Invariants de certains groupes d'origine dynamique

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Summary

In this thesis we study various groups of piecewise continuous transformations, notably of the interval, consisting of piecewise translations. One of these is the group IET of interval exchange transformations. A larger one is the group $\text{IET}^{\bowtie}$ of interval exchange transformations with flips. For every subgroup Γ of $\mathbb{R}$, we consider a subgroup of IET, denoted by $\text{IET}(\Gamma)$, consisting of elements of IET which are continuous outside Γ. We similarly define the group $\text{IET}^{\bowtie}(\Gamma)$. This is a countable group when Γ is countable.

The preliminary Chapter 1 introduces these groups in detail.

In Chapter 2, we exhibit a natural isomorphism between the abelianization of $\text{IET}(\Gamma)$ and the second skew-symmetric power of Γ over $\mathbb{Z}$. The case $\Gamma = \mathbb{R}$ was treated by Arnoux-Fathi-Sah. The case when Γ is equal to 2Γ can be treated similarly, but otherwise there is a new difficulty, and indeed 2-torsion then appears in the abelianization (the above skew-symmetric power not just being the second exterior power). The idea is to construct a group homomorphism which "measures", for every $f \in \text{IET}(\Gamma)$, the set of pairs flipped by f. This group homomorphism is inspired of the classical signature on finite permutation groups. In Chapter 3 we exhibit a natural isomorphism between the abelianization of $\text{IET}^{\bowtie}(\Gamma)$ and an explicit 2-elementary abelian group. In the case $\Gamma = \mathbb{R}$, Arnoux proved that $\text{IET}^{\bowtie}$ is a perfect group.

In Chapter 4, we prove the vanishing of the Kapoudjian class of the group of piecewise continuous self-transformations of $[0,1[$. This result is equivalent to the existence of a group homomorphism onto $\mathbb{Z}/2\mathbb{Z}$ which extends the classical signature on finitely supported permutations.

In Chapter 5 we study a generalization of IET in higher dimension. For $d \geq 1$, we consider the group Rec_d of rectangle exchange transformations of $[0,1[^d$. We introduce a family of transformations, called restricted shuffles, and we prove that they form a generating subset of Rec_d. In dimension 1, this result is already known and easy, but the proof for $d \geq 2$ is much more complicated, the underlying combinatorics being richer. We use this generation result to identify the abelianization of Rec_d by generalizing the Arnoux-Fathi-Sah homomorphism to this higher-dimensional context.

Table des matières

Cette thèse porte sur l'étude de certains groupes d'origine dynamique en s'intéressant à leur abélianisé, ce qui revient à classifier leurs invariants additifs (c'est-à-dire, leurs morphismes vers des groupes abéliens). Elle a été réalisée sous la supervision d'Yves Cornulier entre octobre 2018 et juin 2021. Les groupes d'origine dynamique en question sont des groupes agissant par isométries par morceaux sur un intervalle. Un exemple plutôt connu est le groupe des échanges d'intervalles, noté IET, constitué des translations par morceaux de l'intervalle $[0,1[$.

1) Bref historique

Les éléments du groupe IET sont introduits par M. Keane [Kea75] en 1975 et ce sont principalement leur dynamique et leurs propriétés ergodiques qui sont étudiés. L'étude de IET en tant que groupe est initiée par Arnoux-Fathi [Arn81a] et Sah [Sah81] par l'identification de l'abélianisé de IET. Durant ces quinze dernières années, cette étude s'est poursuivie et renforce notre compréhension des sous-groupes de IET. On peut citer en particulier les travaux de C. Novak [Nov09], Dahmani-Fujiwara-Guirardel [DFG13, DFG17], Boshernitzan [Bos16] et Vorobets [Vor17]. Ces travaux incluent des progrès sur la question de l'existence ou non d'un sous-groupe libre de rang 2 dans IET de Katok. Une autre question, posée par Y. Cornulier, porte sur le caractère moyennable ou non moyennable du groupe IET. Pour cette question aussi, des avancées ont été faites en particulier avec le travail de Juschenko-Monod [JM13] et un peu plus tard rejoints par Matte Bon et de la Salle [JMMS18].
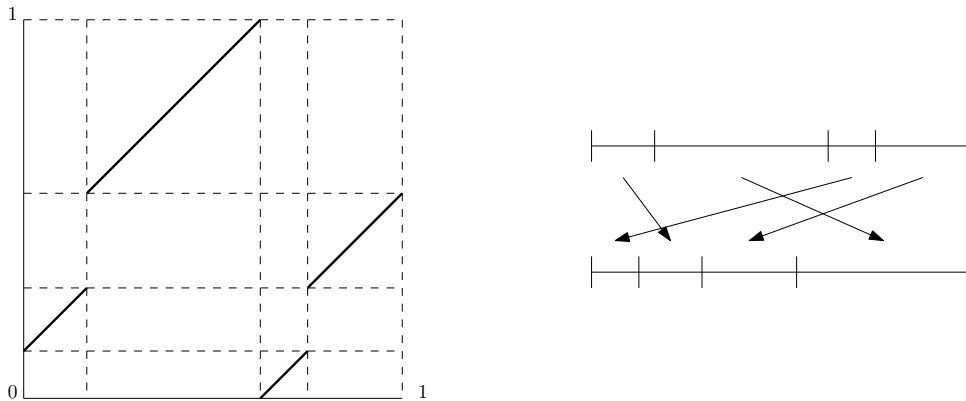
Gauche : Le graphe d'un élément de IET (sans précisions sur les valeurs en les points de discontinuité). **Droite** : Son "action" sur $[0,1[$.

Si on autorise toutes les transformations qui sont isométriques par morceaux, alors on obtient le groupe des échanges d'intervalles avec renversements, noté IET$^{\bowtie}$. Ce groupe possède de nombreuses connexions avec les feuilletages mesurés non orientés sur des surfaces et billards. L'étude de la dynamique et des propriétés ergodiques d'un élément a intéressé Gutierrez [Gut78], Arnoux [Arn81b], Nogueira [Nog89] et Danthony-Nogueira [DN90] et plus récemment O. Paris-Romaskevich et P. Hubert [PRH18] ou Skripchenko-Troubetzkoy [ST18]. Peu de choses sont connus sur le groupe lui-même, Arnoux [Arn81b] a prouvé que c'est un groupe simple et N. Guelman et I. Liousse [GL19a] ont prouvé que ce groupe est uniformément parfait (tout élément est produit d'au plus 6 commutateurs).
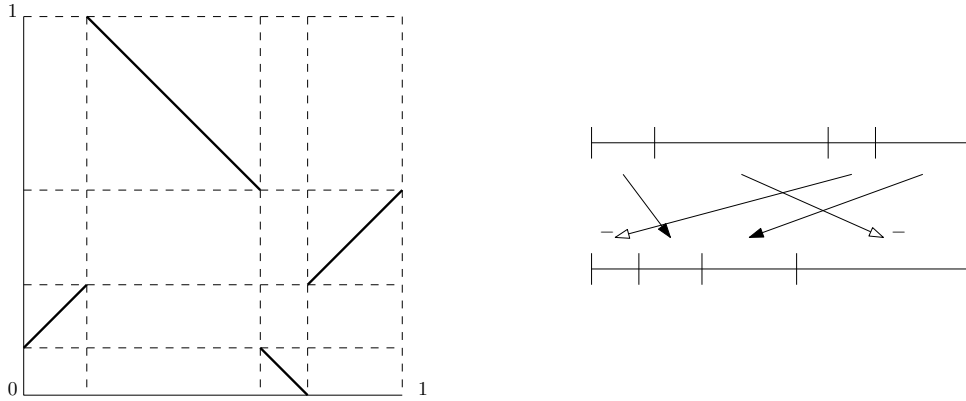
Gauche : Le graphe d'un élément de IET$^{\bowtie}$ qui contients des renversements (sans précisions sur les valeurs en les points de discontinuité). **Droite** : Son "action" sur $[0, 1[$.

Le groupe des transformations continues par morceaux et affines par morceaux apparaît dans différents travaux en particulier à cause de l'inclusion naturelle des groupes de Thompson. Il a été initialement définit par M. Stein [Ste92] et récemment N. Guelman et I. Liousse ont prouvé que son sous-groupe des éléments qui préservent l'ordre par morceaux est simple.

Le groupe de tous les éléments continus par morceaux, dénoté par PC$^{\bowtie}$, a aussi été étudié. Arnoux [Arn81b] a prouvé que son sous-groupe PC$^+$ des éléments qui préservent l'orientation des morceaux est simple.
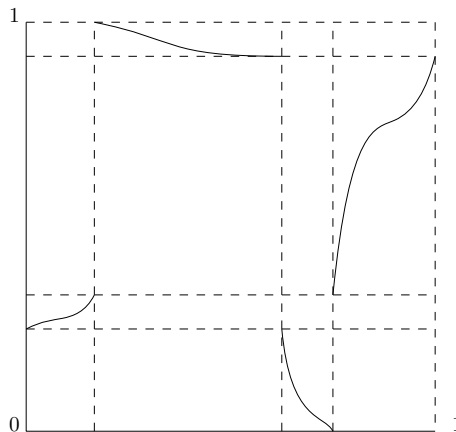
Le graphe d'un élément de PC$^{\bowtie}$ (sans précisions sur les valeurs en les points de discontinuité).

2) Abélianisés des groupes $\mathrm{IET}(\Gamma)$ **et** $\mathrm{IET}^{\bowtie}(\Gamma)$

Le chapitre 1 introduit tous les groupes considérés dans cette thèse.

On décrit des sous-familles d'éléments qui jouent un rôle important dans l'étude de ces groupes, par exemple l'ensemble des rotations restreintes (voir définition 1.2.7) est un système générateur du groupe IET.

On introduit aussi des familles de sous-groupes des groupes IET et $\mathrm{IET}^{\bowtie}$ qui sont à l'origine de cette thèse. Pour tout sous-groupe Γ de $\mathbb{R}/\mathbb{Z}$, on définit $\mathrm{IET}(\Gamma)$, respectivement $\mathrm{IET}^{\bowtie}(\Gamma)$, comme le sous-groupe de IET, respectivement de $\mathrm{IET}^{\bowtie}$, des éléments qui sont continus en dehors de la pré-image $\tilde{\Gamma}$ de Γ dans $\mathbb{R}$. L'identification de leur abélianisé est le premier problème étudié dans cet thèse. Le cas où Γ est fini peut être réduit à l'étude d'un groupe de permutation fini, ainsi on comprend directement son sous-groupe dérivé et son abélianisé. Pour $\tilde{\Gamma} = \mathbb{R}$, Arnoux-Fathi et Sah ont démontré le théorème suivant :

Théorème 1 (Arnoux-Fathi-Sah [Arn81a, Sah81]). *Il existe un morphisme de groupes surjectif (explicite)* $\varphi : \mathrm{IET} \to \bigwedge_{\mathbb{Q}}^{2} \mathbb{R}$ *dont le noyau est égal au sous-groupe dérivé* $D(\mathrm{IET})$. *Il est appelé le* SAF-*invariant.*

On s'intéresse ici au cas de la restriction de ce morphisme aux sous-groupes $\mathrm{IET}(\Gamma)$ pour Γ infini. Que peut-on tirer de la preuve originale ? Celle-ci utilise le fait qu'on peut couper un intervalle en son milieu. Ainsi avec la condition $\tilde{\Gamma} = 2\tilde{\Gamma}$ on peut toujours couper un intervalle en son milieu et ainsi il est possible d'adapter la preuve originale pour obtenir que φ induit un isomorphisme :

$$\mathrm{IET}(\Gamma)_{\mathrm{ab}} \simeq \bigwedge_{\mathbb{Z}}^{2} \tilde{\Gamma}.$$

Quand cette condition n'est pas satisfaite on ne peut plus forcément couper un intervalle en son milieu. Pour Γ quelconque on a encore un morphisme surjectif

$$\mathrm{IET}(\Gamma)_{\mathrm{ab}} \to \bigwedge_{\mathbb{Z}}^{2} \tilde{\Gamma}$$

mais il n'est pas forcément injectif ; son noyau est en fait un 2-groupe. Pour le décrire plus précisément, on introduit la seconde puissance anti-symétrique de $\tilde{\Gamma}$ au dessus de $\mathbb{Z}$ noté $^{\ominus}\!\bigwedge_{\mathbb{Z}}^{2} \tilde{\Gamma}$ qui est généralement différente de la seconde puissance extérieure $\bigwedge_{\mathbb{Z}}^{2} \tilde{\Gamma}$. Dans le chapitre 2 on prouve le résultat suivant :

Théorème 2 (Voir théorème 2.3.24). *Il existe un morphisme de groupes surjectif (explicite)* $\varepsilon_{\Gamma} : \mathrm{IET}(\Gamma) \to {}^{\ominus}\!\bigwedge_{\mathbb{Z}}^{2} \tilde{\Gamma}$ *dont le noyau est égal au sous-groupe dérivé* $D(\mathrm{IET}(\Gamma))$.

L'idée est de construire un morphisme de groupes qui mesure, pour tout $f \in \mathrm{IET}(\Gamma)$, son ensemble d'inversions : $\{(x, y) \mid x \leq y, \ f(x) \geq f(y)\}$. Il est inspiré du morphisme signature des groupes de permutations finis. Nous obtenons alors un morphisme de groupes qui est la "moitié" du SAF-invariant. Pour cela on considère une algèbre de Boole noté A_{Γ} telle que l'ensemble des inversions appartient à $A_{\Gamma} \otimes A_{\Gamma}$. On construit aussi une mesure sur l'algèbre de Boole $A_{\Gamma} \otimes A_{\Gamma}$, à valeurs dans $\bigotimes_{\mathbb{Z}}^{2} \tilde{\Gamma}$, pour mesurer l'ensemble des inversions.

Dans le cas où $\tilde{\Gamma}$ est un groupe abélien libre de rang d, on obtient que l'abélianisé de $\mathrm{IET}(\Gamma)$ est naturellement isomorphe à $\mathbb{Z}^{\frac{d(d-1)}{2}} \times (\mathbb{Z}/2\mathbb{Z})^{d}$.

Arnoux [Arn81b] a prouvé que le groupe $\mathrm{IET}^{\bowtie}$ est simple et sa preuve utilise aussi le fait qu'on peut couper un intervalle en son milieu. On montre que si on a la condition $\tilde{\Gamma} = 2\tilde{\Gamma}$ alors $\mathrm{IET}^{\bowtie}(\Gamma)$ est toujours un groupe simple. Dans le cas général on montre le résultat suivant :

Théorème 3 (Voir théorème 3.4.5). *L'abélianisé* $\mathrm{IET}^{\bowtie}(\Gamma)_{\mathrm{ab}}$ *est naturellement isomorphe au groupe* $\langle \{a \otimes a \ [\mathrm{mod}\ 2] \mid a \in \tilde{\Gamma}\} \rangle \times \langle \{\ell \wedge \ell \ [\mathrm{mod}\ 2] \mid \ell \in \tilde{\Gamma}\} \rangle$, *où le terme de gauche du produit appartient à* $\bigotimes_{\mathbb{Z}}^{2} \tilde{\Gamma}/(2\bigotimes_{\mathbb{Z}}^{2} \tilde{\Gamma})$ *et le terme de droite à* $^{\ominus}\!\bigwedge_{\mathbb{Z}}^{2} \tilde{\Gamma}/(2^{\ominus}\!\bigwedge_{\mathbb{Z}}^{2} \tilde{\Gamma})$.

L'ensemble des inversions d'un élément de IET$^{\bowtie}(\Gamma)$ n'est plus à chaque fois dans $A_\Gamma \otimes A_\Gamma$ et ne peut donc plus être mesuré comme dans le cas de IET(Γ). On remarque alors qu'en considérant l'union de cet ensemble avec son symétrique (par rapport à l'axe $y = x$) on obtient un ensemble qui appartient toujours à $A_\Gamma \otimes A_\Gamma$. Cela donne un premier morphisme de groupes mais dont le noyau est plus gros que le sous-groupe dérivé $D(\text{IET}^{\bowtie}(\Gamma))$. La deuxième idée est d'"approximer", en un certain sense, les éléments de IET$^{\bowtie}(\Gamma)$ par des éléments de IET(Γ). Cela nous permet d'utiliser le morphisme construit pour IET(Γ) pour obtenir un autre morphisme sur IET$^{\bowtie}(\Gamma)$. On prouve alors que l'intersection des noyaux des deux morphismes construits est égal au sous-groupe dérivé $D(\text{IET}^{\bowtie}(\Gamma))$.

3) Classe de Kapoudjian

Soit X un ensemble (pour la suite on aura $X = [0,1[$) et soit $\mathfrak{S}(X)$ le groupe de toutes les permutations de X. On note $\mathfrak{S}_{\text{fin}} = \mathfrak{S}_{\text{fin}}(X)$ son sous-groupe des permutations à support fini. On rappelle ici qu'avec la même preuve utilisée pour les groupes de permutations finis, on peut construire un morphisme de groupes surjectif de $\mathfrak{S}_{\text{fin}}$ sur $\mathbb{Z}/2\mathbb{Z}$ que l'on appelle la signature classique. Son noyau, noté $\mathfrak{A}_{\text{fin}}$, est l'unique sous-groupe d'indice 2 de $\mathfrak{S}_{\text{fin}}$; il est appelé le sous-groupe alterné.

On définit $\widehat{\text{PC}^{\bowtie}}$ le sous-groupe de $\mathfrak{S}(X)$ constitué des transformations qui sont continues en dehors d'un nombre fini de points et on note $\text{PC}^{\bowtie}$ sa projection dans $\mathfrak{S}(X)/\mathfrak{S}_{\text{fin}}$. Ainsi pour tout sous-groupe G de $\mathfrak{S}(X)/\mathfrak{S}_{\text{fin}}$ on note $\widehat{G}$ sa pré-image dans $\mathfrak{S}(X)$.

Dans cette partie, il est important de faire la distinction entre les groupes continus par morceaux (sous-groupes de $\widehat{\text{PC}^{\bowtie}}$) et leur projection dans $\text{PC}^{\bowtie}$. En effet, dans [Cor19b], Y. Cornulier a prouvé que le groupe $\text{PC}^{\bowtie}$ ne se relève pas dans le groupe des permutations de l'intervalle $[0,1[$. Cela motive alors sa question sur l'annulation d'un élément spécifique du second groupe de cohomologie appelé la classe de Kapoudjian ; en effet la non-annulation de cette classe impliquerait le résultat de non-relèvement ci-dessus. Cette classe apparaît dans le travail de Kapoudjian et Sergiescu [Kap02, KS05] où ils considèrent la suite exacte suivante :

$$0 \to \mathbb{Z}/2\mathbb{Z} = \mathfrak{S}_{\text{fin}}/\mathfrak{A}_{\text{fin}} \to \mathfrak{S}/\mathfrak{A}_{\text{fin}} \to \mathfrak{S}(X)/\mathfrak{S}_{\text{fin}} \to 1$$

De plus, pour tout sous-groupe G de $\mathfrak{S}(X)/\mathfrak{S}_{\text{fin}}$ on obtient la suite exacte suivante :

$$0 \to \mathbb{Z}/2\mathbb{Z} = \mathfrak{S}_{\text{fin}}/\mathfrak{A}_{\text{fin}} \to \widehat{G}/\mathfrak{A}_{\text{fin}} \to G \to 1$$

Cette suite exacte correspond en fait à une extension centrale et donc à un élément du second groupe de cohomologie $H^2(G, \mathbb{Z}/2\mathbb{Z})$ appelé la classe de Kapoudjian de G.

Dans le chapitre 4, on répond à la question posée par Cornulier [Cor19b] en prouvant l'annulation de la classe de Kapoudjian pour $\text{PC}^{\bowtie}$. La première chose à noter est que cette question est équivalente à la construction d'un morphisme de groupes sur $\widehat{\text{PC}^{\bowtie}}$ qui étend la signature classique. C'est le théorème suivant :

Théorème 4 (Voir théorème 4.1.2). *Il existe un morphisme de groupes $\varepsilon : \widehat{\text{PC}^{\bowtie}} \to \mathbb{Z}/2\mathbb{Z}$ qui étend la signature classique de $\mathfrak{S}_{\text{fin}}$.*

Ainsi la classe de Kapoudjian de $\text{PC}^{\bowtie}$ est égale à la classe triviale et cela implique le même résultat pour tous les sous-groupes de $\text{PC}^{\bowtie}$. L'idée derrière ce théorème est d'écrire tous les éléments $f \in \widehat{\text{PC}^{\bowtie}}$ comme un produit d'un élément de IET avec un produit de renversements et une permutation à support fini. En imposant un certain paramètre on arrive à avoir une telle décomposition qui est unique et on prouve que la somme modulo 2 du nombre de renversements apparaissant dans la décomposition avec la signature classique de la permutation à support fini est un nombre invariant de ce paramètre. Il reste alors à prouver que l'on a bien un morphisme de groupes.

Une application de ce résultat est la description complète des sous-groupes normaux de certains sous-groupes de $\widehat{\mathrm{PC}^{\bowtie}}$:

Théorème 5 (Voir théorème 4.1.4). *Soit G un sous-groupe de $\mathrm{PC}^{\bowtie}$ qui est simple et non abélien. Soit $\widehat{G}$ sa pré-image dans $\widehat{\mathrm{PC}^{\bowtie}}$. Alors $\widehat{G}$ a exactement 5 sous-groupes normaux, qui forment le diagramme suivant :*

$$
\begin{array}{ccc}
 & \widehat{G} & \\
 \diagup & & \diagdown \\
 \mathfrak{S}_{\mathrm{fin}} & & \mathrm{Ker}(\varepsilon|_{\widehat{G}}) \\
 \diagdown & & \diagup \\
 & \mathfrak{A}_{\mathrm{fin}} & \\
 & | & \\
 & \{1\} & \\
\end{array}
$$
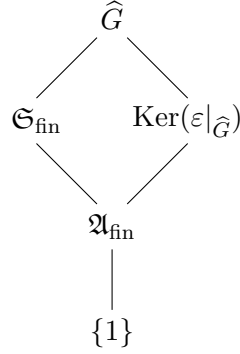
Un point commun entre cette partie et la précédente est la construction d'un morphisme de groupes à valeurs dans un 2-groupe abélien élémentaire. Toutefois, une différence importante est que dans la partie précédente, le morphisme est construit sur le groupe $\mathrm{IET}^{\bowtie}(\Gamma)$. En comparaison, le morphisme est ici défini sur une pré-image de $\mathrm{IET}^{\bowtie}$, et montre qu'un élément naturellement donné du groupe $H^2(\mathrm{IET}^{\bowtie}, \mathbb{Z}/2\mathbb{Z})$ s'annule.

4) Echanges de rectangles

Une manière de généraliser le groupe IET en dimension d pour $d \geq 1$ est de remplacer l'intervalle $[0,1[$ par le rectangle $[0,1[^d$. On peut alors considérer le groupe Rec_d des bijections f de $[0,1[^d$ telles qu'il existe une partition finie en sous-rectangles de $[0,1[^d$ telle que f est une translation sur tous les rectangles de cette partition. On remarque alors que pour $d = 1$ on a l'égalité entre Rec_1 et IET.
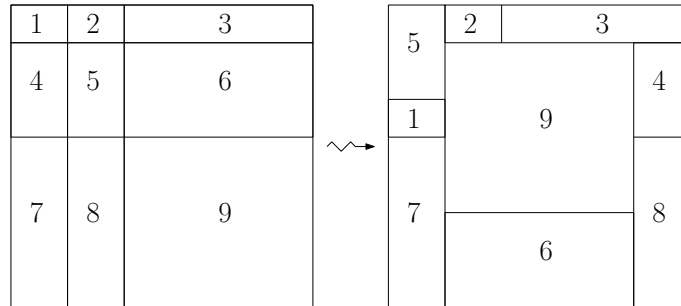
Un élément de Rec_2.

Dans le chapitre 5, on commence l'étude du groupe Rec_d par donner un analogue des rotations restreintes appelé battage restreint (voir définition 5.1.3).

On prouve le théorème suivant :

Théorème 6 (Voir théorème 5.1.5). *L'ensemble des battages restreints est un système générateur de Rec_d.*

En dimension 1, on peut interpreter un échange d'intervalle comme une permutation de segments et le résultat se ramène au fait que le groupe symétrique $\mathcal{S}_n$ est engendré par les transpositions $(i, i+1)$. En dimension supérieure ou égale à 2, on a pas une interprétation

combinatoire si simple : par exemple il n'existe pas forcément de quadrillage envoyé sur un autre quadrillage par translation des pièces. L'idée est alors de démontrer qu'avec certaines conditions sur une partition il est possible de la transformer en quadrillage grâce à des battages restreints. Ces conditions portent sur de la $\mathbb{Q}$-liberté de sous-ensembles de $\mathbb{R}$ qui sont liés aux longueurs des côtés des pièces de la partition.

