcal{B}$. Denote by ∇_j, $j = 1, \cdots, n$, commuting closed derivations defined by the action and suppose that $\mathcal{B}' = \{A \in \bigcap_{j=1}^n \mathrm{dom}(\nabla_j) \,|\, \exists j : \nabla_j A$ traceclass$\}$ is dense in $\mathcal{B}$. Then $(\Omega, d, \int)$ is an unbounded n-cycle over $\mathcal{B}$, where*

$$\Omega := \mathcal{B} \otimes \Lambda\mathbb{C}^n ,$$

the tensor product of $\mathcal{B}$ with the Grassmann algebra $\Lambda\mathbb{C}^n$ with generators e_j, $j = 1, \dots, n$,

$$d(A \otimes v) = \sum_{j=1}^n \nabla_j A \otimes e_j v ,$$

and $\int = \mathcal{T} \otimes \imath$ with $\imath(e_1 \cdots e_n) = 1$, explicitly

$$\int A_0 dA_1 \cdots dA_n = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\, \mathcal{T}(A_0 \nabla_{\sigma(1)} A_1 \cdots \nabla_{\sigma(n)} A_n) .$$

Proof. The algebraic aspects of this proposition are straightforward to show, see e.g. [KRS02]. Since trace class operators form an ideal, $\mathcal{B}'$ is a sub-algebra. Then (9) follows from

$$|\mathcal{T}((X_1 \nabla_1 A_1) \cdots (X_n \nabla_n A_n))| \le \|X_1\| \cdots \|X_n\| \, \|\nabla_1 A_1\| \cdots \|\nabla_{n-1} A_{n-1}\| \, \mathcal{T}(|\nabla_n A_n|) ,$$

and the cyclicity of $\mathcal{T}$. $\square$

The following is an extension of the above construction; it corresponds to an iteration of Lemma 16, p. 258 of [C94].

Proposition 4. *Let* $(\Omega, d, \int)$ *be a (possibly unbounded) k-cycle over the Banach algebra* $\mathcal{B}$ *which is invariant under a differentiable action of* $\mathbb{R}^n$ *in the sense that this action commutes with d and leaves* $\int$ *invariant. Denote by* ∇_j, $j = 1, \cdots, n$, *commuting closed derivations defined by the action and suppose that* $\bigcap_{j=1}^n \mathrm{dom}(\nabla_j) \cap \mathcal{B}'$ *is a dense sub-algebra of* $\mathcal{B}$ *such that on* $\mathcal{B}' \subset \mathcal{B}$ *the character of* $(\Omega, d, \int)$ *is fully defined. Taking* $\Omega' = \Omega \hat{\otimes} \Lambda\mathbb{C}^n$, *the graded tensor product,* $d' = d\hat{\otimes}1 + \delta$ *with* $\delta(w\hat{\otimes}v) = (-1)^{\partial w}\sum_j \nabla_j w\hat{\otimes}e_j v$ *and* $\int' = \int\hat{\otimes}\iota$, *one obtains a* $k+n$*-cycle* $(\Omega', d', \int')$ *over* $\mathcal{B}$.

Proof. The algebraic aspects are straightforward and again given in [KRS02]. The only point to settle is condition (9). It follows iteratively from the case $n = 1$. For $n = 1$, using cyclicity,

$$\left|\int'(X_1 d'A_1)\dots(X_{k+1}d'A_{k+1})\right| \le \sum_{j=1}^{k+1}\left|\int(X_j\nabla_1 A_j)(X_{j+1}\delta A_{j+1})\cdots(X_{j-1}\delta A_{j-1})\right|$$
$$\le \sum_{j=1}^{k+1}\|X_1\|\cdots\|X_{k+1}\|\;\left\|\nabla_1 A_j\right\| C_j,$$

where C_j depends only on $A_1, \dots, A_{j-1}, A_{j+1}, \dots, A_{k+1}$. This inequality shows also that the character of the cycle is defined on $\bigcap_{j=1}^n \mathrm{dom}(\nabla_j) \cap \mathcal{B}'$. $\square$

4.2. Cyclic cocycles for crossed products with $\mathbb{R}$. An action of $\mathbb{R}$ on a graded differential algebra (Ω, d) is a homomorphism $\alpha : \mathbb{R} \to \mathrm{Aut}(\Omega)$ such that $\forall\, t \in \mathbb{R}$, α_t has degree 0 and commutes with d. If Ω is a Banach algebra or even a C^*-algebra, we require in addition that for all $A \in \mathcal{B}$, $t \mapsto \alpha_t(A)$ is continuous and $\|\alpha_t\| = 1$. Therefore we can form $L^1(\Omega, \mathbb{R}, \alpha)$ as well as the crossed product $\Omega \rtimes_\alpha \mathbb{R}$.

Definition 3. *A n-cycle* $(\Omega, d, \int)$ *over* $\mathcal{B}$ *is called invariant under an action* α *of* $\mathbb{R}$ *on* Ω *if the graded trace* $\int$ *is invariant under it. If* $(\Omega, d, \int)$ *is unbounded, we require in addition that the norms* $p(A_1, \dots, A_n)$ *(cf.* Definition 2) *satisfy that*

$$Q(A_1, \dots, A_n) \;:=\; \sup_{t_i\in\mathbb{R}} \; p(\alpha_{t_1}(A_1), \dots, \alpha_{t_n}(A_n)) \tag{10}$$

is finite for all $A_j \in \mathcal{B}' \subset \mathcal{B}$, *where* $\mathcal{B}'$ *is a dense sub-algebra on which the character of the n-cycle is fully defined. An n-trace of* $\mathcal{B}$ *is invariant under an action* α *of* $\mathbb{R}$ *if it is the character of an* α*-invariant cycle* $(\Omega, d, \int)$.

We note that, by cyclicity of the graded trace, the above additional condition is equivalent to demanding that $\sup_{t_1\in\mathbb{R}} p(\alpha_{t_1}(A_1), A_2, \dots, A_n)$ exists for all $A_j \in \mathcal{B}'$. Furthermore, Q inherits the properties of p, i.e.

$$Q(A_1, \dots, \lambda A_j + \lambda' A'_j, \dots, A_n) \le |\lambda|\, Q(A_1, \dots, A_j, \dots, A_n)$$
$$+|\lambda'|\, Q(A_1, \dots, A'_j, \dots, A_n)\,, \tag{11}$$

$$Q(A_1, \dots, A_j A'_j, \dots, A_n) \le \|A'_j\|\, Q(A_1, \dots, A_j, \dots, A_n)$$
$$+\|A_j\|\, Q(A_1, \dots, A'_j, \dots, A_n)\,. \tag{12}$$

Theorem 2. *Let* $(\Omega, d, \int)$ *be an* α*-invariant (possibly unbounded)* n*-cycle over the Banach algebra* $\mathcal{B}$ *and* $\nabla : L^1(\mathbb{R}, \mathcal{B}, \alpha) \to L^1(\mathbb{R}, \mathcal{B}, \alpha)$ *be the derivation* $\nabla f(x) = \imath x f(x)$*. Then* $(\Omega_\alpha, d_\alpha, \int_\alpha)$ *is an unbounded* $n+1$*-cycle over* $L^1(\mathbb{R}, \mathcal{B}, \alpha)$ *where*

$$\Omega_\alpha = L^1(\mathbb{R}, \Omega, \alpha) \hat{\otimes} \Lambda\mathbb{C},$$
$$d_\alpha(\omega \hat{\otimes} v) = d'\omega \hat{\otimes} v + (-1)^{\deg(\omega)} \nabla\omega \hat{\otimes} e_1 v \,, \qquad d'\omega(x) \;=\; d(\omega(x))\,,$$
$$\nabla\omega(x) \;=\; \imath x \omega(x)\,,$$

and $\int_\alpha = \int \mathrm{ev}_0 \hat{\otimes}\, \imath$*, i.e.*

$$\int_\alpha f_0 d_\alpha f_1 \cdots d_\alpha f_{n+1} = \sum_{j=1}^{n+1} (-1)^j \int \big(f_0 df_1 \cdots df_{j-1} (\nabla f_j) df_{j+1} \cdots df_{n+1}\big)(0).$$

Proof. We first show that the triple $(L^1(\mathbb{R}, \Omega, \alpha), d', \int \mathrm{ev}_0)$ defines an unbounded n-cycle over $L^1(\mathbb{R}, \mathcal{B}, \alpha)$. The required algebraic properties are straightforwardly checked (*cf.* [KRS02]) and we focus here on the continuity aspects (9). Let $\mathcal{B}' \subset \mathcal{B}$ be a dense sub-algebra on which the character of $(\Omega, d, \int)$ is fully defined and

$$\mathcal{V}^{\mathrm{fin}} \;:=\; \bigcup_{V \subset \mathcal{B}',\, \dim V < \infty} C_c(\mathbb{R}, \overline{V})\,,$$

the union being over all finite dimensional linear sub-spaces of $\mathcal{B}'$ and $\overline{V} := \bigcup_{t \in \mathbb{R}} \alpha_t(V)$, the orbit of V under the action. The space $\mathcal{V}^{\mathrm{fin}}$ is linear and since $Q(A_1, \ldots, A_n)$ is finite for all $A_j \in \mathcal{B}'$, we obtain from (11) that

$$\overline{Q}(f_1, \ldots, f_n) \;:=\; \sup_{t_j \in \mathbb{R}} Q(f_1(t_1), \ldots, f_n(t_n))$$

is finite for all $f_j \in \mathcal{V}^{\mathrm{fin}}$. Furthermore, $\mathcal{V}^{\mathrm{fin}}$ is clearly dense in $C_c(\mathbb{R}, \mathcal{B}')$ in the L^1-norm. Now let $\mathcal{A}^{\mathrm{fin}}$ be the sub-algebra of $L^1(\mathbb{R}, \mathcal{B}, \alpha)$ generated algebraically by $\mathcal{V}^{\mathrm{fin}}$, *i.e.* it consists of finite twisted convolution products of elements of $\mathcal{V}^{\mathrm{fin}}$. Since, by (12),

$$Q(f_1 g_1(t), f_2(t) \ldots) \leq \int ds\, (\|f_1(s)\|\, Q(g_1(t-s), f_2(t) \ldots)$$
$$+ \|g_1(t-s)\|\, Q(f_1(s), f_2(t) \ldots))\,,$$

we obtain

$$\overline{Q}(f_1 g_1, f_2 \ldots, f_n) \;\leq\; \|f_1\|_{L^1}\, \overline{Q}(g_1, f_2 \ldots) + \|g_1\|_{L^1}\, \overline{Q}(f_1, f_2 \ldots)\,.$$

Therefore $\overline{Q}(f_1, \ldots, f_n)$ is finite for all $f_j \in \mathcal{A}^{\mathrm{fin}}$. The character of $(L^1(\mathbb{R}, \Omega, \alpha), d', \int \mathrm{ev}_0)$ is now restricted to the dense sub-algebra $\mathcal{A}^{\mathrm{fin}}$. Then we obtain

$$\left| \int \mathrm{ev}_0 (X_1 d' f_1) \ldots (X_n d' f_n) \right| \leq \|X_1\|_{L^1} \cdots \|X_n\|_{L^1} |$$
$$\mathrm{supp}(f_1)| \cdots |\mathrm{supp}(f_n)|\; \overline{Q}(f_1, \ldots, f_n)\,. \quad (13)$$

Here $|\mathrm{supp}(f)|$ is the (finite) length of the support of f. This shows that there exists a dense sub-algebra of the Banach algebra $L^1(\mathbb{R}, \mathcal{B}, \alpha)$ on which the character of $(L^1(\mathbb{R}, \Omega, \alpha), d', \int \mathrm{ev}_0)$ satisfies (9).

