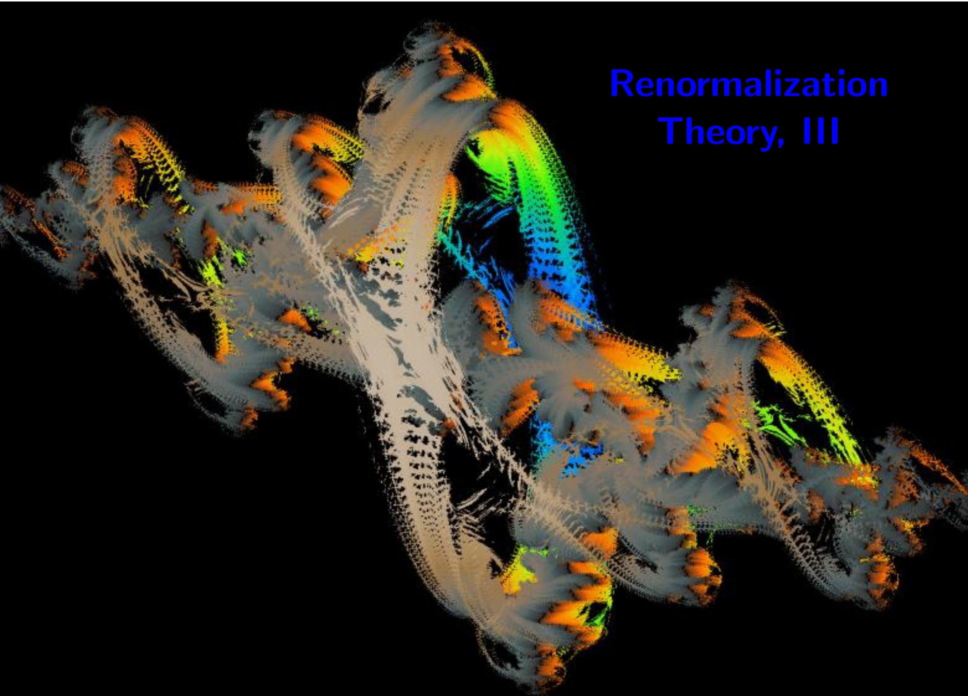
Renormalization Theory, III



Scales in the universe

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Physical phenomena occur over a wide range of scales from the Planck scale $\ell_P = \sqrt{\hbar G/c^3} \simeq 1.6\ 10^{-35}\ m$ to the radius of the observable universe, in practice about 45 billion light-years, hence around $4.4\ 10^{26}\ m$ or better, $2.7\ 10^{61} \ell_P$.

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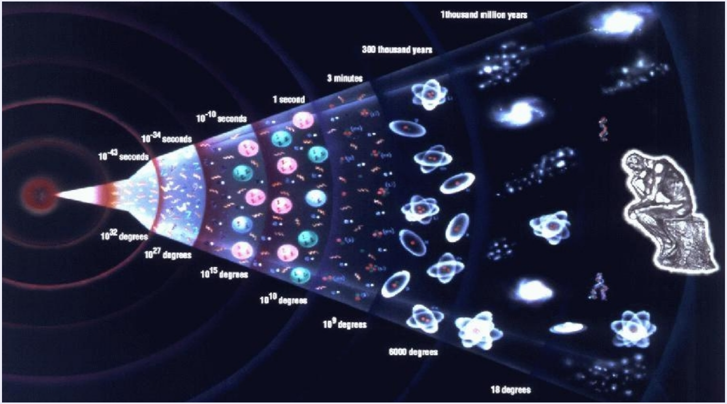

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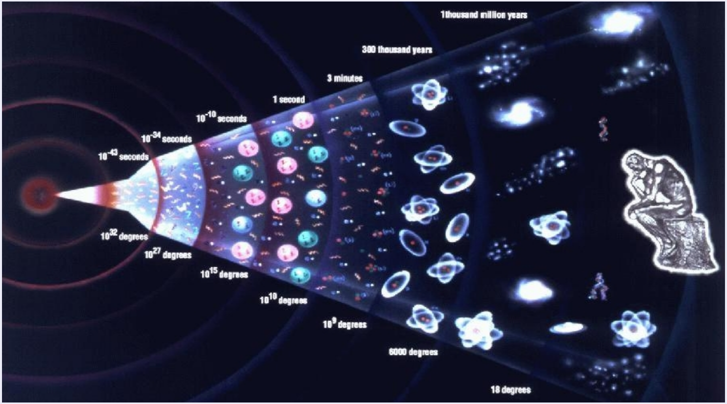

The universe therefore is made of roughly 61 powers of 10 or 140 powers of e.