Après cela, on poursuit avec l'étude du sous-groupe dérivé $D(\mathrm{Rec}_d)$. On dit qu'un élément $f \in \mathrm{Rec}_d$ est une transposition de rectangles s'il existe un sous-rectangle P de $[0, 1[^d$ tel que $P \cap f(P) = \emptyset$ et f permute P avec $f(P)$ en fixant le reste de $[0, 1[^d$ (voir définition 5.1.3) ; on note par $\mathscr{T}_d$ l'ensemble de toutes les transpositions de rectangles et on prouve le résultat suivant :

Théorème 7 (Voir théorème 5.1.6). *Le sous-groupe dérivé $D(\mathrm{Rec}_d)$ est simple et est engendré par son sous-ensemble $\mathscr{T}_d$. De plus il est contenu dans tous les sous-groupes normaux non triviaux de Rec_d.*

Pour le cas $d = 1$, la simplicité est due à Sah [Sah81] (non-publié) et le résultat d'engendrement s'en déduit facilement. Une preuve détaillée est due à Arnoux [Arn81b], une preuve plus simple est presentée par Vorobets dans [Vor17]. Ce théorème repose en partie sur le théorème qui le précède. En effet on déduit de ce dernier que le groupe $D(\mathrm{Rec}_d)$ est engendré par les conjugés des commutateurs de deux battages restreints.

Le dernier théorème du chapitre 5 est une généralisation du théorème d'Arnoux-Fathi-Sah [Arn81a, Sah81] à propos de l'identification de l'abélianisé de IET. On note $\mathbb{R}^{\otimes k}$ la k-ième puissance tensorielle de $\mathbb{R}$ au dessus de $\mathbb{Q}$.

Théorème 8 (Voir théorème 5.7.21). *Le groupe $(\mathrm{Rec}_d)_{\mathrm{ab}}$ est naturellement isomorphe à $(\mathbb{R}^{\otimes d-1} \otimes (\bigwedge_{\mathbb{Q}}^2 \mathbb{R}))^d$.*

L'idée est que chaque direction a une contribution qui est indépendante des autres directions et que cette contribution est calculée grâce à un analogue du morphisme construit par Arnoux-Fathi-Sah. Un exemple d'application est que le groupe $\langle \mathrm{IET}^d \cup \mathscr{T}_d \rangle \neq \mathrm{Rec}_d$ si $d \geq 2$. En effet, on vérifie que la restriction de l'isomorphisme $(\mathbb{R}^{\otimes d-1} \otimes (\bigwedge_{\mathbb{Q}}^2 \mathbb{R}))^d$ n'est pas surjective.

This thesis focuses on the study of some groups with dynamical origins by looking at the abelianization, it is the same as classifying their additive invariants (in other words, group homomorphisms from these groups to abelian groups). It has been realized under the supervision of Yves Cornulier between October 2018 and June 2021. More precisely, we deal with groups which act piecewise isometrically on an interval. An important example of such group is the group of interval exchange transformations denoted by IET, consisting of piecewise translations.

1) Brief historical account.

The elements of IET were introduced by M. Keane [Kea75] in 1975 and it is mostly their dynamics and ergodic properties which have been studied. The study of the group itself was initiated by Arnoux-Fathi [Arn81a] and Sah [Sah81] by determinating its abelianization. In the past fifteen years, this study has been pursued and rises our understanding of IET's subgroups, notably with the work of C. Novak [Nov09], Dahmani-Fujiwara-Guirardel [DFG13, DFG17], Boshernitzan [Bos16] and Vorobets [Vor17]. These works contain some progress about the question, due to Katok, of the existence of a free subgroup of rank 2 in IET. The related question of the amenability of IET has been raised by Y. Cornulier and here also, some progress has been made by Juschenko-Monod [JM13] and further the two of them with Matte Bon and de la Salle [JMMS18].
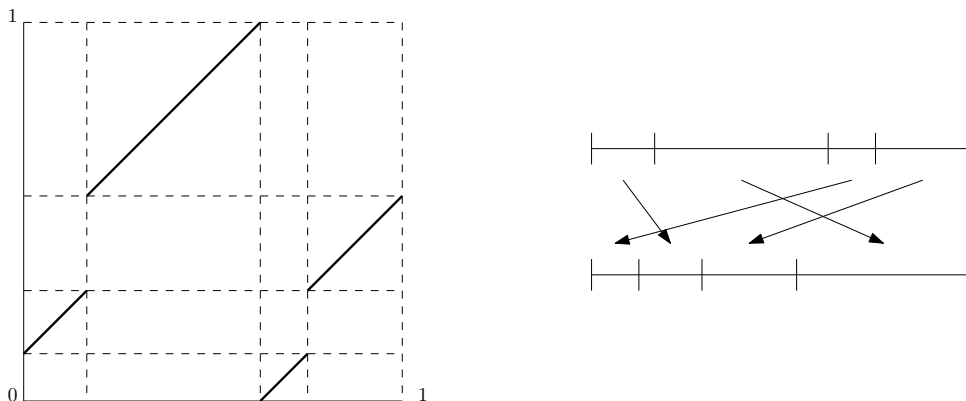
Left: A graph of an element of IET (without specifying values at discontinuity points). **Right**: Its "action" on $[0, 1[$.

If one allows all piecewise isometric transformations, we obtain the interval exchange

transformations group with flips, denoted IET$^{\bowtie}$. This group has strong connections with non oriented measured foliations on surfaces and billiards and the study of the dynamical and ergodic properties of a single element has interested Gutierrez [Gut78], Arnoux [Arn81b], Nogueira [Nog89] and Danthony-Nogueira [DN90] and more recently by O. Paris-Romaskevich and P. Hubert [PRH18] or Skripchenko-Troubetzkoy [ST18]. Few things are known about the group itself, Arnoux [Arn81b] proved that it is a simple group and N. Guelman and I. Liousse [GL19a] proved that this group is uniformly perfect (every element is a product of 6 commutators).
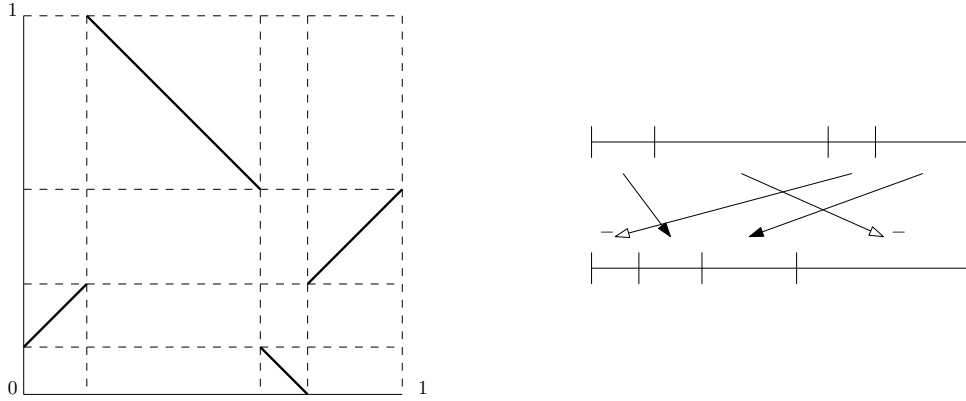
Left: A graph of an element of IET$^{\bowtie}$ that contains flips (without specifying values at discontinuity points). **Right**: Its "action" on $[0, 1[$.

The group of piecewise continuous and affine transformations appears in different works because of the natural inclusion of Thompson's groups. It has been initially defined by M. Stein [Ste92] and recently N. Guelman and I. Liousse [GL19a] proved that its subgroup consisting of piecewise orientation-preserving elements is simple.

The whole group of piecewise continuous elements, denoted by PC$^{\bowtie}$, has been barely studied. Arnoux [Arn81b] proved that its subgroup PC$^+$, consisting of piecewise order-preserving transformations, is a simple group.
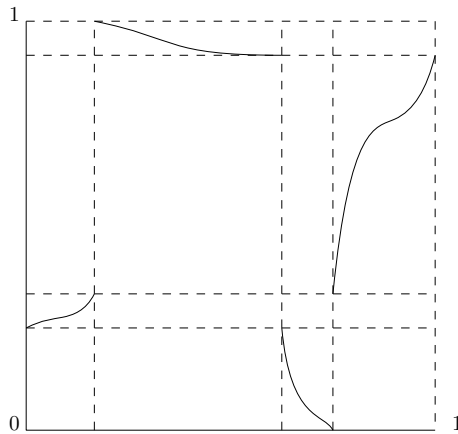
A graph of an element of PC$^{\bowtie}$ (without specifying values at discontinuity points).

2) Abelianization of IET(Γ) and IET$^{\bowtie}(\Gamma)$

Chapter 1 introduces all the groups considered in the thesis.

We describe subfamilies which play a main role in the study of subgroups of PC$^{\bowtie}$, for

instance the set of all restricted rotations (see Definition 1.2.7) is a generating subset of IET.

We also introduce a family of subgroups of IET and IET$^{\bowtie}$ which were the starting point of the thesis. For every subgroup Γ of $\mathbb{R}/\mathbb{Z}$, we define IET(Γ) (resp. IET$^{\bowtie}(\Gamma)$) as the subgroup of IET (resp. IET$^{\bowtie}$) consisting of all elements that are continuous outside the preimage $\tilde{\Gamma}$ of Γ in $\mathbb{R}$. The identification of their abelianization was the first problem of my thesis. The case where Γ is finite can be reduced to the study of a finite permutation group and thus we already well-understand its derived subgroup and abelianization. For $\tilde{\Gamma} = \mathbb{R}$, Arnoux-Fathi and Sah prove the following theorem:

Theorem 1 (Arnoux-Fathi-Sah [Arn81a, Sah81])**.** *There exists an explicit surjective group homomorphism $\varphi : \mathrm{IET} \to \bigwedge^2_{\mathbb{Q}} \mathbb{R}$ whose kernel is equal to the derived subgroup $D(\mathrm{IET})$. It is called the SAF-invariant.*

Here we try to understand the restriction of this morphism to subgroups IET(Γ) for Γ infinite. What can we deduce from the original proof? This proof uses the fact that we can cut an interval at its middle point, then with the condition $\tilde{\Gamma} = 2\tilde{\Gamma}$ it is still possible to cut an interval at its middle point thus it is possible to adapt the original proof to obtain that φ induces an isomorphism

$$\mathrm{IET}(\Gamma)_{\mathrm{ab}} \simeq \bigwedge^2_{\mathbb{Z}} \tilde{\Gamma}.$$

This is no longer true when this condition is not satisfied. For a general Γ, we still have a sujective group homomorphism

$$\mathrm{IET}(\Gamma)_{\mathrm{ab}} \to \bigwedge^2_{\mathbb{Z}} \tilde{\Gamma}$$

but it is not always injective; its kernel is in fact a 2-group. In order to describe it more precisely, we introduce the second skew-symmetric power of $\tilde{\Gamma}$ over $\mathbb{Z}$ denoted by $^{\ominus}\bigwedge^2_{\mathbb{Z}} \tilde{\Gamma}$ which is generally different from the second exterior power $\bigwedge^2_{\mathbb{Z}} \tilde{\Gamma}$. In Chapter 2 we prove the following:

Theorem 2 (See Theorem 2.3.24)**.** *There exists an explicit surjective group homomorphism $\varepsilon_{\Gamma} : \mathrm{IET}(\Gamma) \to {}^{\ominus}\bigwedge^2_{\mathbb{Z}} \tilde{\Gamma}$ whose kernel is equal to the derived subgroup $D(\mathrm{IET}(\Gamma))$.*

The idea is to construct a group homomorphism which for every $f \in \mathrm{IET}(\Gamma)$ measures the set of inversions: $\{(x, y) \mid x \leq y, \ f(x) \geq f(y)\}$. It is inspired by the signature group homomorphism for finite permutation groups. This brings us to obtain a group homomorphism that is the "half" of the SAF-invariant. For this we consider a Boolean algebra denoted by A_{Γ} such that the set of inversions belongs to $A_{\Gamma} \otimes A_{\Gamma}$. We construct a measure on the Boolean algebra $A_{\Gamma} \otimes A_{\Gamma}$, with value in $\bigotimes^2_{\mathbb{Z}} \tilde{\Gamma}$, to measure the set of inversions.

In the case where $\tilde{\Gamma}$ is an abelian free group of rank d, we obtain that the abelianization of IET(Γ) is naturally isomorphic to $\mathbb{Z}^{\frac{d(d-1)}{2}} \times (\mathbb{Z}/2\mathbb{Z})^d$.

Arnoux [Arn81b] proved that the group IET$^{\bowtie}$ is simple and its proof uses the fact that we can cut an interval at its middle point. The proof then extends to if $\tilde{\Gamma} = 2\tilde{\Gamma} \neq \{0\}$ then IET$^{\bowtie}(\Gamma)$ is simple. In the general case we prove the following theorem:

Theorem 3 (See Theorem 3.4.5)**.** *The abelianization group IET$^{\bowtie}(\Gamma)_{\mathrm{ab}}$ is naturally isomorphic to the group $\langle \{a \otimes a \ [\mathrm{mod} \ 2] \mid a \in \tilde{\Gamma}\} \rangle \times \langle \{\ell \wedge \ell \ [\mathrm{mod} \ 2] \mid \ell \in \tilde{\Gamma}\} \rangle$, where the left term of the product is in $\bigotimes^2_{\mathbb{Z}} \tilde{\Gamma}/(2 \bigotimes^2_{\mathbb{Z}} \tilde{\Gamma})$ and the right one is in $^{\ominus}\bigwedge^2_{\mathbb{Z}} \tilde{\Gamma}/(2^{\ominus}\bigwedge^2_{\mathbb{Z}} \tilde{\Gamma})$.*

The set of inversions of an element of IET$^{\bowtie}(\Gamma)$ is not always in $A_{\Gamma} \otimes A_{\Gamma}$. But, if we considered the union of this set with its mirror (its symmetric image according to the axis $y = x$) we obtain a new set that belongs to $A_{\Gamma} \otimes A_{\Gamma}$. This gives a first group homomorphism

but its kernel is larger than the derived subgroup $D(\text{IET}^{\bowtie}(\Gamma))$. The second idea here is to "approximate" in some sense an element of $\text{IET}^{\bowtie}(\Gamma)$ by elements of $\text{IET}(\Gamma)$. From this we manage to construct a second group homomorphism and we prove that the intersection of their kernels is equal to the derived subgroup $D(\text{IET}^{\bowtie}(\Gamma))$.

3) Kapoudjian class

Let X be a set (in the following we will have $X = [0, 1[$) and let $\mathfrak{S}(X)$ be the group of all permutations of X. Let $\mathfrak{S}_{\text{fin}} = \mathfrak{S}_{\text{fin}}(X)$ be its subgroup consisting of all finitely supported permutations. We recall that with the same proof used for finite permutation groups, we can construct a surjective group homomorphism called the classical signature from $\mathfrak{S}_{\text{fin}}$ onto $\mathbb{Z}/2\mathbb{Z}$. Its kernel, denoted by $\mathfrak{A}_{\text{fin}}$, is the alternating subgroup and it is the unique subgroup of index 2 of $\mathfrak{S}_{\text{fin}}$.

We denote by $\widehat{\text{PC}^{\bowtie}}$ the subgroup of $\mathfrak{S}(X)$ consisting of all piecewise continuous transformations and we denote by $\text{PC}^{\bowtie}$ its projection in $\mathfrak{S}(X)/\mathfrak{S}_{\text{fin}}$. Then for every subgroup G of $mfS(X)/\mathfrak{S}_{\text{fin}}$ we define $\widehat{G}$ its preimage in $\mathfrak{S}(X)$.

In this part, it is important here to distinguish between piecewise continuous groups (subgroups of $\widehat{\text{PC}^{\bowtie}}$) and their image in $\text{PC}^{\bowtie}$. Indeed, in [Cor19b], Y. Cornulier proved that the group $\text{PC}^{\bowtie}$ cannot be lifted to a group of permutations of the interval $[0, 1[$. This motivates his question on the vanishing of a specific element of the second cohomology group called the Kapoudjian class; precisely the vanishing of this class implies the above nonlifting result. This class appears in the work of Kapoudjian and Sergiescu [Kap02, KS05] where they considered the following exact sequence:

$$0 \to \mathbb{Z}/2\mathbb{Z} = \mathfrak{S}_{\text{fin}}/\mathfrak{A}_{\text{fin}} \to \mathfrak{S}/\mathfrak{A}_{\text{fin}} \to \mathfrak{S}(X)/\mathfrak{S}_{\text{fin}} \to 1$$

For every subgroup G of $\mathfrak{S}(X)/\mathfrak{S}_{\text{fin}}$, we obtain similarly the following exact sequence:

$$0 \to \mathbb{Z}/2\mathbb{Z} = \mathfrak{S}_{\text{fin}}/\mathfrak{A}_{\text{fin}} \to \widehat{G}/\mathfrak{A}_{\text{fin}} \to G \to 1$$

This exact sequence is in fact a central extension and thus can be seen as an element of the second cohomology group $H^2(G, \mathbb{Z}/2\mathbb{Z})$ called the Kapoudjian class of G.

In Chapter 4, we answer the question of Cornulier [Cor19b] by proving that the Kapoudjian class of $\text{PC}^{\bowtie}$ vanishes. The first thing to notice is that this question is equivalent to the construction of a group homomorphism from $\widehat{\text{PC}^{\bowtie}}$ which extends the classical signature on $\mathfrak{S}_{\text{fin}}$, it is the following theorem:

Theorem 4 (See Theorem 4.1.2). *There exists a group homomorphism $\varepsilon : \widehat{\text{PC}^{\bowtie}} \to \mathbb{Z}/2\mathbb{Z}$ that extends the classical signature on $\mathfrak{S}_{\text{fin}}$.*

Hence the Kapoudjian class of $\text{PC}^{\bowtie}$ vanishes and this implies the result for every subgroup of $\text{PC}^{\bowtie}$. The idea behind this theorem is to write every $f \in \widehat{\text{PC}^{\bowtie}}$ into a product of an element of IET with a product of flips and a finitely supported permutation. Up to a parameter we manage to get that this decomposition is unique and we prove that the sum modulo 2 of the number of flips in the decomposition with the signature of the finitely supported permutation is independent of this parameter. It remains to prove that it is a group homomorphism.

An application of this result is the complete description of normal subgroups of some subgroups of $\widehat{\text{PC}^{\bowtie}}$:

Theorem 5 (see Theorem 4.1.4). *Let G be a subgroup of $\text{PC}^{\bowtie}$ which is simple nonabelian and let $\widehat{G}$ be its preimage in $\mathfrak{S}$. Then $\widehat{G}$ has exactly 5 normal subgroups, which constitute the following diagram:*
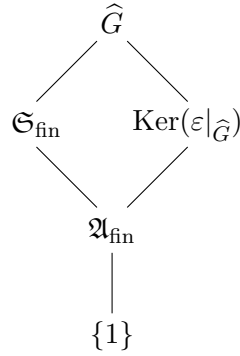
One common point between this part the previous one is the construction of a homomorphism into an elementary abelian 2-group. An important difference, however, is that such homomorphism was constructed in the group $\mathrm{IET}^{\bowtie}(\Gamma)$. In contrast, here the homomorphism is defined on a certain preimage of $\mathrm{IET}^{\bowtie}$, and actually shows that some given natural element of $H^2(\mathrm{IET}^{\bowtie}, \mathbb{Z}/2\mathbb{Z})$ vanishes.

4) Rectangle exchanges

A way to generalize the group IET in dimension d for every $d \geq 1$, is to replace the interval $[0,1[$ by the rectangle $[0,1[^d$. Then we can consider the group Rec_d consisting of all bijections f of $[0,1[^d$ such that there exists a finite partition into subrectangles of $[0,1[^d$ such that f is a translation on every rectangle of this partition. Then for $d = 1$ we have the equality $\mathrm{Rec}_1 = \mathrm{IET}$.
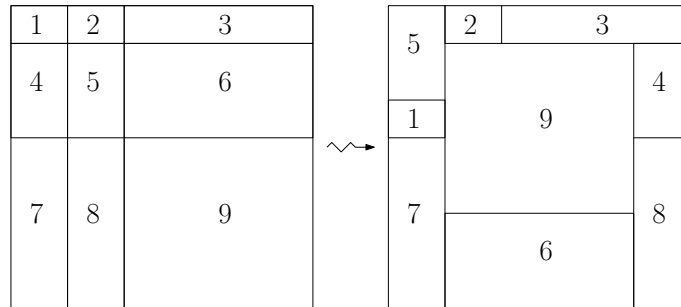
An element of Rec_2.

In Chapter 5, we begin the study of Rec_d by giving an analogue of a restricted rotation called restricted shuffle (see Definition 5.1.3) and proving the following:

Theorem 6 (See Theorem 5.1.5). *The set of all restricted shuffles is a generating subset of* Rec_d.

In dimension 1, we can interpret an interval exchange transformation as a permutation of segments and the result reduces to the fact that the symmetric group $\mathfrak{S}_n$ is generated by transpositions $(i, i+1)$. In dimension superior or equal to 2 , there is not such an easy combinatorial description: for instance there does not always exist a grid-pattern sent onto another grid-pattern by translation of the pieces. The idea is to prove that, with some conditions on a partition, it is possible de move pieces of this partition, thanks to restricted shuffles, in order to obtain a grid-pattern. These conditions are related to the $\mathbb{Q}$-freeness of some subsets of $\mathbb{R}$ which are linked with the length of the side of pieces of the partition.

After that we pursue the study of the derived subgroup $D(\mathrm{Rec}_d)$ by giving a generating subset of the derived subgroup. We called an element $f \in \mathrm{Rec}_d$ a rectangle transposition if there exists a subrectangle P of $[0,1[^d$ such that $P \cap f(P) = \emptyset$ and f permutes P with

$f(P)$ while fixing the rest of $[0, 1[^d$ (see Definition 5.1.3); we denote by $\mathscr{T}$ the set of all rectangle transpositions and we prove the following theorem:

Theorem 7 (See theorem 5.1.6)**.** *The derived subgroup $D(\mathrm{Rec}_d)$ is simple and is generated by its subset $\mathscr{T}_d$. It is contained in every nontrivial normal subgroup of Rec_d.*

For the case $d = 1$, the simplicity is due to Sah [Sah81] (unpublished) and the result of generation is deduced easily. A detailed proof is due to Arnoux [Arn81b], and a simpler one is presented by Vorobets in [Vor17]. This theorem relies on the previous one because we obtain that $D(\mathrm{Rec}_d)$ is generated by conjugates of commutators of two restricted shuffles.

The last theorem of Chapter 5 is a generalization of the theorem of Arnoux-Fathi-Sah [Arn81a, Sah81] about the identification of the abelianization of IET. We denote by $\mathbb{R}^{\otimes k}$ the k-th tensor power of $\mathbb{R}$ over $\mathbb{Q}$.

Theorem 8 (See Theorem 5.7.21)**.** *The group $(\mathrm{Rec}_d)_{\mathrm{ab}}$ is naturally isomorphic to $(\mathbb{R}^{\otimes d-1} \otimes (\bigwedge_{\mathbb{Q}}^2 \mathbb{R}))^d$.*

The idea is that every direction has a contribution which is independent from other directions. This contribution is calculated thanks to an analogue of the group homomorphism constructed by Arnoux-Fathi-Sah [Arn81a, Sah81]. An application is that the group $\langle \mathrm{IET}^d \cup \mathscr{T}_d \rangle \neq \mathrm{Rec}_d$ if $d \geq 2$. Indeed, we check that the restriction of the isomorphism $(\mathbb{R}^{\otimes d-1} \otimes (\bigwedge_{\mathbb{Q}}^2 \mathbb{R}))^d$ is not surjective.

Piecewise continuous groups

Résumé en français. Dans ce chapitre nous construisons les groupes continus par morceaux. On donne aussi des systèmes générateurs de ces groupes ainsi que de leur sousgroupe dérivé. On présente des résultats connus qui sont des préliminaires pour les autres chapitres.

English abstract. This chapter contains a formal definition for piecewise continuous groups. We define subfamilies which allow us to have generating subset for these groups or their derived subgroups. Also, we give known results about these groups that are preliminaries for next chapters.

Contents

1.1 Definitions

1.1.1 Construction

The group of interval exchanges IET can be easily define as the group of all bijections of $[0, 1[$ that are continuous outside a finite set, right-continuous and piecewise a translation. This implies that we always send a right-open and left-closed interval to another. However if we want to allow flips then a right-open and left-closed interval should be sent onto a right-closed and left-open interval. Thus the definition of $\text{IET}^{\bowtie}$ is not immediate. One way to deal with this is to considered such elements up to a finitely supported permutation. The following is the formal construction to do this.

We denote by $\mathfrak{S}([0, 1[)$ the group of bijections of $[0, 1[$ to $[0, 1[$. This group contains a natural normal subgroup denoted by $\mathfrak{S}_{\text{fin}}$ consisting of all finitely supported permutations. For every subgroup G of $\mathfrak{S}([0, 1[)/\mathfrak{S}_{\text{fin}}$, we denote by $\widehat{G}$ its preimage in $\mathfrak{S}([0, 1[)$. We use this notation because our interests is on subgroups of $\mathfrak{S}([0, 1[)/\mathfrak{S}_{\text{fin}}$, also it is easier to define subgroups of $\mathfrak{S}([0, 1[)$.