Now on $L^1(\mathbb{R}, \mathcal{B}, \alpha)$ and $L^1(\mathbb{R}, \Omega, \alpha)$ we have the dual action of $\mathbb{R}$ (identified here with the dual group of $\mathbb{R}$) and its corresponding derivation is ∇. It is densely defined and satisfies the conditions of Proposition 4. Applying that proposition, one obtains $(L^1(\mathbb{R}, \Omega, \alpha), d_\alpha, \int_\alpha)$, which is an unbounded $n+1$-cycle. □

Definition 4. *If η is the character of an α-invariant n-cycle $(\Omega, d, \int)$ over $\mathcal{B}$, we define $\#_\alpha\eta$ to be the character of $(\Omega_\alpha, d_\alpha, \int_\alpha)$ constructed in* Theorem 2*, i.e. for $f_j \in L^1(\mathbb{R}, \mathcal{B}, \alpha)$,*

$$\#_\alpha\eta(f_0, \dots, f_{n+1}) = \sum_{k=1}^{n+1} (-1)^k \int \mathrm{ev}_0\,(f_0 df_1 \cdots \nabla f_k \cdots df_{n+1})\,,$$
$$\nabla f(x) = \imath x f(x)\;.$$

Restricted to the context of smooth $\mathbb{R}$-actions on smooth sub-algebras of C^*-algebras, Theorem 2 can be compared with a result in [ENN88]. In fact, it coincides with a construction given in that article for general n-cycles over a smooth sub-algebra of $\mathcal{B}$ in the case that these n-cycles are α-invariant in our sense.

Whenever we have several commuting $\mathbb{R}$ actions leaving a cycle invariant, we can iterate this construction, since we can extend a second action β in Ω to an action on Ω_α commuting with the differential d_α and leaving $\int_\alpha$ invariant by evaluating it pointwise on functions $f : \mathbb{R} \to \Omega$ and keeping the new Grassmann generator of Ω_α fixed.

Example 6 (*Suspension of n-traces*). The suspension $S\mathcal{B}$ of a C^*-algebra $\mathcal{B}$ is via Fourier transform isomorphic to the crossed product $\mathcal{B} \rtimes_{\mathrm{id}} \mathbb{R}$. For a given n-trace η over $\mathcal{B}$, the above construction yields an $n+1$-trace $\#_{\mathrm{id}}\eta$ over $L^1(\mathbb{R}, \mathcal{B}, \mathrm{id})$. When intertwined with the Fourier transform one obtains a $n+1$-trace which we denote by η^s over a dense Banach sub-algebra of $S\mathcal{B}$. However, since the conditions of Definition 3 are trivially satisfied in that case and the linear space $\mathcal{V}^{\mathrm{fin}}$ used in the proof of Theorem 2 is a dense sub-algebra of $C_0(\mathbb{R}, \mathcal{B})$ under pointwise multiplication we can simplify the arguments of Theorem 2 thereby improving (13), namely, for $f_i \in \mathcal{V}^{\mathrm{fin}}$,

$$\left|\int_{\mathbb{R}} ds \int (X_1 d' f_1)(s) \dots (X_n d' f_n)(s)\right| \;\leq\; \|X_1\| \cdots \|X_n\|\; \tilde{Q}(f_1, \dots, f_n), \tag{14}$$

where $\tilde{Q}(f_1, \dots, f_n) = |\mathrm{supp}(f_1)| \cdots |\mathrm{supp}(f_n)|\, \sup_{t_j \in \mathbb{R}} p(f_1(t_1), \dots, f_n(t_n))$ and the norm is here the C^*-norm on $S\mathcal{B}$.

As a result, if $(\Omega, d, \int)$ is an n-cycle over a Banach algebra $\mathcal{B}$ whose character is η and $\partial_s : S\mathcal{B} \to S\mathcal{B}$ the derivative w.r.t. the suspension variable, then η^s is the character of the unbounded $n+1$-cycle $(\Omega^s, d^s, \int^s)$ over $S\mathcal{B}$, where $\Omega^s := S\Omega\hat{\otimes}\Lambda\mathbb{C}$ and $(\omega \in S\Omega)$

$$d^s(\omega\hat{\otimes}v) \;=\; d'\omega\hat{\otimes}\, v \;+\; (-1)^{\deg(\omega)}\,\partial_s\omega\hat{\otimes}\, e_1 v\;,$$

where $d'\omega(s) = d(\omega(s))$ and $\int^s = \int_{\mathbb{R}} ds \int \hat{\otimes}\, \imath$.

Example 7 (*Canonical 3-trace for the Heisenberg group algebra*). We first construct the canonical 2-trace of the group algebra of $\mathbb{R}^2$ which is equal to $\mathbb{C} \rtimes_{\mathrm{id}} \mathbb{R} \rtimes_{\mathrm{id}} \mathbb{R} \cong SS\mathbb{C}$. On $\mathbb{C}$ we consider the trace Tr, a 0-cocycle which is the character of the 0-cycle $(\mathbb{C}, 0, \mathrm{id})$. Then we apply the construction of Example 6 twice to obtain the double suspension of Tr for the algebra $SS\mathbb{C} \cong \mathbb{C} \rtimes_{\mathrm{id}} \mathbb{R} \rtimes_{\mathrm{id}} \mathbb{R}$, namely $(\mathbb{C}^{ss}, d^{ss}, \mathrm{Tr}^{ss})$, where $\mathbb{C}^{ss} = SS\mathbb{C}\hat{\otimes}\Lambda\mathbb{C}^2$,

$d^{ss} = \partial_{a_3}\hat{\otimes}e_1 + \partial_{a_2}\hat{\otimes}e_2$ and $\mathrm{Tr}^{ss} = \int_{\mathbb{R}^2} da_2 da_3 \mathrm{Tr} \otimes \imath$ (we use the notation of Example 1). If restricted to the smooth group algebra of $\mathbb{R}^2$ this 2-trace becomes a genuine cocycle referred to as the canonical cocycle of the group algebra of $\mathbb{R}^2$. Its character is the well-known Chern character.

Recall that the Heisenberg group algebra is isomorphic to $SS\mathbb{C} \rtimes_{\tilde{\tau}} \mathbb{R}$ and so we seek to apply Theorem 2 to the action $\tilde{\tau}$ on the above double suspension. Since this action does not commute with ∂_{a_3}, we cannot extend it trivially to the Grassmann generators. Instead we set

$$\begin{aligned}(\tilde{\tau}_{a_1}(f\hat{\otimes}1))(a_2)(a_3) &= f(a_2)(a_3 - a_2 a_1)\hat{\otimes}1 \ , \\ \tilde{\tau}_{a_1}(1\hat{\otimes}e_1) &= 1\hat{\otimes}e_1 \ , \\ \tilde{\tau}_{a_1}(1\hat{\otimes}e_2) &= 1\hat{\otimes}e_2 - a_1(1\hat{\otimes}e_1) \ .\end{aligned}$$

We claim that $\tilde{\tau}_{a_1}$ commutes with d^{ss}: it suffices to check this for elements of degree 0 in e_1 and e_2 where we get

$$\begin{aligned}\tilde{\tau}_{a_1} d^{ss}(f\hat{\otimes}1)(a_2)(a_3) &= \partial_1 f(a_2)(a_3 - a_2 a_1)\hat{\otimes}e_1 + \partial_2 f(a_2)(a_3 - a_2 a_1)\hat{\otimes}(e_2 - a_1 e_1) \ , \\ d^{ss}\tilde{\tau}_{a_1}(f\hat{\otimes}1)(a_2)(a_3) &= (\partial_1 f(a_2)(a_3 - a_2 a_1) - a_1 \partial_2 f(a_2)(a_3 - a_2 a_1))\,\hat{\otimes}\, e_1 \\ &\quad + \partial_2 f(a_2)(a_3 - a_2 a_1)\hat{\otimes}e_2 \ .\end{aligned}$$

Furthermore, $\tilde{\tau}$ leaves the graded trace Tr^{ss} invariant, because of $\tilde{\tau}_{a_1}(1\hat{\otimes}e_1 e_2) = 1\hat{\otimes}e_1 e_2$ and the translation invariance of the Lebesgue measure on $\mathbb{R}^2$. In order to apply Theorem 2, we show that the 2-cycle $(\mathbb{C}^{ss}, d^{ss}, \mathrm{Tr}^{ss})$ satisfies the uniform bound (10) w.r.t. the dense sub-algebra $C^1_c(\mathbb{R}^2, \mathbb{C}) \subset SS\mathbb{C}$ given by continuously differentiable functions with compact support. One finds for the norms $p(f_1, f_2)$ (*cf.* Definition 2), using $\partial_{a_2}\tilde{\tau}_t(f_1) = \tilde{\tau}_t(\partial_1 f_1) - t\,\tilde{\tau}_t(\partial_2 f_1)$ and $\partial_{a_3}\tilde{\tau}_t(f_1) = \tilde{\tau}_t(\partial_2 f_1)$,

$$\begin{aligned}p(\tilde{\tau}_t(f_1), f_2) \leq{}& (\|\tilde{\tau}_t(\partial_1 f_1)\|\,\|\partial_2 f_2\| \ + \ \|\tilde{\tau}_t(\partial_2 f_1)\|\,\|\partial_1 f_2\|)\,|\mathrm{supp}(f_2)| \\ &+ \|\tilde{\tau}_t(\partial_2 f_1)\|\,\|\partial_2 f_2\|\ |t|\ |\mathrm{supp}(\tilde{\tau}_t f_1) \cap \mathrm{supp}(f_2)| \ .\end{aligned}$$

Here $\|.\|$ is the supremum norm on $SS\mathbb{C}$. Since $|t|\,|\mathrm{supp}(\tilde{\tau}_t f_1) \cap \mathrm{supp}(f_2)|$ is bounded in t for any two compactly supported functions f_1, f_2, we obtain the desired result, namely that $p(\tilde{\tau}_t(f_1), f_2)$ is bounded in t. Hence we are in a position to apply Theorem 2 from which we then infer a 3-cycle over $L^1(\mathbb{R}, SS\mathbb{C}, \tilde{\tau})$ given by

$$\left(L^1(\mathbb{R}, SS\mathbb{C}\hat{\otimes}\Lambda\mathbb{C}^2, \tilde{\tau})\hat{\otimes}\Lambda\mathbb{C}, d^{ss}_{\tilde{\tau}}, \int_{\mathbb{R}^2} da_2 da_3\, \mathrm{Tr}\,\mathrm{ev}_0 \otimes \imath\right) \ .$$

Its character is the canonical 3-trace of the L^1-crossed product $L^1(\mathbb{R}, SS\mathbb{C}, \tilde{\tau})$. The latter is dense in the Heisenberg group algebra and closed under holomorphic functional calculus.

