# Gap labelling and the pressure on the boundary

Johannes Kellendonk

Institut Girard Desargues, Université Claude Bernard Lyon 1, F-69622 Villeurbanne

November 9, 2004

**Abstract**

In quantum systems described by covariant families of 1-particle Schrödinger operators on half-spaces the pressure on the boundary per unit energy is topologically quantised if the Fermi energy lies in a gap of the bulk spectrum. Its relation with the integrated density of states can be expressed in an integrated version of Streda's formula. This leads also to a gap labelling theorem for systems with constant magnetic field. The proof uses non-commutative topology.

# 1 Introduction

In [KS04a, KS04b] a theory was developed which relates bulk topological invariants of aperiodic quantum mechanical systems to boundary topological invariants. Such invariants describe the topological part of response coefficients in transport theory. The theory has two aspects, topological quantization and equalities between invariants. The latter has predictive character: given a bulk topological invariant there should be a topological invariant associated with the boundary behaviour which has the same numerical value and vice versa. The above-mentionned work was motivated by one application, the Integer Quantum Hall Effect. Here we present the proof of another application which was anounced in [Ke04]: To a gap label, a bulk invariant, corresponds a response coefficient related to the pressure on the boundary. Physically this means that the integrated density of states (IDS) at the Fermi level is equal to the pressure on the boundary per unit energy if the Fermi level lies in a gap of the bulk spectrum. The associated boundary force compensates two forces, the gradient force from the electrical potential and the Lorentz force which arises in the presence of a magnetic field. We obtain hence an equation of the type (Theorem 2)

$$\mathrm{IDS} = \Pi + b\,\sigma^{\parallel}\,, \tag{1}$$

all taken at the Fermi energy. Here $\Pi$ is the gradient pressure per unit energy (i.e. the gradient force per unit area and energy), $\sigma^{\parallel}$ the conductivity of the current along the edge in a direction determined by the magnetic field, and $b$ is proportional to the magnetic field strength. More precisely, for two-dimensional systems $b = \frac{B}{q}$ ($B$ is the magnetic field understood as scalar and $q$ the charge of the particle) and $\sigma^{\parallel} = \sigma_{\mathrm{Hall}}$ is the (edge) Hall conductivity. In three dimensions

$qb$ is the projection of $B$ along the boundary and $\sigma^{\|}$ is the direct conductivity of the component of the boundary current which is perpendicular to $B$. All these quantities are understood as averaged as in the usual approach to disordered systems.

All three quantities are topologically quantised. Whereas in two-dimensions $\frac{h}{q^2}\sigma^{\|}$ is integral this is not the case for $\Pi$ which takes typically values in a countable dense sub-group of $\mathbb{R}$. On the other hand, we show that $\Pi$ is locally constant in $B$.

Equation 1 should be compared with Streda's formula (12) of [St82] which yields an expression of the Hall conductivity of a two-dimensional sample in the bulk. Under the gap condition the direct conductivity vanishes in the bulk and Streda's formula can be obtained as the derivative of (1) w.r.t. the magnetic field,

$$\sigma_{\mathrm{Hall}} = q\frac{\partial}{\partial B}\mathrm{IDS}.$$

$\Pi$ is thus the constant of integration of Streda's formula.

There are two simple situations in which (1) can be verified without the machinery of non-commutative topology. The first is the Landau operator which describes a free electric particle in two dimensions in a homogeneous perpendicular magnetic field. Since the potential is zero in this case $\Pi$ vanishes and the identity can be obtained by explicitly calculating the trace (per unit volume) and Chern number of the projection onto the $n$th Landau level. The result is independently of $n$ equal to $\frac{qB}{h}$ and 1, respectively, and so (1) follows from the additivity of the trace and the Chern number. The second example is the one-dimensional situation which in the periodic case has been treated in [Ke04] and in the aperiodic one in [KZ04]. Here $B = 0$ so that the integrated density of states equals the gradient force per unit energy.

Equation 1 suggests that the gap labelling group for a system with non-vanishing magnetic field has one more generator which accounts for the second term. We will present a gap-labelling theorem (Theorem 3) for systems with magnetic field which demonstrates that this is indeed the case in $d = 2$. It follows from these arguments that the first term $\Pi$ belongs to a sub-group of the gap labelling group which is independent of the magnetic field depending only on the spatial structure (long range order) of the aperiodic system.

Some functional analytic results are proven here only for $d \leq 2$, because they relie heavily on results of [KS04a]. We consider this limitation as technical but not conceptual. The purely topological results of Section 4 are not restricted in dimension.

The article is organized as follows. We begin with a short discussion of response theory and introduce the transport coefficients which are of interest in this article (Examples of Section 2). In Section 3 we recall the model used to describe aperiodic systems with and without a boundary and adapt the formulation of the transport coefficients to that framework. This allows us to see that they are topological and paves the way for a $C^*$-algebraic description in Section 4. To prove Equation 1 we identify the coefficients as results of pairings between higher traces and elements of the $K$-groups and use the boundary maps of non-commutative topology to relate them. Section 4 ends with a gap labelling theorem for systems with non-vanishing magnetic field.

## 2 Response coefficients

The general set up of [KS04b] is aimed at relating topological properties such as topological transport properties of the system in the bulk with those on the boundary. The usual framework for discussing transport is reponse theory, a phenomenological theory in which transport coefficients are determined by investigating how the system responds to a small perturbation which brings it out of thermodynamic equilibrium. We suppose that an infinitesimal perturbation of the system leads to a variation $\delta H$ of the Hamiltonian which can be expressed by a (possibly vector valued) derivation $\delta$. We are interested in an expectation value $\langle \delta H \rangle$ which is the result of evaluating a positive linear functional on $\delta H$. This functional depends on the particular circumstances of the system and what one wants to observe. It can usually be expressed in the form $\langle A \rangle = \mathrm{tr}(\rho A)$ where tr is a trace and $\rho$ a one-particle density matrix both of which may carry physical units. $\langle \delta H \rangle$ is essentially a mechanical force if the variation is induced by a variation in configuration space, it is a current if the variation is induced by a variation in momentum space.

Transport coefficients are coefficients of tensors which relate the strength of a perturbation to the above (vector-valued) force or current. More precisely, one considers the expectation of the operator $\delta H$ w.r.t. the density matrix which has evolved under the perturbed dynamics to time $\infty$. This is equivalent to

$$\langle \delta H \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T \langle \eta_{-t}^{per}(\delta H) \rangle dt$$

where $\eta_t^{per} = \exp t(\imath\mathcal{L} + \lambda\delta)$, $\mathcal{L} = [H, \cdot]$ is the Liouvillian and $\lambda$ the of strength of the perturbation. (As $\delta$ we consider $\lambda$ vectorial and thus $\lambda\delta$ as a scalar product.) The above quantity is called response function; it is the generating function for the transport coefficients. From a Dyson-Phillips expansion of $\eta_t^{per}$ one obtains a power series in $\lambda$ and the transport coefficients are the coefficients of the tensors in the expansion around $\lambda = 0$. The $n$th order term contains then $n + 1$ derivatives w.r.t. $\delta$. The 0th coefficient is a pure equilibrium phenomenon, as in the 0th approximation the time evolution is the unperturbed one.

In the context of aperiodic systems (whether randomly disordered or long range ordered) it has become customary to consider not single systems but ensembles of systems which may be considered as physically indistinguishable. Then the expectation above includes an average over the ensemble (disorder average). Whereas for bulk systems this average is often covered by the trace per unit volume, this is no longer the case for bulk boundary systems.

A transport coefficient becomes topological if, from a mathematical point of view, it depends only on the homotopy class of a certain operator. In the most famous case of the Hall-conductivity the conductivity tensor can be written as the index of a Fredholm operator provided certain localisation properties are satisfied. It depends thus only on the homotopy class of that Fredholm operator which physically leads to the remarkable stability of the coefficient under perturbations. But it is neither neccessary nor perhaps possible to write topological coefficients as indices of Fredholm operators. It suffices to write them as (Connes-) pairings between higher traces (unbounded cyclic cocycles) and $K$-group elements. They then depend only on the homotopy class of a projection or a unitary which also leads to the stability result under perturbations.

## 2.1 Bulk coefficients

Transport in the bulk is described by a model without boundary. We will think of that as a model where the boundary has been pushed to infinity and denote the Hamiltonian (or later the family of Hamiltonians) by $H_\infty$. The one-particle approximation is usually considered at the zero temperature limit and therefore the expectation value is taken over all states of energy less or equal the Fermi energy $E_F$. Thus one takes $\mathrm{tr} = \mathcal{T}$, the trace per unit volume, and

$$\langle A \rangle = \mathcal{T}(\rho A)\,, \qquad \rho = P_{E_F}(H_\infty) \tag{2}$$

where $P_{E_F}(H_\infty)$ is the projection onto the states of $H_\infty$ below the Fermi energy. If $\delta$ is given by a commutator, let's say with $B$, one typically expects that the 0-order term, $\mathcal{T}(\rho[B, H_\infty])$ vanishes due to the cyclicity of the trace. The dominant contribution is then given by the linear term and neglecting higher terms one speaks of linear response theory.

**Example 1 (conductivity tensor)** The perhaps best known example arises if one takes the derivations $\nabla_j = \imath[X_j, \cdot]$ so that $q\nabla_j H_\infty$ is the $j$-component of the usual current operator and $\lambda_j = E_j$, the $j$th component of an external electric field. The 0-order term vanishes and the result of the first order approximation is a 2-tensor, the conductivity tensor. This tensor becomes anti-symmetric and topologically quantised if $E_F$ belongs to a mobility gap [BES94]. Its off-diagonal component for two-dimensional systems is then $\sigma_{\mathrm{Hall}}$, the Hall conductivity.

