Renormalization Theory, IV

Vincent Rivasseau

LPT Orsay

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- **Any loops:** M. Disertori, R. Gurau, J. Magnen and V. Rivasseau,
  Vanishing of Beta Function of Non Commutative Φ_4^4 to all orders,
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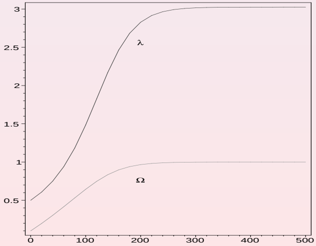

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Asymptotic safeness is truly a new unexpected phenomenon in NCQFT!

The $(\bar\phi \star \phi)^{\star 2}_4$ in the Matrix Base

We consider a slightly different model with complex (charged) fields, which distinguishes left and right, because the proof is slightly easier to explain. But it holds also for the real model. The action of the complex model is

$$S = \int \bar\phi(-\Delta + x^2 + \mu_0)\phi + \frac{\lambda}{2}\int \bar\phi \star \phi \star \bar\phi \star \phi$$

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We work in the matrix base at $\Omega = 1$ (because the flow for Ω goes exponentially fast to $\Omega = 1$ in the ultraviolet regime, see above).

At $\Omega = 1$

$$
\begin{aligned}
S &= \int \bar\phi(-\Delta + x^2 + \mu_0)\phi + \frac{\lambda}{2}\int \bar\phi \star \phi \star \bar\phi \star \phi \\
&= \bar\phi X \phi + \phi X \bar\phi + A\bar\phi\phi + \frac{\lambda}{2}\bar\phi\phi\bar\phi\phi \qquad X = m\delta_{mn}
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$$

Let Σ denote the amputated 1PI two point function (sum of two point connected graphs which cannot be cut in two by removing a single line, with external propagators omitted). The propagator at $\Omega = 1$ is:

$$
C_{mn} = \frac{1}{m+n+A}; \qquad G_{2,mn} = \frac{C_{mn}}{1 - C_{mn}\Sigma(m,n)}
$$

Death of the ghost

$$[G_{2,mn}]^{-1} = m + n + A - \Sigma(m,n) \approx (m+n)(1-\partial\Sigma) + (A - \Sigma(0,0))$$

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- **Theorem:** The equation:

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  holds (up to finite terms) at all orders of perturbation in the UV region.
- The proof: inspired by the work of G. Benfatto and V. Mastropietro on the Thirring model.

The change of variables

$$Z(\eta,\bar\eta) = \int d\phi d\bar\phi \; e^{-(\bar\phi X\phi + \phi X\bar\phi + A\bar\phi\phi + \frac{\lambda}{2}\phi\bar\phi\phi\bar\phi) + \bar\phi\eta + \bar\eta\phi}$$

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Let $U = e^{\imath M}$. One performs the "left" change of variables:

$$\phi \to \phi^U = \phi U \qquad \bar\phi \to \bar\phi^U = U^\dagger \bar\phi$$

which leads to

$$\partial_\eta \partial_{\bar\eta} \frac{\delta \ln Z}{\delta M_{ba}} = 0$$

The Ward identities

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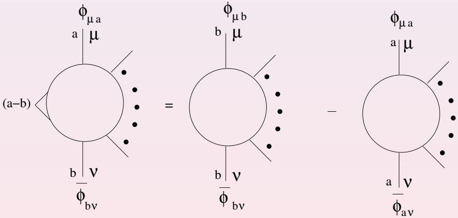
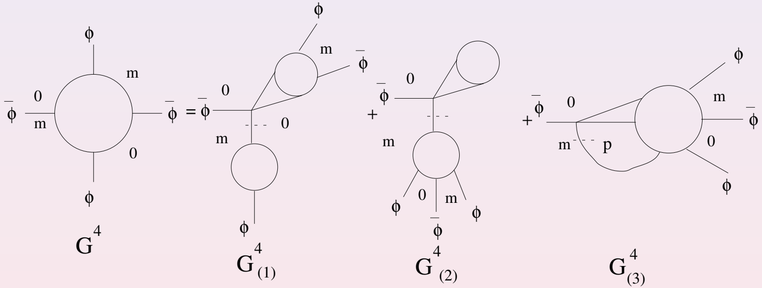
One obtains the Ward identities:

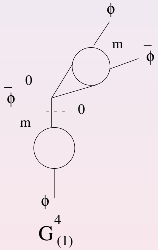


Dyson's equations



- This is a classification of graphs (no combinatoric to check)!
- The second term has one "left tadpole insertion". It vanishes after mass renormalization.