Example 8. If $(\Omega, d, \int)$ is an α-invariant n-cycle over $\mathcal{B}$, then the above construction straightforwardly generalizes to the C^*-field $SS\mathcal{B} \rtimes_{\tilde{\tau}\otimes\alpha} \mathbb{R}$ from Example 2 and we obtain the $n+3$-cycle

$$\left(L^1(\mathbb{R}, SS\Omega\hat{\otimes}\Lambda\mathbb{C}^2, \tilde{\tau}\otimes\alpha)\hat{\otimes}\Lambda\mathbb{C}, d^{ss}_{\tilde{\tau}\otimes\alpha}, \int_{\mathbb{R}^2} da_2 da_3 \int \mathrm{ev}_0 \otimes \imath\right)$$

whose character is an $n+3$-trace on the Banach algebra $L^1(\mathbb{R}, SS\mathcal{B}, \tilde{\tau}\otimes\alpha)$. The details of the proof that the intermediate $n+2$-cycle is $\tilde{\tau}\otimes\alpha$-invariant are worked out as above and we only indicate how to show the bound (10) for the $n+2$-cycle,

$$\left(SS\Omega\hat{\otimes}\Lambda\mathbb{C}^2, d^{ss}, \int_{\mathbb{R}^2} da_2 da_3 \int \otimes \iota \right) .$$

For that consider the dense sub-algebra $\mathcal{V}^{\text{fin}} = \bigcup_{V\cup\mathcal{B}',\, \dim V<\infty} C^1_c(\mathbb{R}^2, V)$ of $SS\mathcal{B}$. Then

$$\tilde{Q}(f_1, \ldots, f_n) := |\text{supp}(f_1)| \ldots |\text{supp}(f_n)| \sup_{s_j, t_j \in \mathbb{R}^2} p(\alpha_{s_1}(f_1(t_1)), \ldots, \alpha_{s_n}(f_n(t_n)))$$

is finite for all $f_j \in \mathcal{V}^{\text{fin}}$, and we get the desired bound (10) from

$$\begin{aligned} p(\tilde{\tau}\otimes\alpha_t(f_1), f_2, \ldots, f_{n+2}) \leq & \sum_{k\neq j} \|\partial_1 f_k\|\, \|\partial_2 f_j\|\, \tilde{Q}(\ldots \text{no } f_k \,\&\, f_j \ldots) \\ & + \sum_{j\neq 1} \|\partial_2 f_1\|\, \|\partial_2 f_j\|\, \tilde{Q}(\ldots \text{no } f_1 \,\&\, f_j \ldots) \\ & |t||\text{supp}(\tilde{\tau}_t f_1) \cap \text{supp}(f_j)| . \end{aligned}$$

4.3. Chains and boundaries. The interest in the following definition is that the graded traces are not supposed to be closed.

Definition 5. *An n-dimensional chain $(\Omega, d, \int, \partial\Omega, r)$ is a graded differential algebra (Ω, d) of top degree n together with a graded trace $\int$ on Ω^n and a surjective homomorphism of graded algebras of degree zero $r : \Omega \to \partial\Omega$ onto a graded algebra of top degree $n-1$ such that $d\ker(r) \subset \ker(r)$ and $\int d\omega = 0$ if $r(\omega) = 0$. The chain is called a chain over $\mathcal{B}$ if there exists an algebra homomorphism $\mathcal{B} \to \Omega^0$.*

For a chain $(\Omega, d, \int, \partial\Omega, r)$ over a Banach algebra we require $r : \Omega \to \partial\Omega$ to be a continuous map between Banach algebras. Such a chain is called unbounded if d and $\int$ and the character of the chain, $(A_0, \cdots, A_n) \mapsto \int A_0 dA_1 \cdots dA_n$, are densely defined but satisfy condition (9).

As for cycles, we consider here only the case that $\mathcal{B} \subset \Omega^0$.

An example of an unbounded chain for a commutative algebra $C(M)$ of functions over a compact manifold is obtained when one looks at the algebra of exterior forms with its usual differential and integration structure, but M has a boundary ∂M. The map r is then simply the restriction to the boundary and $\partial\Omega = \Omega(\partial M)$. In this context, Stokes' Theorem relates integration of exact $\dim(M)$-forms over M to the integration of a form over the boundary ∂M. The following definition is motivated so that such a theorem holds automatically in the non-commutative setting.

Definition 6. *The boundary of an n-dimensional chain $(\Omega, d, \int, \partial\Omega, r)$ is the $n-1$-dimensional cycle $(\partial\Omega, d', \int')$ where*

$$d'\omega' = r d\omega , \qquad \int' \omega' = \int d\omega ,$$

for some $\omega \in r^{-1}(\omega')$.

The following example is important for the construction of the dual of the ENN-map in (8).

Example 9. Consider the continuous C^*-field $SS\mathcal{B} \rtimes_{\tilde{\tau}\otimes\alpha} \mathbb{R}$ from Example 2 together with an α invariant n-cycle $(\Omega, d, \int)$ over $\mathcal{B}$. In Example 8 we have constructed an $n+3$-cycle for a dense Banach sub-algebra of the field. Now restrict this C^*-field to the interval $[\hbar_0, \hbar_1] \subset \mathbb{R}$, i.e. consider $C_0([\hbar_0, \hbar_1] \times \mathbb{R}, \mathcal{B}) \rtimes_{\tilde{\tau}\otimes\alpha} \mathbb{R}$. If we repeat the construction of the $n+3$-cycle, we end up with a graded differential algebra with graded trace

$$\left(L^1(\mathbb{R}, C_0([\hbar_0, \hbar_1] \times \mathbb{R}, \Omega)\hat{\otimes}\Lambda\mathbb{C}^2, \tilde{\tau} \otimes \alpha)\hat{\otimes}\Lambda\mathbb{C}\,, d^{ss}_{\tilde{\tau}\otimes\alpha}, \int_{[\hbar_0,\hbar_1]\times\mathbb{R}} d\hbar\, ds \int \mathrm{ev}_0 \otimes \iota\right)$$

which is not closed. This is a chain if we set

$$\begin{aligned}&\partial\left(L^1(\mathbb{R}, C_0([\hbar_0, \hbar_1] \times \mathbb{R}, \Omega)\hat{\otimes}\Lambda\mathbb{C}^2, \tilde{\tau} \otimes \alpha)\hat{\otimes}\Lambda\mathbb{C}\right)\\ &= \bigoplus_{j=0,1} L^1(\mathbb{R}, C_0(\mathbb{R}, \Omega)\hat{\otimes}\Lambda\mathbb{C}, \tau^{\hbar_j} \otimes \alpha)\hat{\otimes}\Lambda\mathbb{C},\end{aligned}$$

the first Grassmann part $\Lambda\mathbb{C}$ being the sub-algebra of $\Lambda\mathbb{C}^2$ of elements not containing e_1, and

$$r(\omega) = \begin{cases} \omega|_{\hbar=\hbar_0} \oplus \omega|_{\hbar=\hbar_1} & \text{if } \omega e_1 \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

To verify this let us split $d^{ss}_{\tilde{\tau}\otimes\alpha} = \delta_1 + \delta_2$, where $\delta_1(F\hat{\otimes}1) = \partial_\hbar F\hat{\otimes}e_1$. Then

$$d^{ss}_{\tilde{\tau}\otimes\alpha}F_1 \cdots d^{ss}_{\tilde{\tau}\otimes\alpha}F_{n+3} = \delta_1(F_1 d^{ss}_{\tilde{\tau}\otimes\alpha}F_2 \cdots d^{ss}_{\tilde{\tau}\otimes\alpha}F_{n+3}) + \delta_2(F_1 d^{ss}_{\tilde{\tau}\otimes\alpha}F_2 \cdots d^{ss}_{\tilde{\tau}\otimes\alpha}F_{n+3})\,. \tag{15}$$

The graded trace $\int_{[\hbar_0,\hbar_1]\times\mathbb{R}} \int \mathrm{ev}_0$ applied to the second term vanishes, because $\int_{\mathbb{R}} \int \mathrm{ev}_0$ is closed w.r.t. δ_2. The graded trace applied to the first term yields, when first integrated over $[\hbar_0, \hbar_1]$, the boundary term,

$$\int_{[\hbar_0,\hbar_1]\times\mathbb{R}} \int \mathrm{ev}_0 \left(d^{ss}_{\tilde{\tau}\otimes\alpha}F_1 \cdots d^{ss}_{\tilde{\tau}\otimes\alpha}F_{n+3}\right) = \int_{\mathbb{R}} \int \mathrm{ev}_0 \left(F_1\delta_2F_2 \cdots \delta_2F_{n+3}\right)\Big|_{\hbar_0}^{\hbar_1}.$$

The integrand vanishes if $F_1 d^{ss}_{\tilde{\tau}\otimes\alpha}F_2 \cdots d^{ss}_{\tilde{\tau}\otimes\alpha}F_{n+3}$ lies in the kernel of r. This proves that the above is indeed a chain, namely it shows that Stokes' Theorem holds. Since this chain is essentially the restriction of the cycle of Example 8 to a closed interval, its character satisfies (9) w.r.t. to the Banach sub-algebra $L^1(\mathbb{R}, C_0([\hbar_0, \hbar_1] \times \mathbb{R}, \mathcal{B}), \tilde{\tau} \otimes \alpha)$.