**Example 2 (Gap Labelling)** A gap label is a topological invariant which is strictly speaking not a coefficient in the above sense, but nevertheless it is formally very close to one. It arrises if one considers the expection of the identity operator id, i.e. $\langle \mathrm{id} \rangle = \mathcal{T}(P_{E_F}(H_\infty))$. This is not equal to 1 since the functional is not normalized but equal to the integrated density of states at the Fermi level. If $E_F$ belongs to a gap of the spectrum then $\mathcal{T}(P_{E_F}(H_\infty))$ (which is then called the gap label of the gap) becomes topological as well. It is the aim of this article to identify the boundary coefficient which corresponds to the gap label of a gap at the Fermi level.

## 2.2 Boundary coefficients

To describe boundary effects we consider a model on the half space $\{x \in \mathbb{R}^d | x_d \leq s\}$. We will also denote $x^\perp = x_d$ for the component perpendicular to the boundary $\{x \in \mathbb{R}^d | x_d = s\}$. Now the Hamilton operator (and later the family) is denoted by $H$ and supposed to satisfy Dirichlet boundary conditions. It is convenient to think of $H$ as the restriction of a Hamiltonian $H_\infty$ on $\mathbb{R}^d$ (the bulk Hamiltonian) to the half space with Dirichlet boundary conditions.

Boundary coefficients are response coefficients which arise when the expectation is taken with respect to edge states, i.e. states located at the boundary. It is known that such states occur near the Fermi level provided the latter lies in a gap of the spectrum of the bulk Hamiltonian. In that case we may consider

$$\langle A \rangle = \hat{\mathcal{T}}(\hat{\rho} A)\,, \qquad \hat{\rho} = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} P_{[E_F-\epsilon, E_F+\epsilon]}(H) \tag{3}$$

where $\hat{\mathcal{T}}$ is the trace per unit area ($d-1$-dimensional volume) taken parallel to the boundary times the usual trace perpendicular to the boundary (followed by an ensemble average specified in Section 3), and $P_{[E_F-\epsilon,E_F+\epsilon]}(H)$ the spectral projection of $H$ onto the states of energy in the interval $[E_F-\epsilon, E_F+\epsilon]$. Note that we need a trace per unit area to average over edge states and that as a consequence the above expression would be undefined if the density of states of the bulk Hamiltonian at the Fermi level were non-zero. We note also that $\langle \mathrm{id} \rangle$ can be interpreted as the surface density of states.

The choice of $\hat{\rho}$ is motivated by the example of a sample with strip geometry $\mathbb{R}^{d-1} \times [0, L]$. If $L$ is large enough so that tunneling effects between the two boundaries can be neglected one may describe the boundary effects separately in two half-spaces, $\mathbb{R}^{d-1} \times \mathbb{R}^{\leq 0}$ for the right boundary and $\mathbb{R}^{d-1} \times \mathbb{R}^{\geq 0}$ for the left. Using symmetry the second case can be related to the first and therefore $\hat{\mathcal{T}}(P_{[E_F-\epsilon,E_F+\epsilon]}(H)A)$ interpreted as the difference between the expectation of $A$ at the right and its expectation at the left boundary when an external voltage induces a difference in the chemical potential of $2\epsilon$ between the boundaries. As for small $\epsilon$

$$\hat{\mathcal{T}}(P_{[E_F-\epsilon,E_F+\epsilon]}(H)\delta H) \cong 2\epsilon \hat{\mathcal{T}}(\hat{\rho}\delta H)$$

we may regard $q\hat{\mathcal{T}}(\hat{\rho}\delta H)$ as the linear response coefficient between the voltage and $\delta H$. In the above framework it appears however as a 0-order term.

Our examples below are all 0-order terms and arguments based on the cyclicity of the trace in order to show that $\langle [B, H] \rangle = 0$ for some operator $B$ have to be taken with great care, since they are generally wrong if the trace is undefined on $\hat{\rho}BH$ or $\hat{\rho}HB$.

**Example 3 (edge conductivity)** As first example of a boundary coefficient we consider again $\nabla_j = \imath[X_j, \cdot]$ so that $q\nabla_j H$ is the $j$-component of the usual current operator and $\lambda_j = E_j$, but we suppose the $x_j$ is parallel. This example was considered in [KS04a, KS04b] for $d = 2$ where it was shown in particular that

$$\sigma_{\mathrm{Hall}} = \sigma^{\parallel} := \frac{q^2}{\hbar} \hat{\mathcal{T}}(\hat{\rho}\, \nabla_1 H)$$

and that it is $\frac{q^2}{h}$ times an index (it does not vanish in general, since $\hat{\mathcal{T}}$ is undefined on $\hat{\rho}X_1 H$). It is therefore to lowest order the conductivity of the edge current.

**Example 4 (gradient pressure per energy)** A second example arises if one considers a perturbation of the system by shifting the potential but keeping the boundary fixed, i.e. $\delta = \frac{\partial}{\partial x_d}$. Then the 0th order term is

$$\Pi := -\hat{\mathcal{T}}(\hat{\rho}\, \frac{\partial V}{\partial x^{\perp}}),$$

which is minus the electric force per unit area and unit energy exhibited by the edge states on the boundary. We refer to $\Pi$ as the gradient pressure per unit energy. We will see that, at zero magnetic field the pressure on the boundary is that gradient pressure. In a sample with strip geometry we may then view $q^{-1}\Pi$ as the linear response coefficient yielding the ratio between the pressure at zero magnetic field and an external voltage between the boundaries of the strip.

**Example 5 (pressure on the boundary)** Related to the above two examples is one where the perturbation of the system arrises from shifting the boundary but keeping the potential fixed. This yields a force on the boundary per unit area and energy. Formally $\delta$ amounts to a derivation in $s$, the position of the boundary, but it is not obvious how to formulate this. In the absence of a magnetic field one can invoke Newton's principle (as in [Ke04]) to say that the arising force balances the above electrical force on the boundary. In the presence of a magnetic field there is a contribution from a current induced by the shifting of the electric particles at the boundary which turns out to be proportional to $\sigma^{\|}$.

In [KS04b] the coefficient of Ex. 3 was identified with the off-diagonal coefficient of Ex. 1, provided the Fermi-level lies in a gap of the bulk Hamiltonian. Below we show that the "coefficient" of Ex. 2 equals the one of Ex. 5 which, in turn, is a linear combination of those in Ex. 3 and Ex. 4. This is Eq. 1.

# 3 One-particle description by covariant families of Hamiltonians

Following [Pa80, Be85, Be93, KS04a] the bulk behaviour of $d$-dimensional aperiodic media is described by an ensemble of operators. Such an ensemble is a covariant family of operators $\{H_\omega\}_{\omega\in\Omega}$ on $L^2(\mathbb{R}^d)$ indexed by the elements of a probability space $(\Omega, \mathbf{P})$ of configurations in $\mathbb{R}^d$. Observables are then to be averaged over the probability space to be physically meaningful. For an aperiodic system with boundary the concept of covariant families was extended in [KS04a] to a larger space $\hat{\Omega}$ which incorporates the position of the boundary and requires to work with operators satisfying a boundary condition. While setting up the notation we explain some of these developments below.

## 3.1 Pure bulk systems

Here we recall the model we use for particles moving in aperiodic systems without boundary. Consider $(\Omega, \mathbf{P}, \mathbb{R}^d)$, a compact metrizable probability space with a continuous action of $\mathbb{R}^d$, denoted $\omega \mapsto x\cdot\omega$, and an ergodic invariant probability measure $\mathbf{P}$. For example, we may think of an element $\omega \in \Omega$ as a set of atomic positions decorated if necessary with the information of what kind of atom there is and the action of $\mathbb{R}^d$ be simply by translation of the set. The invariant measure $\mathbf{P}$ yields the probability that such a configuration of atoms is realized in the physical sample and observable quantities are $\mathbf{P}$-averages. The ergodicity of the measure is needed for the physical interpretation of these averages but not for the topological arguments of Section 4. The motion of a (charged) particle is in the one-particle picture described by a Schrödinger equation with Hamiltonian (in units in which $\frac{\hbar^2}{2m} = 1$)

$$H_\omega = \sum_{j=1}^{d} (\imath \frac{\partial}{\partial x_j} - qA_j)^2 + V_\omega$$

where $A$ is the (linear) vector potential for the magnetic field and $V_\omega$ a background potential. How exactly $V_\omega$ looks is not important, but it should depend on the background configuration $\omega$ in a covariant way, namely $V_\omega(x-y) = V_{y\cdot\omega}(x)$ (translation of the configuration amounts to translating the potential). $\{H_\omega\}_{\omega\in\Omega}$ is then a family of operators which satisfies

$$U(x)H_\omega U(x)^* = H_{x\cdot\omega} \tag{4}$$

where the $U(x)$ are the magnetic translation operators. Since there is no boundary we speak of a bulk Hamiltonian. Often $\Omega$ is the hull of a single configuration.

The spectrum of the family $\{H_\omega\}_{\omega\in\Omega}$ is defined to be the union of the spectra of $H_\omega$ and also referred to as the bulk spectrum. If $\omega$ has a dense orbit then the spectrum of $H_\omega$ is equal to the bulk spectrum.