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But the fifteen to sixteen scales between ℓ_P and $2\ 10^{-19}$ meters (about 1 Tev), are still terra incognita for physics.

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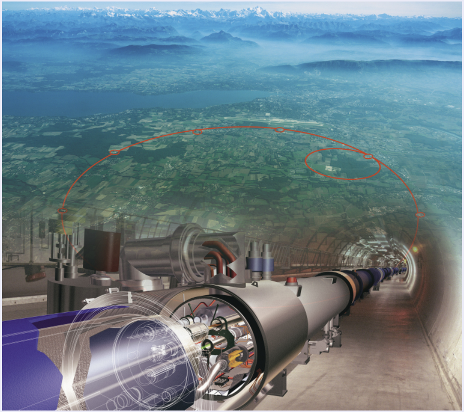

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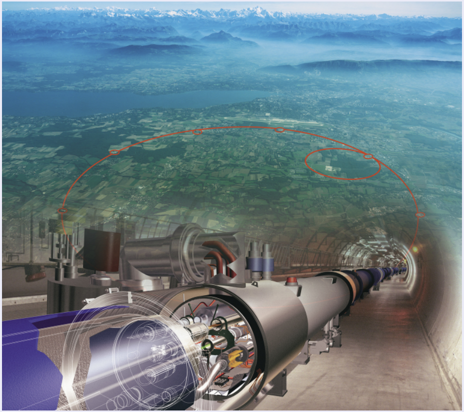

After that treat no new spectacular advance of this type is planned on terra incognita, hence we have some time to deepen our theoretical and mathematical understanding.

Noncommutative Geometry

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Space-time itself could be of this type; for instance at a certain scale new uncertainty relations could appear between length and width which would generalize Heisenberg's relations.

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where $\Theta^{\mu\nu}$ is an antisymmetric constant tensor which in the simplest case can be written as:

$$\Theta^{\mu\nu} = \theta \begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & (0) \\ (0) & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{pmatrix}$$

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The unique product (associative, but noncommutative) generated by these relations on (Schwarz class) functions is called the Moyal-Weyl product and writes:

$$(f \star g)(x) \;=\; \int \frac{d^4y}{(2\pi)^4} d^4z \, f(x+y) g(x+z) e^{2\imath y \wedge z}$$

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In $d = 4$ every index m, n... is a pair $m = (m_1, m_2)$ ...

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  - Alternative to current ideas, eg on supersymmetry

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- The relevant species is now the species of ribbon graphs.

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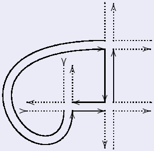

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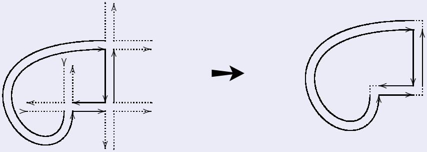

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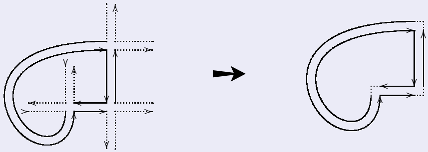

V=3
L=3
F=2
E=6
EF=2

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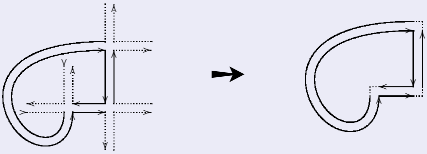

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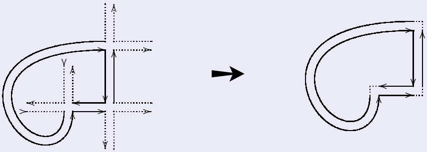

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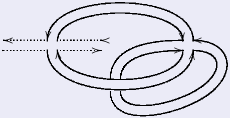