5. Pairings Between K-Theory and Higher Traces

A cyclic n-cocycle η over $\mathcal{B}$ extends to one over $\mathcal{B}^+ = \mathcal{B} \times \mathbb{C}$ via $\eta((A_0, \lambda_0), \ldots, (A_n, \lambda_n)) = \eta(A_0, \ldots, A_n)$. Moreover, $\mathrm{Tr} \otimes \eta$ is a cyclic cocycle over $M_m(\mathcal{B}) \cong M_m(\mathbb{C}) \otimes \mathcal{B}$, where Tr is the standard matrix trace. Define, for a projection $p \in M_m(\mathcal{B}^+)$ or a $u \in U(M_m(\mathcal{B}))$, respectively,

$$\langle \eta, p\rangle = c_n \,\mathrm{Tr} \otimes \eta(p, \ldots, p)\,, \qquad n \text{ even}, \tag{16}$$

$$\langle \eta, u\rangle = c_n \,\mathrm{Tr} \otimes \eta(u^* - 1, u - 1, \ldots)\,, \qquad n \text{ odd}\,, \tag{17}$$

the last formula with alternating entries. The normalization constants are (chosen as in [P83], but not as in [C94]):

$$c_{2k} = \frac{1}{(2\pi\imath)^k}\frac{1}{k!}, \qquad c_{2k+1} = \frac{1}{(2\pi\imath)^{k+1}}\frac{1}{2^{2k+1}}\frac{1}{(k+\frac{1}{2})(k-\frac{1}{2})\cdots\frac{1}{2}}.$$

5.1. General properties of pairings. The map $\langle .,.\rangle$ defined in (16) and (17) is referred to as Connes' pairing between the K-theory and cyclic cohomology of $\mathcal{B}$ because of the following property:

Theorem 3 [C86, C94]. *Let $\mathcal{B}$ be a Banach algebra. The map $\langle .,.\rangle$ induces bi-additive maps $K_n(\mathcal{B}) \times \bigoplus_{m\geq 0} HC^{2m+n}(\mathcal{B}) \to \mathbb{C}$. In particular, if η a cyclic n-cocycle and x a projection in $M_m(\mathcal{B}^+)$ if n is even, or a unitary of $U(M_m(\mathcal{B}))$ if n is odd, then $\langle \eta, x\rangle$ depends only on the homotopy class of x.*

Another point of view of this theorem is that an even (odd) cyclic cocycle defines a functional $K_n(\mathcal{B}_0) \to \mathbb{C}$, $n = 0$ ($n = 1$). In Sect. 4.1 we mentioned that there are not enough (bounded) cyclic cocycles on a C^*-algebra and therefore discussed n-traces on Banach algebras. These do equally well in this context as the following proposition shows.

Proposition 5 [C86]. *Any n-trace on a Banach algebra $\mathcal{B}$ defines by extension of the formulas* (16) *and* (17) *a functional on $K_n(\mathcal{B})$.*

The reason for this is that the algebra $\mathcal{B}''$ mentionned in Sect. 4.1 to which the n-trace can be extended by continuity is closed under holomorphic functional calculus. This implies that $K_n(\mathcal{B}'') \cong K_n(\mathcal{B})$ with isomorphisms induced by $\mathcal{B}'' \subset \mathcal{B}$ and therefore any class in $K_n(\mathcal{B})$ contains a representative descending from $\mathcal{B}''$ which can be used to determine the pairing. By [B85], the $n+1$-trace $\#_\alpha\eta$ constructed in Theorem 2 yields a well-defined functional on the K-groups of the C^*-crossed products.

Furthermore, Connes analysis of [C86] in which he shows that the character of a higher trace on $\mathcal{B}$ can be extended by continuity to a dense sub-algebra $\mathcal{B}''$ which is closed under holomorphic functional calculus does not require $\int$ to be closed and therefore extends to the case of chains. This implies that the character of the boundary of an unbounded chain $(\Omega, d, \int, \partial\Omega, r)$ over $\mathcal{B} = \Omega^0$ is fully defined on $r(\mathcal{B}'')$. By continuity of the surjection r, $r(\mathcal{B}'')$ is dense in $\partial\Omega^0$ and closed under holomorphic functional calculus. Hence $K_n(r(\mathcal{B}'')) \cong K_n(\partial\Omega^0)$ with isomorphism induced by inclusion. Therefore, the character η of the boundary of the unbounded chain defines by extension of the formulas (16) and (17) a functional on $K_n(\partial\Omega^0)$.

Proposition 6. *Let $(\Omega, d, \int, \partial\Omega, r)$ be an n-dimensional (possibly unbounded) chain over a Banach algebra Ω^0 and consider $x' \in \partial\Omega^0$, a projection if $n-1$ is even (a unitary if $n-1$ is odd). If there exists a projection $x \in \Omega^0$ if $n-1$ is even (a unitary if $n-1$ is odd) such that x' is homotopic to $r(x)$ then x' pairs trivially with the character η of the boundary of the chain.*

Proof. Consider first the case in which $n-1 = 2k$ and x' is homotopic to $r(p)$ for a projection $p \in \Omega^0$. Since the chain is unbounded p can be found in a sub-algebra of Ω^0 which is closed under holomorphic functional calculus and to which the character of the chain extends by continuity. Then the pairing reads

$$\frac{1}{c_{2k}}\langle \eta, x'\rangle = \int' r(p)(d'r(p))^{2k} = \int (dp)^{2k+1}.$$

As $\int$ is a graded trace, this vanishes because

$$(dp)^{2k+1} \;=\; p(dp)^{2k+1}(1-p) + (1-p)(dp)^{2k+1}p\,.$$

If the degree is $n-1 = 2k+1$ we let u be a unitary in the above dense sub-algebra such that x' is homotopic to $r(u)$. Then

$$\frac{1}{c_{2k+1}}\langle \eta, x'\rangle \;=\; \int (du^*du)^{k+1} \;=\; -\int (dudu^*)^{k+1}\,.$$

The last equality follows because $\int$ is a graded trace and the degree of du and $(du^*du)^k du^*$ are both odd. On the other hand, recall $du^* = -u^*duu^*$ so that, using cyclicity again

$$\int (du^*du)^{k+1} \;=\; \int (-u^*duu^*du)^{k+1} \;=\; \int (-duu^*duu^*)^{k+1} \;=\; \int (dudu^*)^{k+1}\,,$$

which shows that $\langle \eta, x'\rangle$ has to vanish. $\square$

Under pairing with K-theory, cyclic cohomology behaves like a periodic cohomology theory. This means, in particular, that there exists a map (denoted S in [C94]) that assigns to each cyclic n-cocycle a cyclic $n+2$-cocycle which pairs in the same way with K-theory. In our context this reads as follows.

Recall that an n-trace η on $\mathcal{B}$ extends to an n-trace on the matrix algebras over $\mathcal{B}$ or to $\mathcal{K}\otimes\mathcal{B}$ by the operator trace on the left factor, $\mathrm{Tr}\otimes\eta$. Furthermore, following [R82] we construct in the appendix an isomorphism $S\mathcal{B}\rtimes_{\tau\otimes\alpha}\mathbb{R}\cong\mathcal{K}\otimes\mathcal{B}$ for any $\mathbb{R}$-action α on $\mathcal{B}$. If η is α-invariant then under this isomorphism $\mathrm{Tr}\otimes\eta$ gets identified with the character η^e of the n-cycle $(S\Omega\rtimes_{\tau\otimes\alpha}\mathbb{R}, d^e, \int^e)$,

$$(d^e f)(x)(s) \;=\; d(f(x)(s))\,, \qquad \int^e \omega \;=\; \int ds \int \omega(0)(s)\,.$$

Although the following result is known, we provide a proof of it in the appendix, filling in some details left to reader in [C94].

Theorem 4. *Let η be an α-invariant n-trace on $\mathcal{B}$ and x be a representative for an element in $K_i(S\mathcal{B}\rtimes_{\tau\otimes\alpha}\mathbb{R})$. Then*

$$\langle \#_{\tau\otimes\alpha}\eta^s, x\rangle \;=\; -\frac{1}{2\pi}\,\langle \eta^e, x\rangle\,. \tag{18}$$

5.2. The duality equation. In the following proposition we construct the dual to the ENN-maps (8).

Proposition 7. *Let η be an α-invariant n-trace on $\mathcal{B}$ and $SS\mathcal{B}\rtimes_{\tilde\tau\otimes\alpha}\mathbb{R}$ the C^*-field from* Example 2 *with fibre map $\varphi_\hbar$. Furthermore, $x\in M_m((SS\mathcal{B}\rtimes_{\tilde\tau\otimes\alpha}\mathbb{R})^+)$ is a projection if n is even, or $x\in U(M_m(SS\mathcal{B}\rtimes_{\tilde\tau\otimes\alpha}\mathbb{R})$ if η is odd. Denote $x_\hbar = \varphi_\hbar(x)$. Then the pairings*

$$\langle \#_{\tau^\hbar\otimes\alpha}\eta^s, x_\hbar\rangle$$

(for a given $\hbar$ over the algebra $S\mathcal{B}\rtimes_{\tau^\hbar\otimes\alpha}\mathbb{R}$) are independent of $\hbar$. In other words, the map

$$\#_{\tau^1\otimes\alpha}\eta^s \mapsto \#_{\tau^0\otimes\alpha}\eta^s$$

is the dual to the ENN-*map* (8).

Proof. We apply Proposition 6 to the chain constructed in Example 9. The boundary of this chain is a cycle for $S\mathcal{B} \rtimes_{\tau^{\hbar_0}\otimes\alpha} \mathbb{R} \oplus S\mathcal{B} \rtimes_{\tau^{\hbar_1}\otimes\alpha} \mathbb{R}$ and its character is given by $\#_{\tau^{\hbar_1}\otimes\alpha}\eta^s \oplus -\#_{\tau^{\hbar_0}\otimes\alpha}\eta^s$, *cf.* Eq. (15). Furthermore $r(x) = (x_{\hbar_0}, x_{\hbar_1})$ is a projection (unitary) in that algebra. By Proposition 6 $(x_{\hbar_0}, x_{\hbar_1})$ therefore pairs trivially with the character of the boundary of the chain, i.e. $0 = \langle \#_{\tau^{\hbar_1}\otimes\alpha}\eta^s, x_{\hbar_1}\rangle - \langle \#_{\tau^{\hbar_0}\otimes\alpha}\eta^s, x_{\hbar_0}\rangle$. □

Theorem 5. *Let x be a representative for an element in $K_i(\mathcal{B} \rtimes_\alpha \mathbb{R})$ and η be an α-invariant n-trace over $\mathcal{B}$. Then*

$$\langle \#_\alpha \eta, x\rangle \;=\; -\langle \#_{\tau\otimes\alpha}\eta^s, \partial_i(x)\rangle\,, \tag{19}$$

where $\partial_0 = \exp : K_0(\mathcal{B} \rtimes_\alpha \mathbb{R}) \to K_1(S\mathcal{B} \rtimes_{\tau\otimes\alpha} \mathbb{R})$ *and* $\partial_1 = \mathrm{ind} : K_1(\mathcal{B} \rtimes_\alpha \mathbb{R}) \to K_0(S\mathcal{B} \rtimes_{\tau\otimes\alpha} \mathbb{R})$.

Proof. Applying Proposition 7 to the points $\hbar = 0$ and $\hbar = 1$ and Proposition 1, it suffices to show that $\langle \#_\alpha\eta, x\rangle = -\langle \#_{\mathrm{id}\otimes\alpha}\,\eta^s, \partial_i(x)\rangle$, where now $\partial_0 = \beta$ and $\partial_1 = \Theta$.