## 3.2 Bulk and boundary

We add the data of the position of a possible boundary of the systems by enlarging $\Omega$ to the non-compact space $\hat\Omega = \Omega\times(\mathbb{R}\cup\infty)$. $\infty$ means here $+\infty$ and so $\mathbb{R}\cup\infty = (-\infty,+\infty]$. The point $(\omega, s)\in\hat\Omega$ has now the interpretation of a configuration $\omega$ with boundary at $\{x\in\mathbb{R}^d|x_d = s\}$, if $s<\infty$, or without a boundary if $s=\infty$. Configuration and boundary can both be translated and so we define $x\cdot(\omega, s) = (x\cdot\omega, s+x_d)$ $(\infty+x_d=\infty)$. We call the first $d-1$ components of $x$ its parallel components, as they are parallel to the boundary, and the last one its perpendicular component. The measure $\hat{\mathbf{P}} = \mathbf{P}\times\lambda$ ($\lambda$ the Lebesgues measure) is invariant under this action of $\mathbb{R}^d$ and so we obtain again a Borel dynamical system $(\hat\Omega, \hat{\mathbf{P}}, \mathbb{R}^d)$. Now define $H_{\omega,s}$ to be the restriction of $H_\omega$ to $\{x\in\mathbb{R}^d|x_d\le s\}$ with Dirichlet boundary conditions at the boundary $\{x\in\mathbb{R}^d|x_d = s\}$. This gives us a covariant family[1]

$$H = \{H_{\hat\omega}\}_{\hat\omega\in\hat\Omega},$$

namely it satisfies

$$U(x)H_{\hat\omega}U(x)^* = H_{x\cdot\hat\omega}. \tag{5}$$

We assume that the magnetic field is homogeneous and hence equal to a constant 2-form which we split as

$$B = B^\parallel + \frac{\hbar\gamma}{q}dx^\parallel\wedge dx^\perp \tag{6}$$

where $B^\parallel$ is the constant extension of a 2-form on the boundary. This determines the linear coordinate $x^\parallel$, which parametrises a 1-dimensional subspace of the boundary, and the constant $\gamma$ up to a sign. Then in a gauge in which $A_d = 0$ (Landau gauge) the magnetic translation in the perpendicular direction $e_d$ reads (for $\psi\in L^2(\mathbb{R}^d)$)

$$(U(te_d)\psi)(x) = e^{-\imath\gamma t x^\parallel}\psi(x-te_d)\,. \tag{7}$$

We denote the generator of this magnetic translation by $K^\perp$. It is thus given by

$$K^\perp = \imath\frac{\partial}{\partial x^\perp} - \gamma x^\parallel. \tag{8}$$

[1]For the remaining sections $H$ and $H_\infty$ will be covariant families of operators and not single operators as in Section 2.

Note that the family $H = \{H_{\hat\omega}\}_{\hat\omega\in\hat\Omega}$ contains the pure bulk family if we set $s = \infty$. We denote the bulk Hamiltonians $H_\omega$ therefore also by $H_{\omega,\infty}$ and the pure bulk family by

$$H_\infty = \{H_{\omega,\infty}\}_{\omega\in\Omega}.$$

At this point the reader might wonder why we have chosen the family $\{H_{\hat\omega}\}_{\hat\omega\in\hat\Omega}$ to describe the bulk-boundary system and not simply $\{H_{\omega,0}\}_{\omega\in\Omega}$, after all the latter is indexed by a compact space. In fact, apart from the subfamily $H_\infty$ the two families contain the same information but the first one proved to be useful in connection with the Wiener Hopf extension which arises when sending the boundary to infinity.

## 3.3 Response coefficients for covariant families

Here we adapt the response coefficients defined in the examples of Section 2 to the formalism of covariant families of operators. The precision we will gain will allow us also to tackle the problem of formulating the perturbation of Ex. 5 as a derivation. We will do this by, first, defining accurately the traces and derivations involved, and second, by regularising the definitions of the 0th order boundary coefficients as in [KS04a].

### 3.3.1 Derivations and traces

Traces and derivations are the ingredients which enter not only into the construction of response coefficients but also in that of higher traces, objects used for the construction of invariants in non commutative topology and employed later on.

**Definition 1** *Let $F = \{F_{\hat\omega}\}_{\hat\omega\in\hat\Omega}$ be a covariant family of bounded integral operators. Define the (unbounded) derivations*

$$\begin{aligned}
(\delta_i F)_{(\omega,s)} &:= \lim_{t\to 0} \frac{F_{(te_i\cdot\omega,s)} - F_{(\omega,s)}}{t}\,, \quad e_i \textit{ unit vector in } i\textit{-direction}\\
(\partial_s F)_{(\omega,s)} &:= \lim_{t\to 0} \frac{F_{(\omega,s+t)} - F_{(\omega,s)}}{t}\\
(\nabla_j F)_{(\omega,s)} &:= \imath[X_j, F_{(\omega,s)}]\,, \quad j \neq d
\end{aligned}$$

*provided on the r.h.s. are again bounded integral operators. We denote $\delta^\perp = \delta_d$.*

**Lemma 1** *With the above notation, $(\delta^\perp + \partial_s)F = \imath[K^\perp, F]$ where $K^\perp$ is the generator of the magnetic translations perpendicular to the boundary (8).*

**Proof:** We have $(\delta_d + \partial_s)F = \lim_{t\to 0} \frac{F_{te_d\cdot\hat\omega} - F_{\hat\omega}}{t}$ and so the lemma follows by differentiation of the covariance relation (5). $\square$

For the sequel, $\chi$ is a positive compactly supported function on $\mathbb{R}^d$ satisfying $\int_{\mathbb{R}^d} dx\, \chi(x) = 1$ and $\chi^{||}$ a positive function on $\mathbb{R}^d$ which is constant in the perpendicular component, has compact support in the parallel components, and satisfies $\int_{\mathbb{R}^{d-1}} dy\, \chi^{||}(y,0) = 1$ ($\chi^{||} = 1$ if $d = 1$). Moreover let $\|T\|_1$ denote the (Schatten) traceclass norm of an operator $T$ on $L^2(\mathbb{R}^d)$.

**Definition 2 ([KS04a])** *A bulk covariant family $\{F_\omega\}_{\omega\in\Omega}$ is called $\mathcal{T}$-traceclass if $\|\chi F_\omega\|_1$ is integrable w.r.t.* $\mathbf{P}$*, and we define for traceclass families*

$$\mathcal{T}(F_\infty) \;=\; \int_\Omega d\mathbf{P}(\omega)\;\mathrm{Tr}(\chi F_{\omega,\infty}),$$

*where* Tr *is the usual trace on $L^2(\mathbb{R}^d)$. A covariant family $F = \{F_{\hat\omega}\}_{\hat\omega\in\hat\Omega}$ will be called $\hat{\mathcal{T}}$-traceclass if $\|\chi^{\|}F_{\hat\omega}\|_1$ is integrable w.r.t.* $\mathbf{P}$*, and we define for traceclass families,*

$$\hat{\mathcal{T}}(F) = \int_\Omega d\mathbf{P}(\omega)\;\mathrm{Tr}(\chi^{\|}F_{\omega,s}).$$

We mention as an aside that also $\hat{\mathcal{T}}(F) = \int_{\hat\Omega} d\hat{\mathbf{P}}(\hat\omega)\;\mathrm{Tr}(\chi F_{\hat\omega})$ and so both traces are formally defined in the same way. Proposition 1 shows that $\mathcal{T}(F_\infty)$ does not depend on the choice of $\chi$ and $\hat{\mathcal{T}}(F)$ not on the choice of $\chi^{\|}$ and $s$.

**Proposition 1 ([KS04a])** *For covariant traceclass families on $L^2(\mathbb{R}^d)$ with jointly continuous integral kernels (or a finite number of isolated point singularities)*

$$\begin{aligned}\mathcal{T}(F_\infty) &= \int_\Omega d\mathbf{P}(\omega)\,\langle 0\,|F_{\omega,\infty}|0\,\rangle\;,\\ \hat{\mathcal{T}}(F) &= \int_{\hat\Omega} d\hat{\mathbf{P}}(\hat\omega)\,\langle 0\,|F_{\hat\omega}|0\,\rangle\;.\end{aligned}$$

*Furthermore, $\hat{\mathcal{T}}$ is closed under the derivations $\delta_j$, $\nabla_j$ and $\partial_s$, i.e.*

$$\hat{\mathcal{T}}\circ\delta_j = \hat{\mathcal{T}}\circ\nabla_j = \hat{\mathcal{T}}\circ\partial_s = 0.$$

**Proof:** The only statements which are not covered by [KS04a] are $\hat{\mathcal{T}}\circ\delta_j = 0$ and $\hat{\mathcal{T}}\circ\partial_s = 0$. These follow directly from the invariance of $\mathbf{P}$ under the action of $\mathbb{R}^d$ and of the Lebesgue measure under translation. □

### 3.3.2 Bulk coefficients

It follows from Birkhoff's ergodicity theorem that $\mathcal{T}(F_\infty)$ can be interpreted as the $\mathbf{P}$-averaged trace per unit ($d$-dimensional) volume of $F_\infty$ which coincides almost surely with trace per unit volume of $F_{\omega,\infty}$. The expectation $\langle A\rangle$ from (2) has to be understood as follows. The observable $A$ as well as the density matrix $\rho = P_{E_F}(H_\infty)$ are no longer single operators but pure bulk covariant families and therefore $\langle A\rangle = \mathcal{T}(P_{E_F}(H_\infty)A)$ in case $P_{E_F}(H_\infty)A$ is traceclass. In particular the average contains a $\mathbf{P}$-average.