Definition 1.1.1. We denote by $\widehat{\text{PC}^{\bowtie}}(X)$ the subgroup of $\mathfrak{S}([0, 1[$ consisting of all elements that are continuous outside a finite set. Its subgroup consisting of all piecewise orientation-preserving transformations is denoted $\widehat{\text{PC}^{+}}$.

The group $\widehat{\text{PC}^{\bowtie}}$ contains the subgroup $\mathfrak{S}_{\text{fin}}$ thus the quotient $\text{PC}^{\bowtie} = \widehat{\text{PC}^{\bowtie}}/\mathfrak{S}_{\text{fin}}$ is well-defined.

Definition 1.1.2. We denote by $\widehat{\text{IET}^{\bowtie}}$ the subgroup of $\mathfrak{S}([0, 1[)$ consisting of all elements that are continuous outside a finite set and piecewise isometric. We also define $\widehat{\text{IET}^{+}}$ as the intersection $\widehat{\text{IET}^{\bowtie}} \cap \widehat{\text{PC}^{+}}$.

Then we notice that the group IET^{+} is isomorphic to IET thus we freely use either both notations. The group $\text{IET}^{\bowtie}$ is the group of interval exchanges with flips up to a finitely supported permutation.

Remark 1.1.3. Here the "$+$" means that we preserve locally the order of $[0, 1[$. We can also consider PC^{-} the subset of $\text{PC}^{\bowtie}$ consisting of piecewise-reversing elements. It is not a group however if we consider $\text{PC}^{\pm} = \text{PC}^{+} \sqcup \text{PC}^{-}$ we obtain a proper subgroup of $\text{PC}^{\bowtie}$. Then PC^{+} is a subgroup of index 2 in $\text{PC}^{\pm}$. The same remark can be done by defining similarly IET^{-} and $\text{IET}^{\pm}$ and we obtain the following property:

Proposition 1.1.4. *The normalizer of* IET^{+} *in* $\text{IET}^{\bowtie}$ *is* $\text{IET}^{\pm}$. *The same is true when we replace* IET *by* PC.

Proof. Let g be an element of $\widehat{\text{IET}^{\bowtie}}$ which is not the identity. There are two cases:

1. If $g \in \widehat{\text{IET}^{+}} \smallsetminus \text{IET}$ then $g = \sigma g'$ with $\sigma \in \mathfrak{S}_{\text{fin}} \smallsetminus \{\text{Id}\}$ and $g' \in \text{IET}$. Then for every $f \in \text{IET}$ we have $gfg^{-1} = \sigma g'fg'^{-1}\sigma^{-1}$. Thus it is enough to treat the case of $\mathfrak{S}_{\text{fin}}$. Let us assume $g \in \mathfrak{S}_{\text{fin}}$ then let x in the support of g. There exist two consecutive right-open and left-closed intervals I and J of the same length such that x is the right endpoint of I (and the left endpoint of J). Up to reduce I and J we can assume that I does not intersect the support of g. Then let $f \in \text{IET}$ which exchanges I and J while fixing the rest of $[0, 1[$. Then gfg^{-1} exchanges the interior of I with the interior of J but $gfg^{-1}(x)$ is not equal to $f(x)$ because $f(x)$ is the left endpoint of I and I does not intersect the support of g. Then we deduce that gfg^{-1} is not right-continuous on J.

2. If $g \in \widehat{\mathrm{IET}^{\bowtie}} \setminus \widehat{\mathrm{IET}^+}$. Then we can find two consecutive subinterval I and J where g is continuous and order-reversing on $I \cup J$. Let a be the right endpoint of J. Let f be the element in IET which exchanges I and J. Then gfg^{-1} exchanges the interior of $g(J)$ with the interior of $g(I)$. However the left endpoint of $g(J)$ is send by g^{-1} on a which is fixed by f. Then gfg^{-1} fixes the left endpoint of $g(J)$, thus gfg^{-1} is not right-continuous on $g(J)$.

$\square$

A natural family of subgroups of IET^+ and $\mathrm{IET}^{\bowtie}$ can be defined thanks to subgroups of $\mathbb{R}/\mathbb{Z}$. Let Γ be a subgroup of $\mathbb{R}/\mathbb{Z}$ and let $\tilde{\Gamma}$ be its preimage in $\mathbb{R}$.

Definition 1.1.5. We denote by $\widehat{\mathrm{IET}^{\bowtie}}(\Gamma)$ (resp. $\widehat{\mathrm{IET}^+}(\Gamma)$) the subgroup of $\widehat{\mathrm{IET}^{\bowtie}}$ (resp. $\widehat{\mathrm{IET}^+}$) consisting of all elements that are continuous outside $\tilde{\Gamma}$.

We describe below the case where Γ is finite. After that we always assume that the group Γ is dense in $\mathbb{R}/\mathbb{Z}$.

Remark when Γ is finite

We assume that Γ is a finite subgroup of $\mathbb{R}/\mathbb{Z}$. Then there exists $n \in \mathbb{N}_{\geq 1}$ such that $\tilde{\Gamma}$ is equal to $\frac{1}{n}\mathbb{Z}$. We deduce that for every $1 \leq i \leq n$, every element of $\widehat{\mathrm{IET}^{\bowtie}}(\Gamma)$ is continuous on the interval $[\frac{i-1}{n}, \frac{i}{n}[$ up to a finitely supported permutation.

Then the group $\mathrm{IET}(\Gamma)$ is naturally isomorphic to the finite permutation group $\mathfrak{S}_n$. It is a Coxeter group of type A_{n-1} so its abelianization is $\{1\}$ if $n = 1$ and it is $\mathbb{Z}/2\mathbb{Z}$ if $n > 1$.

The group $\mathrm{IET}^{\bowtie}(\Gamma)$ is isomorphic to the signed symmetric group $\mathbb{Z}/2\mathbb{Z} \wr \mathfrak{S}_n$. It is a Coxeter group of type B_n so its abelianization is $\mathbb{Z}/2\mathbb{Z}$ for $n = 1$ and it is $(\mathbb{Z}/2\mathbb{Z})^2$ if $n > 1$.

1.1.2 Topological-full groups

A way to understand groups is to understand their actions on some spaces. Here we explicit how piecewise continuous groups and their image in $\mathrm{PC}^{\bowtie}$ act on a Stone space.

Definition 1.1.6. A Stone space is a totally disconnected compact space.

For example every Cantor space is a Stone space. In this context we define topological-full groups.

Definition 1.1.7. Let X be a Stone space and G be a subgroup of $\mathrm{Homeo}(X)$. The group G is a topological-full group if for every n in $\mathbb{N}$, every $\{X_1, X_2, \ldots, X_n\}$ and $\{Y_1, Y_2, \ldots, Y_n\}$ two partitions into clopen subsets of X and for every g_i in G such that $g_i(X_i) = Y_i$; the element $g \in \mathrm{Homeo}(X)$ which satisfies $g = g_i$ on X_i is an element of G. The group G is said to be minimal if for every $x \in X$ the orbit of x is dense in X.

Topological-full groups have connection with groupoids and they are intensively studied. We do not explain more these notions in this thesis and for more background about it we redirect to the work of Matui [Mat06, Mat15], of Nekrashevych [Nek19] and the work of Matte Bon [MB18]. We explicit a result of Nekrashevych about the smallest normal subgroups appearing in minimal topological-full groups, adapted to our cases. It gives the simplicity of a lot of derived subgroups.

Theorem (Nekrashevych [Nek19]). *Let X be an infinite Stone space and let G be a subgroup of $\mathrm{Homeo}(X)$ such that G is a topological-full group acting minimally on X. Then the subgroup of G generated by the subset of all elements of order 3 is simple and contained in every normal subgroup of G.*

Next, we present two Stone spaces and we explain how $\widehat{\mathrm{PC}^{\bowtie}}$ and $\mathrm{PC}^{\bowtie}$ act on one of them.

Examples.   1. We denote by $\mathbf{S}^{(-,+)}$ the space obtained from $\mathbb{R}/\mathbb{Z}$ by "doubling" all points. We replace every point $x \in \mathbb{R}/\mathbb{Z}$ by two copies x^-, x^+ where $x^- < x^+$ and where we endow $\mathbf{S}^{(-,+)}$ with the topology induced by the natural circular order on it. For more clarity we think about $[0,1[$ as $\mathbf{S}^{(-,+)}$ where every interval $[a,b[$ has to be understand as the interval $[a^+, b^-]$ in $\mathbf{S}^{(-,+)}$. As we have the order topology then the set of every such intervals is a basis of the topology, hence the space is Hausdorff and totally disconnected. The compactness is a consequence of the compactness of $\mathbb{R}/\mathbb{Z}$. Indeed let I be a directed set and $\psi : I \to \mathbf{S}^{(-,+)}$ be a net. Let $p : \mathbf{S}^{(-,+)} \to \mathbb{R}/\mathbb{Z}$ be the natural projection. Then $\xi := p(\psi)$ is a net for $\mathbb{R}/\mathbb{Z}$ which is compact so there exists a cluster point $\alpha \in \mathbb{R}/\mathbb{Z}$ for ξ. By contradiction, assume that α^- and α^+ are not cluster points for ψ. Hence there exist $a \in I$ and U^-, U^+ open neighbourhoods of respectively α^- and α^+ such that for every $b \in I$ with $b \geq a$ we have $\psi(b) \notin U^- \cup U^+$. We deduce that there exists $w, x \in \mathbb{R}/\mathbb{Z}$ different from α such that $\alpha^-, \alpha^+ \in [w^+, x^-]$ and $\psi(b) \notin [w^+, x^-]$ for every $b \geq a$. Hence for every $b \geq a$ we have $\xi(b) = p(\psi(b)) \notin \,]w, x[\, \subset p([w^+, x^-])$. But $]w, x[$ is an open neighbourhood of α which is in contradiction with the fact that α is a cluster point for ξ. We deduce that either α^- or α^+ is a cluster point for ψ.

2. The second example is when we triple every point. We denote by $\mathbf{S}^{(-,\bullet,+)}$ the space obtained from $\mathbb{R}/\mathbb{Z}$ by "tripling" all points. We replace every point $x \in \mathbb{R}/\mathbb{Z}$ by three copies $x^-, x^\bullet, x^+$ where $x^- < x^\bullet < x^+$ and where we endow $\mathbf{S}^{(-,\bullet,+)}$ with the topology induced by the natural circular order on it. Here an $[a,b[$ has to be understand as $\{a^\bullet\} \sqcup [a^+, b^-]$ in $\mathbf{S}^{(-,\bullet,+)}$. Similar proof can be done to prove that it is a Stone space. We just point that $\{a^\bullet\}$ is also a clopen subset; thus $\mathbf{S}^{(-,\bullet,+)}$ is a Stone space with isolated points.

Then we can see that $\widehat{\mathrm{PC}^{\bowtie}}$ and its subgroups can also be seen as subgroups of $\mathrm{Homeo}(\mathbf{S}^{(-,\bullet,+)})$. For every $f \in \widehat{\mathrm{PC}^{\bowtie}}$ and for every $a \in [0,1[$ if f is continuous on a neighbourhood of a then if f is order-preserving on this neighbourhood we define $f(a^\varepsilon) = (f(a))^\varepsilon$ with $\varepsilon \in \{+,-\}$ and if f is order-reversing we define $f(a^\varepsilon) = (f(a))^{-\varepsilon}$. Also for every $a \in [0,1[$, we put $f(a^\bullet) = (f(a))^\bullet$.

Similarly, $\mathrm{PC}^{\bowtie}$ and its subgroups can be seen as subgroups of $\mathrm{Homeo}(\mathbf{S}^{(-,+)})$.

With this point of view it is immediate that every group defined with the symbol PC or IET are topological-full groups. However only those that are subgroups of $\mathrm{Homeo}(\mathbf{S}^{(-,+)})$ are minimal topological-full groups. Indeed for subgroups of $\mathrm{Homeo}(\mathbf{S}^{(-,\bullet,+)})$, the orbit of any point $a^\bullet$ with $a \in [0,1[$ is never dense.

1.2   Subfamilies and tools

We introduce the notion of partition associated to an element, this tool is really useful to describe how act an element. We pursue by describing different subfamilies and how they manage to describe some groups. After that we focus on the group IET by expliciting the decomposition into minimal and periodic elements done by Arnoux-Fathi [Arn81a]. We finish by some properties between elements of finite order and those of order 2. We denote by λ the Lebesgue measure on $\mathbb{R}$.

1.2.1   Partition associated and combinatorial description

From now on every partition into intervals of $[0,1[$ is supposed to be finite and into right-open and left-closed intervals. For every subinterval I of $\mathbb{R}$ we denote by I° its interior.

Definition 1.2.1. Let $\widehat{f}$ be an element of $\widehat{\mathrm{PC}^{\bowtie}}$ and $\mathcal{P}$ be a partition into intervals of $[0, 1[$. The partition $\mathcal{P}$ is said to be *a partition associated with* $\widehat{f}$ if for every interval $I \in \mathcal{P}$, the element $\widehat{f}$ is continuous on I°. We denote by $\Pi_{\widehat{f}}$ the set of all partitions associated with $\widehat{f}$. It is an *essential partition associated with* $\widehat{f}$ if there exists a finitely supported permutation σ such that $\mathcal{P}$ is a partition associated with $\sigma \widehat{f}$. For every essential partition associated with $\widehat{f}$ we denote by $\widehat{f}(P)$ the *arrival partition of* $\widehat{f}$ *associated with* $\mathcal{P}$ the partition into right-open and left-closed intervals of $[0, 1[$ such that for every $J \in f(\mathcal{P})$ there exists $I \in \mathcal{P}$ such that $J^{\circ} = \sigma \widehat{f}(I^{\circ})$.

It is necessary to define essential partitions in order to define partition associated with an element of $\mathrm{PC}^{\bowtie}$. Because two representatives have the same set of essential partitions associated but not the same set of partitions associated.

Definition 1.2.2. Let $f \in \mathrm{PC}^{\bowtie}$ and let $\widehat{f}$ be a representative of f in $\widehat{\mathrm{PC}^{\bowtie}}$. Let $\mathcal{P}$ be a partition into intervals of $[0, 1[$. The partition $\mathcal{P}$ is said to be *a partition associated with* f if it is an essential partition associated with $\widehat{f}$. We denote by Π_f the set of all partitions into intervals associated with f. We denote by $f(\mathcal{P})$ the *arrival partition of* f *associated with* $\mathcal{P}$ the arrival partition of $\widehat{f}$ associated with $\mathcal{P}$.

Let $\widehat{f} \in \widehat{\mathrm{PC}^{\bowtie}}$ and f be its image in $\mathrm{PC}^{\bowtie}$. We precise that the set Π_f is not equal to $\Pi_{\widehat{f}}$, it is more larger.

Remark 1.2.3. For every f in $\mathrm{PC}^{\bowtie}$ there exists a unique partition $\mathcal{P}_f^{\min}$ associated with f which has a minimal number of intervals. It is actually minimal in the sense of refinement: Π_f consists precisely of the set of partitions refining $\mathcal{P}_f^{\min}$. This partition is the partition define by the set of discontinuities of the unique representative of f in $\widehat{\mathrm{PC}^{\bowtie}}$ that has the minimal number of discontinuities.

Sometimes we will want to know where the length of the intervals live:

Definition 1.2.4. Let S be a subset of $\mathbb{R}$. Let $f \in \mathrm{PC}^{\bowtie}$ and $\mathcal{P}$ be a partition associated with f. The partition $\mathcal{P}$ is called a S-partition if for every $I \in \mathcal{P}$ the length of I is in S.

Remark 1.2.5. Let Γ be a subgroup of $\mathbb{R}/\mathbb{Z}$ and let $\tilde{\Gamma}$ be its preimage in $\mathbb{R}$. For the sake of notations, a $\tilde{\Gamma}$-partition is also called a Γ-partition.

Another description strongly used in IET is a description that gives the dynamic of the element:

Definition 1.2.6. Let $f \in \mathrm{IET}$ and let $\mathcal{P} = \{I_1, \ldots, I_k\}$ be a partition associated with f where we order the intervals to be consecutive. Let $f(\mathcal{P}) = \{J_1, \ldots, J_k\}$ be the arrival of f associated with $\mathcal{P}$ where we order the intervals to be consecutive. Let $\mu := \{\lambda(I_1), \ldots, \lambda(I_k)\}$ and σ be the finitely supported permutation such that $f(I_i) = J_{\sigma(i)}$ up to a finite number of points. Then (μ, σ) is called *a combinatorial description of* f. If every component of μ is in $\tilde{\Gamma}$ then (μ, σ) is a Γ-*combinatorial description of* f. The partition $\mathcal{P}$ is also called *the partition associated with* (μ, σ).

1.2.2 Subfamilies

One of the first thing we want to get, when we study a group, is a nice generating subset. We start in IET to finish in $\mathrm{PC}^{\bowtie}$.

1.2.2.1 In IET

Definition 1.2.7. For every $a, b \in \mathbb{R}$, a *restricted rotation of type* (a, b) is an element r of IET such that there exists two consecutive intervals I and J of length a and b respectively with $\sup(I) = \inf(J)$, where r is the translation by $+b$ on I and $-a$ on J. The intervals I and J are called the intervals associated with r.

For every subgroup Γ of $\mathbb{R}/\mathbb{Z}$, a Γ-*restricted rotation* is a restricted rotation in IET(Γ).
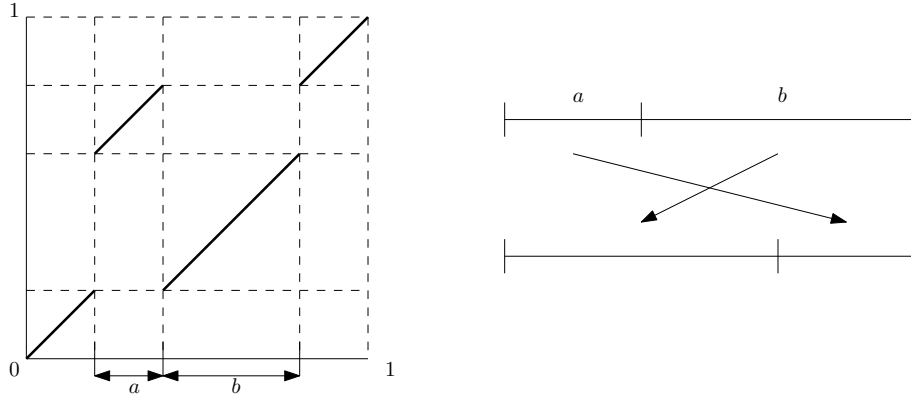
Figure 1.2.1 – **Left:** Graph of a restricted rotation. **Right:** representation of a restricted rotation in pictures.

We cite Arnoux [Arn81b] and Vorobets [Vor17] for the following theorem.

Theorem 1.2.8. *Let $f \in$ IET and let $\mathcal{P}$ be a partition associated with f. Let $S = \{\lambda(I) \mid I \in \mathcal{P}\}$. Then f is a product of restricted rotations with type inside $S \times S$.*

Proof. Let $f, \mathcal{P}$ and S be as in the statement. We numbered the intervals of $\mathcal{P}$, then f act like a permutation on this numbering. Such a permutation can be written as a product of transposition and such a transposition on the numbering coincide with a restricted rotation with type inside $S \times S$. $\qquad\square$

Corollary 1.2.9. *The group IET is generated by its subset of all restricted rotations and for every subgroup Γ of $\mathbb{R}/\mathbb{Z}$, the group IET(Γ) is generated by its subset of all Γ-restricted rotations.*

This gives one decomposition with a property on the length of the intervals. Another useful property is the decomposition into elements of small support. To obtain this result we introduce some elements of order 2 and we need to understand how we can reduce a restricted rotation. This is done by Vorobets in [Vor17] in IET.

Definition 1.2.10. Let $a \in [0, \frac{1}{2}]$ and $f \in$ IET. The element f is an IET-*transposition of type a* if it permutes two non-overlapping subintervals of $[0, 1[$ of length a while fixing the rest of $[0, 1[$. For every subgroup Γ of $\mathbb{R}/\mathbb{Z}$, if in addition $f \in$ IET(Γ) then f is a Γ-*transposition of type a*.
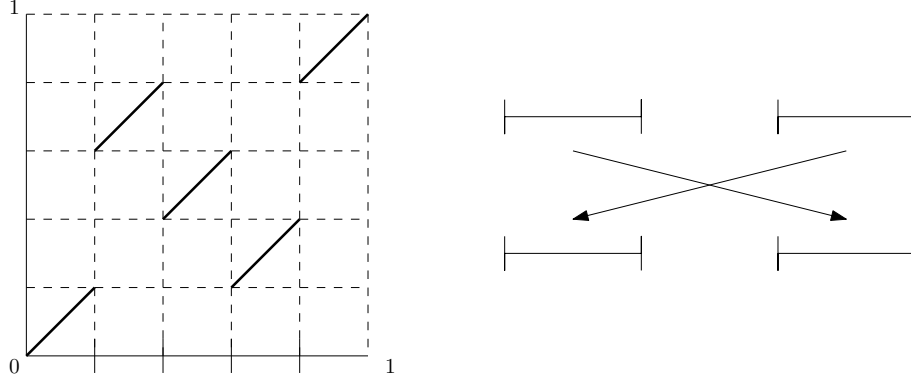
Figure 1.2.2 – **Left:** Graph of a transposition. **Right:** representation of a transposition in pictures.

Proposition 1.2.11. *Let Γ be a subgroup of $\mathbb{R}$ and let $\tilde{\Gamma}$ be its preimage in $\mathbb{R}$. Let r be a Γ-restricted rotation of type (a, b). Then r is the product of a Γ-transposition of type $\min(a, b)$ with a Γ-restricted rotation r' of type $(a - b, b)$ ou $(a, b - a)$. We can also choose r' such that $\inf(\mathrm{Supp}(r')) = \inf(\mathrm{Supp}(r))$.*

Proof. This is immediate with Figure 1.2.3. If $a \leq b$ then it appears a Γ-restricted rotation of type $(a, b - a)$ and if $a \geq b$ it is a Γ-restricted rotation of type $(a - b, b)$. $\qquad\square$
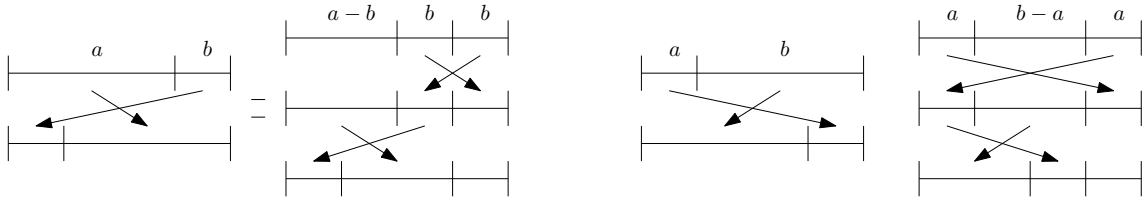
Figure 1.2.3 – **Left:** Case where $\min(a, b) = b$. **Right:** Case where $\min(a, b) = a$.

Lemma 1.2.12. *For every $a, b > 0$ and $\varepsilon > 0$ there exists sequences $(a_i)_{i \in \mathbb{N}}$ and $(b_i)_{i \in \mathbb{N}}$ such that:*

1. *$(a_0, b_0) = (a, b)$;*

2. *either $(a_{i+1}, b_{i+1}) = (a_i, b_i - a_i)$ or $(a_i - b_i, b_i)$;*

3. *both $(a_i)_i$ and $(b_i)_i$ converge to 0.*

Proof. By induction we define $(a_{i+1}, b_{i+1}) = (a_i - b_i, b_i)$ if $a_i > b_i$ and $(a_{i+1}, b_{i+1}) = (a_i, b_i - a_i)$ if $a_i \leq b_i$. Then we notice that both $(a_i)_i$ and $(b_i)_i$ are decreasing sequences of positive real numbers. By construction we have $a_{i+1} + b_{i+1} = a_i + b_i - \min(a_i, b_i)$. By iterating the equality we deduce that the sum of the minimum converges thus this minimum tends to 0. As we reduce at each step the maximum by the minimum we deduce that the minimum encounters both an infinite number of times $(a_i)_i$ and $(b_i)_i$ thus these two sequences tend to 0. $\qquad\square$

Proposition 1.2.13. *Let Γ be a subgroup of $\mathbb{R}/\mathbb{Z}$. For every $\varepsilon > 0$ and every $f \in \mathrm{IET}$, the element f is a product of Γ-restricted rotations and Γ-transpositions with a support's length smaller than ε.*

Proof. Let Γ be a subgroup of $\mathbb{R}/\mathbb{Z}$ and let $\tilde{\Gamma}$ be its preimage in $\mathbb{R}$. Let ε and f as in the statement. If f is a Γ-transposition, let I be the interval such that f permutes I and $f(I)$ while fixing the rest of $[0,1[$. The interval I has endpoints in $\tilde{\Gamma}$ hence we can partition I into intervals with endpoints in $\tilde{\Gamma}$ and with support's length smaller than $\frac{\varepsilon}{2}$. Then we obtain that f is a product of Γ-transpositions with support's length smaller than ε. By Theorem 1.2.8 it is enough to do the case where f is a restricted rotation. Thanks to Proposition 1.2.11 and Lemma 1.2.12 and the previous case we deduce the result. $\qquad\square$

1.2.2.2 In $\widehat{\mathrm{IET}^{\bowtie}}$ and $\mathrm{IET}^{\bowtie}$

The only difference between the definition of $\widehat{\mathrm{IET}^{+}}$ and $\widehat{\mathrm{IET}^{\bowtie}}$ is that we have piecewise isometric elements and not only piecewise translations. This means that the only thing we add is the possibility to reverse an interval.