Let us look here at the case $i = 0$ in which x is represented by a projection p. Then $\beta[p]$ is represented by the unitary $Gp + 1$, where $G(s) = e^{2\pi\imath\chi(s)} - 1$. Since G commutes with p, the calculation is simple. If η is the character of the $2k+1$ cycle $(\Omega, d, \int)$, then $\#_{\mathrm{id}\otimes\alpha}\,\eta^s$ is the character of $(\Omega^s_\alpha, d^s_\alpha, \int_{\mathbb{R}}\int \mathrm{ev}_0)$ and we write again $d^s_\alpha = \delta_1 + d_\alpha$, where $\delta_1(f\hat{\otimes}1) = \partial_s f\hat{\otimes}e_2$. Then $d^s_\alpha(Gp) = (\partial_s G)p\hat{\otimes}e_2 + G d_\alpha p$ so that, using $p(d_\alpha p)^m p = p(d_\alpha p)^m$ for even m and $p(d_\alpha p)^m p = 0$ for odd m, one gets

$$\begin{aligned}\langle \#_{\mathrm{id}\otimes\alpha}\eta^s, Gp+1\rangle &= c_{2k+3}\int_{\mathbb{R}}\int \mathrm{ev}_0\left(\bar{G}p\, d^s_\alpha(Gp)\left(d^s_\alpha(\bar{G}p)d^s_\alpha(Gp)\right)^{k+1}\right)\\ &= -\,(k+2)c_{2k+3}\int_{\mathbb{R}} G'G^{k+1}\bar{G}^{k+2}\int \mathrm{ev}_0\left(p(d_\alpha p)^{2k+2}\right)\\ &= -\,\langle \#_\alpha\eta, p\rangle,\end{aligned}$$

since $\int_{\mathbb{R}} G'G^{k+1}\bar{G}^{k+2} = 2\pi\imath\;\frac{(2k+3)!}{(k+1)!(k+2)!}$.

As the case $i = 1$ is not needed for our application to the quantum Hall effect, we refer the reader to [P83] for the corresponding calculation. □

Theorem 6. *Consider an $\mathbb{R}$-action α on a C^*-algebra $\mathcal{B}$ together with an α invariant n-trace η on $\mathcal{B}$. Then $\#_\alpha\eta$ satisfies the duality equation*

$$\langle \#_\alpha\eta, x\rangle \;=\; -\frac{1}{2\pi}\,\langle \eta, \partial_i x\rangle\,, \qquad x \in K_i(\mathcal{B} \rtimes_\alpha \mathbb{R})\,,$$

where ∂_i are the boundary maps of K-theory associated to (1).

Proof. Combine Theorem 5 and Theorem 4. □

6. Topology of the Integer Quantum Hall Effect

In quantum Hall samples, there are two current carrying mechanisms: edge currents flow along the boundaries due to intercepted cyclotron orbits and bulk currents result from the Lorentz drift in presence of an exterior electric field. Important in the present context is that both these currents are topologically quantized by a Fredholm index resulting from a pairing of adequate elements of K-groups and higher traces (unbounded cyclic cocyles). While for bulk currents, this was known for a long time [Be86, AS85, K87, Be88, ASS94, BES94], quantisation of edge currents was only proven more recently and under a gap condition on the bulk Hamiltonian, namely, in a tight binding context in [SKR00, KRS02, EG02], and for continuous magnetic (differential) operators in [KS03]. In the continuous case, the two pairings turn out to be over two of the algebras in the Wiener-Hopf extension (1) and Theorem 6 then implies equality of bulk and edge Hall conductivities. To explain this result and the quantities involved in more detail is the subject of this section.

Within the tight-binding approximation an analogous result was proven in [KRS02] and re-derived by ad-hoc methods by Elbau and Graf [EG02]. Equality of bulk and edge conductivities appeared already in various other guises, for instance, in the framework of scaling theory [P85] and that of classical mechanics as resulting from a simple conservation law [F94].

6.1. Bulk and edge-Hall conductivity. Here we summarize the framework and the results of [KS03] and then state the main theorem for quantum Hall systems. We described a quantum Hall system with disorder and a boundary by means of covariant families of integral operators on $L^2(\mathbb{R}^2)$. For this we combined a Borel probability space $(\Omega, \mathbf{P})$ whose elements describe disorder configurations[2] in $\mathbb{R}^2$ with the space $\hat{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ (topologically a half open interval). Points $s \in \hat{\mathbb{R}}$ describe the position of the boundary of the half-space[3] $\mathbb{R} \times \mathbb{R}^{\leq s}$, where $\mathbb{R}^{\leq s} = \{x \in \mathbb{R} | x \leq s\}$. The space Ω carries an $\mathbb{R}^2$-action which is reminiscent of the translation of a disorder configuration. We denote this action by $\omega \mapsto \boldsymbol{x} \cdot \omega$. For concreteness we may think of $\boldsymbol{x} \cdot \omega$ as the configuration ω being shifted by $\boldsymbol{x}$, *i.e.* $\boldsymbol{x} \cdot \omega$ looks at $\boldsymbol{y} + \boldsymbol{x}$ like ω at $\boldsymbol{y}$. The probability measure $\mathbf{P}$ is required to be invariant and ergodic under this action. Furthermore, Ω carries a compact metric topology w.r.t. which the action is continuous. The $\mathbb{R}^2$-action is extended to an action on $\hat{\Omega} = \Omega \times (\mathbb{R} \cup \{+\infty\})$ by $(\omega, s) \mapsto (\boldsymbol{x} \cdot \omega, s + x_2)$, that is the extended action shifts the boundary in the same direction as the configuration. A family $A = (A_{\hat{\omega}})_{\hat{\omega} \in \hat{\Omega}}$ of integral operators on $L^2(\mathbb{R}^2)$ is called covariant if

$$U(\boldsymbol{\xi}) A_{\hat{\omega}} U(\boldsymbol{\xi})^* = A_{\boldsymbol{\xi} \cdot \hat{\omega}} , \qquad \boldsymbol{\xi} \in \mathbb{R}^2 ,$$

where $U(\boldsymbol{\xi})$ are the magnetic translation operators defined in Sect. 6.2 below. The quantum Hall system is described by a random family $H = (H_{\hat{\omega}})_{\hat{\omega} \in \hat{\Omega}}$ of Hamiltonians,

$$H_{\hat{\omega}} = \frac{\hbar^2}{2m}((\imath \partial_1 - \gamma X_2)^2 + \imath \partial_2^2) + V_\omega$$

acting on $L^2(\mathbb{R} \times \mathbb{R}^{\leq s})$ with Dirichlet boundary conditions at s. Here V_ω is the potential depending on the random variable $\omega \in \Omega$ and γ is the strength of the magnetic field. It is shown in [KS03] that sufficiently regular bounded and compactly supported functions of H yield covariant families of integral operators. Moreover, if we push the boundary to $+\infty$, then we describe a disordered system without boundary by a family of Hamiltonians denoted $H_\infty = (H_{(\omega,\infty)})_{\omega \in \Omega}$. In this framework [BES94, ASS94], the bulk Hall conductivity of a gas of independent electrons described by the planar Hamiltonian at zero-temperature and with chemical potential μ belonging to a mobility gap in the spectrum of the Hamiltonian is given by

[2] We use here the customory notation Ω for the space of disorder configurations. It should not be confused with the notation for graded algebras used earlier.

[3] It is easier in the present context to work with the left half space instead of the right half space we used in [KS03].

$$\sigma_b^{\perp}(\mu) = \frac{q^2}{h}\,\mathrm{ch}(P_\mu, P_\mu, P_\mu)\;, \quad \mathrm{ch}(A, B, C) \\ = -2\pi\imath\;\mathcal{T}(A([X_1, B][X_2, C] - [X_2, B][X_1, C]))\;. \qquad (20)$$

Here $P_\mu = \chi_{(-\infty,\mu]}(H_\infty)$ is the covariant family of associated Fermi projections, X_j the j-component of the position operator and $\mathcal{T}$ the trace per unit volume (precise definition given in Eq. (23) below).

The main result of [KS03] is that the edge Hall conductivity of a gas of independent electrons described by the Hamiltonian H with Dirichlet boundary conditions at the edge at zero-temperature and with chemical potential μ belonging to a gap Δ in the spectrum of the *planar* Hamiltonian H_∞ is given by

$$\sigma_e^{\perp}(\mu) \;=\; -\frac{q^2}{h}\,\xi(\mathcal{U}^*(\Delta) - 1, \mathcal{U}(\Delta) - 1)\;, \qquad \xi(A, B) = \hat{\mathcal{T}}(A[X_1, B])\;. \quad (21)$$

Here $\mathcal{U}(\Delta)$ is constructed from the half-planar Hamiltonian via functional calculus

$$\mathcal{U}(\Delta) \;=\; \exp(-2\pi\imath\, G(H)) \qquad (22)$$

for a monotonously decreasing smooth function $G : \mathbb{R} \to \mathbb{R}$ with $G(-\infty) = 1$, $G(\infty) = 0$, and $\mathrm{supp}(G') \subset \Delta\backslash G^{-1}(\frac{1}{2})$, and $\hat{\mathcal{T}}$ is the trace per unit length along the boundary combined with the operator trace perpendicular to the boundary (its precise definition is given in Eq. (24) below). The corollary of Theorem 6 is then:

Theorem 7. *Suppose that E is in a gap of H_∞, the Hamiltonian on the plane without boundary. Then*

$$\sigma_e^{\perp}(E) \;=\; \sigma_b^{\perp}(E)\;.$$

In the Landau model and its restriction to the half-space, a proof can be given by explicitly calculating both pairings and then seeing that both numbers are the same [SKR00]. This calculation makes use of the translation invariance in the direction along the boundary. Not only does it exclude any type of disorder which breaks that invariance, it is also not very satisfactory in that it does not show directly that both pairings are the same. Moreover, the explicit calculation of both sides becomes already very complicated if a periodic potential is added.

To prove the theorem we need to describe how to view covariant families of operators as elements of C^*-algebras and how (20) and (21) can be interpreted as pairings.

6.2. Observable algebra with disorder and boundary. The first component of the action $\omega \mapsto \boldsymbol{x} \cdot \omega$ of $\mathbb{R}^2$ on Ω corresponds to translating the disorder configuration along the boundary and yields an $\mathbb{R}$-action on $C(\Omega)$ given by

$$\beta^{\parallel}_{x_1}(f)(\omega) = f((x_1, 0)\cdot\omega)\;.$$

The second component of the action $\omega \mapsto \boldsymbol{x} \cdot \omega$ yields for given $\gamma \in \mathbb{R}$ an $\mathbb{R}$-action on the crossed product $C(\Omega) \rtimes_{\beta^\parallel} \mathbb{R}$ by ($f : \mathbb{R} \to C(\Omega)$)

$$\beta^{\perp}_{x_2}(f)(x_1)(\omega) \;=\; e^{\imath\gamma x_1 x_2} f(x_1)((0, x_2)\cdot\omega)\;.$$

This corresponds to translating the disorder configuration perpendicular to the boundary together with a phase shift depending on γ. We can interpret γ as the strength of the magnetic field and $\mathcal{A}_\infty = C(\Omega) \rtimes_{\beta^\parallel} \mathbb{R} \rtimes_{\beta^\perp} \mathbb{R}$ as the observable algebra for the planar model.