Examples 1,2 have been considered in this context. If $E_F$ lies in a gap of the bulk spectrum then $P_{E_F}(H_\infty)$ is traceclass and so the gap label of the gap at the Fermi energy well-defined. The ergodicity of $\mathbf{P}$ combined with a Shubin type formula yields that for almost all $\omega\in\Omega$ the integrated density of states of $H_{\omega,\infty}$ at $E$ is equal to the gap label $\mathcal{T}(P_{E_F}(H_\infty))$ [Be93]. We will not consider single systems with specific configuration $\omega$ but denote by IDS the integrated density of the covariant family, i.e. the $\mathbf{P}$-average.

As for Ex. 1, the component $\frac{1}{2}(\sigma_{ij}-\sigma_{ji})$ of the totally antisymmetric part of the conductivity tensor of Ex. 1 can be written as $\frac{q^2}{h}2\pi\imath\mathcal{T}(P_{E_F}[\nabla_i P_{E_F},\nabla_j P_{E_F}])$ ($\nabla_j$ can be defined by the above formula for all $j$ for pure bulk covariant families). This has been discussed in detail in [BES94].

### 3.3.3 Boundary coefficients

$\hat{\mathcal{T}}(F)$ is the **P**-averaged trace per unit ($d-1$-dimensional) area of $F$ and the expectation $\langle A\rangle$ from (3) has to be understood in such a way that $A$ and $\hat{\rho} = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}P_{[E_F-\epsilon,E_F+\epsilon]}(H)$ are covariant families and $\langle A\rangle = \hat{\mathcal{T}}(\hat{\rho}A)$. The area is taken parallel to the boundary and so $\hat{\mathcal{T}}$ is the **P**-average of the product between the trace per unit ($d-1$-dimensional) volume on the parallel components $\mathbb{R}^{d-1}$ and the usual trace on $L^2(\mathbb{R})$.

For motivational reasons we now present some formal arguments. We defined the total boundary force to be minus the variation of the energy under a variation of the position $s$ of the boundary. It has only non-vanishing component perpendicular to the boundary and we concentrate on this component. One is thus tempted to write

$$F_b = -\hat{\mathcal{T}}(\hat{\rho}\,\partial_s H)$$

for the total boundary force per unit area and unit energy. But this is formal as its stands, since it is not clear what $\partial_s H$ means. A formal application of Lemma 1 together with $\delta^\perp H = \delta^\perp V = -\frac{\partial V}{\partial x^\perp}$ yields in the Landau gauge

$$F_b = -\hat{\mathcal{T}}(\hat{\rho}\frac{\partial V}{\partial x^\perp}) \,+\, \gamma\hat{\mathcal{T}}(\hat{\rho}\nabla^\parallel H) \,-\, \imath\hat{\mathcal{T}}(\hat{\rho}[\frac{\partial}{\partial x^\perp}, H]). \tag{9}$$

The first term corresponds to the gradient force of the electric potential $V$ per unit area and energy of Ex. 4. The second term is proportional to the conductivity of the edge current of Ex. 3. The third term formally vanishes by cyclicity. The r.h.s. of (9) is thus the r.h.s. of (1) but our reasoning was non rigorous (also $[\frac{\partial}{\partial x^\perp}, H]$ is not well defined). We will arrive at the same conclusion below using well-defined expressions.

## 3.4 Regularized expressions for the boundary forces

Here we propose alternative expressions for the components of the boundary force in case the Fermi energy lies in a gap $\Delta$ of the bulk spectrum.

Define via functional calculus the unitary operator

$$\mathcal{U} \;=\; \exp(-2\pi\imath\, G(H)) \tag{10}$$

where $G \in C^\infty(\mathbb{R})$ is a monotonically decreasing function with $G(-\infty) = 1$, $G(\infty) = 0$ and $\operatorname{supp} G' \subset \Delta$. The following theorem is essentially covered by Theorem 3 of [KS04a].

**Theorem 1** *Let $V$ be a covariant family of bounded potentials on $\mathbb{R}^d$, $d \le 2$, which are bounded differentiable. Let $\Delta$ be in a gap of the spectrum of $H_\infty$.*

*Then $\mathcal{U}-1$, $\delta^\perp\mathcal{U}$, $\partial_s\mathcal{U}$, $\nabla^\parallel\mathcal{U}$, and $[\frac{\partial}{\partial x^\perp},\mathcal{U}]$ are bounded covariant families of integral operators. Moreover, $\mathcal{U}-1$ is $\hat{\mathcal{T}}$-traceclass and*

$$\hat{\mathcal{T}}((\mathcal{U}^*-1)\,\delta^\perp\mathcal{U}) \;=\; -2\pi\imath\hat{\mathcal{T}}(G'(H)\frac{\partial V}{\partial x^\perp})\;, \tag{11}$$

$$\hat{\mathcal{T}}((\mathcal{U}^*-1)\,\nabla^\parallel\mathcal{U}) \;=\; 2\pi\imath\hat{\mathcal{T}}(G'(H)\nabla^\parallel H)\;,\quad d>1\;, \tag{12}$$

$$\hat{\mathcal{T}}((\mathcal{U}^*-1)\,[\frac{\partial}{\partial x^\perp},\mathcal{U}]) \;=\; 0\;. \tag{13}$$

**Proof:** To show the traceclass property of $\mathcal{U}-1$ we refine an argument of [KS04a].[2] We write $U-1 = \cos(2\pi G) - 1 - \imath\sin(2\pi G) = \cos(2\pi G) - 1 - 2\imath\left((\cos(\pi G)-1)\sin(\pi G) + \sin(\pi G)\right)$ which is a linear combination of three smooth positive functions having support contained in $\Delta$, namely $1-\cos(2\pi G)$, $(1-\cos(\pi G))\sin(\pi G)$, and $\sin(\pi G)$. The traceclass property of $\mathcal{U}-1$ follows therefore from the estimates of Theorem 1 of [KS04a].

That $\delta^\perp \mathcal{U}$ is a covariant family of bounded integral operators can be seen upon applying Duhamel's formula and the hypothesis that $\delta^\perp H = -\frac{\partial V}{\partial x^\perp}$ is bounded. The arguement leading to (11) is then exactly as in Theorem 3 of [KS04a] if one replaces the derivation $\nabla_1$ there by $\delta^\perp$.

We indicate why $\frac{\partial}{\partial x^\perp}\mathcal{U}$, $\mathcal{U}\frac{\partial}{\partial x^\perp}$, and $[X^\parallel, \mathcal{U}]$ are covariant families of bounded integral operators. In fact, the (slightly cumbersome) arguments leading to the sufficient estimates on the integral kernels of $\frac{\partial}{\partial x^\perp}\mathcal{U}$ and $\mathcal{U}\frac{\partial}{\partial x^\perp}$ are outlined at the end of the proof of Proposition 3 of [KS04a] and the term $[X^\parallel, \mathcal{U}]$ is literally treated in Theorem 3 there. By Lemma 1 also $\partial_s \mathcal{U}$ is a covariant family of bounded integral operators.

Since $\frac{\partial}{\partial x^\perp}\mathcal{U}$ and $\mathcal{U}\frac{\partial}{\partial x^\perp}$ are covariant families of bounded integral operators we can employ the cyclicity of the trace to conclude $\hat{\mathcal{T}}((\mathcal{U}^*-1)[\frac{\partial}{\partial x^\perp}, \mathcal{U}]) = 0$ so that Theorem 3 of [KS04a] yields (12) ($\nabla_1 = \imath[X^\parallel, \cdot]$). It should be stressed that the cyclicity of the trace cannot be used to conclude $\hat{\mathcal{T}}((\mathcal{U}^*-1)[X^\parallel, \mathcal{U}]) = 0$ since $X^\parallel \mathcal{U}$ is not covariant. $\square$

**Remark 1** The above theorem is restricted to $d \leq 2$ but its extension to arbitrary dimension should, at least without magnetic field not pose great difficulties. The crucial traceclass property mentionned in the theorem follows from the decay of integral kernels of $F(H)$ for sufficiently regular functions $F$ which have support contained in $\Delta$. Both, $\mathcal{U}-1$ and $G'$ are of that type. The required decay was shown in Theorem 1 of [KS04a]. Without magnetic field the analysis is simpler since one can work with the well-known exact expression for the complex heat kernel and treat the potential perturbatively.

The r.h.s. in the formulas of the theorem hold for all choices of $G$ subject to the above conditions and we may view $\hat{\rho}$ as being approximated by $-G'(H)$ if $G$ approximates the indicator function on $(-\infty, E_F]$. We therefore define the regularised expression for the boundary force as follows.

**Definition 3** *Let the Fermi energy $E_F$ lie in a gap $\Delta$ of the spectrum of $H_\infty$. The total boundary force per unit area and unit energy is*

$$F_b = \frac{1}{2\pi\imath}\,\hat{\mathcal{T}}((\mathcal{U}^*-1)\,\partial_s\mathcal{U}).$$

[2] The stronger condition $\operatorname{supp} G' \subset \Delta \backslash G^{-1}(\frac{1}{2})$ used in [KS04a, KS04b] is erroneous. With the refined argument presented here it can be replaced by $\operatorname{supp} G' \subset \Delta$ in [KS04a, KS04b] as well.