Definition 1.2.14. Let I be a subinterval of $[0,1[$. We define the I-reflection map as the element $\widehat{r}_I$ of $\widehat{\mathrm{IET}^{\bowtie}}$ that reverses the interior I° while fixing the rest of $[0,1[$. The type of an I-reflection map is the length of I. We define the I-reflection as the image of the I-reflection map in $\mathrm{IET}^{\bowtie}$. The type of an I-reflection is the length of I. For every subgroup Γ of $\mathbb{R}/\mathbb{Z}$, a Γ-reflection is an I-reflection for some subinterval of $[0,1[$ with endpoints in Γ.

Remark 1.2.15. A reflection is also called a flip and that is why $\mathrm{IET}^{\bowtie}$ is called the group of Interval Exchange Transformations with flips.

It is immediate that $\widehat{\mathrm{IET}^{\bowtie}}$ is generated by $\widehat{\mathrm{IET}^{+}}$ and its subset of reflection maps. The similar is true for $\widehat{\mathrm{IET}^{\bowtie}}(\Gamma)$. Hence the group $\mathrm{IET}^{\bowtie}$ is generated by IET^{+} and its subset of reflections. Also we can reduce to the subset of reflection:

Proposition 1.2.16. *For every subgroup Γ of $\mathbb{R}/\mathbb{Z}$, the group $\mathrm{IET}^{\bowtie}(\Gamma)$ is generated by its subset of Γ-reflections.*

Proof. It is enough to show that $\mathrm{IET}^{+}(\Gamma)$ is generated by flips, thus to show that every Γ-restricted rotation is a product Γ-reflections. Let r be a Γ-restricted rotation and let I and J be the two intervals associated to r. We assume $\sup(I) = \inf(J)$. For every right-open and left-closed subinterval I of $[0,1[$, we define s_I the I-reflection. Then $r_{I,J} = s_I s_J s_{I \cup J}$. $\qquad\square$

We deduce that we have generating subsets consisting of elements with small support:

Corollary 1.2.17. *For every $\varepsilon > 0$ and every subgroup Γ of $\mathbb{R}/\mathbb{Z}$, the group $\mathrm{IET}^{\bowtie}(\Gamma)$ is generating by its subset of Γ-reflections and Γ-transpositions whose support's measure is less than ε.*

1.2.2.3 In $\widehat{\mathrm{PC}^{\bowtie}}$

The main thing to see here is that we can modify the length of every interval by an element of the homeomorphism group of $[0,1[$ which preserves the orientation (this group is denoted by $\mathrm{Homeo}^{+}([0,1[)$). This allows us to link the group $\widehat{\mathrm{PC}^{\bowtie}}$ with $\widehat{\mathrm{IET}^{\bowtie}}$ and to give a generating subset consisting of elements which have small support.

Proposition 1.2.18. *For every $\widehat{f} \in \widehat{\mathrm{PC}^{\bowtie}}$ there exist $\phi, \psi \in \mathrm{Homeo}([0,1[)$ such that $\widehat{f} \circ \phi$ and $\psi \circ \widehat{f}$ are two elements of $\widehat{\mathrm{IET}^{\bowtie}}$.*

Proof. Let $\widehat{f} \in \widehat{\mathrm{PC}^{\bowtie}}$ and let $\mathcal{P} := \{I_1, \ldots, I_n\}$ be a partition associated with $\widehat{f}$ where we sorted the intervals such that for every $1 \leq i \leq n-1$ we have $\sup(I_i) = \inf(I_{i+1})$. We denote by a_i the left endpoint of I_i. Let $f(\mathcal{P}) = \{J_1, \ldots, J_n\}$ and let σ be the permutation of $\{1, \ldots, n\}$ such that $f(I_i^{\circ}) = J_{\sigma(i)}^{\circ}$. We denote by b_i the left endpoint of J_i. There exists

a finitely supported permutation whose support is included in $\{a_i \mid 1 \le i \le n\}$ such that $f(a_{\tau(i)}) = b_i$. Let $\{K_1, \ldots, K_n\}$ be the partition of $[0,1[$ into intervals such that for every $1 \le i \le n$ the length of K_i is equal to $I_{\sigma^{-1}(i)}$. Let $\hat{g}$ be the element of $\widehat{\mathrm{IET}^{\bowtie}}$ which sends K_i° on $I_{\sigma^{-1}(i)}^\circ$ and sends the left endpoint of K_i on the left one of $I_{\sigma^{-1}(i)}$. We also ask $\hat{g}$ to be order-reversing on K_i if $\hat{f}$ is order-reversing on $I_{\sigma^{-1}(i)}$, and order-preserving otherwise. Then the composition $\hat{f} \circ \tau \circ \hat{g}$ is an element of $\mathrm{Homeo}^+([0,1[)$ by sending K_i on J_i. Hence there exists $\psi \in \mathrm{Homeo}^+([0,1[)$ such that $\psi \circ f \in \widehat{\mathrm{IET}^{\bowtie}}$. The other case is similar. $\qquad\square$

Corollary 1.2.19. *We have the following equalities:*

1. $\widehat{\mathrm{PC}^{\bowtie}} = \mathrm{Homeo}^+([0,1[)\widehat{\mathrm{IET}^{\bowtie}}$;

2. $\widehat{\mathrm{PC}^+} = \mathrm{Homeo}^+([0,1[)\widehat{\mathrm{IET}^+}$.

Hence, to obtain that there exists a generating subset of $\widehat{\mathrm{PC}^{\bowtie}}$ consisting of elements of small support it is enough to show it for $\mathrm{Homeo}^+([0,1[)$, a proof is given by Arnoux in [Arn81b].

Proposition 1.2.20. *Let $\varepsilon > 0$, then every $f \in \mathrm{Homeo}^+([0,1[)$ is a product of elements of $\mathrm{Homeo}^+([0,1[)$ with support's length less than ε.*

Proof. Let $f \in \mathrm{Homeo}^+([0,1[)$. Let $I = [a, a+b[$ be a subinterval of $[0,1[$ such that $\mathrm{Supp}(f) \subset I$. Let us assume that $f(a + \frac{b}{2}) \in [a, a + \frac{b}{2}[$. Then there $g \in \mathrm{Homeo}^+([0,1[)$ such that $g(f(a + \frac{b}{2})) = a + \frac{b}{2}$ and $\mathrm{Supp}(g) \in [a, a + \frac{3}{4}b[$. Then $a + \frac{b}{2}$ is a fixed point of $g \circ f$ hence there exist $f_1, f_2 \in \mathrm{Homeo}^+([0,1[)$ such that $\mathrm{Supp}(f_1) \subset [a, a + \frac{b}{2}[$ and $\mathrm{Supp}(f_2) \subset [a + \frac{b}{2}, a + b[$. Then we deduce that $f = g^{-1} \circ f_1 \circ f_2$ and every homeomorphism of this product have a support's length less than $\frac{\lambda(I)}{3}$. The same can be done if we assume that $f(a + \frac{b}{2}) \in [a + \frac{b}{2}, a + b[$, thus by iterating this process a finite number of time we obtain the result. $\qquad\square$

Corollary 1.2.21. *For every $\varepsilon > 0$, the group $\mathrm{PC}^{\bowtie}$ and PC^+ have a generating subset consisting of elements which have support's length less than ε.*

1.2.3 Minimal and finite order elements in IET

The study of finite order elements appears to be useful for the study of the derived subgroup, see Subsection 2.3.3. First we give a nice decomposition into two kinds of elements done by Arnoux in [Arn81b] and then we describe how finite order elements and elements order 2 are related.

1.2.3.1 Decomposition into elements with disjoint support

Definition 1.2.22. Let U be a subset of $[0,1[$. An element f of IET is said to be *minimal on U* if U is invariant by f and for every $x \in U$, the orbit of x by f is dense in U.

The following decomposition into minimal and periodic elements done by Arnoux in [Arn81b] is a decomposition into disjoint elements. Arnoux proved it in the case of IET and the result in the case of IET(Γ) is just a consequence.

Proposition 1.2.23. *Let Γ be a subgroup of $\mathbb{R}/\mathbb{Z}$ ad let $\tilde{\Gamma}$ be its preimage in $\mathbb{R}$. Let $f \in$ IET(Γ). There exist unions of right-open and left-closed disjoint intervals $U_1, \ldots, U_n, V_1, \ldots, V_k$ with endpoints in $\tilde{\Gamma}$, such that:*

1. *they form a partition of $[0,1[$ into f-invariant subsets;*

2. *For every $1 \leq i \leq n$ the element f is minimal on U_i;*

3. *For every $1 \leq i \leq k$ there exists $p_i \in \mathbb{N}$ such that every element $x \in V_i$ has minimal period p_i.*

1.2.3.2 Finite order elements

A consequence of Proposition 1.2.23 is the following Proposition proved by Vorobets [Vor17] in the case of IET:

Proposition 1.2.24. *Let Γ be a subgroup of $\mathbb{R}/\mathbb{Z}$. Every finite order element f of $\mathrm{IET}(\Gamma)$ is a finite product of Γ-transpositions. In particular there exist $n \in \mathbb{N}$ and $\mathcal{P} := \{I_1, \ldots, I_n\}$ a partition associated with f and a permutation σ of $\{1, \ldots, n\}$ such that $f(I_i) = I_{\sigma(i)}$.*

Proof. Let $f \in \mathrm{IET}(\Gamma)$ be a finite order element. By Proposition 1.2.23 there exists $k \in \mathbb{N}$ and for every $1 \leq i \leq k$ there exist $p_i \in \mathbb{N}$ and an element $f_i \in \mathrm{IET}(\Gamma)$ of minimal period p_i such that:

1. for every $1 \leq i \neq j \leq k$ the support of f_i and the support of f_j do not overlap;

2. $f = f_1 \ldots f_k$.

Then it is enough to assume f to have minimal period $p \in \mathbb{N}$. In this case, the union of the set of discontinuities of powers of f is finite, we denote by $\mathcal{P}$ the partition of $[0,1[$ into intervals defined by these points. We notice that $\mathcal{P}$ is a Γ-partition associated with f. By construction there exists a partition of $\mathcal{P}$ into subsets p intervals such that f is a permutation of the intervals in every of these subsets. We deduce that f is a product of Γ-transpositions. $\qquad\square$

Now we want to understand how the product of two IET-transpositions is a finite order element. This is not immediate because for every $n \in \mathbb{N}$ we can construct examples of such a product with order n:

Proposition 1.2.25. *For every n in $\mathbb{N}_{\geq 1}$ there exist two IET-transpositions f and g such that the product gf has order n.*

Proof. We distinguish the case where n is even or odd. In both cases we illustrate the proof with Figure 1.2.4. The case $n = 1$ is given by the equality $f^2 = \mathrm{Id}$ for any IET-transposition f. Let $n \in \mathbb{N}_{\geq 1}$.

Let I and J be two consecutive intervals of the same length $\ell \in [0, \frac{1}{2}]$ and let g be the IET-transposition that swaps I and J. Let $A_1, A_2, \ldots, A_{n-1}$ and C be consecutive intervals of length $\frac{\ell}{n}$ such that the left endpoint of A_1 is the left endpoint of I (hence the right endpoint of C is the right endpoint of I). Let D and $B_1, B_2, \ldots, B_{n-1}$ be consecutive intervals of length $\frac{\ell}{n}$ such that the right endpoint of B_{n-1} is the right endpoint of J (hence the left endpoint of D is the left endpoint of J). Let f be the IET-transposition that swaps A_i and B_i for every $1 \leq i \leq n-1$. Hence by definition we get $g(A_1) = D$, $g(A_i) = B_{i-1}$ for every $2 \leq i \leq n-1$ and $g(C) = B_{n-1}$. So the composition gf is equal to the permutation $(A_1\ A_2\ \ldots\ A_{n-1}\ C\ B_{n-1}\ B_{n-2}\ \ldots\ B_1\ D)$. Thus gf has order $2n$.

It remains the case of order $2n - 1$. Let I, J and K be three consecutive intervals with I and J of length $\ell \in [0, \frac{1}{3}[$ and K of length $\ell' \in\ ]\frac{\ell}{n}, \ell[$. Let g be the IET-transposition that swaps I and J. We define $A_1, A_2, \ldots, A_{n-1}$ consecutive intervals of length $\frac{\ell}{n}$ such that the right endpoint of A_{n-1} is the right endpoint of I. We define also D and $B_1, B_2, \ldots, B_{n-1}$ consecutive intervals of length $\frac{\ell}{n}$ such that the left endpoint of B_{n-1} is the left endpoint of K. Let f be the IET-transposition that swaps A_i and B_i for every $1 \leq i \leq n-1$. One can check that the product gf in this case is the permutation $(A_1\ A_2\ \ldots\ A_{n-1}\ B_{n-1}\ B_{n-2}\ \ldots\ B_1\ D)$ so gf has order $2n - 1$. $\qquad\square$
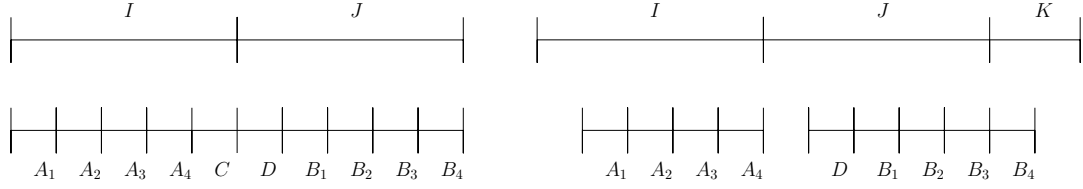
Figure 1.2.4 – Illustration of Proposition 1.2.25 with $n = 5$. **Left**: For the case "fg has order $2n$". **Right**: For the case "fg has order $2n - 1$".

To prove that every product of IET-transpositions is a fintie order element, we use the notion of reversible maps and some of their properties proved by N. Guelman and I. Liousse in [GL19b].

Definition 1.2.26. Let h, k be two elements of IET. We said that h reverses k if $hkh^{-1} = k^{-1}$.

A direct consequence of the definition is:

Proposition 1.2.27. *Let $h, k \in$ IET such that h reverses k. Then for every $p \in \mathbb{Z}$ both h and h^{-1} reverse k^p. Thus $k^p h = hk^{-p}$.*

For every $f \in$ IET we denote by $\mathrm{Per}(f)$ the set of periodic points of f and by $\mathrm{Fix}(f)$ the set of fixed points of f.

Lemma 1.2.28. *Let $h, k \in$ IET such that h reverses k. Then:*

1. *The set $\mathrm{Per}(k)$ is invariant by h;*

2. *The set $\mathrm{Fix}(h)$ is a subset of $\mathrm{Per}(k)$.*

Proof. 1. Let $x \in \mathrm{Per}(k)$ and let $p \in \mathbb{Z}$ such that $k^p(x) = x$. By Proposition 1.2.27 we have $k^p(h(x)) = hk^p(x) = h(x)$, thus $h(x) \in \mathrm{Per}(k)$.

2. By contradiction let $x \in \mathrm{Fix}(h) \smallsetminus \mathrm{Per}(k)$. As h is in IET we deduce that there exists $c \in [0, 1[$ such that the whole interval $[x, c[\subset \mathrm{Fix}(h)$. Also x is not a periodic point of k thus the point x is a right cluster point of the k-orbit of x. This means that there exists a sequence $(p_n)_n$ of positive integers such that $k^{p_n}(x) > x$ and $\lim_{n \to \infty} k^{p_n}(x) = x$. Hence there exists an integer $p > 0$ such that $k^p(x) \in [x, c[$ thus it is a fixed point of h. Then $k^p(x) = h(k^p(x)) = k^{-p}(h(x)) = k^{-p}(x)$ and we deduce that $x \in \mathrm{Per}(k)$ which is a contradiction.

$\square$

Corollary 1.2.29. *Let f and g be two IET-transpositions. Then f and g reverse gf and for every $x \notin \mathrm{Per}(gf)$ the $\langle f, g \rangle$-orbit of x is contained in $[0, 1[\smallsetminus \mathrm{Per}(gf) \subset \mathrm{Supp}(f) \cap \mathrm{Supp}(g)$.*

Lemma 1.2.30. *Let f and g be two IET-transpositions. Then gf has finite order.*

Proof. Let I, J, A, B be the intervals such that $\sup(I) \leq \inf(J)$ and f swaps I with J while fixing the rest of $[0, 1[$ and $\sup(A) \leq \inf(B)$ and g swaps A with B while fixing the rest of $[0, 1[$. Let $\alpha, \beta \in [0, 1[$ such that $J = I + \alpha$ and $B = A + \beta$. A crucial property is that either $J \cap A$ or $I \cap B$ is the empty set.

By contradiction let x be a point that is not in $\mathrm{Per}(gf)$. By Corollary 1.2.29 we know that the $\langle f, g \rangle$-orbit of x is included in :

$$\mathrm{Supp}(f) \cap \mathrm{Supp}(g) = (I \cap A) \sqcup (I \cap B) \sqcup (J \cap A) \sqcup (J \cap B)$$

We distinguish three cases:

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1. If the $\langle f, g \rangle$-orbit of x does not intersect $(I \cap A) \sqcup (J \cap B)$ then it is included in $(I \cap B) \cap (J \cap A)$ and as one of them is empty it is included either in I or J which is impossible.

2. Now assume that the $\langle f, g \rangle$-orbit of x intersects $I \cap A$. Up to change x by a point of its gf-orbit we can assume that $x \in I \cap A$. Then we deduce that $f(x) \in (J \cap A) \sqcup (J \cap B)$.

   a If $J \cap A = \emptyset$ then $f(x) \in J \cap B$ thus $gf(x) \in I \cap A$ hence for every $n \in \mathbb{N}$ we have $(gf)^n(x) \in I \cap A$. Then $(gf)^n(x) = x + n(\alpha - \beta)$ hence x has to be inside $\mathrm{Per}(gf)$ which is a contradiction.

   b If $J \cap A \neq \emptyset$ then $I \cap B = \emptyset$ thus $g(x) \in J \cap B$ and we deduce that $fg(x) \in I \cap A$. Then for ever $n \in \mathbb{N}$ we obtain that $(fg)^n(x) = x + n(\beta - \alpha)$ hence $x \in \mathrm{Per}(fg)$. Also $\mathrm{Per}(fg) = \mathrm{Per}(g^{-1}f^{-1}) = \mathrm{Per}(gf)$ thus $x \in \mathrm{Per}(gf)$ which is a contradiction.

3. Let now assume that the $\langle f, g \rangle$-orbit of x intersects $J \cap B$ and does not intersect $I \cap A$. Up to change x by a point of its gf-orbit we can assume that $x \in J \cap B$. Then $f(x) \in I \cap B$, in particular $I \cap B \neq \emptyset$ thus $J \cap A = \emptyset$ and we deduce that $g(x) \in I \cap A$ which is a contradiction.

$\square$

Remark 1.2.31. In the case where $\alpha = \beta$ every point of $[0, 1[$ has a gf-orbit of at most cardinal three. Thus gf is at most of order 6.

1.3 Derived subgroup and abelianization

1.3.1 Simplicity of the derived subgroup

For a group G, its derived subgroup, denoted $D(G)$, is the subgroup generated by its subset of all commutators. The quotient $G_{\mathrm{ab}} = G/D(G)$ is the abelianization of G. It is the biggest quotient of G which is an abelian group. A group is said to be perfect if it is equal to its derived subgroup and it is said to be simple if the only normal subgroups of G are $\{1\}$ and G.

   We recall how a generating set of a group G can give a generating subset of $D(G)$.

Proposition 1.3.1. *Let G be a group and S be a generating set. Let N be the normal closure of the set of commutators of the elements of S. Then N is a generating subset of $D(G)$.*

Proof. The proof is only a consequence of the following equalities between commutators:

1. $[f_1 f_2, g] = f_1 [f_2, g] f_1^{-1} [f_1, g]$;

2. $[f, g_1 g_2] = [f, g_1] g_1 [f, g_2] g_1^{-1}$.

$\square$

   Thanks to Theorem 1.1.2 It is enough to prove that the derived subgroup is the smallest normal subgroup to obtain that the derived subgroup is simple. This come from the Epstein's simplicity criterion [Eps70].

Theorem 1.3.2. *Let Γ be a dense subgroup of $\mathbb{R}/\mathbb{Z}$. Let G be a subgroup of $\mathrm{PC}^{\bowtie}$ which contains $\mathrm{IET}^+(\Gamma)$ and such that for every $\varepsilon > 0$ there exists a generating subset of G consisting of elements with support's length less than ε. Then $D(G)$ is the smallest normal subgroup of G.*

Proof. Let G be a subgroup of $\text{PC}^{\bowtie}$ which contains $\text{IET}^+(\Gamma)$ and let N be a normal subgroup of G. It is enough to show that there exists $\varepsilon > 0$ such that every commutator of two element in G with a support's length less than ε is in N. Let $f \in N \smallsetminus \{\text{Id}\}$. There exists an interval I such that f is continuous on I and $I \cap f(I) = \emptyset$ and $I \cup f(I) \neq [0, 1[$. Let $\varepsilon = \frac{\lambda(I)}{3}$ and let $g_1, g_2 \in \text{PC}^+$ (resp. IET^+) with support's length less than ε. By density of Γ, there exists $t \in \text{IET}^+(\Gamma)$ that translates $\text{Supp}(g_1) \cup \text{Supp}(g_2)$ inside I. Let $h = t^{-1} \circ f \circ t$. As N is normal we deduce that $h \in N$. Also we notice that $h(\text{Supp}(g_1) \cup \text{Supp}(g_2)) \cap (\text{Supp}(g_1) \cup \text{Supp}(g_2)) = \emptyset$ thus $\text{Supp}(g_1) \cap \text{Supp}(hg_2h^{-1}) = \emptyset$ thus these two elements commute. We deduce that the commutator $[g_1, [g_2, h]] = [g_1, g_2]$ also $[g_2, h] \in N$ and so $[g_1, [g_2, h]] \in N$. Then $[g_1, g_2] \in N$ and we deduce that $D(G)$ is a subgroup of N. $\qquad\square$

Corollary 1.3.3. *Let G be a subgroup of $\text{PC}^{\bowtie}$. Assume that there exists a dense subgroup Γ of $\mathbb{R}/\mathbb{Z}$ such that $\text{IET}^+(\Gamma)$ is a subgroup of G. Then the derived subgroup $D(G)$ is simple. In particular this is true for $\text{PC}^{\bowtie}, \text{PC}^+, \text{IET}^{\bowtie}(\Gamma)$ and $\text{IET}^+(\Gamma)$ for every dense subgroup Γ of $\mathbb{R}/\mathbb{Z}$.*

1.3.2 Abelianization

1.3.2.1 Simple groups

We explicit here some groups that are perfect and thanks to Corollary 1.3.3 these groups will be simple.

Theorem 1.3.4 (Arnoux [Arn81b]). *Both PC^+ and $\text{IET}^{\bowtie}$ are simple.*

The case of $\text{IET}^{\bowtie}$ results of the fact that a reflection is a commutator thanks to Figure 1.3.1. For PC^+ it results from the simpleness of the group $\text{Homeo}^+(\mathbb{R}/\mathbb{Z})$ (see Schreier and Ulam [SU34]); this implies that a restricted rotation is a commutator as well as every homeomorphism of $[0, 1[$ with small support. Theorem 1.3.4 has for direct consequence:

Corollary 1.3.5. *The group $\text{PC}^{\bowtie}$ is perfect and thus simple.*
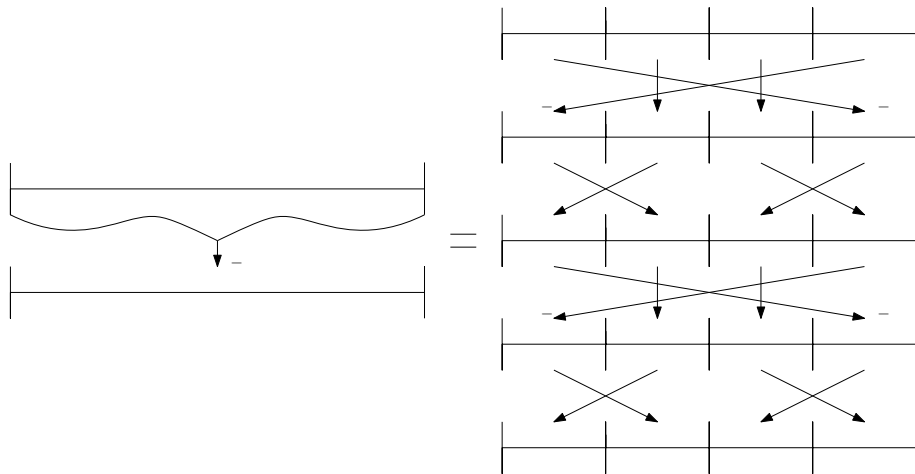
Figure 1.3.1 – How a reflection can be seen as a commutator.