Following the philosophy in [KS03], we can view $\mathcal{A}_\infty$ as being obtained from a larger algebra $\mathcal{A} = C_0(\hat{\Omega}) \rtimes_{\beta^\parallel} \mathbb{R} \rtimes_{\tau\otimes\beta^\perp} \mathbb{R}$, where we extend $\beta^\parallel$ trivially to the second factor of $\hat{\Omega}$ and

$$(\tau \otimes \beta^\perp)_{x_2}(f)(x_1)(\hat{\omega}) = e^{\imath\gamma x_1 x_2} f(x_1)((0, x_2) \cdot \hat{\omega}) \ .$$

$\mathcal{A}$ can be interpreted as the observable algebra for the system with boundary. Its relation with the covariant operator families of [KS03] is as follows.

A point $\hat{\omega} \in \hat{\Omega}$ defines a one-dimensional representation of $C_0(\hat{\Omega})$, $\rho_{\hat{\omega}} : C_0(\hat{\Omega}) \to \mathbb{C}$: $\rho_{\hat{\omega}}(f) = f(\hat{\omega})$. Applying (4) twice, we get a representation $\pi_{\hat{\omega}}$ of $\mathcal{A}$ on $L^2(\mathbb{R}^2)$. If $F : \mathbb{R} \to (\mathbb{R} \to C_0(\hat{\Omega}))$, then the integral kernel of $\pi_{\hat{\omega}}(F)$ is

$$\begin{aligned}\langle \boldsymbol{x} \,|\pi_{\hat{\omega}}(F)|\boldsymbol{y}\,\rangle &= \rho_{\hat{\omega}}\beta^\parallel_{-x_1}\Big((\tau \otimes \beta^\perp)_{-x_2}\,(F(x_2 - y_2))\,(x_1 - y_1)\Big)\\ &= e^{-\imath\gamma(x_1-y_1)x_2} F(x_2 - y_2)(x_1 - y_1)(-\boldsymbol{x} \cdot \hat{\omega}) \ .\end{aligned}$$

It follows that $\langle \boldsymbol{x} - \boldsymbol{\xi} \,|\pi_{\hat{\omega}}(F)|\boldsymbol{y} - \boldsymbol{\xi}\,\rangle = e^{\imath\gamma(x_1-y_1)\xi_2}\langle \boldsymbol{x} \,|\pi_{\boldsymbol{\xi}\cdot\hat{\omega}}(F)|\boldsymbol{y}\,\rangle$ and hence

$$U(\boldsymbol{\xi}\,)\pi_{\hat{\omega}}(F)U(\boldsymbol{\xi}\,)^* \;=\; \pi_{\boldsymbol{\xi}\cdot\hat{\omega}}(F) \ ,$$

where the magnetic translation operators $U(\boldsymbol{\xi}\,)$ are defined by

$$(U(\boldsymbol{\xi}\,)\psi)(\boldsymbol{x}\,) = \hat{\Phi}(\boldsymbol{\xi}, \boldsymbol{x} - \boldsymbol{\xi}\,)\,\psi(\boldsymbol{x} - \boldsymbol{\xi}\,) \ , \qquad \hat{\Phi}(\boldsymbol{\xi}, \boldsymbol{x}\,) \;=\; e^{-\imath\gamma\xi_2 x_1} \ .$$

The collection $\pi(F) = (\pi_{\hat{\omega}}(F))_{\hat{\omega}\in\hat{\Omega}}$ forms therefore a covariant family of bounded integral operators. By construction the operators are weakly continuous in $\hat{\omega}$ and the norm $\|\pi(F)\|_\infty$ which we defined in [KS03] to be the essential supremum over $\|\pi_{\hat{\omega}}(F)\|$ is bounded by the C^*-norm $\|F\|$. If we apply the above construction to each summand in the direct sum representation $\rho = \bigoplus_{\hat{\omega}\in\hat{\Omega}} \rho_{\hat{\omega}}$ we obtain a representation π which also decomposes into a direct sum representation, namely on $\bigoplus_{\hat{\omega}\in\hat{\Omega}} L^2(\mathbb{R}^2)$, and we can interpret the covariant family $(\pi_{\hat{\omega}}(F))_{\hat{\omega}\in\hat{\Omega}}$ as the representative $\pi(F)$ of F. Since the direct sum representation ρ is faithful, also π is faithful and so, first, identifies $\mathcal{A}$ with a sub-algebra of the completion of the algebra of weakly continuous covariant families of bounded integral operators denoted $\mathcal{A}$ in [KS03], and second, implies $\|\pi(F)\|_\infty = \|F\|$. The estimates established in [KS03] now show that for potentials $V_\omega(\boldsymbol{x}) := V(-\boldsymbol{x} \cdot \omega)$ with $V \in C(\Omega)$ and differentiable along the flow of the $\mathbb{R}^2$ action and $F \in C^k_c(\mathbb{R})$ with $k > 6$, the covariant families $F(H)$ and $D_j F(H)$ can be viewed as elements of $L^1(\mathbb{R}, L^1(\mathbb{R}, C_0(\hat{\Omega}), \beta^\parallel), \tau \otimes \beta^\perp)$ and hence of $\mathcal{A}$.

6.3. Pushing the boundary to infinity. The second component in $(\omega, s) \in \hat{\Omega}$ describes the position of the boundary and is allowed to take the value $+\infty$. The evaluation $\mathrm{ev}_\infty(F)(x_2)(x_1)(\omega) = F(x_2)(x_1)(\omega, +\infty)$ has the effect of pushing the boundary to $+\infty$. The algebra $\mathcal{A}_\infty = C(\Omega) \rtimes_{\beta^\parallel} \mathbb{R} \rtimes_{\beta^\perp} \mathbb{R}$ is therefore the observable algebra of the model with disorder, but without a boundary (planar model). Now the crucial observation is that the pushing of the boundary to infinity defines a surjective algebra morphism

$$\mathrm{ev}_\infty : \mathcal{A} \to \mathcal{A}_\infty \ .$$

Therefore $\mathcal{A}$ is an extension of $\mathcal{A}_\infty$ by the edge algebra $\mathcal{E} := \ker(\mathrm{ev}_\infty)$ which can be understood as the algebra of observables which are located at the boundary. This gives an exact sequence precisely of the form (1).

6.4. Chern class and non-commutative winding number. The Chern class ch of (20) and the 1-trace ξ of (21) can be obtained by application of Proposition 3. In the first case the algebra is $\mathcal{A}_\infty = C(\Omega) \rtimes_{\beta^\parallel} \mathbb{R} \rtimes_{\beta^\perp} \mathbb{R}$, with trace $\mathcal{T} : \mathcal{A}_\infty \to \mathbb{C}$:

$$\mathcal{T}(F) \;=\; \int_\Omega d\mathbf{P}(\omega)\; F(0)(0)(\omega) \tag{23}$$

and derivations

$$\nabla_j(F)(x_2)(x_1) \;=\; \imath x_j F(x_2)(x_1)\,.$$

Then ch is then $-2\pi\imath$ times the character of the 2-cycle constructed from these data using Proposition 3. Its domain includes the dense sub-algebra $C_c(\mathbb{R}, C_c(\mathbb{R}, C(\Omega)))$.

In the second case, the algebra is the ideal $\mathcal{E} = \ker(\mathrm{ev}_\infty) \subset \mathcal{A} = C_0(\hat{\Omega}) \rtimes_{\beta^\parallel} \mathbb{R} \rtimes_{\tau\otimes\beta^\perp} \mathbb{R}$ with trace $\hat{\mathcal{T}} : \mathcal{A} \to \mathbb{C}$:

$$\hat{\mathcal{T}}(F) \;=\; \int_{\hat{\Omega}} d\mathbf{P}(\omega) ds\; F(0)(0)(\omega, s) \tag{24}$$

and derivation

$$\nabla_1(F)(x_2)(x_1) \;=\; \imath x_1 F(x_2)(x_1).$$

The character of the 1-trace constructed from these data using Proposition 3 is ξ and its domain contains $C_c(\mathbb{R}, C_c(\mathbb{R}, C(\hat{\Omega}))) \cap \mathcal{E}$.

6.5. Proof of Theorem 7. Theorem 7 is obtained by application of Theorem 6 to the algebras and actions involved in Sect. 6.2 and the higher traces from Sect. 6.4. Specifically, we take $\mathcal{B} = C(\Omega) \rtimes_{\beta^\parallel} \mathbb{R}$ with action $\alpha = \beta^\perp$ and 1-trace η on $\mathcal{B}$ given by $\eta(f, g) = \int_\Omega d\mathbf{P}(\omega)\,(f\nabla_1 g)(0)(\omega)$. Then ξ from (21) is given by $\xi = \imath\eta^e$ and ch from (20) by ch $= -2\pi\imath\ \#_\alpha\eta$. Now let $P_\mu \in \mathcal{A}_\infty = \mathcal{B} \rtimes_\alpha \mathbb{R}$ be the element corresponding to the Fermi projection. The exponential map associated with the extension defined by $\mathcal{A} \xrightarrow{\mathrm{ev}_\infty} \mathcal{A}_\infty$ yields $[\mathcal{U}(\Delta)]_1 = \exp[P_\mu]_0$. Thus

$$\frac{1}{2\pi\imath}\;\langle \mathrm{ch}, P_\mu\rangle \;=\; \langle \#_{\tau\otimes\alpha}\eta^s, \mathcal{U}(\Delta)\rangle \;=\; -\,\frac{1}{2\pi\imath}\;\langle \xi, \mathcal{U}(\Delta)\rangle\;,$$

the first equality following from Theorem 5 and the second from Theorem 4. Since $c_1 = c_2$ we have

$$\mathrm{ch}(P_\mu, P_\mu, P_\mu) \;=\; -\,\xi(\mathcal{U}(\Delta)^* - 1, \mathcal{U}(\Delta) - 1)\;,$$

which proves Theorem 7. $\square$

A. Periodicity in Cyclic Cohomology

For the convenience of the reader we present a detailed proof of Theorem 4. The following two isomorphisms [R82] allow to reduce this proof to a calculation for compact operators:

$$\Psi : S\mathcal{B} \rtimes_{\tau\otimes\alpha} \mathbb{R} \to S\mathcal{B} \rtimes_{\tau\otimes \mathrm{id}} \mathbb{R}\,, \qquad \Psi(f)(x)(s) \;=\; \alpha_s(f(x)(s))\,, \tag{25}$$

and (identifying $S\mathcal{B} \rtimes_{\tau\otimes\mathrm{id}} \mathbb{R}$ with $S\mathbb{C} \rtimes_\tau \mathbb{R} \otimes \mathcal{B}$)

$$\Phi : S\mathcal{B} \rtimes_{\tau\otimes\mathrm{id}} \mathbb{R} \to \mathcal{K}(L^2(\mathbb{R})) \otimes \mathcal{B}\,, \qquad \Phi \;=\; \rho \otimes \mathrm{id}\,, \tag{26}$$

where ρ is the representation of $S\mathbb{C} \rtimes_\tau \mathbb{R}$ on $L^2(\mathbb{R})$ given by

$$(\rho(f)\psi)(x) \;=\; \int dy\, f(y)(x)\,\psi(x-y) \;=\; \int dy\, f(x-y)(x)\,\psi(y)\,.$$

Hence the integral kernel of $\rho(f)$ is $\langle x|\rho(f)|y\rangle = f(x-y)(x)$ so that $\mathrm{Tr}(\rho(f)) = \int_{\mathbb{R}} dx f(0)(x)$.