By Lemma 1 and Theorem 1 the total boundary force has the two contributions

$$\begin{aligned} F_b &= \frac{1}{2\pi\imath}\Big(-\hat{\mathcal{T}}((\mathcal{U}^*-1)\,\delta^\perp\mathcal{U}) - \gamma\,\hat{\mathcal{T}}((\mathcal{U}^*-1)\,\nabla^\parallel\mathcal{U})\Big) && (14)\\ &= \hat{\mathcal{T}}(G'(H)\frac{\partial V}{\partial x^\perp}) \;-\; \gamma\hat{\mathcal{T}}(G'(H)\nabla^\parallel H) && (15)\\ &= \Pi + \frac{\gamma\hbar}{q^2}\sigma^\parallel\,. && (16)\end{aligned}$$

Here $\sigma^\parallel$ is the direct conductivity of the current in the direction $x^\parallel$ (in $d=1$ this term is absent). We stress that the above expressions have there physical interpretation of Section 2 not for a single system defined by a fixed $\hat{\omega}\in\hat{\Omega}$ but for their $\mathbf{P}$-averages.

It is a general fact of non commutative topology that expressions like $\hat{\mathcal{T}}((\mathcal{U}^*-1)\,\delta\mathcal{U})$, where $\delta$ is a closed derivation leaving the trace invariant, depend only on the homotopy class of the unitary $\mathcal{U}$ in the $C^*$-algebra on which the trace and the derivation are (densely) defined. They are thus a topological invariants. If the algebra is separable (as is the case below) there are only countably many homotopy classes of unitaries and we may say that $\hat{\mathcal{T}}((\mathcal{U}^*-1)\,\delta^\perp\mathcal{U})$ is topologically quantised. Equations 14–16 imply therefore that $F_b$, $\Pi$ and $\sigma^\parallel$ are topologically quantised. This implies that they are stable against (covariant) perturbations of the potential which do not lead to a closing of the gap at the Fermi energy. It means also that they take values in specific discrete (but perhaps dense) subgroups of $\mathbb{R}$ which depend only on the topology of the system. In fact, Theorem 3 of [KS04a] contains as well an identification of $\frac{h}{q^2}\sigma^\parallel$ with an index and hence an integer. This relies on the specific form of the derivation $\nabla_1$ and cannot be expected for the derivation $\delta^\perp$. We will see below that, at least for a large class of systems, $\Pi$ lies in the gap labelling group without magnetic field.

# 4 Relation between the boundary force and the integrated density of states

The following theorem is the main result and leads together with (14–16) to Equation 1.

**Theorem 2** *For energies $E$ in gaps*

$$F_b(E) = \mathrm{IDS}(E).$$

Its proof will be given below in the framework of non-commutative topology and requires a reformulation in terms of operator algebras.

## 4.1 The one-particle approximation in the operator algebraic formulation

When studying topological properties of covariant families of operators it proved most useful to view them as representations of crossed product $C^*$-algebras. In the gauge we use the $C^*$-algebra for the bulk-boundary system is

$$\mathcal{A} = (C_0(\hat{\Omega}) \rtimes \mathbb{R}^{d-1})_{\tau\otimes\beta^\perp}\mathbb{R}$$

where $C_0(\hat{\Omega})\rtimes\mathbb{R}^{d-1} = C_0(\hat{\Omega})\rtimes_{\alpha^{\|},B^{\|}}\mathbb{R}^{d-1}$ is the crossed product w.r.t. to the action $\alpha^{\|}_{te_j}(h)(\omega,s) = h(te_j\cdot\omega,s)$, $j\neq d$, twisted by $B^{\|}$, and $(f:\mathbb{R}^{d-1}\to C_0(\hat{\Omega}))$

$$
\begin{aligned}
(\tau\otimes\beta^{\perp})_t(f)(y)(\omega,s) &= \beta^{\perp}_t(f)(y)(\omega,s+t)\ , &(17)\\
\beta^{\perp}_t(f)(y)(\omega,s) &= e^{\imath\gamma t y^{\|}} f(y)(te_d\cdot\omega,s)\ . &(18)
\end{aligned}
$$

The component $y^{\|}$ of $y$ is defined by the splitting of the magnetic field in (6). We will also need to consider the actions $\alpha^{\perp}$ and $\tau$ on $\mathcal{A}$ given on functions $A:\mathbb{R}^d\to C_0(\hat{\Omega})$ by

$$
\begin{aligned}
\tau_t(A)(x)(\omega,s) &= A(x)(\omega,s+t)\ , &(19)\\
\alpha^{\perp}_t(A)(x)(\omega,s) &= A(x)(te_d\cdot\omega,s)\ . &(20)
\end{aligned}
$$

The algebra for the bulk system $\mathcal{A}_\infty$ is the image of the surjective algebra homomorphism

$$
\mathrm{ev}_\infty:\mathcal{A}\to\mathcal{A}_\infty := (C(\Omega)\rtimes\mathbb{R}^{d-1})\rtimes_{\beta^{\perp}}\mathbb{R}
$$

given by evaluating the second component of $(\omega,s)\in\hat{\Omega}$ at infinity.

The covariant families $H$ and $H_\infty$ are associated with $\mathcal{A}$ and $\mathcal{A}_\infty$, respectively, in the following sense. The evaluation representations $\mathrm{ev}_{\hat{\omega}}: C_0(\hat{\Omega})\to\mathbb{C}$ and $\mathrm{ev}_\omega: C(\Omega)\to\mathbb{C}$ induce families of representations $\{\pi_{\hat{\omega}}\}_{\hat{\omega}\in\hat{\Omega}}$ of $\mathcal{A}$ and $\{\pi_\omega\}_{\omega\in\Omega}$ of $\mathcal{A}_\infty$ on $L^2(\mathbb{R}^d)$ [KS04b]. Let $F:\mathbb{R}\to\mathbb{C}$ be a continuous function which vanishes at 0 and $\infty$. Then $F(H)=\{F(H_{\hat{\omega}}\}_{\hat{\omega}\in\hat{\Omega}}$ belongs to $\mathcal{A}$ in the sense that there exists an $A\in\mathcal{A}$ such that $\pi_{\hat{\omega}}(A)=F(H_{\hat{\omega}})$. Likewise, $\pi_\omega(\mathrm{ev}_\infty(A))=F(H_{\omega,\infty})$. This has been proven rigorously for $d\leq 2$ and to indicate the reasoning we provide here some details in case $B^{\|}=0$. In this case the magnetic translations parallel to the boundary are ordinary translations and phases appear only in $U(x^{\perp}e_d)$ (7). Moreover, if $A:\mathbb{R}\to(\mathbb{R}^{d-1}\to C_0(\hat{\Omega}))$ then the integral kernel of $\pi_{\hat{\omega}}(A)$ is

$$
\langle(x,x^{\perp})\,|\pi_{\hat{\omega}}(A)|(y,y^{\perp})\,\rangle = e^{-\imath\gamma(x^{\|}-y^{\|})x^{\perp}}A(x^{\perp}-y^{\perp})(x-y)(-(x,x^{\perp})\cdot\hat{\omega})\ . \tag{21}
$$

It follows that $U(x\,)\pi_{\hat{\omega}}(A)U(x\,)^* = \pi_{x\cdot\hat{\omega}}(A)$ and so $\{\pi_{\hat{\omega}}(A)\}_{\hat{\omega}\in\hat{\Omega}}$ is a covariant family of operators. Therefore, if $\{F_{\hat{\omega}}\}_{\hat{\omega}\in\hat{\Omega}}$ is a covariant family of bounded integral operators then $F_{\hat{\omega}}=\pi_{\hat{\omega}}(A)$ with $A(x^{\perp})(x)(\hat{\omega}) = e^{\imath\gamma x^{\|}x^{\perp}}\langle(x,x^{\perp})\,|F_{x\cdot\hat{\omega}}|(0,0)\,\rangle$ provided $A$ satisfies the continuity properties neccessary to belong to the crossed product. Such continuity properties have been established in $d=2$ for half-sided operators [KS04a] (the case without boundary can also be found in [Be93] for any $d$).

The derivations and traces defined in Section 3.3.1 are obtained from the following derivations and traces on $\mathcal{A}$ or $\mathcal{A}_\infty$ ($A:\mathbb{R}^d\to C_0(\hat{\Omega})$, $B:\mathbb{R}^d\to C(\Omega)$):

$$
\begin{aligned}
\delta^{\perp}A &= \lim_{t\to 0}\frac{\alpha^{\perp}_t(A)-A}{t},\\
\partial_s A &= \lim_{t\to 0}\frac{\tau_t(A)-A}{t},\\
(\nabla_j A)(x) &= \imath x_j A(x),\quad j\neq d,\\
\hat{\mathcal{T}}(A) &= \int_{\hat{\Omega}} d\hat{\mathbf{P}}(\hat{\omega})\ A(0)(\hat{\omega}),\\
\mathcal{T}(B) &= \int_{\Omega} d\mathbf{P}(\omega)\,B(0)(\omega).
\end{aligned}
$$

Here we have used the same notation as earlier (although it should strictly speaking read for instance $\delta_j(\pi(A))$, etc.).

#### 4.1.1 Bulk and boundary topological response coefficients as pairings

According to [KS04b] the topological invariants we are interested in are obtained from the $K$-groups and higher traces of the algebras in the Wiener Hopf extension. Bulk invariants are obtained if one pairs $K$-group elements with higher traces over the bulk algebra $\mathcal{A}_\infty$ and boundary invariants steem from such pairings over the ideal $\mathcal{E}$ of $\mathcal{A}$ given by the kernel of $\mathrm{ev}_\infty$.