1.3.2.2 The second skew-symmetric power

Let A be a commutative ring (we only deal with $\mathbb{Z}$ and $\mathbb{Q}$) and let V be a A-module. The second tensor product of V over A is simply $\bigotimes_A^2 V$. The second symmetric power $\mathsf{S}_A^2 V$ is the quotient of $\bigotimes_A^2 V$ by the submodule W generated by elements $x \otimes y - y \otimes x$ when x and y range over V. The second exterior power $\bigwedge_A^2 V$ is the quotient of $\bigotimes_A^2 V$ by the submodule generated by elements $x \otimes x$ when x ranges over V. These notions are very classical. We need a less usual one, namely the second skew-symmetric power $^\ominus\!\!\bigwedge_A^2 V$, defined as the quotient of the second tensor power $\bigotimes_A^2 V$ by the submodule W' generated by the set $\{x \otimes y + y \otimes x \mid x, y \in V\}$.

Thanks to the inclusion $W' \subset W$, there is a canonical quotient map $^\ominus\!\!\bigwedge_A^2 V \to \bigwedge_A^2 V$; if 2 is invertible in A, this is an isomorphism, but we will typically deal with the ring $\mathbb{Z}$ in which case this kernel can be a nontrivial elementary abelian 2-group.

When 2 is invertible in A, we can identify $\bigwedge^2 V$ with the submodule of $\bigotimes_A^2 V$ generated by elements of the form $x \otimes y - y \otimes x$.

For instance, if $V \simeq \mathbb{Z}^d$, then $\bigwedge_{\mathbb{Z}}^2 V$ is isomorphic to $\mathbb{Z}^{d(d-1)/2}$, while this kernel is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^d$.

1.3.2.3 Abelianization of IET

Arnoux-Fathi [Arn81a] and Sah [Sah81] proved the following theorem that identifies the abelianization of IET.

Theorem 1.3.6 (Arnoux-Fathi-Sah [Arn81a, Sah81])**.** *There exists a surjective group homomorphism from* IET *onto the second skew-symmetric algebra of $\mathbb{R}$ over $\mathbb{Q}$ whose kernel is the derived subgroup of* IET.

Definition 1.3.7. This group homomorphism is called the SAF-invariant, or the SAF-homomorphism, and it is define by the following:

$$\varphi := \begin{array}{ccc} \text{IET} & \longrightarrow & \bigwedge_{\mathbb{Q}}^2 \mathbb{R} \\ f & \longmapsto & \sum_{a \in \mathbb{R}} a \wedge \lambda((f - \mathrm{Id})^{-1}(\{a\})) \end{array}$$

In particular, for every restricted rotation f of type (a, b) we have $\varphi(f) = 2b \wedge a$.

Remark 1.3.8. Let Γ be a subgroup of $\mathbb{R}/\mathbb{Z}$ and let $\tilde{\Gamma}$ be its preimage in $\mathbb{R}$. If $\tilde{\Gamma} = 2\tilde{\Gamma}$ then the restriction of the SAF-invariant to $\mathrm{IET}^+(\Gamma)$ is surjective on $^\ominus\!\!\bigwedge_{\mathbb{Z}}^2 \tilde{\Gamma}$ and its kernel is equal to the derived subgroup $D(\mathrm{IET}^+(\Gamma))$.

We explicit how this homomorphism can be expressed thanks to a combinatorial description:

Proposition 1.3.9. *Let $f \in$* IET *and (μ, σ) be a combinatorial description of f. Let k be the number of coordinates of μ. Then we have:*

$$\varphi(f) = \sum_{i=1}^{k} \left(\sum_{\substack{i: \\ \sigma(i) < \sigma(j)}} \mu_i - \sum_{i<j} \mu_i \right) \wedge \mu_j$$

Proof. Let $f \in$ IET and (α, τ) be a combinatorial description of f. Let n be the length of α and let $\{I_1, I_2, \ldots, I_n\}$ be the partition associated with (α, τ). For each j we denote by $v(j)$ the value of $f - \mathrm{Id}$ on I_j. Thus we deduce that $\varphi(f) = \sum_{j=1}^{n} v(j) \wedge \lambda(I_j) = \sum_{j=1}^{n} v(j) \wedge \alpha_j$. Also we know that $v(j) = \sum_{\substack{i \\ \tau(i) < \tau(j)}} \alpha_i - \sum_{i<j} \alpha_i$ (see Figure 1.3.2 below), and this gives the conclusion. $\square$
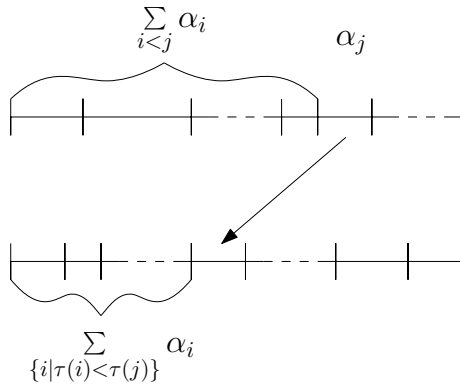
Figure 1.3.2 – Illustration for the value of the SAF-homomorphism in Proposition 1.3.9

The injectivity of the SAF-homomorphism relies on the fact that we can always cut an interval into two intervals with the same size. This is the main obstruction to obtain the abelianization of $\mathrm{IET}(\Gamma)$. This is the same obstruction to understand the abelianization of $\mathrm{IET}^{\bowtie}(\Gamma)$ because to show that a reflection is a commutator we cut an interval into four intervals of the same size.

Abelianization of IET(Γ)

Résumé en français.   Soit Γ un sous-groupe dense de $\mathbb{R}/\mathbb{Z}$ et $\tilde{\Gamma}$ sa pré-image dans $\mathbb{R}$. Dans ce chapitre on établit un isomorphisme naturel entre l'abélianisé de IET(Γ) et la seconde puissance anti-symétrique de $\tilde{\Gamma}$ sur $\mathbb{Z}$ notée par ${}^\ominus\bigwedge^2_{\mathbb{Z}} \tilde{\Gamma}$. Ce groupe possède souvent de la 2-torsion qui n'est pas détectée par le morphisme de groupes SAF. L'idée est de construire un morphisme de groupes qiu est inspiré par le morphisme signature sur les groupes de permutations finis. On introduit la notion d'inversion pour un élément du groupe IET et on essaye de mesurer l'ensemble de ses inversions. Ce chapitre correspond à la première moitié de l'article [Lac20a].

English abstract.   Let Γ be a dense subgroup of $\mathbb{R}/\mathbb{Z}$ and $\tilde{\Gamma}$ be its preimage in $\mathbb{R}$. In this chapter, we establish a natural isomorphism between the abelianization of IET(Γ) and the second skew-symmetric power of $\tilde{\Gamma}$ over $\mathbb{Z}$ denoted by ${}^\ominus\bigwedge^2_{\mathbb{Z}} \tilde{\Gamma}$. This group often has non-trivial 2-torsion, which is not detected by the SAF-homomorphism. The idea is to construct a group homomorphism that is inspired by the classical signature on finite permutation group. We introduce the notion of inversions for an element of IET and we try to measure the set of inversions. This chapter corresponds with the first half of the article [Lac20a].

Contents

2.1 Positive cone of $\tilde{\Gamma}$

One difficulty encountered to understand the abelianization of IET(Γ) is that $\mathbb{R}$ is a $\mathbb{Q}$-vector space and $\tilde{\Gamma}$ is only a $\mathbb{Z}$-module. Hence understanding the positive cone of $\tilde{\Gamma}$ will give informations about how cutting an interval into smaller intervals.

Let us introduce some classical terminology from the theory of ordered abelian groups. An ordered abelian group is an abelian group endowed with an invariant partial ordering.

For any subgroup H of $\mathbb{R}$ we denote by $H_+ := \{x \in H \mid x \geq 0\}$ the positive cone of H. A difficulty is that H_+ is not, in general, finitely generated as a subsemigroup. For every subset B of $\mathbb{R}$ we denote by $\text{Vect}_{\mathbb{N}}(B)$ the subsemigroup generated by B.

A subsemigroup of an abelian group is *simplicial* if it is generated, as a subsemigroup, by a finite $\mathbb{Z}$-independent subset, and it is *ultrasimplicial* if it is the direct limit of an upward directed set of simplicial subsemigroups. An ordered abelian group is *simplicially ordered* if its positive cone is simplicial and is *ultrasimplicially ordered* if its positive cone is ultrasimplicial. The next theorem is proved by G.A. Elliott in [Ell79]:

Theorem 2.1.1. *Every totally ordered abelian group is ultrasimplicially ordered.*

We deduce the following corollary for the totally ordered abelian group $\tilde{\Gamma}$:

Corollary 2.1.2. *There exist an upward directed set $(M_\Gamma, \leq)$ and a finite $\mathbb{Z}$-linearly independent subset S_a of $\tilde{\Gamma}$ for every $a \in M_\Gamma$ such that for every $a, b \in M_\Gamma$ with $a \leq b$ we have $\text{Vect}_{\mathbb{N}}(S_a) \subset \text{Vect}_{\mathbb{N}}(S_b)$ and $\tilde{\Gamma}_+$ is equal to the direct limit $\varinjlim \text{Vect}_{\mathbb{N}}(S_a)$.*

We will use the Theorem 2.1.1 in the form of the following corollary which specifies the finite rank case:

Corollary 2.1.3. *Let H be an abelian subgroup of $\mathbb{R}$ of finite rank d. Then there exists a sequence $(B_n)_{n \in \mathbb{N}}$ of $\mathbb{Z}$-basis of H such that for each n we have $\text{Vect}_{\mathbb{N}}(B_n) \subset \text{Vect}_{\mathbb{N}}(B_{n+1})$ and H_+ is equal to the increasing union $\bigcup_n \text{Vect}_{\mathbb{N}}(B_n)$. Furthermore for every $k \in \mathbb{N}$ and $L_1, L_2, \ldots, L_k \in H_+$ there exists a basis $\{\ell_1, \ell_2, \ldots, \ell_d\} \subset H_+$ of H such that for every $1 \leq i \leq k$ the element L_i is a linear combination of $\ell_1, \ell_2, \ldots, \ell_d$ with coefficients in $\mathbb{N}$.*

2.2 Kernel of the SAF-homomorphism's restriction to IET(Γ)

See Definition 1.3.7 for the definition of the SAF-homomorphism. We denote by φ_Γ the restriction of the SAF-homomorphism to IET(Γ). We follow the idea of Y. Vorobets in [Vor17] and introduce the notion of balanced product of restricted rotations. The aim is to show that the kernel of φ_Γ is generated by its subset of all balanced products of Γ-restricted rotations.

2.2.1 Balanced product of Γ-restricted rotations

Definition 2.2.1. Let $n \in \mathbb{N}$ and let r_i be a restricted rotation for every $1 \leq i \leq n$. For every $a, b \in \tilde{\Gamma}_+$ let $n_{a,b}$ be the number of restricted rotation r_i of type (a, b). The tuple $(r_1, r_2, \ldots, r_n)$ is said to be a *balanced tuple of restricted rotations* if $n_{a,b} = n_{b,a}$ for every $a, b \in \tilde{\Gamma}_+$. We say that a product g of restricted rotations is a *balanced product of restricted rotations* if there exists a balanced tuple of restricted rotations $(r_1, \ldots, r_n)$ such that $g = r_1 \ldots r_n$.

Example 2.2.2. Let a be an element in $\tilde{\Gamma}_+$ with $a \leq \frac{1}{2}$. Every Γ-restricted rotation of type (a, a) is a balanced product of restricted rotations. It is also a Γ-transposition which swapped two consecutive intervals of same length a.

Example 2.2.3. Let $a, b \in \tilde{\Gamma}_+$ with $a + b \leq 1$. If h is a Γ-restricted rotation of type (a, b) then h^{-1} is a Γ-restricted rotation of type (b, a). Thus every element of $D(\mathrm{IET}(\Gamma))$ is a balanced product of Γ-restricted rotations.

In order to get the decomposition of the elements of $\mathrm{Ker}(\varphi_\Gamma)$, we need to know the freeness of some families of $^\ominus\bigwedge^2_{\mathbb{Z}} V$.

Lemma 2.2.4. *Suppose V is a subgroup of $\mathbb{R}$ (then it is a $\mathbb{Z}$-module). Let k in $\mathbb{N}_{\geq 1}$ and $v_1, v_2, \ldots v_k$ elements of V which are $\mathbb{Z}$-linearly independent. Then the wedge products $v_i \wedge v_j$ for $1 \leq i < j \leq k$ are $\mathbb{Z}$-linearly independent in $^\ominus\bigwedge^2_{\mathbb{Z}} V$.*

Proof. Let $v_1, v_2, \ldots v_k$ in V which are $\mathbb{Z}$-linearly independent. It is sufficient to prove the lemma for $V = \mathbb{R}$ because being $\mathbb{Z}$-linearly independent in $^\ominus\bigwedge^2_{\mathbb{Z}} \mathbb{R}$ implies being $\mathbb{Z}$-linearly independent in $^\ominus\bigwedge^2_{\mathbb{Z}} V$. We know that $^\ominus\bigwedge^2_{\mathbb{Z}} \mathbb{R}$ is isomorphic to $\bigwedge^2_{\mathbb{Z}} \mathbb{R}$. Let us assume that $v_1, v_2, \ldots, v_k$ are in $\mathbb{R}$. Then being $\mathbb{Z}$-linearly independent is the same that being $\mathbb{Q}$-linearly independent. Indeed if there exist $p_1, p_2, \ldots, p_k$ in $\mathbb{Z}$ and $q_1, q_2, \ldots, q_k$ in $\mathbb{N}_{\geq 1}$ such that $\sum_{i=1}^{k} \frac{p_i}{q_i} v_i = 0$ then $\sum_{i=1}^{k} (\prod_{\substack{j=1 \\ j \neq i}} q_j p_i) v_i = 0$ is an equality in $\mathbb{Z}$. Thus for each $i \in \{1, 2, \ldots, k\}$ we have $\prod_{\substack{j=1 \\ j \neq i}} q_j p_i = 0$. Or $q_j \neq 0$ for every j then $p_i = 0$ for every i.

The $\mathbb{Q}$-vector space generated by all the v_i for $1 \leq i \leq k$ is isomorphic to $\mathbb{Q}^k$. We can complete the $\mathbb{Q}$-linearly independent set $\{v_1, v_2, \ldots, v_k\}$ in a basis S of $\mathbb{R}$ seen as a $\mathbb{Q}$-vector space. Thus in $\bigwedge^2_{\mathbb{Z}} \mathbb{R}$ the elements $v_i \wedge v_j$ for $1 \leq i < j \leq k$ are $\mathbb{Q}$-linearly independent so they are $\mathbb{Z}$-linearly independent in $^\ominus\bigwedge^2_{\mathbb{Z}} \mathbb{R}$ and this gives the conclusion. $\qquad\square$

Lemma 2.2.5. *Any transformation f in $\mathrm{Ker}(\varphi_\Gamma)$ can be written as a balanced product of Γ-restricted rotations.*

Proof. Let $f \in \mathrm{Ker}(\varphi_\Gamma)$. This is trivial if $f = \mathrm{id}$; assume otherwise. Let (μ, σ) be a Γ-combinatorial description of f (see Definition 1.2.6), let $k \in \mathbb{N}$ and $\{I_1, I_2, \ldots, I_k\}$ be the partition into intervals associated to (μ, σ) (we have $k \geq 2$ as f is not the identity). We recall that μ_i is the length of I_i for every $1 \leq i \leq k$.

We treat the case where Γ is finitely generated. Then $\tilde{\Gamma}$ is also finitely generated and we denote by d its rank. By Corollary 2.1.3 there exist $\ell_1, \ell_2, \ldots, \ell_d$ in $\tilde{\Gamma}_+$ such that $\mathcal{L} := \{\ell_1, \ell_2, \ldots, \ell_d\}$ is a basis of $\tilde{\Gamma}$ and such that μ_i is a linear combination of $\ell_1, \ell_2, \ldots, \ell_d$ with non-negative integer coefficients for every $1 \leq i \leq k$. Then I_i can be partitioned into smaller intervals with length in $\mathcal{L}$ for every $1 \leq i \leq k$. We obtain a partition associated with f whose intervals have length in $\mathcal{L}$. By Theorem 1.2.8 there exist $n \in \mathbb{N}$ and a restricted rotation f_i of type (a_i, b_i) with $a_i, b_i \in \mathcal{L}$ for $1 \leq i \leq n$ such that $f = f_1 f_2 \ldots f_n$. For any $i, j \in \{1, 2, \ldots, d\}$ let s_{ij} be the number of Γ-restricted rotation of type (ℓ_i, ℓ_j) in the tuple $(f_1, f_2, \ldots, f_n)$. As $\varphi_\Gamma(f_i) = \ell_j \wedge \ell_i - \ell_i \wedge \ell_j = 2\ell_j \wedge \ell_i$ (see Definition 1.3.7), we obtain that:

$$\varphi_\Gamma(f) = \sum_{i=1}^{d} \sum_{j=1}^{d} 2 s_{ij} (\ell_j \wedge \ell_i) = \sum_{1 \leq i < j \leq d} 2(s_{ij} - s_{ji})(\ell_j \wedge \ell_i)$$

We know that $\{\ell_1, \ell_2, \ldots, \ell_d\}$ is a basis of $\tilde{\Gamma}$ thus by Lemma 2.2.4 we obtain that $\{\ell_j \wedge \ell_i\}_{1 \leq i < j \leq d}$ is a free family of $^\ominus\bigwedge^2_{\mathbb{Z}} \tilde{\Gamma}$. Then the assumption $\varphi_\Gamma(f) = 0$ implies $s_{ij} = s_{ji}$ for every $1 \leq i < j \leq d$. This means that the product of Γ-restricted rotations $f_1 f_2 \ldots f_n$ is balanced.

We do not assume Γ finitely generated any more. Hence we only know that $\varphi_\Gamma(f) = \sum_{j=1}^{k} \big(\sum_{\substack{i: \\ \sigma(i) < \sigma(j)}} \mu_i - \sum_{i < j} \mu_i \big) \wedge \mu_j = 0$ in $^\ominus\bigwedge^2_{\mathbb{Z}} \tilde{\Gamma}$ (see Proposition 1.3.9). We denote by $\overline{\varphi_\Gamma(f)} :=$

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$\sum_{j=1}^{k} \big(\sum_{\substack{i: \\ \sigma(i)<\sigma(j)}} \mu_i - \sum_{i<j} \mu_i \big) \otimes \mu_j$. It is a representative of $\varphi_\Gamma(f)$ in $\bigotimes_{\mathbb{Z}}^2 \tilde{\Gamma}$. Then there exist a finite set J and $x_j, y_j \in \tilde{\Gamma}$ for every $j \in J$, such that $\overline{\varphi_\Gamma(f)} = \sum_{j\in J} x_j \otimes y_j + y_j \otimes x_j$. We denote by $\tilde{A}$ the subgroup of $\tilde{\Gamma}$ generated by $\{\mu_i\}_{1\le i\le k} \cup \{x_j, y_j\}_{j\in J}$. Then $\tilde{A}$ is a finitely generated subgroup of $\mathbb{R}$ which contains $\mathbb{Z}$. Its image A in $\mathbb{R}/\mathbb{Z}$ is a finitely generated subgroup of $\mathbb{R}/\mathbb{Z}$. Also we know that f is in $\mathrm{IET}(A)$ and (μ, σ) is also a A-combinatorial description of f and $\overline{\varphi_\Gamma(f)}$ is an element of $\bigotimes_{\mathbb{Z}}^2 \tilde{A}$. Thus in $^\ominus\bigwedge_{\mathbb{Z}}^2 \tilde{A}$ we have:

$$\varphi_A(f) = \sum_{j=1}^{k} \big(\sum_{\substack{i: \\ \sigma(i)<\sigma(j)}} \mu_i - \sum_{i<j} \mu_i \big) \wedge \mu_j = [\overline{\varphi_\Gamma(f)}]_{\ominus\bigwedge_{\mathbb{Z}}^2 \tilde{A}} = [\sum_{j\in J} x_j \otimes y_j + y_j \otimes x_j]_{\ominus\bigwedge_{\mathbb{Z}}^2 \tilde{A}} = 0$$

Then we can applied the previous case and conclude that f is a balanced product of A-restricted rotations, thus a balanced product of Γ-restricted rotations. $\qquad\square$

2.2.2 $\mathrm{Ker}(\varphi_\Gamma)$ is generated by Γ-transpositions

The work of Y.Vorobets [Vor17] done for IET can be adapted to show the next two lemmas. We reproduce here their proof.

Lemma 2.2.6. *Let f and g be two Γ-restricted rotations. If they have the same type then $f^{-1}g$ is finite a product of Γ-transpositions.*

Proof. The case $f = g$ is immediate and we assume that $f \neq g$. Thanks to Proposition 1.2.11, we can reduce to the case where f and g have a support whose length is less than $\frac{1}{5}$ thanks to a finite number of Γ-transpositions. Then there exists an interval I disjoint from $\mathrm{Supp}(f) \cup \mathrm{Supp}(g)$ and such that $\lambda(I) = \lambda(\mathrm{Supp}(f))$. Let h be the restricted rotation which has the type of f and such that its support is equal to I. Then $f^{-1}g = f^{-1}hh^{-1}g$. Then f and h has non-overlapping support, same is true for g and h, and Figure 2.2.1 proves that $f^{-1}h$ and $h^{-1}g$ are products of three Γ-transpositions. $\qquad\square$
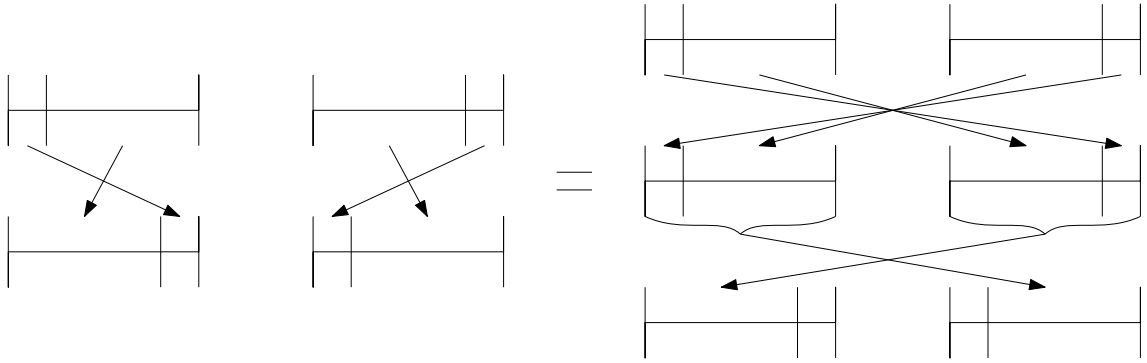
Figure 2.2.1 – Proof of Lemma 2.2.6 in the case where f and g have non-overlapping support.

Lemma 2.2.7. *Let f be a Γ-restricted rotation and g be any transformation in $\mathrm{IET}(\Gamma)$. Then the commutator $[f, g]$ is a product of Γ-transpositions.*

Proof. Let I be the support of f. If g is continuous on I then $gf^{-1}g^{-1}$ is a Γ-restricted rotation of support $g(I)$ with the same type of f^{-1} and by Lemma 2.2.6 we obtain the

result. Thanks to Proposition 1.2.11 there exists a product h of Γ-transposition such that hf is a Γ-restricted rotation such that g is continuous on $\operatorname{Supp}(hf)$. Then it is enough to see that we have the following equalities:

$$[f,g] = fgf^{-1}g^{-1} = h^{-1}(hfgf^{-1}h^{-1}g^{-1})(ghg^{-1}) = h^{-1}[hf,g]ghg^{-1}$$

Hence by the previous case $[hf,g]$ is a product of Γ-transpositions. Also we can always write a Γ-transposition into a product of Γ-transpositions with smaller supports. Hence the conjugate ghg^{-1} is a product of Γ-transpositions. $\qquad\square$

These lemmas with Proposition 1.2.13 give us the next theorem. It is proved by Y.Vorobets in the case $\Gamma = \mathbb{R}/\mathbb{Z}$ in [Vor17].