Lemma 1. *Let η be a α-invariant cyclic cocycle over $\mathcal{B}$. Then*

$$\#_{\tau\otimes\alpha}\, \eta^s \;=\; \Psi^* \#_{\tau\otimes\mathrm{id}}\, \eta^s\,, \tag{27}$$

$$\eta^e \;=\; \Psi^*\Phi^* \mathrm{Tr} \otimes \eta\,. \tag{28}$$

Proof. Let η be the character of $(\Omega, d, \int)$. One has

$$\#_{\tau\otimes\alpha}\, \eta^s(f_0,\ldots,f_n) \;=\; \int_{\mathbb{R}} ds \int \left(f_0 d^s_{\tau\otimes\alpha} f_1 \cdots d^s_{\tau\otimes\alpha} f_n\right)(0)(s)\,,$$

where the product is that in $S\Omega \rtimes_{\tau\otimes\alpha} \mathbb{R}$. On the other hand

$$(\Psi^*\#_{\tau\otimes\mathrm{id}}\, \eta^s)(f_0,\ldots,f_n) = \int_{\mathbb{R}} ds \int \left(\Psi(f_0) d^s_{\tau\otimes\mathrm{id}} \Psi(f_1) \cdots d^s_{\tau\otimes\mathrm{id}} \Psi(f_n)\right)(0)(s),$$

with product in $S\Omega \rtimes_{\tau\otimes\mathrm{id}} \mathbb{R}$. Now (27) follows from $d^s_{\tau\otimes\mathrm{id}}\Psi(f) = \Psi(d^s_{\tau\otimes\alpha} f)$ and

$$\left(\Psi(f_0) d^s_{\tau\otimes\mathrm{id}} \Psi(f_1) \cdots d^s_{\tau\otimes\mathrm{id}} \Psi(f_n)\right)(0)(s) \;=\; \alpha_s\left(f_0 d^s_{\tau\otimes\alpha} f_1 \cdots d^s_{\tau\otimes\alpha} f_n\right)(0)(s)$$

and the α-invariance of η.

For (28), let $f_j = \Psi^{-1}(g_j \otimes b_j)$, where $g_j \otimes b_j \in S\mathbb{C} \rtimes_\tau \mathbb{R} \otimes \mathcal{B} \cong S\mathcal{B} \rtimes_{\tau\otimes\mathrm{id}} \mathbb{R}$. Then

$$\begin{aligned}
\Psi^*\Phi^*\, \mathrm{Tr} \otimes \eta(f_0,\ldots,f_n) &= \mathrm{Tr} \otimes \eta(\rho(g_0)\otimes b_0, \ldots, \rho(g_n)\otimes b_n)\\
&= \mathrm{Tr}(\rho(g_0\cdots g_n)) \int b_0 db_1 \cdots db_n\,,\\
\eta^e(f_0,\ldots,f_n) &= \int_{\mathbb{R}} ds \int \mathrm{ev}_0\, \Psi^{-1}(g_0\otimes b_0) d\Psi^{-1}\\
&\quad (g_1 \otimes b_1) \cdots d\Psi^{-1}(g_n \otimes b_n)\\
&= \int_{\mathbb{R}} ds\, \mathrm{ev}_0(g_0\cdots g_n) \int b_0 db_1 \cdots b_n\,,
\end{aligned}$$

and the lemma follows from the fact that $\mathrm{Tr}(\rho(g)) = \int_{\mathbb{R}} ds\, g(0)(s)$ for $g \in S\mathbb{C} \rtimes_\tau \mathbb{R}$. □

Lemma 2. *Let η be an α-invariant cyclic cocycle over $\mathcal{B}$. Then for a projection $X \in \mathcal{K} \otimes \mathcal{B}$ or a unitary $X \in U(\mathcal{K} \otimes \mathcal{B})$ one has*

$$\langle \mathrm{Tr} \otimes \eta, X \rangle \;=\; -2\pi \;\langle \#_{\tau\otimes \mathrm{id}}\eta^s, \Phi^{-1}(X)\rangle \;.$$

Proof. We start with the case $\mathcal{B} = \mathbb{C}$ and $\eta = \mathrm{Tr}$ which is the character of $(\mathbb{C}, 0, \mathrm{Tr})$. Then $\mathrm{Tr} \otimes \eta = \mathrm{Tr}$ and $\#_{\tau\otimes\mathrm{id}}\eta^s = \#_\tau \mathrm{Tr}^s$ is the character of $(S\mathbb{C} \rtimes_\tau \mathbb{R} \otimes \Lambda\mathbb{C}^2, \delta, \int_\tau^s)$ where, for $f : \mathbb{R} \to S\mathbb{C}$, $\delta = \delta_1 + \delta_2$ with $\delta_1(f) = \partial_s f \otimes e_1$, $\delta_2(f) = \nabla_x f \otimes e_2$, $\nabla_x f(x) = \imath x f(x)$, and $\int_\tau^s f \otimes e_1 e_2 = \int_{\mathbb{R}} ds f(0)(s) = \mathrm{Tr}(\rho(f))$. Let M and D be operators given by $M\psi(x) = \imath x\psi(x)$ and $D\psi(x) = \psi'(x)$ with the usual common domain $C_c^1(\mathbb{R})$. They satisfy $[D, M] = \imath$. Then, for differentiable f,

$$\langle x|[M, \rho(f)]|y\rangle \;=\; \imath(x-y)f(x-y)(x) \;=\; \langle x|\rho(\nabla_x f)|y\rangle \;,$$

$$\langle x|[D, \rho(f)]|y\rangle \;=\; (\partial_x + \partial_y)f(x-y)(x) \;=\; \langle x|\rho(\partial_s f)|y\rangle \;,$$

and therefore, if $p \in S\mathbb{C} \rtimes_\tau \mathbb{R}$ is a differentiable projection,

$$\langle \#_\tau \mathrm{Tr}^s, p\rangle \;=\; \frac{1}{2\pi\imath}\,\mathrm{Tr}\left(\rho(p)[[D, \rho(p)], [M, \rho(p)]]\right) \;=\; -\frac{1}{2\pi}\,\mathrm{Tr}(\rho(p)) \;. \tag{29}$$

In the last equation, we used $P[[D, P], [M, P]]P = -P[D, M]P + [PDP, PMP]$, $P = \rho(p) = P^2$. This proves the statement for $\mathcal{B} = \mathbb{C}$ and $\eta = \mathrm{Tr}$.

In the general case, η is the character of some n-cycle $(\Omega, d, \int)$ over $\mathcal{B}$. Then $d^s_{\tau\otimes\mathrm{id}} = d' + \delta$, where δ, is as above and $d'(f)(x)(s) = d(f(x)(s))$.

We apply this first to a projection of the form $X = \rho(p) \otimes x \in \mathcal{K} \otimes \mathcal{B}^+$, where x and p are projections. Then $d^s_{\tau\otimes\mathrm{id}}(\Phi^{-1}(X)) = p \otimes dx + \delta p \otimes x$. Let $n = 2k$, $k \geq 0$. Using $p(\delta p)p^j = 0$ if $j > 0$, $x(dx)^{l-1}x = 0$ for $l > 1$ and $\int_{\mathbb{R}} ds\, \mathrm{ev}_0(p(\delta p)^2) = 2\pi\imath\;\langle \#_\tau \mathrm{Tr}^s, p\rangle = -\imath\;\mathrm{Tr}(\rho(p))$, we get

$$\begin{aligned}
\langle \#_{\tau\otimes\mathrm{id}}\,\eta^s, \Phi^{-1}(X)\rangle \;&=\; c_{2k+2}\int^s_{\tau\otimes\mathrm{id}} \Phi^{-1}(X(d^s_{\tau\otimes\mathrm{id}}X)^{2k+2}) \\
&= c_{2k+2}\sum_{0<l<j\leq 2k+2}\int_{\mathbb{R}} ds\,\mathrm{ev}_0\left(p(\delta p)p^{j-l-1}(\delta p)\right) \\
&\int x(dx)^{l-1}x(dx)^{j-i-1}x(dx)^{2k+2-j} \\
&= -\imath\;\frac{(k+1)c_{2k+2}}{c_{2k}}\;\mathrm{Tr}(\rho(p))\;\langle \eta, x\rangle \\
&= -\frac{1}{2\pi}\;\langle \mathrm{Tr}\otimes\eta, X\rangle \;.
\end{aligned}$$

Next we apply this to a unitary $X = \rho(p) \otimes x + \rho(p)^\perp \otimes 1$ for $x \in U(\mathcal{B})$ and a projection p. Then $d^s_{\tau\otimes\mathrm{id}}(\Phi^{-1}(X)) = p \otimes dx + \delta p \otimes (x-1)$. If $n = 2k+1$, $k > 0$, one obtains, taking into account that $p(\delta p)p^k = 0$ for $k > 0$,

$$\begin{aligned}
\langle \#_{\tau\otimes\mathrm{id}}\,\eta^s, \Phi^{-1}(X)\rangle \;&=\; c_{2k+3}\int^s_{\tau\otimes\mathrm{id}} \Phi^{-1}((X^*-1)d^s_{\tau\otimes\mathrm{id}}X(d^s_{\tau\otimes\mathrm{id}}X^*d^s_{\tau\otimes\mathrm{id}}X)^{k+1}) \\
&= c_{2k+3}\int_{\mathbb{R}} ds\,\mathrm{ev}_0\left(p(\delta p)^2\right)\sum_{0<j<2k+3} \\
&\int (x^*-1)dx^{(1)}\cdots dx^{(j-1)}(x^*-1)(x-1)dx^{(j+2)}\cdots dx^{(2k+3)},
\end{aligned}$$

where $x^{(j)} = x$ if j is odd and $x^{(j)} = x^*$ if j is even. We claim that

$$\sum_{j=1}^{2k+2} \int (x^* - 1)dx^{(1)} \cdots dx^{(j-1)}(x^* - 1)(x - 1)dx^{(j+2)} \cdots dx^{(2k+3)}$$
$$= 2(2k+3) \int (x^* - 1)dx(dx^*dx)^k, \tag{30}$$

which then implies

$$\langle \#_{\tau\otimes\mathrm{id}}\, \eta^s, \Phi^{-1}(X)\rangle \;=\; -\imath\, \frac{2(2k+3)c_{2k+3}}{c_{2k+1}}\, \mathrm{Tr}(\rho(p))\langle \eta, x\rangle \;=\; -\frac{1}{2\pi}\, \langle \mathrm{Tr}\otimes\eta, X\rangle\ .$$