For gap labels this is the $K$-theoretic formulation of the gap labelling due to Bellissard [Be85, Be93]. The label of a gap $\Delta$ in the bulk spectrum is given by $\langle \mathcal{T}, [P_E]\rangle$, $E \in \Delta$, where $\langle \cdot, \cdot \rangle$ denotes Connes pairing with the normalisation used in [KS04b],

$$\langle \mathcal{T}, [P_E]\rangle = \mathcal{T}(P_E) = \mathrm{IDS}(E).$$

To formulate the boundary force per unit area and energy as a pairing we consider 1-traces constructed from the trace $\hat{\mathcal{T}}$ and derivations which leave this trace invariant [Co94]. Let $\delta$ be a (closed) derivation on $\mathcal{E}$ such that $\hat{\mathcal{T}} \circ \delta = 0$. Then

$$\hat{\mathcal{T}}_\delta(A, B) := \hat{\mathcal{T}}(A \delta B)$$

is a 1-trace on $\mathcal{E}$ and Connes pairing of it with a $K_1$-class of $\mathcal{E}$ represented by a unitary $U$ reads in the normalisation we use

$$\langle \hat{\mathcal{T}}_\delta, [U]\rangle = \frac{1}{2\pi\imath} \hat{\mathcal{T}}_\delta(U^* - 1, U)\,.$$

Thus the response coefficients at energy $E \in \Delta$ of Examples 3–5 read

$$\sigma^{\|} = -\langle \hat{\mathcal{T}}_{\nabla^{\|}}, [\mathcal{U}]\rangle \tag{22}$$

$$\Pi = -\langle \hat{\mathcal{T}}_{\delta^{\perp}}, [\mathcal{U}]\rangle \tag{23}$$

$$F_b = \langle \hat{\mathcal{T}}_{\partial_s}, [\mathcal{U}]\rangle\,. \tag{24}$$

In particular they depend only on the $K_1$-class of $\mathcal{U}$.

#### 4.1.2 Linking bulk and boundary invariants

The algebra homomorphism $\mathrm{ev}_\infty : \mathcal{A} \to \mathcal{A}_\infty$ corresponds to pushing the boundary to $\infty$. The kernel of $\mathrm{ev}_\infty$ is generated by elements which are represented by covariant families of operators which have integral kernels decaying away from the boundary and therefore may be associated with the observables located near the boundary. We denote this algebra by $\mathcal{E}$ (for edge). The resulting exact sequence

$$0 \to \mathcal{E} \hookrightarrow \mathcal{A} \xrightarrow{\mathrm{ev}_\infty} \mathcal{A}_\infty \to 0 \tag{25}$$

ties the topological properties of $\mathcal{A}_\infty$ and $\mathcal{E}$, i.e. of bulk and boundary, together. $\mathcal{A}$ is called the Wiener Hopf extension of $\mathcal{A}_\infty$ by $\mathcal{E}$. Such an extension can always be constructed if one has a crossed product algebra with $\mathbb{R}$ [Ri82].

By functoriality of $K$ the exact sequence (25) gives rise to a boundary map $\partial : K(\mathcal{A}_\infty) \to K(\mathcal{E})$. The part on degree 0, $\partial_0 = \exp : K_0(\mathcal{A}_\infty) \to K_1(\mathcal{E})$, also called exponential map, can be evaluated on $K_0$-classes of spectral projections of $H_\infty$ on gaps. In fact, if $\Delta$ is an interval in a gap of the spectrum of the (pure bulk) Hamiltonian $H_\infty$ and $E \in \Delta$ then

$$\exp([P_E(H_\infty)]) = [\mathcal{U}] \tag{26}$$

where $\mathcal{U} = \exp(-2\pi\imath\, G(H))$ is as in (10) constructed from the bulk-boundary Hamiltonian.

In [KS04b] results of [ENN88] concerning the cyclic cohomology of a smooth variant $\mathcal{B}^\infty$ of $\mathcal{B}$ and its smooth crossed product were extended to $\beta$-invariant $n$-traces over $\mathcal{B}$: If $\eta$ is an $n$-trace over the $C^*$-algebra $\mathcal{B}$ which is the character of a $\beta$-invariant cycle $(\Omega, \int, d)$ then

$$\#_\beta\eta(f_0,\ldots,f_{n+1}) \;=\; \sum_{k=1}^{n+1} (-1)^k \int (f_0 df_1 \cdots \nabla f_k \cdots df_{n+1})(0)\;, \quad \nabla f(x) = \imath x f(x), \tag{27}$$

$f \in L^1(\mathbb{R}, \mathcal{B}, \beta)$, defines a $n+1$-trace on the $L^1$-crossed product $L^1(\mathbb{R}, \mathcal{B}, \beta)$ of $\mathcal{B}$ with $(\mathbb{R}, \beta)$ whose pairing extends to the $K$-groups of $\mathcal{B} \rtimes_\beta \mathbb{R}$ and satisfies

$$\langle \#_\beta\eta, x\rangle \;=\; -\frac{1}{2\pi}\,\langle \eta, \partial x\rangle\;, \qquad x \in K(\mathcal{B} \rtimes_\beta \mathbb{R})\;. \tag{28}$$

In our more specific context ($\mathcal{B} = C(\Omega) \rtimes \mathbb{R}^{d-1}$, $\beta = \beta^\perp$) the last equation relates a bulk invariant (l.h.s.) to a boundary invariant (r.h.s.). In particular

$$\langle \#_{\beta^\perp}\eta, [P_E(H_\infty)]\rangle = -\frac{1}{2\pi}\,\langle \eta, [\mathcal{U}]\rangle$$

and we need the inverse of $\#_{\beta^\perp}$ to solve $\langle \#_{\beta^\perp}\eta, \cdot\rangle = \langle \mathcal{T}, \cdot\rangle$ for $\eta$.

As was shown in [ENN88], $\#_\beta$ has an inverse in periodic cyclic cohomology and this inverse is essentially given by $\#_{\hat\beta}$. Here $\hat\beta_\omega : \mathcal{B} \rtimes_\beta \mathbb{R} \to \mathcal{B} \rtimes_\beta \mathbb{R}$: $\hat\beta_\omega(f)(t) = e^{-it\omega} f(t)$ is the action of the dual group $\hat{\mathbb{R}} \cong \mathbb{R}$ on the crossed product, and the double crossed product $\mathcal{B} \rtimes_\beta \mathbb{R} \rtimes_{\hat\beta} \hat{\mathbb{R}}$ is identified with $\mathcal{B} \otimes \mathcal{K}$. We need the expression of this map on higher traces whose image is a higher trace on the ideal of the Wiener-Hopf extension and therefore present some details. Fourier transformation of the suspension variable induces an isomorphism $\mathcal{F} : S\mathcal{B} \rtimes_{\tau\otimes\beta} \mathbb{R} \cong \mathcal{B} \rtimes_{\mathrm{id}} \hat{\mathbb{R}} \rtimes_{\hat\tau\otimes\beta} \mathbb{R}$ where $(\hat\tau\otimes\beta)_t(f)(\omega) = e^{-it\omega}\beta_t(f(\omega))$. Combined with $\phi : \mathcal{B} \rtimes_{\mathrm{id}} \hat{\mathbb{R}} \rtimes_{\hat\tau\otimes\beta} \mathbb{R} \to \mathcal{B} \rtimes_\beta \mathbb{R} \rtimes_{\hat\beta} \hat{\mathbb{R}}$: $\phi(f)(t)(\omega) = e^{-it\omega} f(\omega)(t)$ we obtain the isomorphism $\psi = \phi \circ \mathcal{F} : S\mathcal{B} \rtimes_{\tau\otimes\beta} \mathbb{R} \to \mathcal{B} \rtimes_\beta \mathbb{R} \rtimes_{\hat\beta} \hat{\mathbb{R}}$. Then, for a $\beta$-invariant $n$-trace $\eta$ on $\mathcal{B}$, $\psi_*\#_{\hat\beta}\#_\beta\eta = \#_{\tau\otimes\beta}\mathcal{F}_*\#_{\mathrm{id}}\eta$ is a $n+2$-trace on $S\mathcal{B} \rtimes_{\tau\otimes\beta} \mathbb{R}$ which by periodicity of the pairing (c.f. Appendix A of [KS04b] for our normalisation) satisfies

$$-2\pi\langle \psi_*\#_{\hat\beta}\#_\beta\eta, x\rangle = \langle \eta, x\rangle\;, \qquad x \in K(\mathcal{B})\;. \tag{29}$$

Applying (28,29) to $\xi = \#_\beta\eta$ yields

$$\langle \xi, x\rangle \;=\; \langle \psi_*\#_{\hat\beta}\xi, \partial x\rangle\;, \qquad x \in K(\mathcal{B} \rtimes_\beta \mathbb{R})\;. \tag{30}$$

Since (27) involves an evaluation of the integrand at 0 the image of $\#_\beta$ on $\beta$-invariant $n$-traces is $\hat\beta$-invariant and (27) can be employed for computing $\#_{\hat\beta}$.