Theorem 2.2.8. *Every balanced product of Γ-restricted rotations can be written as a product of Γ-transpositions.*

Proof. Let $(f_1, f_2, \ldots, f_n)$ be a balanced tuple of restricted rotations. The proof is by strong induction on the length n of the tuple. If $n = 1$ then (f_1) is a balanced tuple of Γ-restricted rotations, thus f_1 is a Γ-restricted rotation of type (a,a) with $a \in \tilde{\Gamma}$ so it is also a Γ-transposition.
For the general case, let (a,b) be the type of f_1. If $a = b$ then f_1 is a Γ-transposition and $(f_2, f_3, \ldots, f_n)$ is a balanced tuple of restricted rotations. By the induction assumption we obtain the result. If $a \neq b$ then there exists $k \in \{2, \ldots, n\}$ such that f_k is a Γ-restricted rotation of type (b,a). Let $g_1 = f_2 \ldots f_{k-1}$ or $g_1 = \operatorname{Id}$ if $k = 2$. Let $g_2 = f_{k+1} \ldots f_n$ or $g_2 = \operatorname{Id}$ if $k = n$. Then we can write

$$f_1 f_2 \ldots f_n = f_1 g_1 f_k g_2 = (f_1 f_k)(f_k^{-1} g_1 f_k g_1^{-1})(g_1 g_2)$$

Hence, the induction assumption and Lemmas 2.2.6 and 2.2.7 give the result. $\qquad\square$

Corollary 2.2.9. *The kernel $\operatorname{Ker}(\varphi_\Gamma)$ is generated by the set of all Γ-transpositions.*

2.3   Description of the abelianization of $\operatorname{IET}(\Gamma)$

In this section we construct a surjective group homomorphism $\varepsilon_\Gamma : \operatorname{IET}(\Gamma) \to {}^\ominus\bigwedge_{\mathbb{Z}}^2 \tilde{\Gamma}$ whose kernel is the derived subgroup $D(\operatorname{IET}(\Gamma))$.

2.3.1   Boolean measures

In finite permutation groups there is a natural signature. One way to describe the signature is as follows: the signature of a finite permutation f, viewed in $\mathbb{Z}/2\mathbb{Z}$ is the number modulo 2 of pairs (x,y) such that $x < y$ and $f(x) > f(y)$. In our context where $f \in \operatorname{IET}(\Gamma)$, while this set is infinite, the idea is to measure it in a meaningful sense. We denote by $\operatorname{Itv}(\Gamma)$ the set of subintervals of $\mathbb{R}$ whose endpoints are in $\tilde{\Gamma}$.

Definition 2.3.1. Let A be a Boolean algebra and G be an abelian group. Let $\mu : A \to G$ be a finitely additive map: $\forall\, U, V \in A$ disjoint, $\mu(U \sqcup V) = \mu(U) + \mu(V)$. Such a μ is called a *Boolean algebra measure for A in G*.

Notation 2.3.2. We recall that $\operatorname{Itv}(\Gamma)$ is the set of all intervals $[a,b[$ with a and b in $\tilde{\Gamma}$ and $0 \leq a < b \leq 1$. Let A_Γ be the Boolean algebra of subsets of $[0,1[$ generated by $\operatorname{Itv}(\Gamma)$. Then A_Γ is a Boolean subalgebra of $\{0,1\}^{[0,1[}$. By noting λ the Lebesgue measure on $[0,1[$ we get that λ is a Boolean measure for A_Γ in $\tilde{\Gamma}$.

It might be useful to notice that for k in $\mathbb{N}_{\geq 1}$ and every $I_1, I_2, \ldots, I_k$ intervals in $\text{Itv}(\Gamma)$, the intersection $\bigcap_i I_i$ is still an element of $\text{Itv}(\Gamma)$. Moreover for every I in $\text{Itv}(\Gamma)$, the complement of I is the disjoint union of two elements of $\text{Itv}(\Gamma)$. Thus any Boolean combination of elements of $\text{Itv}(\Gamma)$ is a finite disjoint union of such elements.

Proposition 2.3.3. *Let X and Y be two sets, let A be a Boolean subalgebra of $\{0,1\}^X$ and let B be a Boolean subalgebra of $\{0,1\}^Y$. Let G and H be two abelian groups (we see them as $\mathbb{Z}$-modules), let $\mu : A \to G$ be a Boolean algebra measure for A in G and $\nu : B \to H$ be a Boolean algebra measure for B in H. Let $C := A \otimes B$ be the Boolean algebra product (generated by subsets of the form $a \times b$ with a in A and b in B). Then there exists a unique Boolean algebra measure $\omega : C \to G \otimes H$ for C in $G \otimes H$ such that for every a in A and b in B we have $\omega(a \times b) = \mu(a) \otimes \nu(b)$.*

Proof. Let ω_1 and ω_2 be two such Boolean algebra measures, thus they are equal on every $a \times b$ for $a \in A$ and $b \in B$. Let c be an element of C, then there exist k in $\mathbb{N}$ and $a_1, \ldots, a_k$ in A and $b_1, \ldots, b_k$ in B such that $c = \bigsqcup_{i=1}^{k} a_i \times b_i$. So $\omega_1(c) = \sum_{i=1}^{k} \omega_1(a_i \times b_i) = \sum_{i=1}^{k} \omega_2(a_i \times b_i) = \omega_2(c)$. Thus $\omega_1 = \omega_2$ and the unicity is proved.

It is sufficient to prove the existence for every finite Boolean subalgebra of C. Indeed if we assume that for every D finite Boolean subalgebra of C there exists a Boolean algebra measure m_D for D in $G \otimes H$ such that $m_D(a \times b) = \mu(a) \otimes \nu(b)$ for every a in A and b in B with $a \times b$ in D. Let c be an element of C. Then $\{0_C, c, \neg c, 1_C\}$ is a finite Boolean subalgebra of C non-trivial. Moreover if c is in $D_1 \cap D_2$ where D_1 and D_2 are two finite Boolean subalgebras of C then by noting D the Boolean subalgebra generated by D_1 and D_2 we get that D is a finite Boolean subalgebra of C containing c. Thus $m_D|_{D_1}$ is a Boolean measure for D_1 in $G \otimes H$ which satisfies $m_D|_{D_1}(a \times b) = \mu(a) \otimes \nu(b)$ for every a in A and b in B with $a \times b$ in D_1. By unicity we get $m_D|_{D_1} = m_{D_1}$ and the same argument gives $m_D|_{D_2} = m_{D_2}$ thus $m_{D_1}(c) = m_D(c) = m_{D_2}(c)$. So by putting $\omega(c) = m_D(c)$, the map ω is well-defined. also if we take two disjoint elements c and c' in C. Then by taking any finite Boolean subalgebra D of C which contains c and c' we get $m_D(c + c') = m_D(c) + m_D(c') = \omega(c) + \omega(c')$ and the value does not depend on D. Thus ω is the wanted Boolean algebra measure.

Let now D be a finite Boolean subalgebra of C. Then there exist $k, \ell \in \mathbb{N}$ and $a_1, \ldots, a_k \in A$ and $b_1, \ldots, b_\ell \in B$ such that every $d \in D$ is a Boolean combination of $a_i \times b_j$. Then let D' be the finite Boolean algebra generated by all the $a_i \times b_j$ with $1 \leq i \leq k$ and $1 \leq j \leq \ell$. Let U be the finite Boolean subalgebra of A generated by all a_i and let V be the finite Boolean subalgebra of B generated by all b_j. Then U and V are atomic. Let $u_1, \ldots u_n$ be the atoms of U and $v_1, \ldots v_m$ be the atoms of V. Hence D' is atomic with atoms given by $u_i \times v_j$ for every $1 \leq i \leq k$ and $1 \leq j \leq \ell$. Then for each element d in D' there exists a unique $J_d \subset \{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$ such that $d = \bigsqcup_{(i,j) \in J_d} u_i \otimes v_j$. Hence the map $m_{D'}$ defined by $m_{D'}(d) = m_{D'}(\bigsqcup_{(i,j) \in J_d} u_i \times v_j) = \sum_{(i,j) \in J_d} \mu(u_i) \otimes \nu(v_j)$ is well-defined, finitely additive and satisfies $m_{D'}(a \times b) = \mu(a) \otimes \nu(b)$ for every $a \in A$, $b \in B$ such that $a \times b \in D'$. $\qquad \square$

Notation 2.3.4. By applying the previous proposition with $X = Y = [0,1[$ and $A = B = A_\Gamma$ and $\mu = \nu = \lambda$, there exists a unique Boolean algebra measure $\omega_\Gamma : A_\Gamma \otimes A_\Gamma \to \bigotimes_{\mathbb{Z}}^2 \tilde{\Gamma}$ such that for every a, b, c and d in $\tilde{\Gamma}_+$ with $a < b \leq 1$ and $c < d \leq 1$ we have $\omega_\Gamma([a,b[\times [c,d[) = (b-a) \otimes (d-c)$.

We need to check some $\text{IET}(\Gamma)$-invariance for the measure ω_Γ. For this we define an action of $\text{IET}(\Gamma)$ on $[0,1[\times [0,1[$ by $f.(x,y) = (f(x), f(y))$. Hence for every P in A_Γ we have $f.P$ in A_Γ, this gives us a new Boolean algebra measure $f_*\omega_\Gamma$.

Proposition 2.3.5. *For every f in $\mathrm{IET}(\Gamma)$ and every P in $A_\Gamma \otimes A_\Gamma$ we have $f.P :=$ $\{(f(x), f(y)) \mid (x,y) \in P\}$ in $A_\Gamma \otimes A_\Gamma$. Furthermore we have $f_* \omega_\Gamma = \omega_\Gamma$.*

Proof. Let $f \in \mathrm{IET}(\Gamma)$ and (μ, σ) be a Γ-combinatorial description of f and let $\{I_1, \ldots, I_n\}$ be the partition into intervals associated. Let P be an element of $A_\Gamma \otimes A_\Gamma$. There exist m in $\mathbb{N}$ and pairwise disjoint elements $p_1, p_2, \ldots, p_m$ of $\mathrm{Itv}(\Gamma) \times \mathrm{Itv}(\Gamma)$ such that $P = \bigsqcup_{i=1}^{k} p_i$. As f is a permutation of $[0,1[$ we get $f. \bigsqcup_{i=1}^{m} p_i = \bigsqcup_{i=1}^{m} f.p_i$, so it is enough to show that $f.p$ belongs to $A_\Gamma \otimes A_\Gamma$. For $i \in \{1, \ldots, k\}$, let $a_i, b_i, c_i, d_i \in \tilde{\Gamma}$ such that $p_i = [a_i, b_i[\times [c_i, d_i[$. Then $f.p_i = \bigsqcup_{(k,l)} f([a_i, b_i[\cap I_k) \times f([c_i, d_i[\cap I_l)$ which is a finite disjoint union of elements of $\mathrm{Itv}(\Gamma) \times \mathrm{Itv}(\Gamma)$ because a_i, b_i, c_i, d_i and extremities of I_l are in $\tilde{\Gamma}$ for each $1 \leq l \leq n$. In conclusion $f.p_i$ is in $A_\Gamma \otimes A_\Gamma$ thus $f.P$ is in $A_\Gamma \otimes A_\Gamma$.

Also f is piecewise a translation and λ is the Lebesgue measure, so for any J in $\mathrm{Itv}(\Gamma)$ we have $\lambda(J) = \lambda(f(J))$. Thus:

$$\omega_\Gamma(f.p_i) = \sum_{(k,l)} \omega_\Gamma(f([a_i, b_i[\cap I_k) \times f([c_i, d_i[\cap I_l))$$

$$= \sum_{(k,l)} \lambda(f([a_i, b_i[\cap I_k)) \otimes \lambda(f([c_i, d_i[\cap I_l))$$

$$= (\sum_k \lambda([a_i, b_i[\cap I_k)) \otimes (\sum_l \lambda([c_i, d_i[\cap I_l))$$

$$= \lambda([a_i, b_i[) \otimes \lambda([c_i, d_i[)$$

$$= \omega_\Gamma(p_i)$$

This gives us $\omega_\Gamma(f.P) = \sum_{i=1}^{m} \omega_\Gamma(f.p_i) = \sum_{i=1}^{m} \omega_\Gamma(p_i) = \omega_\Gamma(P)$. Hence $\omega_\Gamma = f_*^{-1} \omega_\Gamma$. As f^{-1} is also in $\mathrm{IET}(\Gamma)$ we can do the same to deduce $\omega_\Gamma = f_* \omega_\Gamma$. $\qquad \square$

2.3.2 Creation of a signature

Definition 2.3.6. Let f be a transformation in $\mathrm{IET}(\Gamma)$. Every pair (x, y) in $[0,1[\times [0,1[$ such that $x < y$ and $f(x) > f(y)$ is called an *inversion of f*. We denote by $\mathcal{E}_f$ the set of all inversions of f.

Proposition 2.3.7. *Let f be a transformation in $\mathrm{IET}(\Gamma)$ and (μ, τ) be a combinatorial description of f. Let $\{I_1, I_2, \ldots, I_k\}$ be the partition into intervals associated with (μ, τ) (see Definition 1.2.6). We have $\mathcal{E}_f = \bigsqcup_{j=1}^{n} \bigsqcup_{\substack{i < j \\ \tau(i) > \tau(j)}} I_i \times I_j$.*

Proof. Let (μ, τ) be a Γ-combinatorial description of f and let $\{I_1, I_2, \ldots, I_k\}$ be the partition into intervals associated. Let (x_0, y_0) be an element of $\mathcal{E}_f$. Then there exist $i, j \in \{1, 2, \ldots, k\}$ such that $x_0 \in I_i$ and $y_0 \in I_j$. As $x_0 < y_0$ we have $i \leq j$. Furthermore if $i = j$ then as f is an isometry which preserves the order on I_i we get $f(x_0) < f(y_0)$ which is a contradiction, we deduce that $i < j$. By definition of f we have $f(I_i), f(I_j) \in \mathrm{Itv}(\Gamma)$ and they are disjoint. Thus as $f(x_0) > f(y_0)$ we deduce that for every $x \in I_i$ and $y \in I_j$ we have $x < y$ and $f(x) > f(y)$, so $I_i \times I_j \subset \mathcal{E}_f$. Also, this implies $\sigma(i) > \sigma(j)$ and we deduce that $\bigsqcup_{j=1}^{k} \bigsqcup_{\substack{i < j \\ \tau(i) > \tau(j)}} I_i \times I_j = \mathcal{E}_f$. $\qquad \square$

Corollary 2.3.8. *For every $f \in \mathrm{IET}(\Gamma)$ we have $\mathcal{E}_f \in A_\Gamma \otimes A_\Gamma$.*

We denote by p the projection from $\bigotimes^2_{\mathbb{Z}} \tilde{\Gamma}$ into ${}^{\ominus}\bigwedge^2_{\mathbb{Z}} \tilde{\Gamma}$.

Definition 2.3.9. The *signature for* IET(Γ) is the following map:

$$\varepsilon_\Gamma : \begin{array}{ccc} \text{IET}(\Gamma) & \longrightarrow & {}^{\ominus}\bigwedge^2_{\mathbb{Z}} \tilde{\Gamma} \\ f & \longmapsto & p \circ \omega_\Gamma(\mathcal{E}_f) \end{array}$$

Proposition 2.3.10. *For every A and B in A_Γ we have:*

$$p \circ \omega_\Gamma(A \times B) = -p \circ \omega_\Gamma(B \times A)$$

Proof. Let $A, B \in A_\Gamma$ then:

$$\begin{aligned} p \circ \omega_\Gamma(A \times B) &= \lambda(A) \wedge \lambda(B) \\ &= -\lambda(B) \wedge \lambda(A) \\ &= -p \circ \omega_\Gamma(B \times A) \end{aligned}$$

$\square$

Theorem 2.3.11. *The map ε_Γ is a group homomorphism.*

Proof. Let $f, g \in \text{IET}(\Gamma)$. We denote by s the symmetry of axis $y = x$. We remark that every element I of $A_\Gamma \otimes A_\Gamma$ satisfies $s(I) \in A_\Gamma \otimes A_\Gamma$. Then $\{(x,y) \mid x < y,\ g(x) > g(y),\ fg(x) < fg(y)\} = \mathcal{E}_g \cap sg^{-1}(\mathcal{E}_f)$ is an element of $A_\Gamma \otimes A_\Gamma$. We also notice that $\{(x,y) \mid x < y,\ g(x) < g(y),\ fg(x) > fg(y)\} = \mathcal{E}_{f \circ g} \cap \mathcal{E}_g^c$ and $\{(x,y) \mid x < y,\ g(x) > g(y),\ fg(x) > fg(y)\} = \mathcal{E}_{f \circ g} \cap \mathcal{E}_g$ are two elements of $A_\Gamma \otimes A_\Gamma$.

For more clarity we do some calculus first. By Proposition 2.3.5 and Proposition 2.3.10 we get:

$$\begin{aligned} &- p \circ \omega_\Gamma(\{(x,y) \mid x < y,\ g(x) > g(y),\ fg(x) < fg(y)\}) \\ =\ &- p \circ \omega_\Gamma(\{(g(x), g(y)) \mid x < y,\ g(x) > g(y),\ fg(x) < fg(y)\}) \\ =\ & p \circ \omega_\Gamma(\{(g(y), g(x)) \mid x < y,\ g(x) > g(y),\ fg(x) < fg(y)\}) \\ =\ & p \circ \omega_\Gamma(\{(u,v) \mid g^{-1}(u) > g^{-1}(v),\ u < v,\ f(u) > f(v)\}) \end{aligned}$$

and

$$\begin{aligned} & p \circ \omega_\Gamma(\{(x,y) \mid x < y,\ g(x) < g(y),\ fg(x) > fg(y)\}) \\ =\ & p \circ \omega_\Gamma(\{(g(x), g(y)) \mid x < y,\ g(x) < g(y),\ fg(x) > fg(y)\}) \\ =\ & p \circ \omega_\Gamma(\{(u,v) \mid g^{-1}(u) < g^{-1}(v),\ u < v,\ f(u) > f(v)\}) \end{aligned}$$

In addition:

$$\begin{aligned} & p \circ \omega_\Gamma(\{(u,v) \mid g^{-1}(u) > g^{-1}(v),\ u < v,\ f(u) > f(v)\}) \\ & \quad + p \circ \omega_\Gamma(\{(u,v) \mid g^{-1}(u) < g^{-1}(v),\ u < v,\ f(u) > f(v)\}) \\ =\ & p \circ \omega_\Gamma(\mathcal{E}_f) \\ =\ & \varepsilon_\Gamma(f) \end{aligned}$$

Hence by adding and remove the same quantity at the fourth equality we obtain:

$$\begin{aligned}
\varepsilon_\Gamma(f \circ g) &= p \circ \omega_\Gamma(\{(x,y) \mid x < y,\ fg(x) > fg(y)\} \\
&= p \circ \omega_\Gamma(\{(x,y) \mid x < y,\ g(x) > g(y),\ fg(x) > fg(y)\} \\
&\qquad \sqcup \{(x,y) \mid x < y,\ g(x) < g(y),\ fg(x) > fg(y)\}) \\
&= p \circ \omega_\Gamma(\{(x,y) \mid x < y,\ g(x) > g(y),\ fg(x) > fg(y)\}) \\
&\qquad + p \circ \omega_\Gamma(\{(x,y) \mid x < y,\ g(x) < g(y),\ fg(x) > fg(y)\}) \\
&= p \circ \omega_\Gamma(\{(x,y) \mid x < y,\ g(x) > g(y),\ fg(x) > fg(y)\}) \\
&\qquad + p \circ \omega_\Gamma(\{(x,y) \mid x < y,\ g(x) > g(y),\ fg(x) < fg(y)\}) \\
&\qquad - p \circ \omega_\Gamma(\{(x,y) \mid x < y,\ g(x) > g(y),\ fg(x) < fg(y)\}) \\
&\qquad + p \circ \omega_\Gamma(\{(x,y) \mid x < y,\ g(x) < g(y),\ fg(x) > fg(y)\}) \\
&= p \circ \omega_\Gamma(\mathcal{E}_g) + p \circ \omega_\Gamma(\mathcal{E}_f) \\
&= \varepsilon_\Gamma(g) + \varepsilon_\Gamma(f)
\end{aligned}$$

In conclusion, ε_Γ is additive thus it is a group homomorphism. $\qquad\square$

Proposition 2.3.12. *Let $a \in \tilde{\Gamma}_+$ with $a \leq \frac{1}{2}$ and f be a Γ-transposition of type a. Then $\varepsilon_\Gamma(f) = a \wedge a$.*

Proof. Let $u, v, b \in \tilde{\Gamma}$ such that $((u, a, b, a, v), (2\ 4))$ is a Γ-combinatorial description of f (see Figure 2.3.1). Let $\{I_1, \ldots, I_5\}$ be the partition into intervals associated. We already proved in 2.3.7 that it is sufficient to check if a pair $(x, y) \in I_i \times I_j$ is in $\mathcal{E}_f$ to know that $I_i \times I_j$ is in $\mathcal{E}_f$. We also have $I_i \times I_j \notin \mathcal{E}_f$ if $j \leq i$. Thus one can look at the graph of f to find that $\mathcal{E}_f$ is equal to the tiling space on Figure 2.3.1. We deduce that $\varepsilon_\Gamma(f) = a \wedge b + a \wedge a + b \wedge a = a \wedge a$. $\qquad\square$
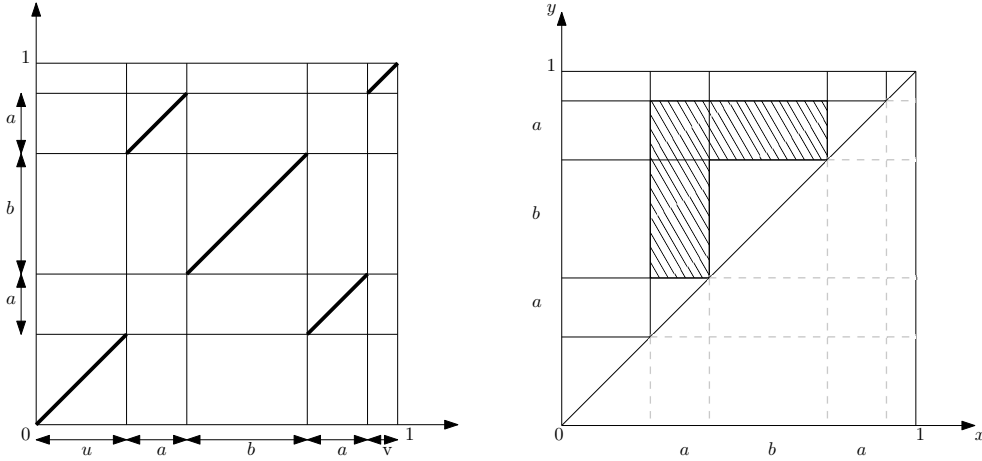
Figure 2.3.1 – Illustration of how to calculate the value of ε_Γ on a Γ-transposition f in Proposition 2.3.12. **Left**: The graph of f. **Right**: The set $\mathcal{E}_f$.

Thanks to Proposition 2.3.7 we can calculate the value of ε_Γ for every $f \in \mathrm{IET}(\Gamma)$:

Proposition 2.3.13. *Let $f \in \mathrm{IET}(\Gamma)$ and (α, τ) be a Γ-combinatorial description of f. Let n be the length of α. Then*

$$\varepsilon_\Gamma(f) = \sum_{j=1}^{n} \sum_{\substack{i<j \\ \tau(i) > \tau(j)}} \alpha_i \wedge \alpha_j$$

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Proposition 2.3.14. *The group homomorphism ε_Γ is surjective.*

Proof. Let $a, b \in \tilde{\Gamma}$. We assume that $0 \leq a, b < 1$ and $0 \leq a + b \leq 1$. Let r be the IET(Γ) restricted rotation of type (a, b), whose intervals associated are $[0, a[$ and $[a, a+b[$. Then we obtain that $\varepsilon_\Gamma(r) = a \wedge b$.

In the general case, let $w \in \tilde{\Gamma}$ with $0 \leq w \leq \frac{1}{2}$. Then there exist $k, \ell \in \mathbb{Z}$ and $a', b' \in \tilde{\Gamma}$ with $0 \leq a', b' < w$ such that $a = kw + a'$ and $b = \ell w + b'$. Then $a \wedge b = k\ell w \wedge w + kw \wedge b' + \ell a' \wedge w + a' \wedge b'$. By the previous case and as ε_Γ is a group homomorphism, we deduce that $a \wedge b$ is in $\mathrm{Im}(\varepsilon_\Gamma)$. $\qquad\square$

2.3.3 Description of $\mathrm{Ker}(\varepsilon_\Gamma)$

The aim of this part is to conclude that $\mathrm{Ker}(\varepsilon_\Gamma) = D(\mathrm{IET}(\Gamma))$ and the induced morphism $\mathrm{IET}(\Gamma)_{\mathrm{ab}} \to {}^{\ominus}\bigwedge_{\mathbb{Z}}^2 \tilde{\Gamma}$ is an isomorphism. We recall that φ_Γ is the restriction of the SAF-homomorphism φ to $\mathrm{IET}(\Gamma)$. The image of φ_Γ is in $\bigwedge_{\mathbb{Q}}^2 \mathbb{R}$ which is equal to ${}^{\ominus}\bigwedge_{\mathbb{Z}}^2 \mathbb{R}$. Let ρ be the canonical map from ${}^{\ominus}\bigwedge_{\mathbb{Z}}^2 \tilde{\Gamma}$ to $\bigwedge_{\mathbb{Q}}^2 \mathbb{R}$ induced by the inclusion $\tilde{\Gamma} \subset \mathbb{R}$.