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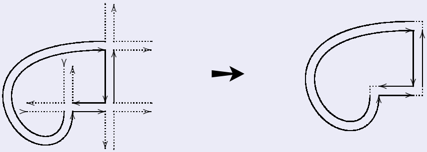

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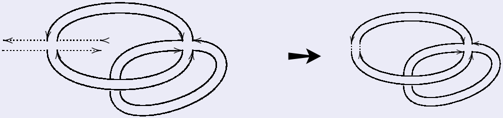

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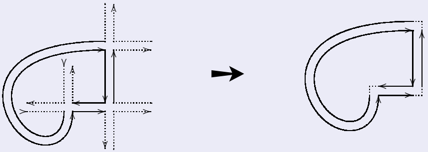

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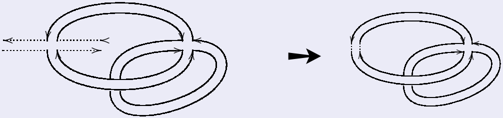

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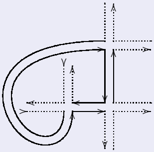

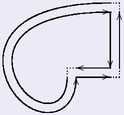

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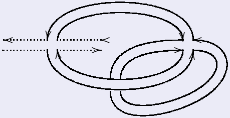

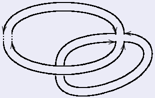

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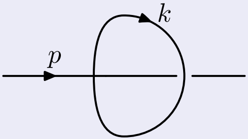
For instance,



$$\propto \lambda \int d^4k \, \frac{e^{ip^\mu k^\nu \theta_{\mu\nu}}}{k^2 + m^2}$$

$$\propto \lambda \sqrt{\frac{m^2}{\tilde{p}^2}} K_1\left(\sqrt{m^2 \tilde{p}^2}\right) \sim_{p \to 0} p^{-2}$$

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The Euclidean theory with an additional harmonic potential

$$S = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \star \partial^\mu \phi + \frac{\Omega^2}{2} (\tilde{x}_\mu \phi) \star (\tilde{x}^\mu \phi) + \frac{\mu_0^2}{2} \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right) (x)$$

where $\tilde{x}_\mu = 2\Theta^{-1}_{\mu\nu} x^\nu$, is covariant under a symmetry $p_\mu \leftrightarrow \tilde{x}_\mu$ called Langmann-Szabo symmetry and is renormalizable at every order in λ!

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where $\tilde{x}_\mu = 2\Theta^{-1}_{\mu\nu} x^\nu$, is covariant under a symmetry $p_\mu \leftrightarrow \tilde{x}_\mu$ called Langmann-Szabo symmetry and is renormalizable at every order in λ!
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- parametric representation and dimensional renormalization

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Among other aspects, the GW model is a theory of non independent, non identically distributed random matrices for which one can understand the large N limit.

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Recall the interaction is very simple namely $\mathrm{Tr}\phi^4$. But the propagator is not.

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$$G_{m,m+h;l+h,l} = \frac{\theta}{8\Omega}\int_0^1 d\alpha \, \frac{(1-\alpha)^{\frac{\mu_0^2\theta}{8\Omega}}}{(1+C\alpha)^2}\left(\frac{\sqrt{1-\alpha}}{1+C\alpha}\right)^{m+l+h}$$
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with $\mathcal{A}(m,l,h,u) = \sqrt{\binom{m}{m-u}\binom{m+h}{m-u}\binom{l}{l-u}\binom{l+h}{l-u}}$ and $C(\Omega) = \frac{(1-\Omega)^2}{4\Omega}$.

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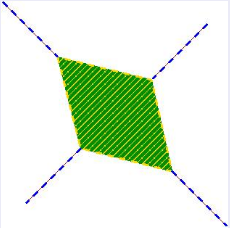
This vertex is non-local and oscillates. It has a parallelogram shape in each symplectic pair, and its oscillation is proportional to its area:



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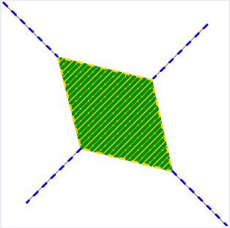

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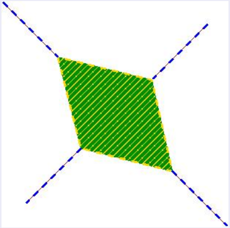

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As in the commutative case the first two elements are quite universal. The third depends on details of the model, such as dimension and interaction.