The first term



$$G^4_{(1)}(0, m, 0, m) = \lambda C_{0m} G_2(0, m) G_{2,ins}(0, 0; m)$$

The first term, II

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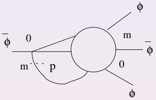
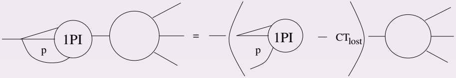
The Ward identity gives:

$$\begin{aligned} G_{2,ins}(0,0;m) &= \lim_{\zeta\to 0} G_{2,ins}(\zeta,0;m) = \lim_{\zeta\to 0} \frac{G_2(0,m) - G_2(\zeta,m)}{\zeta} \\ &= -\partial_L G_2(0,m) \to \\ G^4_{(1)}(0,m,0,m) &= \lambda [G_2(0,m)]^4 \frac{C_{0m}}{G_2(0,m)} [1 - \partial_L \Sigma(0,m)] \end{aligned}$$

The third term





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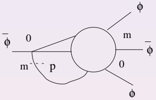

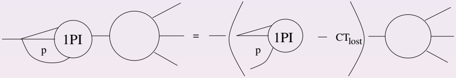

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But in the renormalized theory we must add the missing mass counterterm.

$$G^4_{(3)} = C_{0m} \sum_p G^4_{ins}(0, p; m, 0, m) - C_{0m}(CT_{missing})G^4(0, m, 0, m)$$

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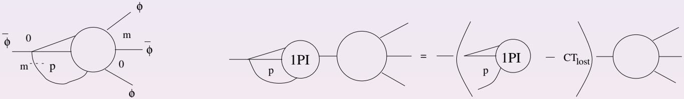

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But one has $CT_{missing} = \Sigma^R(0,0) = \Sigma(0,0) - T^L$, hence one concludes:

$$G^4_{(3)}(0,m,0,m) = -C_{0m} G^4(0,m,0,m) \frac{1}{G_2(0,0)} \frac{\partial\Sigma(0,0)}{1-\partial\Sigma(0,0)}$$

Death of the ghost

One puts $G^4_{(3)}$ on the left side of Dyson's equation:

$$G^4(0,m,0,m)(1 + C_{0m}\frac{1}{G_2(0,0)}\frac{\partial\Sigma(0,0)}{1-\partial\Sigma(0,0)})$$
$$= \lambda[G_2(0,m)]^4\frac{C_{0m}}{G_2(0,m)}[1-\partial_L\Sigma(0,m)]$$

and using $C_{0m} = 1/(m + A_{ren})$; $G_2(0,m) = 1/[m(1-\partial\Sigma) + A_{ren}]$ one gets:

$$\frac{G^4}{1-\partial\Sigma}(1-\partial\Sigma+\frac{A_{ren}}{m+A_{ren}}\partial\Sigma) = \lambda[G_2]^4(1-\partial\Sigma)(1-\frac{m}{m+A_{ren}}\partial\Sigma)$$

hence, since red terms are equal, amputating gives $\Gamma_4 = \lambda(1-\partial\Sigma)^2$ hence $\beta = 0$!

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- $\phi_4^{\star 4}$ cannot be treated through the "classsical", "à la Glimm-Jaffe-Spencer" cluster expansions (roughly because the interaction is non-local).
- The loop-vertices expansion solves this problem.

Sketch of proof: constructive matrix model

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Consider the matrix model

$$d\nu(\Phi) = \frac{1}{Z(\lambda, N)} e^{-\frac{\lambda}{N} \ \mathrm{Tr} \Phi^\star \Phi \Phi^\star \Phi} d\mu(\Phi)$$

where $d\mu$ is the usual GUE measure.

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We can rewrite it with an intermediate matrix field σ as

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$$H = \sqrt{\frac{\lambda}{N}} 1 \otimes \sigma^R \text{ is self-adjoint!}$$

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Expand the exponential as $\sum_n \frac{V^n}{n!}$. Then apply the "swiss knife" forest formula to get

Theorem

$$\log\ Z(\lambda, N) = \sum_{n=1}^{\infty} \sum_{\mathcal{T}} \Big\{ \prod_{\ell \in \mathcal{T}} \big[\int_0^1 dh_\ell \sum_{i_\ell, j_\ell, k_\ell, l_\ell} \big] \Big\} \int d\nu_{\mathcal{T}}(\{\sigma^v\}, \{h\}) \Big\{ \prod_{\ell \in T} [\delta_{i_\ell l_\ell} \delta_{j_\ell k_\ell} \frac{\delta}{\delta \sigma^{v(\ell)}_{i_\ell, j_\ell}} \frac{\delta}{\delta \sigma^{v'(\ell)}_{k_\ell, l_\ell}}] \Big\} \prod_v V_v$$

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Sketch of Proof Left indices provide a particular cyclic order at each loop vertex. The σ field acts only on right indices, hence left indices are conserved, and there is a single global N factor per loop vertex coming from the trace over the left index. But there is a single trace over right indices corresponding to turning around the tree with of a product of resolvents bounded by 1!

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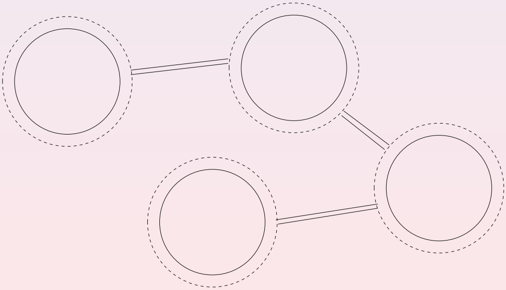

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After Fourier transform we get in space-time dimension D (omitting the trivial external propagators):

$$A_G(p) = \delta(\sum p) \int_0^\infty \frac{e^{-V_G(p,\alpha)/U_G(\alpha)}}{U_G(\alpha)^{D/2}} \prod_l (e^{-m^2\alpha_l} d\alpha_l) \ .$$

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where U_G and V_G are the first and second "Kirchoff-Symanzik" polynomials.