Lemma 2.3.15. *We have $2\rho \circ \varepsilon_\Gamma = -\varphi_\Gamma$.*

Proof. Let $f \in \mathrm{IET}(\Gamma)$ and (α, τ) be a Γ-combinatorial description of f. Let n be the length of α. Thanks to Propositions 1.3.9 and 2.3.13 we have:

$$
\begin{aligned}
\varphi_\Gamma(f) &= \sum_{j=1}^n \Big(\sum_{\substack{i \\ \tau(i) < \tau(j)}} \alpha_i - \sum_{i < j} \alpha_i \Big) \wedge \alpha_j \\
&= \sum_{j=1}^n \sum_{\substack{i > j \\ \tau(i) < \tau(j)}} \alpha_i \wedge \alpha_j + \sum_{j=1}^n \Big(\sum_{\substack{i < j \\ \tau(i) < \tau(j)}} \alpha_i - \sum_{i < j} \alpha_i \Big) \wedge \alpha_j \\
&= \sum_{i=1}^n \sum_{\substack{j < i \\ \tau(j) > \tau(i)}} \alpha_i \wedge \alpha_j - \sum_{j=1}^n \sum_{\substack{i < j \\ \tau(i) > \tau(j)}} \alpha_i \wedge \alpha_j \\
&= -2 \sum_{i=1}^n \sum_{\substack{j < i \\ \tau(j) > \tau(i)}} \alpha_j \wedge \alpha_i \\
&= -2\, \rho \circ \varepsilon_\Gamma(f) \qquad\qquad\qquad \square
\end{aligned}
$$

Corollary 2.3.16. *We have the inclusion $\mathrm{Ker}(\varepsilon_\Gamma) \subset \mathrm{Ker}(\varphi_\Gamma)$.*

By Corollary 2.2.9 we know that $\mathrm{Ker}(\varphi_\Gamma)$ is generated by the set of all Γ-transpositions. Thus it is natural to look at these elements who are also in $\mathrm{Ker}(\varepsilon_\Gamma)$. If σ is a Γ-transposition of type a in $\mathrm{Ker}(\varepsilon_\Gamma)$ then we have the equality $a \wedge a = 0$. We want to prove that σ is in $D(\mathrm{IET}(\Gamma))$ if and only if $a \in 2\tilde{\Gamma}$.

We denote by $\mathsf{S}_{\mathbb{Z}}^2 \tilde{\Gamma}$ the second symmetric power of $\tilde{\Gamma}$ and we denote by $a \odot a$ image of $a \otimes a$ in $\mathsf{S}_{\mathbb{Z}}^2 \tilde{\Gamma}$.

For every group G and every $w \in G$ we use the notation $w \, [\mathrm{mod}\ 2]$ for the image of w in $G/2G$.

Lemma 2.3.17. *For every group G, the group ${}^{\ominus}\bigwedge_{\mathbb{Z}}^2 G / 2({}^{\ominus}\bigwedge_{\mathbb{Z}}^2 G)$ is naturally isomorphic to $\mathsf{S}_{\mathbb{Z}}^2 G / 2(\mathsf{S}_{\mathbb{Z}}^2 G)$.*

Proof. We have a surjective group homomorphism from $G \otimes G$ onto $\mathsf{S}_{\mathbb{Z}}^2 G / 2(\mathsf{S}_{\mathbb{Z}}^2 G)$. We notice that for every $x, y \in G$, the element $x \otimes y + y \otimes x$ is sent to $2(x \odot y) \, [\mathrm{mod}\ 2] = 0$. Hence we deduce a surjective group homomorphism from ${}^{\ominus}\bigwedge_{\mathbb{Z}}^2 G$ onto $\mathsf{S}_{\mathbb{Z}}^2 G / 2(\mathsf{S}_{\mathbb{Z}}^2 G)$. It is

immediate that $2(^\ominus\bigwedge^2_\mathbb{Z} G)$ is sent to zero. Thus we have a surjective group homomorphism from $^\ominus\bigwedge^2_\mathbb{Z} G/2(^\ominus\bigwedge^2_\mathbb{Z} G)$ onto $\mathsf{S}^2_\mathbb{Z} G/2(\mathsf{S}^2_\mathbb{Z} G)$. In order to show that it is injective we take $w \in G \otimes G$ such that $[w]_{\mathsf{S}^2_\mathbb{Z} G} \; [\mathrm{mod}\; 2] = 0 = 0$ and we show that $[w]_{^\ominus\bigwedge^2_\mathbb{Z} G} \; [\mathrm{mod}\; 2] = 0$. There exist $n \in \mathbb{N}$ and $a_1, \dots a_n, b_1, \dots, b_n \in G$ such that $w + \sum_{i=1}^{n} a_i \otimes b_i - b_i \otimes a_i = 0$. Hence $[w]_{^\ominus\bigwedge^2_\mathbb{Z} G} + 2\sum a_i \wedge b_i = 0$. Finally $[w]_{^\ominus\bigwedge^2_\mathbb{Z} G} \; [\mathrm{mod}\; 2] = 0$ thus we have an isomorphism between $^\ominus\bigwedge^2_\mathbb{Z} G/2(^\ominus\bigwedge^2_\mathbb{Z} G)$ and $\mathsf{S}^2_\mathbb{Z} G/2\mathsf{S}^2_\mathbb{Z} G$. $\qquad\square$

Proposition 2.3.18. *Let $a \in \tilde{\Gamma}$, if $a \wedge a \; [\mathrm{mod}\; 2] = 0$ then a belongs to $2\tilde{\Gamma}$.*

Proof. For every group G, the group $^\ominus\bigwedge^2_\mathbb{Z} G/2(^\ominus\bigwedge^2_\mathbb{Z} G)$ is naturally isomorphic to the second symmetric power $\mathsf{S}^2_\mathbb{Z} G/2(\mathsf{S}^2_\mathbb{Z} G)$. This comes from the fact that these groups satisfy the following universal property: for every group G and every abelian elementary 2-group We denote i the natural inclusion of $G \times G$ into $\mathsf{S}^2_\mathbb{Z} G/2(\mathsf{S}^2_\mathbb{Z} G)$. For every bilinear symmetric group homomorphism $b : G \times G \to A$ there exists a unique group homomorphism $f : \mathsf{S}^2_\mathbb{Z} G/2(\mathsf{S}^2_\mathbb{Z} G) \to A$ such that for every $g, h \in G$ we have $b(g, h) = f(i(g, h))$.

Let $a \in \tilde{\Gamma}$ with $a \neq 0$ (because we already have $0 = 2 \times 0$). We denote by $a \odot a$ the image of $a \otimes a$ in $\mathsf{S}^2_\mathbb{Z} G/2(\mathsf{S}^2_\mathbb{Z} G)$, and we assume that $a \odot a \; [\mathrm{mod}\; 2] = 0$. The projection $\tilde{\Gamma} \to \tilde{\Gamma}/2\tilde{\Gamma}$ gives rise to a morphism $\zeta : \mathsf{S}^2_\mathbb{Z} \tilde{\Gamma} \to \mathsf{S}^2_\mathbb{Z}(\tilde{\Gamma}/2\tilde{\Gamma})$. As $2(\mathsf{S}^2_\mathbb{Z} \tilde{\Gamma}) \subset \mathrm{Ker}(\zeta)$ we obtain a morphism $\zeta' : \mathsf{S}^2_\mathbb{Z} \tilde{\Gamma}/2\mathsf{S}^2_\mathbb{Z} \tilde{\Gamma} \to \mathsf{S}^2_\mathbb{Z}(\tilde{\Gamma}/2\tilde{\Gamma})$. Hence if $a \notin 2\tilde{\Gamma}$ then $\zeta(a \odot a) \neq 0$ thus $\zeta'(a \odot a \; [\mathrm{mod}\; 2]) \neq 0$ which is a contradiction with the assumption. In conclusion $a \in 2\tilde{\Gamma}$. $\qquad\square$

Corollary 2.3.19. *Every Γ-transposition f in $\mathrm{Ker}(\varepsilon_\Gamma)$ is in $D(\mathrm{IET}(\Gamma))$.*

Proof. Let $a \in \tilde{\Gamma}_+$, with $a \leq \frac{1}{2}$, be the type of f and let $u, v \in \tilde{\Gamma}_+$ such that $I_1 = [u, u+a[$ and $I_2 = [v, v+a[$ are the two intervals swapped by f. From $f \in \mathrm{Ker}(\varepsilon)$ we deduce that $\varepsilon_\Gamma(f) = a \wedge a = 0$. Hence $a \wedge a \; [\mathrm{mod}\; 2] = 0$ in $^\ominus\bigwedge^2_\mathbb{Z} \tilde{\Gamma}/2(^\ominus\bigwedge^2_\mathbb{Z} \tilde{\Gamma})$. Then by Proposition 2.3.18 there exists $b \in \tilde{\Gamma}$ such that $a = 2b$. Thus if we define g as the unique Γ-transposition of type b that swaps $[u, u+b[$ and $[v, v+b[$ and h as the unique $\mathrm{IET}(\Gamma)$ that swaps $[u, u+b[$ with $[u+b, u+a[$ and $[v, v+b[$ with $[v+b, v+a[$. Then $f = ghgh$ and as $g^2 = h^2 = \mathrm{Id}$ we deduce that $f \in D(\mathrm{IET}(\Gamma))$. $\qquad\square$

In order to show that $\mathrm{Ker}(\varepsilon_\Gamma) = D(\mathrm{IET}(\Gamma))$ we prove that any element $f \in \mathrm{Ker}(\varepsilon_\Gamma)$ can be written as $f = \sigma h$ where $h \in D(\mathrm{IET}(\Gamma))$ and σ is a Γ-transposition. This concludes because we just show that a Γ-transposition which is also in $\mathrm{Ker}(\varepsilon_\Gamma)$ is in $D(\mathrm{IET}(\Gamma))$. We begin by a particular case of a product of Γ-transpositions with pairwise disjoint support. The aim will be to reduce the general case to this one. We recall that the identity is considered as a Γ-transposition.

Lemma 2.3.20. *Let $k \in \mathbb{N}$ and $\tau_1, \tau_2, \dots, \tau_k$ be Γ-transpositions with pairwise disjoint support. Then $\tau_1\tau_2 \dots \tau_k = \sigma h$ where σ is a Γ-transposition and h is an element of $D(\mathrm{IET}(\Gamma))$. Moreover the support of h and σ do not overlap and are included in the union of the supports of the τ_i.*

Proof. By induction it is enough to show the result in the case $k = 2$ Let respectively a_1 and a_2 be the type of τ_1 and τ_2. As their support do not overlap we know that τ_1 and τ_2 commute. Hence we can assume $a_1 \geq a_2$ without loss of generality. If $a_1 = a_2$ then there exists $f \in \mathrm{IET}(\Gamma)$ such that $\tau_1 = f\tau_2 f^{-1}$. Thus $\tau_1\tau_2$ is in $D(\mathrm{IET}(\Gamma))$. If $a_1 > a_2$ let $u, v \in \tilde{\Gamma}$ such that $[u, u+a_1[$ and $[v, v+a_1[$ are the intervals swapped by τ_1. Let g and h be Γ-transpositions such that g swaps the intervals $[u, u+a_2[$ and $[v, v+a_2[$ and h swaps the intervals $[u+a_2, u+a_1[$ and $[v+a_2, v+a_1[$. Thus $\tau_1 = hg$. Moreover g and τ_2 are two Γ-transpositions with same type and non-overlapping support. Then by the previous case, the product $g\tau_2$ is in $D(\mathrm{IET}(\Gamma))$ and its support does not intersect the support of h. Then $f = h(g\tau_2)$ is the wanted decomposition. $\qquad\square$

We can now treat the case of finite order elements:

Lemma 2.3.21. *If $f \in \mathrm{IET}(\Gamma)$ has finite order then there exist a Γ-transposition σ and $h \in D(\mathrm{IET}(\Gamma))$ such that the support of σ and h are inside the support of f and $f = \sigma h$.*

Proof. Let $f \in \mathrm{IET}(\Gamma)$ as in the statement. By Proposition 1.2.24 there exist $n \in \mathbb{N}$ and $\mathcal{P} = \{I_1, I_2, \ldots, I_n\} \in \Pi_f$ and $\sigma \in \mathfrak{S}_n$ such that $f(I_i) = I_{\sigma(i)}$. Let $k \in \mathbb{N}$ and $\sigma = c_1 c_2 \ldots c_k$ be the disjoint cycle decomposition for σ. Let f_i be the element of $\mathrm{IET}(\Gamma)$ that is equal to f on I_j for every $j \in \mathrm{Supp}(c_i)$ while fixing the rest of $[0, 1[$. Then $f = f_1 \ldots f_k$ and f_i commutes with f_j for every $1 \le i \ne j \le k$. Then if the statement is true for every f_i we can write $f_i = \tau_i h_i$ with τ_i a Γ-transposition and $h \in D(\mathrm{IET}(\Gamma))$, both of them with support inside the support of f_i. Then $f = f_1 \ldots f_k = \tau_1 h_1 \ldots \tau_k h_k = \tau_1 \ldots \tau_k h_1 \ldots h_k$ because the support of τ_i does not overlap with the support of τ_j of h_j for every $1 \le j \le k$ and $j \ne i$. We conclude with Lemma 2.3.20 applied to $\tau_1 \ldots \tau_k$.
Let c by a cycle of length $n \ge 2$ and let $I_1, I_2, \ldots, I_n$ be non-overlapping intervals of $\mathrm{Itv}(\Gamma)$ of same length. Let $f \in \mathrm{IET}(\Gamma)$ be the element that permutes the set $\{I_1, I_2, \ldots, I_n\}$ by c. Then if $c \in D(\mathfrak{S}_n) = \mathfrak{A}_n$ we deduce that $f \in D(\mathrm{IET}(\Gamma))$. If $c \notin D(\mathfrak{S}_n)$ then let g be the unique Γ-transposition that swaps I_1 with $I_{c(1)}$ (we notice that the support of g is included in the support of f). By the previous case, $gf \in D(\mathrm{IET}(\Gamma))$ and we conclude that $f = g(gf)$ is a wanted decomposition. $\qquad\square$

Corollary 2.3.22. *Let τ and τ' be two Γ-transpositions. There exist a Γ-transposition σ and $h \in D(\mathrm{IET}(\Gamma))$ such that $\tau\tau' = \sigma h$ and the support of σ and h are included in the union of the support of τ and the one of τ'.*

Proof. By Lemma 1.2.30 we deduce that $f := \tau\tau'$ has finite order. Hence by Lemma 2.3.21 we obtain the result. $\qquad\square$

Lemma 2.3.23. *Let k in $\mathbb{N}$ and $\tau_1, \tau_2, \ldots, \tau_k$ be some Γ-transpositions. Then there exist a Γ-transposition σ and $h \in D(\mathrm{IET}(\Gamma))$ such that $\tau_1 \tau_2 \ldots \tau_k = \sigma h$.*

Proof. The proof is by induction on k. The initialisation $k = 1$ is immediate. The case $k = 2$ is Corollary 2.3.22. Now if we assume the result for $k \ge 2$ let $\tau_1, \tau_2, \ldots, \tau_k, \tau_{k+1}$ be Γ-transpositions. Then by assumption, applied to $\tau_2 \tau_3 \ldots \tau_{k+1}$, there exist a Γ-transposition σ and $h \in D(\mathrm{IET}(\Gamma))$ such that $\tau_2 \tau_3 \ldots \tau_{k+1} = \sigma h$. Hence $\tau_1 \tau_2 \ldots \tau_{k+1} = \tau_1 \sigma h$. By using the case $k = 2$ we deduce that there exist a Γ-transposition σ' and $h' \in D(\mathrm{IET}(\Gamma))$ such that $\tau_1 \sigma = \sigma' h'$. Thus $\tau_1 \tau_2 \ldots \tau_{k+1} = \sigma' g$, with $g = h'h \in D(\mathrm{IET}(\Gamma))$, which is a wanted decomposition. $\qquad\square$

Finally we can prove the main theorem of the section:

Theorem 2.3.24. *We have the equality $\mathrm{Ker}(\varepsilon_\Gamma) = D(\mathrm{IET}(\Gamma))$, and the induced morphism $\mathrm{IET}(\Gamma)_{\mathrm{ab}} \to {}^{\ominus}\bigwedge_{\mathbb{Z}}^2 \tilde{\Gamma}$ is an isomorphism.*

Proof. The inclusion from right to left is immediate. For the other inclusion let $f \in \mathrm{Ker}(\varepsilon_\Gamma)$. By Corollary 2.3.16 we know that $f \in \mathrm{Ker}(\varphi_\Gamma)$, then by Corollary 2.2.9 there exists $k \in \mathbb{N}$ such that f is equal to the product $\tau_1 \tau_2 \ldots \tau_k$ where τ_i is a Γ-transposition. By Lemma 2.3.23 there exist a Γ-transposition σ and $h \in D(\mathrm{IET}(\Gamma))$ such that $f = \sigma h$. Then $\varepsilon_\Gamma(f) = \varepsilon_\Gamma(\sigma)\varepsilon_\Gamma(h) = \varepsilon_\Gamma(\sigma) = 0$. By Corollary 2.3.19 we deduce $\sigma \in D(\mathrm{IET}(\Gamma))$. Hence $f = \sigma h \in D(\mathrm{IET}(\Gamma))$.
We deduce that the induced group homomorphism $\varepsilon'_\Gamma : \mathrm{IET}(\Gamma)_{\mathrm{ab}} \to {}^{\ominus}\bigwedge_{\mathbb{Z}}^2 \tilde{\Gamma}$ is injective. Furthermore ε_Γ is surjective by Proposition 2.3.14 thus ε'_Γ is surjective and we conclude that ε'_Γ is an isomorphism. $\qquad\square$

Abelianization of $\mathrm{IET}^{\bowtie}(\Gamma)$

Résumé en français. Pour tout sous-groupe $\mathrm{IET}^{\bowtie}(\Gamma)$ on établit un isomorphisme naturel entre son abélianisé et le 2-groupe $\langle\{a\otimes a\ [\mathrm{mod}\ 2]\mid a\in\tilde{\Gamma}\}\rangle\times\langle\{\ell\wedge\ell\ [\mathrm{mod}\ 2]\mid\ell\in\tilde{\Gamma}\}\rangle$. Ce groupe est un sous-groupe abélien élémentaire de $\bigotimes^2_{\mathbb{Z}}\tilde{\Gamma}/(2\bigotimes^2_{\mathbb{Z}}\tilde{\Gamma})\times{}^{\ominus}\bigwedge^2_{\mathbb{Z}}\tilde{\Gamma}/(2^{\ominus}\bigwedge^2_{\mathbb{Z}}\tilde{\Gamma})$. Pour obtenir ce résultat, on construit deux morphismes de groupes. L'un des deux est un analaogue du morphisme signature ε_Γ construit dans la section 2.3. L'ensemble des inversions ne peut plus tous le temps être mesuré avec la mesure construite dans la section 2.3.1, toutefois l'union de cet ensemble avec son symétrique peut être mesuré. Le noyau du morphisme signature est légèrement plus gros que le sous-groupe dérivé $D(\mathrm{IET}^{\bowtie}(\Gamma))$; en fait, il manque quelques renversements d'un certain type. On remarque que ces renversements qui manquent sont conjugués à une rotation restreinte. On veut alors utiliser le morphisme de groupes ε_Γ directement. Pour cela, on a besoin de voir un élémént de $\mathrm{IET}^{\bowtie}(\Gamma)$ comme une "limite" d'éléments de $\mathrm{IET}(\Gamma)$ en un certain sens.

Ce chapitre correspond à la seconde moitié de l'article [Lac20a].

English abstract. For every subgroup $\mathrm{IET}^{\bowtie}(\Gamma)$ we establish a natural isomorphism between its abelianization and $\langle\{a\otimes a\ [\mathrm{mod}\ 2]\mid a\in\tilde{\Gamma}\}\rangle\times\langle\{\ell\wedge\ell\ [\mathrm{mod}\ 2]\mid\ell\in\tilde{\Gamma}\}\rangle$ which is a 2-elementary abelian subgroup of $\bigotimes^2_{\mathbb{Z}}\tilde{\Gamma}/(2\bigotimes^2_{\mathbb{Z}}\tilde{\Gamma})\times{}^{\ominus}\bigwedge^2_{\mathbb{Z}}\tilde{\Gamma}/(2^{\ominus}\bigwedge^2_{\mathbb{Z}}\tilde{\Gamma})$. In order to obtain this we construct two group homomorphisms. One is an analogue of the signature homomorphism ε_Γ constructed in Section 2.3. The set of inversions cannot be measured anymore with the measure created in Section 2.3.1, however the union of this with its symmetric still can be measured. The kernel of the signature will be slightly larger than $D(\mathrm{IET}^{\bowtie}(\Gamma))$; in fact, it will miss some reflections of a certain type. We notice that these missing reflections are conjugate to a restricted rotation, thus we will try to use directly the group homomorphism ε_Γ. To do this we need to see that an element of $\mathrm{IET}^{\bowtie}(\Gamma)$ is a "limit" of elements in $\mathrm{IET}(\Gamma)$ in some sense.

This chapter corresponds with the second half of the article [Lac20a].

Contents

3.1 Positive substitute

We refer to Section 1.2.1 for the different kinds of partition associated. Here we introduce the notion of positive substitute. The idea is to get a unique decomposition for every element of $\widehat{\mathrm{IET}^{\bowtie}}$. This can be obtain by having a parameter on which the decomposition will depend and this parameter is the partition associated chosen. Also it will be important to understand in which cases this decomposition satisfies the group homomorphism relation. For this we introduce the notion of a partition associated with a tuple.

3.1.1 Definition

We recall here that for every subinterval I of $\mathbb{R}$ we denote by $\widehat{r_I}$ the I-reflection map and by r_I the I-reflection.

Proposition 3.1.1. *Let $\widehat{f} \in \widehat{\mathrm{PC}^{\bowtie}}$ (resp. $\widehat{\mathrm{IET}^{\bowtie}}$) and let $\mathcal{P}$ be a partition associated with $\widehat{f}$. Then there exist a unique subset $A(\widehat{f}, \mathcal{P}) \subset \widehat{f}(\mathcal{P})$ and a unique finitely supported permutation $\sigma_{(\widehat{f}, \mathcal{P})}$ such that $\sigma_{(\widehat{f}, \mathcal{P})} \circ (\prod_{I \in A(\widehat{f}, \mathcal{P})} \widehat{r_I}) \circ \widehat{f}$ belongs to $\widehat{\mathrm{PC}^+}$ (resp. $\widehat{\mathrm{IET}^+}$) and is right-continuous.*

Proof. Let $\widehat{f}$ and $\mathcal{P}$ as in the statement. We know that $\widehat{f}$ is continuous on every interior I° with $I \in \mathcal{P}$. We define:

$$A(f, \mathcal{P}) = \{J \mid \exists I \in \mathcal{P} \text{ such that } J^{\circ} = \widehat{f}(I^{\circ}) \text{ and } \widehat{f}|_{I^{\circ}} \text{ is order-reversing }\}$$

It is the unique subset of $\widehat{f}(\mathcal{P})$ such that $(\prod_{I \in A(\widehat{f}, \mathcal{P})} \widehat{r_I}) \circ \widehat{f}$ is an element of $\widehat{\mathrm{PC}^+}$ and thus it exists a unique finitely supported permutation $\sigma_{(\widehat{f}, \mathcal{P})}$ such that $\sigma_{(\widehat{f}, \mathcal{P})} \circ (\prod_{I \in A(\widehat{f}, \mathcal{P})} \widehat{r_I}) \circ \widehat{f}$ is right-continuous. $\qquad\square$

Corollary 3.1.2. *Let $f \in \mathrm{PC}^{\bowtie}$ (resp. $\mathrm{IET}^{\bowtie}$) and $\mathcal{P}$ be a partition associated with f. There exists a unique subset $A(f, \mathcal{P}) \subset f(\mathcal{P})$ such that $(\prod_{I \in A(f, \mathcal{P})} r_I) \circ f$ belongs to PC (resp. IET).*

Definition 3.1.3. The finitely supported permutation $\sigma_{(\widehat{f}, \mathcal{P})}$ is called the *default of pseudo right-continuity* of $\widehat{f}$ according to $\mathcal{P}$. The composition $\sigma_{(\widehat{f}, \mathcal{P})} \circ (\prod_{I \in A(\widehat{f}, \mathcal{P})} \widehat{r_I}) \circ \widehat{f}$ is equal to $(\prod_{I \in A(f, \mathcal{P})} r_I) \circ f$. We denote it by $f_{\mathcal{P}}^+$ and it is called the *positive $\mathcal{P}$-substitute* of $\widehat{f}$ and f.