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- In direct space

$$G^i(x, y) = \int_{M^{-2i}}^{M^{-2(i-1)}} dt \cdots \leqslant K M^{2i} e^{-c_1 M^{2i} \|x-y\|^2 - c_2 M^{-2i} (\|x\|^2 + \|y\|^2)}$$

The corresponding new renormalization group corresponds to a completely new mixture of the previous ultraviolet and infrared notions. Furthermore there exists only a half direction which is infinite for this RG.

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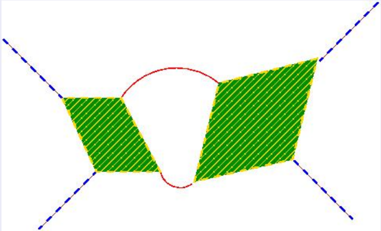

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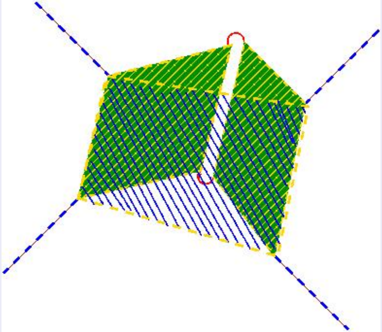

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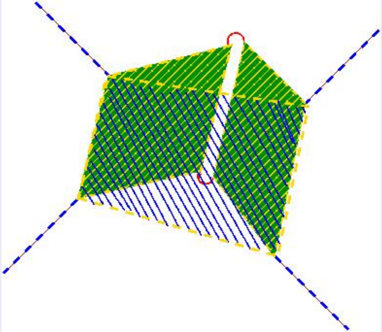

This principle applies only to planar graphs with a single external face.

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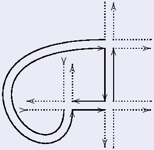

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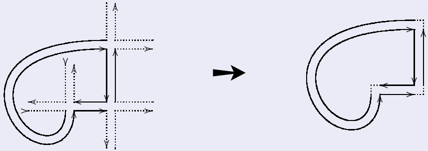

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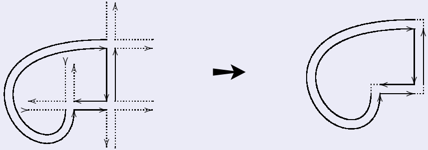

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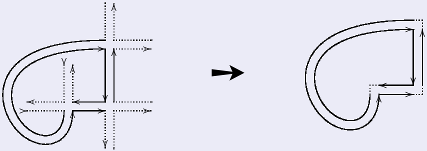

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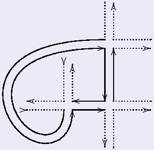

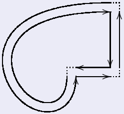

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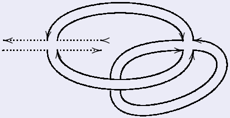

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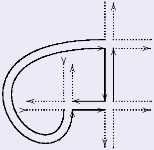

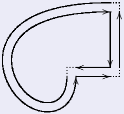

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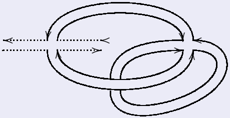

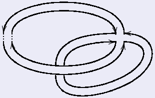

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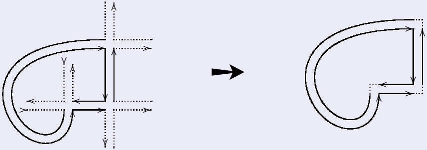

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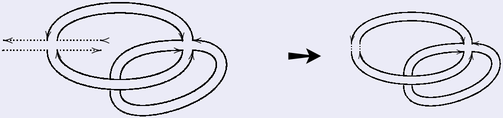

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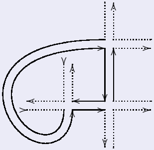

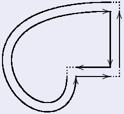

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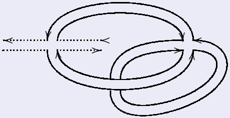

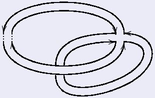

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Unexpected: No Landau ghost, so complete constructive analysis (with loop vertices exansion) should be possible!