Since $(x^* - 1)(x - 1) = 2 - x - x^*$, Eq. (30) is equivalent to

$$\int \sum_{l=0}^{k} (x^* - 1)(dxdx^*)^l \left((x + x^*)dx + dx(x + x^*)\right)(dx^*dx)^{k-l}$$
$$= -2 \int (x^* - 1)(dxdx^*)^k dx\ . \tag{31}$$

Now use $dx^*x = -x^*dx$ and $dxx^* = -xdx^*$ to pull x and x^* of $(x + x^*)$ either to the right or to the left and then use cyclicity in order to obtain

$$\begin{aligned}
\text{l.h.s. of (31)} &= \sum_{l=0}^{k} \int x(x^* - 1)\Big((dx^*dx)^l dx(dx^*dx)^{k-l} - (dxdx^*)^l dx^*(dxdx^*)^{k-l}\\
&\quad -(dx^*dx)^l dx^*(dx^*dx)^{k-l} + (dxdx^*)^l dx(dxdx^*)^{k-l}\Big)\\
&= \sum_{l=0}^{k} \Bigg(-\eta(x, \underbrace{x^*, x, \ldots}_{2l}, x, \underbrace{x^*, x, \ldots}_{2(k-l)}) + \eta(x, \underbrace{x, x^*, \ldots}_{2l}, x^*, \underbrace{x, x^*, \ldots}_{2(k-l)})\\
&\quad + \eta(x, \underbrace{x^*, x, \ldots}_{2l}, x^*, \underbrace{x^*, x, \ldots}_{2(k-l)}) - \eta(x, \underbrace{x, x^*, \ldots}_{2l}, x, \underbrace{x, x^*, \ldots}_{2(k-l)})\Bigg)\ .
\end{aligned}$$

Here the entries under-braced are alternating. For fixed l the first and the fourth term in each summand cancel by cyclic symmetry. The remaining terms are

$$\begin{aligned}
\text{l.h.s. of (31)} &= \sum_{l=0}^{k} \Bigg(-\eta(\underbrace{x, x^*, \ldots}_{2l}, \underbrace{x^*, x, \ldots}_{2(k-l+1)}) + \eta(\underbrace{x, x^*, \ldots}_{2(l+1)}, \underbrace{x^*, x, \ldots}_{2(k-l)})\Bigg)\\
&= -\eta(\underbrace{x^*, x \ldots}_{2(k+1)}) + \eta(\underbrace{x, x^*, \ldots}_{2(k+1)})\\
&= -\int 2(x^* - 1)dx(dx^*dx)^k\ .
\end{aligned}$$

It remains to compute the pairings for unitaries X of the form $X - 1 = \sum_j a_j \otimes b_j \in \mathcal{K} \otimes \mathcal{B}$, where the sum is finite and the a_j have finite rank (for projections of the form

$\sum_j a_j \otimes b_j \in \mathcal{K} \otimes \mathcal{B}^+$ the argument is similar). In that case $X - 1 \in M_n \otimes \mathcal{B}$ ($M_n = M_n(\mathbb{C})$ is the associated sub algebra of $\mathcal{K}$ for some finite n). Let $e \in M_n$ be an arbitrary rank one projection, $e^\perp$ its ortho complement in M_n. Then, by the above,

$$\begin{aligned}\langle \mathrm{Tr} \otimes \eta, X\rangle &= \langle \mathrm{Tr} \otimes \mathrm{Tr} \otimes \eta, e \otimes X + e^\perp \otimes 1\rangle \\ &= -2\pi\ \langle \#_{\tau\otimes\mathrm{id}}(\mathrm{Tr} \otimes \eta)^s, \rho^{-1}(e) \otimes X + \rho^{-1}(e^\perp) \otimes 1\rangle\,.\end{aligned}$$

Further let $U \in \mathcal{U}(M_n \otimes M_n)$ be a unitary such that $\mathrm{Ad}_U \in \mathrm{Aut}(M_n \otimes M_n)$ is the flip, $Ua_1 \otimes a_2 U^* = a_2 \otimes a_1$. Since $\mathcal{U}(M_n \otimes M_n)$ is connected, a path connecting U to the identity gives rise to a homotopy in $M_n \otimes M_n \otimes \mathcal{B}^+$ between $e \otimes X + e^\perp \otimes 1$ and $\mathrm{Ad}_U \otimes \mathrm{id}(e \otimes X + e^\perp \otimes 1)$. Since $\mathrm{Ad}_U \otimes \mathrm{id}(e \otimes X) = \sum_j a_j \otimes e \otimes b_j$, we have by homotopy invariance of the pairings

$$\begin{aligned}&\langle \#_{\tau\otimes\mathrm{id}}(\mathrm{Tr} \otimes \eta)^s, \rho^{-1}(e) \otimes X + \rho^{-1}(e^\perp) \otimes 1\rangle \\ &\quad= \langle \#_{\tau\otimes\mathrm{id}}(\mathrm{Tr} \otimes \eta)^s, \sum_j a_j \otimes \rho^{-1}(e) \otimes b_j + \mathrm{Ad}_U \otimes \mathrm{id}(e^\perp \otimes 1)\rangle \\ &\quad= \langle \#_{\tau\otimes\mathrm{id}}\eta^s, \sum_j a_j \otimes b_j + 1\rangle\,.\end{aligned}$$

This proves the statement. □

Proof of Theorem 4. Combine the last two lemmas with $X = \Phi\Psi(x)$. □

References

[AS85] Avron, J.E., Seiler, R.: Quantization of the Hall Conductance for Gerneral Multiparticle Schrödinger Hamiltonians. Phys. Rev. Lett. **54**, 259–262 (1985)

[ASS94] Avron, J.E., Seiler, R., Simon, B.: Charge Deficiency, Charge Transport and Comparison of Dimensions. Com. Math. Phys. **159**, 399–422 (1994)

[Be86] Bellissard, J.: K-theory of C^*-algebras in solid state physics. In: *Statistical Mechanics and Field Theory: Mathematical Aspects*, *Lecture Notes in Physics* **257**, edited by T. Dorlas, M. Hugenholtz, M. Winnink, Berlin: Springer-Verlag, 1986, pp. 99–156

[Be88] Bellissard, J.: Ordinary quantum Hall effect and non-commutative cohomology. In: *Proc. of the Bad Schandau Conference on Localization, 1986*, eds. P. Ziesche, W. Weller, Teubner Texte Phys. **16**, Leipzig: Teubner-Verlag, 1988

[BES94] Bellissard, J., van Elst, A., Schulz-Baldes, H.: The Non-Commutative Geometry of the Quantum Hall Effect. J. Math. Physics **35**, 5373–5451 (1994)

[Bl86] Blackadar, B.: *K-Theory for Operator Algebras*. New York: Springer-Verlag, 1986

[B85] Bost, J.B.: K-théorie des products croisés et principe d'Oka. C. R. Acad. Sc. Paris, t. 301, Série I, no. 5, 189–192 (1985)

[C81] Connes, A.: An analogue of the Thom isomorphism. Adv. in Math. **39**, 31–55 (1981)

[C86] Connes, A.: Cyclic cohomology and the transverse fundamental class of a foliation. In: *Geometric methods in operator algebras (Kyoto, 1983)*, Pitman Res. Notes in Math., Harlow: Longman, 1986, pp. 52–144

[C94] Connes, A.: *Non-Commutative Geometry*. San Diego: Acad. Press, 1994

[EG02] Elbau, P., Graf, G.-M.: Equality of bulk and edge Hall conductance revisited. Commun. Math. Phys. **229**, 415–432 (2002)

[ENN88] Elliott, G., Natsume, T., Nest, R.: Cyclic cohomology for one-parameter smooth crossed products. Acta Math. **160**, 285–305 (1988)

[ENN93] Elliott, G., Natsume, T., Nest, R.: The Heisenberg group and K-theory. K-Theory **7**, 409–428 (1993)

[F94] Fröhlich, J.: Mathematical aspects of the quantum Hall effect. In: *First European Congress of Mathematics*, Vol. **II** (Paris, 1992), Progr. Math. **120**, Basel: Birkhäuser, 1994, pp. 23–48

[KRS02] Kellendonk, J., Richter, T., Schulz-Baldes, H.: Edge current channels and Chern numbers in the integer quantum Hall effect. Rev. Math. Phys. **14**, 87–119 (2002)

[KS03] Kellendonk, J., Schulz-Baldes, H.: Quantization of edge currents for continuous magnetic operators. J. Funct. Anal. **209**, 388–413 (2004)
[K87] Kunz, H.: The quantum Hall effect for electrons in a random potential. Comm. Math. Phys. **112**, 121–145 (1987)
[L98] Landsman, N.P.: *Mathematical topics between classical and quantum mechanics*. New York, Berlin, Heidelberg: Springer-Verlag, 1998
[P79] Pedersen, G.: C^*-algebras and their automorphism groups. New York: Academic Press, 1979
[P83] Pimsner, M.V.: Ranges of traces on K_0 of reduced crossed products by free groups. In: *Operator Algebras and their Connections with Topology and Ergodic Theory*, Lecture Notes in Math. **1132**, Berlin-Heidelberg-New York: Springer-Verlag, 1983, pp. 374–408
[P85] Pruisken, A.M.M.: Field Theory, Scaling and the Localization Problem. In: Prange, R. Girvin, S. (ed.) *The Quantum Hall Effect*, 2nd Edition, Berlin: Springer-Verlag, 1990
[R82] Rieffel, M.A.: Connes' analogue for crossed products of the Thom isomorphism. Contemp. Math. **10**, 143–154 (1982)
[R89] Rieffel, M.A.: Continuous fields of C^*-algebras coming from group cocycles and actions. Math. Ann. **283**, 631–643 (1989)
[SKR00] Schulz-Baldes, H., Kellendonk, J., Richter, T.: Simultaneous quantization of the edge and bulk Hall conductivity. J. Phys. A: Math. Gen. **33**, L27–L32 (2000)

Communicated by A. Connes