**Proposition 2** *Let $\mathcal{T}$ be a $\hat{\beta}$-invariant 0-trace on $\mathcal{B}\rtimes_\beta\mathbb{R}$ then $\psi_*\#_{\hat{\beta}}\mathcal{T}$ is a 1-trace on $S\mathcal{B}\rtimes_{\tau\otimes\beta}\mathbb{R}$ and*

$$\psi_*\#_{\hat{\beta}^\perp}\mathcal{T}(f_0,f_1) = \mathcal{T}\left(\int_{\mathbb{R}} ds f_0 \partial_s f_1(s)\right).$$

**Proof:** Let $f_j:\mathbb{R}\to S\mathcal{B}$. By (27) $\psi_*\#_{\hat{\beta}}\mathcal{T}(f_0,f_1) = \mathcal{T}((\psi(f_0)\nabla\psi(f_1))(0))$ with $\nabla\psi(f)(\hat{x}) = \imath\hat{x}\psi(f)(\hat{x})$ where $0,\hat{x}\in\hat{\mathbb{R}}$. Applying $\psi$ is essentially Fourier-transforming the dual variable $\hat{x}$ which becomes the suspension variable $s$. Under this transformation $\nabla$ becomes $\partial_s$ and the evaluation on $0\in\hat{\mathbb{R}}$ the integral over $s$. $\square$

**Proof of Theorem 2** In periodic cyclic cohomology $\mathcal{T}$ lies in the image of $\#_{\beta^\perp}$ (it pairs in the same way as $-\frac{1}{2\pi}\#_\beta\#_{\hat{\beta}^\perp}\mathcal{T}$). Equation 30 combined with the last proposition yield therefore

$$\langle\mathcal{T},[P_E(H_\infty)]\rangle = \langle\psi_*\#_{\hat{\beta}^\perp}\mathcal{T},[\mathcal{U}]\rangle = \frac{1}{2\pi\imath}\hat{\mathcal{T}}\left((\mathcal{U}^*-1)\partial_s\mathcal{U}\right) \tag{31}$$

which proves Theorem 2 since the left hand side is $\mathrm{IDS}(E)$ and the right hand side $F_b(E)$.

**Remark 2** This result can also be obtained by applying directly Connes' formula of [Co81], because the exponential map of the Wiener Hopf extension is (in our normalisation minus) the inverse of the Connes Thom isomorphism. Denoting $\check{\mathcal{T}}:\mathcal{B}=C(\Omega)\rtimes\mathbb{R}^{d-1}\to\mathbb{C}$, $\check{\mathcal{T}}(f)=\mathbf{P}(f(0))$ that formula reads, for unitaries $U$ in the unitalisation of $\mathcal{B}$ with $U-1\in\mathcal{B}$,

$$\mathcal{T}(\exp^{-1}[U]) = -\frac{1}{2\pi\imath}\check{\mathcal{T}}((U^*-1)\delta U)$$

where $\delta$ is the derivation generating the action $\beta^\perp$. Under the isomorphism $S\mathcal{B}\rtimes_{\tau\otimes\beta^\perp}\mathbb{R}\cong\mathcal{B}\otimes\mathcal{K}$, $\hat{\mathcal{T}}$ is identified with $\check{\mathcal{T}}\otimes\mathrm{Tr}$. The derivation of (18) yields $\delta=\delta^\perp+\gamma\nabla^\parallel$. Finally (13) implies that the above expression coincides with (31).

## 4.2 Gap-labelling group for systems with magnetic field

The gap labelling group for a system with observable $C^*$-algebra $\mathcal{C}$ and trace tr is $\langle\mathrm{tr},K_0(\mathcal{C})\rangle$. A lot of work has been devoted to computing this group for the type of crossed products considered here. With the notable exception of the rotation algebra (in which the Hofstadter Hamiltonian lies) the known results concern zero magnetic field.

We note that $K_0(\mathcal{A})$ vanishes [Ri82] and so the gap-labelling for the bulk-boundary system is trivial. Indeed, $H$ is not expected to have gaps in its spectrum, the edge states fill the gaps.

We consider here the case of a pure bulk system with homogenous magnetic field and show that this situation can be related to the case without magnetic field. We use the results of the last section making a choice of direction of the boundary such that the first part $B^\parallel$ in the splitting (6) vanishes. Then $\gamma=\frac{qB}{\hbar}$ and

$$\mathcal{A}=\mathcal{A}^\gamma=\mathcal{B}\rtimes_{\beta^\gamma}\mathbb{R}$$

where $\mathcal{B}=C_0(\hat{\Omega})\rtimes_{\alpha^\parallel}\mathbb{R}^{d-1}$ and we have denoted $\tau\otimes\beta^\perp$ now by $\beta^\gamma$ to emphasise its dependence on $\gamma$. If clarity demands it we also write ${\mathcal{A}_\infty}^\gamma$ and $\mathcal{E}^\gamma$ for the quotient and the ideal in the

Wiener Hopf extension and $\mathcal{T}^\gamma$, $\hat{\mathcal{T}}^\gamma$ for the traces. We recall that $\mathcal{E}^\gamma = S\mathcal{B} \rtimes_{\beta^\gamma} \mathbb{R}$ is isomorphic to $\mathcal{B} \otimes \mathcal{K}$ for all $\gamma$ and under this isomorphism $\hat{\mathcal{T}}^\gamma$ is mapped onto $\check{\mathcal{T}} \otimes \mathrm{Tr}$ where $\check{\mathcal{T}} : \mathcal{B} \to \mathbb{C}$, $\check{\mathcal{T}}(f) = \mathbf{P}(f(0))$ and Tr is the standard trace. Thus unlike the trace $\mathcal{T}^\gamma$ on ${\mathcal{A}_\infty}^\gamma$ which depends on $\gamma$ (recall that the trace on the projection of the lowest Landau level is proportional to $\gamma$) $\hat{\mathcal{T}}^\gamma$ is invariant under translation of $\gamma$. This is crucial for the following proposition.

**Proposition 3** $\langle \hat{\mathcal{T}}^\gamma_{\delta^\perp}, K_1(\mathcal{E}^\gamma)\rangle$ *does not depend on* $\gamma$. *In particular, in* $d = 2$, *the gradient pressure per energy* $\Pi$ *is constant in* $\gamma$ *as long as the gap at the Fermi energy stays open.*

**Proof:** We use similar arguments as in [KS04a]. Consider the $C^*$-algebra $SS\mathcal{B} \rtimes_{\tilde{\beta}} \mathbb{R}$, the additional suspension variable playing the role of $\gamma$, where, for $f : \mathbb{R} \to S\mathcal{B}$

$$\tilde{\beta}_{x^\perp}(f)(\gamma) = \beta^\gamma_{x^\perp}(f)(\gamma).$$

This is the total algebra of a $C^*$-field over $\mathbb{R}$ (the space of $\gamma$'s) which is trivial since all fibres are isomorphic algebras. If we restrict the $\gamma$-variable to a closed interval $I = [\gamma_0, \gamma_1]$ we obtain a trivial $C^*$-field over $I$ and the corresponding ENN-map $\mu : K_1(S\mathcal{B} \rtimes_{\beta^{\gamma_0}} \mathbb{R}) \to K_1(S\mathcal{B} \rtimes_{\beta^{\gamma_1}} \mathbb{R})$ is an isomorphism [ENN93, KS04b].

We construct a 2-trace over this algebra as in Prop. 3 of [KS04b]. For that we consider on this algebra the 0-trace $\hat{\mathcal{T}}^s = \int_\mathbb{R} d\gamma \hat{\mathcal{T}}^\gamma$, i.e. for $F : \mathbb{R} \to SS\mathcal{B}$

$$\hat{\mathcal{T}}^s(F) = \int_\mathbb{R} d\gamma \int_\mathbb{R} ds \check{\mathcal{T}}(F(0)(\gamma)(s)).$$

As mentioned above, $\hat{\mathcal{T}}^\gamma$ and hence also $\hat{\mathcal{T}}^s$ is invariant under the $\mathbb{R}$-action given by translating the $\gamma$-variable. Furthermore, this action commutes with the fibrewise extension of the action $\alpha^\perp$. It therefore follows that $\partial_\gamma$ and the fibrewise extension of $\delta^\perp$ (also denoted by the same letter) are two commuting derivations on $SS\mathcal{B} \rtimes_{\tilde{\beta}} \mathbb{R}$ which leave $\hat{\mathcal{T}}^s$ invariant. By Prop. 3 of [KS04b]

$$(SS\mathcal{B} \rtimes_{\tilde{\beta}} \mathbb{R} \otimes \Lambda\mathbb{C}^2, d, \hat{\mathcal{T}}^s \otimes \imath)$$

defines an (unbounded) 2-cycle for $SS\mathcal{B} \rtimes_{\tilde{\beta}} \mathbb{R}$ where $\Lambda\mathbb{C}^2$ is the Grassmann algebra generated by two elements $\xi_1, \xi_2$, $\imath(\xi_1\xi_2) = 1$, and

$$dF \otimes 1 = \delta^\perp(F) \otimes \xi_1 + \partial_s F \otimes \xi_2 \, , \quad d(1 \otimes \xi_j) = 0 \, .$$

If we restrict the $C^*$-field to a sub-interval $I = [\gamma_0, \gamma_1]$ then the above cycle restricts to a chain with boundary

$$(S\mathcal{B} \rtimes_{\beta^{\gamma_0}} \mathbb{R} \otimes \Lambda\mathbb{C} \oplus S\mathcal{B} \rtimes_{\beta^{\gamma_1}} \mathbb{R} \otimes \Lambda\mathbb{C}, d', -\hat{\mathcal{T}}^{\gamma_0} \otimes \imath' \oplus \hat{\mathcal{T}}^{\gamma_1} \otimes \imath')$$

the Grassmann algebra being that generated by $\xi_1$, $\imath'(\xi_1) = 1$, and $d'f \otimes 1 = \delta^\perp(f) \otimes \xi_1$, $d'(1 \otimes \xi_1) = 0$. This is a unbounded 1-cycle over $S\mathcal{B} \rtimes_{\beta^{\gamma_0}} \mathbb{R} \oplus S\mathcal{B} \rtimes_{\beta^{\gamma_1}} \mathbb{R}$ whose character is $-\hat{\mathcal{T}}^{\gamma_0}_{\delta^\perp} \oplus \hat{\mathcal{T}}^{\gamma_1}_{\delta^\perp}$. As in Prop. 7 of [KS04b] follows that

$$\langle \hat{\mathcal{T}}^{\gamma_0}_{\delta^\perp}, [U]\rangle = \langle \hat{\mathcal{T}}^{\gamma_1}_{\delta^\perp}, \mu([U])\rangle$$

which implies the first statement of the proposition.