Remark that space-time has disappeared and the dimension D is now a parameter.

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$$
\begin{aligned}
U_G = & \sum_T \prod_{l \notin T} \alpha_l \ , \\
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where the first sum is over spanning trees T of G and the second sum is over "two-trees" T_2, i.e. forests separating the graph in exactly two connected components $E(T_2)$ and $F(T_2)$.

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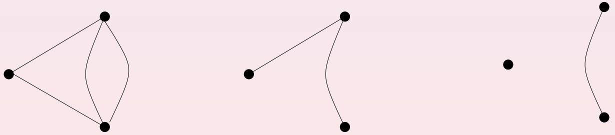

A graph A spanning tree A two-tree

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Theorem *Let $A = A_{ij}$ be an n by n square matrix with $\sum_i A_{ij} = 0$ $\forall j$. and let $A^{11} = A_{ij}, ij \neq 1$, then*

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where the sum is over rooted trees with root at 1 (oriented away from 1).

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$$\mathcal{A}_{G,\bar{v}}(\{x_e\},\ p_{\bar{v}}) = K' \int_0^\infty \prod_l [d\alpha_l (1-t_l^2)^{D/2}] HU_{G,\bar{v}}(t)^{-D/2} e^{-\frac{HV_{G,\bar{v}}(t,x_e,p_{\bar{v}})}{HU_{G,\bar{v}}(t)}},$$

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where $HU_{G,\bar{v}}(t)$ is polynomial in the t variables and $HV_{G,\bar{v}}(t, x_e, p_{\bar{v}})$ is a quadratic form in the external variables $(x_e, p_{\bar{v}})$ whose coefficients are polynomials in the t variables.

HU as a positive sum

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The incidence matrix of a ribbon graph contains roughlly speaking twice as many variables as for an ordinary graph.

$$HU_{G,\bar{v}}(t) = \sum_{I,J} \Omega^{k_{IJ}-2g} (Pf_{IJ})^2 \prod_{\ell \in I} t_\ell \prod_{\ell' \in J} t_{\ell'}$$

$$k_{IJ} = |I| + |J| - L - F + 1$$

where Pf_{IJ} is the Pfaffian of a certain antisymmetric matrix with integer entries where lines and columns corresponding to two sets I and J have been deleted.

The $g = 0$ case

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$$HU_{G,\bar{v}}(t) = \sum_{J \ bitree} (2/\Omega)^{2g(G_J)} \prod_{\ell \in J} t_\ell + \text{subleading terms}$$

where a bitree J is a tree in the dual graph whose complement contains a tree in the direct graph.

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This representation allows to define dimensional regularization and renormalisation.

Canonical Moves for Graphs

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There are two "universal" moves on the species of ordinary graphs

- Deleting a line ℓ ($G \to G - \ell$, $V \to V$, $L \to L - 1$)
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But there are three "universal" moves on the species of ribbon graphs

- Deleting a line ($V \to V$, $L \to L - 1$, $F \to F$, $g \to g$)
- Contracting a line ($V \to V - 1$, $L \to L - 1$, $F \to F$, $g \to g$)
- Genus reduction, or deletion of a pair of "crossing lines" ($V \to V$, $L \to L - 2$, $F \to F - 1$, $g \to g - 1$). This third non trivial move requires a reshuffling of the other lines.

Canonical Topological Polynomials

Canonical Topological Polynomials

There is a canonical polynomial in two variables on the species of ordinary graphs, called the Tutte polynomial, completely defined by the two properties:

- if G has l 1PI lines (bridges) and m tadpoles (loops) and no other edges, then $\mathcal{T}(G; x, y) = x^l y^m$
- If ℓ is a line which is neither a 1PI line nor a tadpole, then $\mathcal{T}(G; x, y) = \mathcal{T}_{G-e}(x, y) + \mathcal{T}_{G/e}(x, y)$.

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The Kirchoff-Symanzik polynomial is a multivariable version of Tutte polynomial. A tree has only 1PI lines, hence $\mathcal{T}_T(x, 0) = x^L = x^{v-1}$, and for a general connected G (without tadpoles) $\mathcal{T}(x, 0) = \sum_{T \in G} x^{v-1}$, where v is the number of vertices.

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$$BR_G(x,y,z) = \sum_{J \in G} x^{v(G)-c(G)-v(J)+c(J)} y^{l(J)-v(J)+c(J)} z^{2c(J)-F(J)+l(J)-v(J)}$$

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- l is the number of lines
- F is the number of faces

so that for $c(J) = 1$, one recognizes $z^{2g(J)}$.

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These universal polynomials are used in graph coloring, knot theory, matroids....