$\Pi = \hat{\mathcal{T}}(G'(H)\frac{\partial V}{\partial x^\perp})$ depends a priori on $\gamma$. By Theorem 1 of [KS04a] we have $|\langle x|G'(H_{\omega,s})|y\rangle| \le \frac{C}{1+|x_2+s|^2+|y_2+s|^2}$ if $G$ is $C^9$. The constant $C$ is uniform in $\omega$ and depends continuously on $\gamma$. Since $\frac{\partial V}{\partial x^\perp}$ is bounded it follows that $\Pi$ depends continuously on $\gamma$. On the other hand $\Pi$ is an element of $\langle\hat{\mathcal{T}}^\gamma_{\delta^\perp}, K_1(\mathcal{E}^\gamma)\rangle$ which, as we just saw, does not depend on $\gamma$ and is a countable subgroup of $\mathbb{R}$ since $\mathcal{E}^\gamma$ is separable. The second statement follows therefore from the continuity of $\Pi$ in $\gamma$. $\square$

**Theorem 3** $\langle\mathcal{T}, K_0(\mathcal{A}_\infty{}^\gamma)\rangle \subset \langle\mathcal{T}, K_0(\mathcal{A}_\infty{}^0)\rangle + \gamma\langle\hat{\mathcal{T}}_{\nabla^\parallel}, K_1(\mathcal{E}^\gamma)\rangle$.

**Proof:** By the results of the last section $\langle\mathcal{T}, K_0(\mathcal{A}_\infty{}^\gamma)\rangle \subset \langle\hat{\mathcal{T}}_{\delta^\perp}, K_1(\mathcal{E}^\gamma)\rangle + \gamma\langle\hat{\mathcal{T}}_{\nabla^\parallel}, K_1(\mathcal{E}^\gamma)\rangle$ and $\langle\mathcal{T}, K_0(\mathcal{A}_\infty{}^0)\rangle = \langle\hat{\mathcal{T}}_{\delta^\perp}, K_1(\mathcal{E}^0)\rangle$. The theorem thus follows from the last proposition. $\square$

It was shown in [KS04a] for $d = 2$ that $\langle\hat{\mathcal{T}}_{\nabla^\parallel}, [U]\rangle$ is an index and hence an integer. The gap-labelling group has thus at most one more generator in $d = 2$.

#### 4.2.1 Systems with totally disconnected transversal

$(\Omega, \mathbb{R}^d)$ is a dynamical system with invariant measure $\mathbf{P}$. In the case where $\Omega$ has a complete transversal $X$ which is a totally disconnected set and reduces the dynamical system to $(X, \mathbb{Z}^d)$ (we simply say that the system has a totally disconnected transversal) one can say a lot more about the gap-labelling group. The algebra $\mathcal{A}_\infty{}^\gamma$ is Morita equivalent to the groupoid $C^*$-algebra $C^*(\mathcal{G}, \gamma)$, twisted by the magnetic field $\gamma$, of an $r$-discrete groupoid $\mathcal{G}$ whose unit space is $X$. $\mathbf{P}$ induces a measure on $X$ and a trace $\tau$ on the groupoid $C^*$-algebra. If the magnetic field is zero the groupoid $C^*$-algebra becomes the crossed product algebra of $(X, \mathbb{Z}^d)$ and, as was recently shown [BBG, BO03, KP03]

$$\langle\mathcal{T}, K_0(\mathcal{A}_\infty{}^0)\rangle = \mathbb{Z}[\mathbf{P}]. \tag{32}$$

$\mathbb{Z}[\mathbf{P}]$ is the $\mathbb{Z}$-module generated by the measures of the clopen subsets of $X$. It is independent of the choice of the transversal and has an interpretation as the module generated by the relative frequencies of atomic clusters appearing in the solid.

**Corollary 1** *For systems with totally disconnected transversal*

$$\mathbb{Z}[\mathbf{P}] + \gamma\mathbb{Z} \ \subset\ \langle\mathcal{T}, K_0(\mathcal{A}_\infty{}^\gamma)\rangle \ \subset\ \mathbb{Z}[\mathbf{P}] + \gamma\langle\hat{\mathcal{T}}_{\nabla^\parallel}, K_1(\mathcal{E}^\gamma)\rangle$$

*the inclusions being equalities if* $d = 2$.

**Proof:** The inclusion $C(X) \subset C^*(\mathcal{G}, \gamma)$ induces an inclusion $\mathbb{Z}[\mathbf{P}] \subset \langle\tau, K_0(C^*(\mathcal{G}, \gamma))\rangle = \langle\mathcal{T}, K_0(\mathcal{A}_\infty{}^\gamma)\rangle$.

The inclusion $\mathbb{C} \rtimes \mathbb{R}^{d-1} \rtimes_{\beta^\gamma} \mathbb{R} \subset C(\Omega) \rtimes \mathbb{R}^{d-1} \rtimes_{\beta^\gamma} \mathbb{R}$ (here we view $\mathbb{C}$ as the constant functions in $C(\Omega)$) can be used to see that $\langle\mathcal{T}, K_0(\mathcal{A}_\infty{}^\gamma)\rangle \cap \langle\hat{\mathcal{T}}^\gamma_{\delta^\perp}, K_1(\mathcal{E}^\gamma)\rangle \neq \emptyset$. In fact, in $d = 2$ the projection on the lowest Landau level gives rise to an element in the intersection and a similar projection can be constructed in higher dimensions.

This together with (32) implies the inclusions stated in the corollary. Moreover, in $d = 2$, $\langle\hat{\mathcal{T}}_{\nabla^\parallel}, K_1(\mathcal{E}^\gamma)\rangle = \mathbb{Z}$. $\square$

# References


[Be85] J. Bellissard, *K-theory of $C^*$-algebras in solid state physics*, in *Statistical Mechanics and Field Theory: Mathematical Aspects*, *Lecture Notes in Physics* **257**, edited by T. Dorlas, M. Hugenholtz, M. Winnink, 99–156 (Springer-Verlag, Berlin, 1986).

[Be93] J. Bellissard, *Gap labelling theorems for Schrödinger operators*, 538–630, in *From Number Theory to Physics*, (Springer, Berlin, 1993).

[BES94] J. Bellissard, A. van Elst, H. Schulz-Baldes, *The Non-Commutative Geometry of the Quantum Hall Effect*, J. Math. Physics **35**, 5373–5451 (1994).

[BBG] J. Bellissard, R. Benedetti, J. Gambaudo, *Spaces of tilings, finite telescopic approximations and gap-labelling*, to appear in Commun. Math. Phys.

[BO03] M. Benameur, H. Oyono-Oyono, *Gap-labelling for quasi-crystals (proving a conjecture by J. Bellissard)*, Proc. Operator-Algebras and Mathematical Physics (Constanta 2001) eds. J.M. Combes *etal.* (2003).

[Co81] A. Connes, *An analogue of the Thom isomorphism*, Adv. in Math. **39**, 31–55 (1981).

[Co94] A. Connes, *Non-Commutative Geometry*, (Acad. Press, San Diego, 1994).

[ENN88] G. Elliott, T. Natsume, R. Nest, *Cyclic cohomology for one-parameter smooth crossed products*, Acta Math. **160**, 285-305 (1988).

[ENN93] G. Elliott, T. Natsume, R. Nest, *The Heisenberg group and $K$-theory*, $K$-Theory **7**, 409–428 (1993).

[KP03] J. Kaminker, I.F. Putnam, *A proof of the Gap Labeling Conjecture*, Michigan J. Math. **51**, 537–547 (2003).

[KS04a] J. Kellendonk, H. Schulz-Baldes, *Quantization of edge currents for continuous magnetic operators*, J. Funct. Anal. **209**, 388–413 (2004).

[KS04b] J. Kellendonk, H. Schulz-Baldes, *Boundary maps for $C^*$-crossed products with $\mathbb{R}$ with an application to the Quantum Hall Effect*, Commun. Math. Phys. **249**, 611–637 (2004).

[Ke04] J. Kellendonk, *Topological quantization of boundary forces and the integrated density of states*, J. Phys. A. **37** L161–L166 (2004).

[KZ04] J. Kellendonk, I.P. Zois, *Rotation numbers, Gap Labelling and Boundary forces*, Preprint, mp-arc 04-217.

[Pa80] L. Pastur, *Spectral properties of disordered systems in the one-body approximation*, Commun. Math. Phys. **75**, 179–196 (1980).

[Ri82] M. A. Rieffel, *Connes' analogue for crossed products of the Thom isomorphism*, Contemp. Math. **10**, 143–154 (1982).

[St82] P. Streda, *Theory of quantised Hall conductivity in two dimensions*, J. Phys. C. **15**, L717–L721 (1982).