The dependance on the partition is really important as we can see on Figures 3.1.1 and 3.1.2.
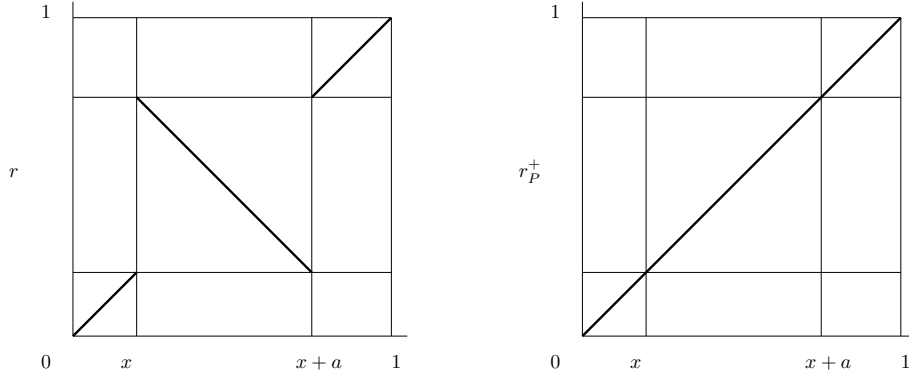
Figure 3.1.1 – Positive substitute for a reflection in the case $S = \{x, a, 1 - (x + a)\}$.
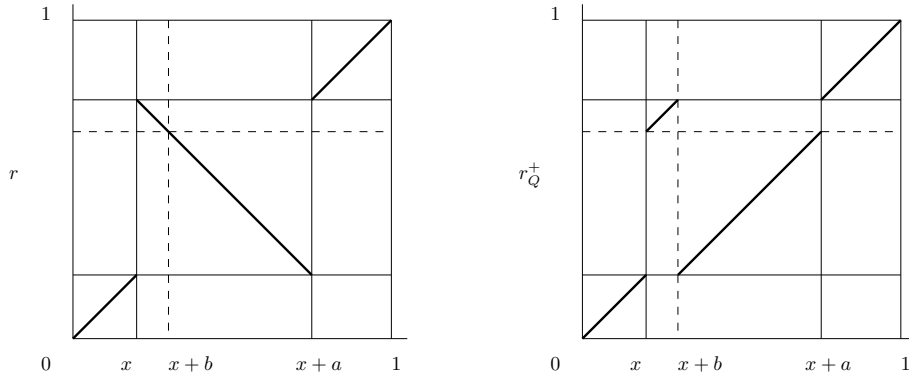
Figure 3.1.2 – Positive substitute for a reflection in the case $S = \{x, b, a - b, 1 - (x + a)\}$.

3.1.2 Compatibility with the composition

We need to define what is a partition associated with a tuple.

Definition 3.1.4. Let $n \in \mathbb{N}$ and $f_1, f_2, \ldots, f_n \in \mathrm{IET}^{\bowtie}(\Gamma)$. Let S be a finite subset of $\mathbb{R}$ and let $\mathcal{P}$ be a partition into intervals of $[0, 1[$, we said that $\mathcal{P}$ is a *partition into S-intervals associated with* $(f_1, f_2, \ldots, f_n)$ if:

1. $\mathcal{P}$ is a partition into S-intervals associated with f_1,

2. For every $2 \leq i \leq n - 1$, $f_i f_{i-1} \ldots f_1(\mathcal{P})$ is a partition into S-intervals associated with f_{i+1}.

Remark 3.1.5. Let S and T be two finite subsets of $\mathbb{R}$ and let $\mathcal{P}$ be a partition into S-intervals associated with $(f_1, f_2, \ldots, f_n)$. Then any refinement of $\mathcal{P}$ into T-intervals is a partition into T-intervals associated with $(f_1, f_2, \ldots, f_n)$.

We also want to talk about order-preserving and order-reversing for elements in $\mathrm{IET}^{\bowtie}$.

Definition 3.1.6. Let $f \in \mathrm{IET}^{\bowtie}$ and $\mathcal{P} \in \Pi_f$. Let $I \in \mathcal{P}$, we say that f is *order-preserving on I* (resp. *order-reversing on I*) if there exists a representative of f in $\widehat{\mathrm{IET}^{\bowtie}}$ that is order-preserving on I (resp. order-reversing on I).

Thanks to this when we have a partition $\mathcal{P}$ associated with an element $f \in \mathrm{IET}^{\bowtie}$ we can always say that f is either order-preserving or order-reversing on every interval of $\mathcal{P}$.

With some conditions we deduce how positive substitute behaves with composition.

Proposition 3.1.7. *Let $n \in \mathbb{N}$ and $f_1, f_2, \ldots, f_n \in \mathrm{IET}^{\bowtie}(\Gamma)$. Let $\mathcal{P}$ be a partition into S-intervals associated with $(f_1, f_2, \ldots, f_n)$. Let $g_1 = (f_1)^+_{\mathcal{P}}$ and $g_i = (f_i f_{i-1} \ldots f_1)^+_{f_{i-1} \ldots f_1(\mathcal{P})}$ for every $2 \leq i \leq n$. Then $(f_n f_{n-1} \ldots f_1)^+_{\mathcal{P}} = g_n g_{n-1} \ldots g_1$.*

Proof. For this proof we denote by r_I the I-reflection for every subinterval I of $[0, 1[$.

By iteration it is sufficient to show the result for $n = 2$. Let $f, g \in \mathrm{IET}^{\bowtie}(\Gamma)$ and $\mathcal{P}$ be a partition into S-intervals associated with (f, g). Let $n \in \mathbb{N}$ such that $\mathcal{P} = \{I_1, I_2, \ldots, I_n\}$ and $f(\mathcal{P}) = \{J_1, J_2, \ldots, J_n\}$ and $f^+_{\mathcal{P}}(I_i) = J_i$ for $1 \leq i \leq n$. Let $1 \leq i \leq n$, we notice that $r_{J_i} \circ f^+_{\mathcal{P}}|_{I_i} \circ r_{I_i} = f^+_{\mathcal{P}}|_{I_i}$. There are 4 cases:

1. If f is order-preserving on I_i and g is order-preserving on J_i then $g \circ f$ is order-preserving on I_i so :
$$(g \circ f)^+_{\mathcal{P}}|_{I_i} = (g \circ f)|_{I_i} = g|_{J_i} \circ f|_{I_i} = g^+_{f(\mathcal{P})}|_{J_i} \circ f^+_{\mathcal{P}}|_{I_i}$$

2. If f is order-preserving on I_i and g is order-reversing on J_i then $g \circ f$ is order-reversing on I_i so :
$$(g \circ f)^+_{\mathcal{P}}|_{I_i} = (g \circ f)_{I_i} \circ r_{I_i} = g|_{J_i} \circ f|_{I_i} \circ r_{I_i} = g^+_{f(\mathcal{P})}|_{J_i} \circ r_{J_i} \circ f^+_{\mathcal{P}}|_{I_i} \circ r_{I_i} = g^+_{f(\mathcal{P})}|_{J_i} \circ f^+_{\mathcal{P}}|_{I_i}$$

3. If f is order-reversing on I_i and g is order-preserving on J_i then $g \circ f$ is order-reversing on I_i so :
$$(g \circ f)^+_{\mathcal{P}}|_{I_i} = (g \circ f)_{I_i} \circ r_{I_i} = g|_{J_i} \circ f|_{I_i} \circ r_{I_i} = g^+_{f(\mathcal{P})}|_{J_i} \circ f^+_{\mathcal{P}}|_{I_i} \circ r_{I_i} \circ r_{I_i} = g^+_{f(\mathcal{P})}|_{J_i} \circ f^+_{\mathcal{P}}|_{I_i}$$

4. If f is order-reversing on I_i and g is order-reversing on J_i then $g \circ f$ is order-preserving on I_i so :
$$(g \circ f)^+_{\mathcal{P}}|_{I_i} = (g \circ f)|_{I_i} = g|_{J_i} \circ f|_{I_i} = g^+_{f(\mathcal{P})}|_{J_i} \circ r_{J_i} \circ f^+_{\mathcal{P}}|_{I_i} \circ r_{I_i} = g^+_{f(\mathcal{P})}|_{J_i} \circ f^+_{\mathcal{P}}|_{I_i}$$

$\square$

3.2 Analogue of the signature

3.2.1 Balanced product of reflections

We give here a first description of $D(\mathrm{IET}^{\bowtie})$. It is inspired of the work done for IET with balanced product restricted rotations.

Definition 3.2.1. Let $n \in \mathbb{N}$ and $r_1, r_2, \ldots, r_n$ be some Γ-reflections. For every $\ell \in \tilde{\Gamma}_+$ let n_ℓ be the number of Γ-reflections of type ℓ among these elements. The tuple $(r_1, r_2, \ldots, r_n)$ is a *balanced tuple of Γ-reflections* if 2 divides n_ℓ for every $\ell \in \tilde{\Gamma}_+$. We say that a product of Γ-reflections is a *balanced product of Γ-reflections* if it can be written as a product of a balanced tuple of Γ-reflections.

Lemma 3.2.2. *The set of all balanced products of Γ-reflections is a generating subset of $D(\mathrm{IET}^{\bowtie}(\Gamma))$.*

Proof. As any element of $\mathrm{IET}^{\bowtie}(\Gamma)$ is a finite product of Γ-reflections (see Proposition 1.2.16) and as a reflection has order 2 we deduce that every element of $D(\mathrm{IET}^{\bowtie}(\Gamma))$ is a balanced product of reflections.

Let r and s be two reflections with the same type. Let t be the transposition that swaps the support of r with the support of s, then $s = trt^{-1}$. As a reflection has order 2, the product rs is a commutator.

Let $n \in \mathbb{N}$ and $r_1, r_2, \ldots r_n$ be Γ-reflections such that $r_1 r_2 \ldots r_n$ is a balanced product of reflections. Then n is even and up to compose with an element of $D(\mathrm{IET}^{\bowtie})$ we can assume that r_{2i-1} and r_{2i} have the same type for every $1 \leq i \leq \frac{n}{2}$. Thus by the previous case we deduce that $r_1 r_2 \ldots r_n$ is in $D(\mathrm{IET}^{\bowtie})$. $\square$

By Proposition 1.2.16 and Lemma 3.2.2 we deduce the following:

Corollary 3.2.3. *Let $f \in \mathrm{IET}^{\bowtie}(\Gamma)$ then $f^2 \in D(\mathrm{IET}^{\bowtie}(\Gamma))$.*

Proposition 3.2.4. *Any balanced product of Γ-restricted rotations with type in $\{(a,b) \in (\tilde{\Gamma}_+)^2 \mid a \neq b\}$ is a balanced product of Γ-reflections.*

Proof. Let $a, b \in \tilde{\Gamma}_+$. Let r be a restricted rotation of type (a,b) an s be a restricted rotation of type (b,a). Let I and J be the two consecutive intervals permuted by r. Then r is the composition of the I-reflection, the J-reflection and the $I \cup J$-reflection thus a product of a reflections of type a with one of type b and one of type $a + b$. The same is true for s thus we obtain that rs is the product of two reflections of type a, two of type b and two of type $a + b$; so rs is a balanced product of Γ-reflections. $\qquad \square$

3.2.2 The group homomorphism

Here we start with the work done in Section 2.3. We remark that if we denote $A'_{\mathbb{R}}$ the Boolean algebra of subsets of $[0,1[$ generated by the set of all intervals $[a,b]$, $A_{\mathbb{R}}$ the one generated by the set of all intervals $[a,b[$ and A_{fin} the one generated by all the singletons $\{x\}$ then $A_{\mathbb{R}}$ is isomorphic to $A^*_{\mathbb{R}} := A'_{\mathbb{R}}/A_{\mathrm{fin}}$. This is why we do not make a difference between A_Γ and its image in $A^*_{\mathbb{R}}$.

The notion of inversions as defined in 2.3.6 is no longer relevant because for every reflection r, we have its set of inversions $\mathcal{E}_r \notin A_\Gamma \otimes A_\Gamma$; indeed if I is the interval reflected by r then $\mathcal{E}_r = \{(x,y) \in I \times I \mid x < y\}$. We need to be more precise:

Definition 3.2.5. For every element $f \in \widehat{\mathrm{IET}^{\bowtie}}(\Gamma)$ we define:

1. $\mathcal{E}_{f,1} := \{(x,y) \mid x < y,\ f(x) > f(y)\}$, the set of all inversions of type 1 of f,

2. $\mathcal{E}_{f,2} := \{(x,y) \mid y < x,\ f(y) > f(x)\}$, the set of all inversions of type 2 of f,

3. $\mathcal{E}_f^{\bowtie} := \mathcal{E}_{f,1} \cup \mathcal{E}_{f,2}$, the set of all inversions of f.

From now on we will write $\mathcal{E}_f^{\bowtie} = \mathcal{E}_f$.

Proposition 3.2.6. *For every $f \in \widehat{\mathrm{IET}^{\bowtie}}(\Gamma)$ we have $\mathcal{E}_f \in A_\Gamma \otimes A_\Gamma$.*

Proof. Let $f \in \widehat{\mathrm{IET}^{\bowtie}}$ and let $\mathcal{P}$ be a partition associated with f. Let $f_{\mathcal{P}}^+$ be the positive $\mathcal{P}$-substitute of f. Let $\mathcal{J} \subset \mathcal{P}$ be the subset of intervals where f is order-reversing. By seeing A_Γ as its image in $A^*_{\mathbb{R}}$ then we deduce that $\mathcal{E}_f = \mathcal{E}_{f_{\mathcal{P}}^+} \cup \bigcup_{I \in \mathcal{J}} I \times I$ is an element of $A_\Gamma \otimes A_\Gamma$. $\qquad \square$

Then this set can be measured with the same measure $\omega_\Gamma : A_\Gamma \otimes A_\Gamma \to \bigotimes_{\mathbb{Z}}^2 \tilde{\Gamma}$ (see Notation 2.3.4).

Example 3.2.7.   1. If we consider a Γ-restricted rotation r of type (a,b) (with $a, b \in \tilde{\Gamma}_+$), we obtain $\omega_\Gamma(\mathcal{E}_r) = a \otimes b + b \otimes a$ (see Figure 3.2.1).

2. Let $I \in \mathrm{Itv}(\Gamma)$ be a subinterval of $[0,1[$ of length a. Let r be the I-reflection map. We have $\omega_\Gamma(\mathcal{E}_r) = a \otimes a$ (see Figure 3.2.1).

3. Let τ a finitely supported permutation. Then $\mathcal{E}_\tau$ is a union of singletons thus $\mathcal{E}_\tau = \emptyset \in A_\Gamma \otimes A_\Gamma$ and $\omega_\Gamma(\mathcal{E}_\tau) = \omega_\Gamma(\emptyset) = 0$.
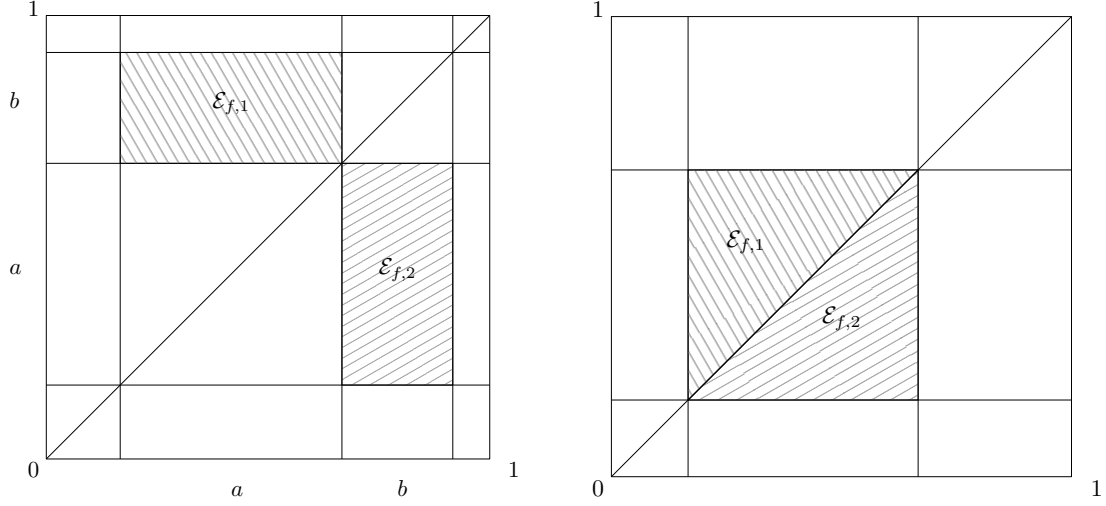
Figure 3.2.1 – **Left**: Set of inversions for a restricted rotation. **Right**: Set of inversions for a I-reflection

Here we expect a 2-group for the image of our group homomorphism. We denote by π the projection of $\bigotimes_{\mathbb{Z}}^2 \tilde{\Gamma}$ onto $\bigotimes_{\mathbb{Z}}^2 \tilde{\Gamma}/2(\bigotimes_{\mathbb{Z}}^2 \tilde{\Gamma})$ and for every $a \in \tilde{\Gamma}$ we denote $\pi(a \otimes a)$ by $a \otimes a \pmod 2$.

Definition 3.2.8. We define *the signature for* $\widehat{\mathrm{IET}^{\bowtie}(\Gamma)}$ as the map:

$$\widehat{\varepsilon_{\Gamma}^{\bowtie}}: \quad \widehat{\mathrm{IET}^{\bowtie}(\Gamma)} \quad \longrightarrow \quad \bigotimes_{\mathbb{Z}}^2 \tilde{\Gamma}/2(\bigotimes_{\mathbb{Z}}^2 \tilde{\Gamma})$$
$$f \quad \longrightarrow \quad \omega_{\Gamma}(\mathcal{E}_f) \pmod 2$$

For more clarity we explicit some equalities used to show that $\widehat{\varepsilon_{\Gamma}^{\bowtie}}$ is a group homomorphism.

Lemma 3.2.9. *Let* $f, g \in \widehat{\mathrm{IET}^{\bowtie}(\Gamma)}$. *We have the following equalities:*

1. $\mathcal{E}_{f \circ g} \cup (\mathcal{E}_g \cap g^{-1}(\mathcal{E}_f)) = \mathcal{E}_g \cup g^{-1}(\mathcal{E}_f)$,

2. $\mathcal{E}_{f \circ g} \cap \mathcal{E}_g \cap g^{-1}(\mathcal{E}_f) = \emptyset$.

Remark 3.2.10. We notice that $\mathcal{E}_g \cap g^{-1}(\mathcal{E}_f)$ is an element of $A_{\Gamma} \otimes A_{\Gamma}$.

Proof. 1. We proceed by double inclusions.
From left to right, we know that $\mathcal{E}_g \cap g^{-1}(\mathcal{E}_f) \subset \mathcal{E}_g \cup g^{-1}(\mathcal{E}_f)$ hence it is sufficient to show the inclusion $\mathcal{E}_{f \circ g} \subset \mathcal{E}_g \cup g^{-1}(\mathcal{E}_f)$. Let (x, y) be in $\mathcal{E}_{f \circ g}$. We can assume that $x < y$, the case $x > y$ is similar. Then we deduce that $f(g(x)) > f(g(y))$. We have two cases, if $g(x) > g(y)$ then $(x, y) \in \mathcal{E}_g$ else we have $g(x) < g(y)$ and $f(g(x)) > f(g(y))$ thus $(x, y) = g^{-1}(g(x), g(y)) \in g^{-1}\mathcal{E}_f$.
From right to left, let $(x, y) \in \mathcal{E}_g \cup g^{-1}(\mathcal{E}_f)$. We can assume that $x < y$ the case $x > y$ is similar. If $(x, y) \in \mathcal{E}_g \cap g^{-1}(\mathcal{E}_f)$ then it is done. We have two cases, if $(x, y) \in \mathcal{E}_g$ and $(x, y) \notin g^{-1}(\mathcal{E}_f)$ then as $x < y$ we have $g(x) > g(y)$ and $f(g(x)) > f(g(y))$ thus $(x, y) \in \mathcal{E}_{f \circ g}$. If $(x, y) \in g^{-1}(\mathcal{E}_f)$ and $(x, y) \notin \mathcal{E}_g$ then as $x < y$ we have $g(x) < g(y)$ and $f(g(x)) > f(g(y))$ thus $(x, y) \in \mathcal{E}_{f \circ g}$.

2. By contradiction let us assume that there exists $(x, y) \in \mathcal{E}_{f \circ g} \cap \mathcal{E}_g \cap g^{-1}(\mathcal{E}_f)$. We can assume that $x < y$, the case $x > y$ is similar. As $x < y$ and $(x, y) \in \mathcal{E}_{f \circ g} \cap \mathcal{E}_g$ we know that $g(x) > g(y)$ and $f(g(x)) > f(g(y))$. However $g.(x, y) = (g(x), g(y)) \in \mathcal{E}_f$ thus as $g(x) > g(y)$ we have $f(g(x)) < f(g(y))$ which is a contradiction.

$\square$

Theorem 3.2.11. *The signature $\widehat{\varepsilon_\Gamma^{\bowtie}}$ is a group homomorphism.*

Proof. Let $f, g \in \widehat{\mathrm{IET}^{\bowtie}(\Gamma)}$. By using the second equality of Thanks to Lemma 3.2.9 on the third line and the first equality on the fourth line, we have:

$$
\begin{aligned}
\widehat{\varepsilon_\Gamma^{\bowtie}}(f \circ g) &= \omega_\Gamma(\mathcal{E}_{f \circ g}) \ [\mathrm{mod}\ 2]\\
&= \omega_\Gamma(\mathcal{E}_{f \circ g}) + \omega_\Gamma(\mathcal{E}_g \cap g^{-1}(\mathcal{E}_f)) - \omega_\Gamma(\mathcal{E}_g \cap g^{-1}(\mathcal{E}_f)) \ [\mathrm{mod}\ 2]\\
&= \omega_\Gamma(\mathcal{E}_{f \circ g} \sqcup (\mathcal{E}_g \cap g^{-1}(\mathcal{E}_f))) - \omega_\Gamma(\mathcal{E}_g \cap g^{-1}(\mathcal{E}_f)) \ [\mathrm{mod}\ 2]\\
&= \omega_\Gamma(\mathcal{E}_g \cup g^{-1}(\mathcal{E}_f)) - \omega_\Gamma(\mathcal{E}_g \cap g^{-1}(\mathcal{E}_f)) \ [\mathrm{mod}\ 2]\\
&= \omega_\Gamma(\mathcal{E}_g) + \omega_\Gamma(g^{-1}(\mathcal{E}_f)) - 2\omega_\Gamma(\mathcal{E}_g \cap g^{-1}(\mathcal{E}_f)) \ [\mathrm{mod}\ 2]\\
&= \omega_\Gamma(\mathcal{E}_g) + \omega_\Gamma(g^{-1}(\mathcal{E}_f)) \ [\mathrm{mod}\ 2]\\
&= \widehat{\varepsilon_\Gamma^{\bowtie}}(g) + \widehat{\varepsilon_\Gamma^{\bowtie}}(f)
\end{aligned}
$$

$\square$

We notice that every finitely supported permutation of support in $\tilde{\Gamma}$ is in $\mathrm{Ker}(\widehat{\varepsilon_\Gamma^{\bowtie}})$.

Corollary 3.2.12. *There exists a group homomorphism $\varepsilon_\Gamma^{\bowtie} : \mathrm{IET}^{\bowtie}(\Gamma) \to \bigotimes_{\mathbb{Z}}^2 \tilde{\Gamma}/2(\bigotimes_{\mathbb{Z}}^2 \tilde{\Gamma})$ such that for every $f \in \mathrm{IET}^{\bowtie}(\Gamma)$ we have $\varepsilon_\Gamma^{\bowtie}(f) = \widehat{\varepsilon_\Gamma^{\bowtie}}(\hat{f})$ for every representative $\hat{f}$ of f in $\widehat{\mathrm{IET}^{\bowtie}(\Gamma)}$.*

Example 3.2.13. We give the value of $\varepsilon_\Gamma^{\bowtie}$ for two kinds of elements:

1. Let r be a Γ-reflection of type a then $\varepsilon_\Gamma^{\bowtie}(r) = a \otimes a \ [\mathrm{mod}\ 2]$,

2. Let s be a Γ-restricted rotation of type (p, q) then:

$$
\varepsilon_\Gamma^{\bowtie}(s) = p \otimes q + q \otimes p \ [\mathrm{mod}\ 2]
$$

As $\mathrm{IET}^{\bowtie}$ is generated by reflections, we deduce the image of $\varepsilon_\Gamma^{\bowtie}$:

Corollary 3.2.14. *The subgroup $\mathrm{Im}(\varepsilon_\Gamma^{\bowtie})$ of $\bigotimes_{\mathbb{Z}}^2 \tilde{\Gamma}/2(\bigotimes_{\mathbb{Z}}^2 \tilde{\Gamma})$ is generated by the subset $\langle \{a \otimes a \ [\mathrm{mod}\ 2] \mid a \in \tilde{\Gamma}\} \rangle$.*

3.2.3   Description of $\mathrm{Ker}(\varepsilon_\Gamma^{\bowtie})$

As $\bigotimes_{\mathbb{Z}}^2 \tilde{\Gamma}/2(\bigotimes_{\mathbb{Z}}^2 \tilde{\Gamma})$ is an abelian group we know that $D(\mathrm{IET}^{\bowtie}(\Gamma))$ is included in $\mathrm{Ker}(\varepsilon_\Gamma^{\bowtie})$. We will see later that the other inclusion is false in general.

With Figure 1.3.1 we already know that for every $a \in \tilde{\Gamma}$ such that $\frac{a}{4} \in \tilde{\Gamma}$ then every Γ-reflection of type a is in the derived subgroup $D(\mathrm{IET}^{\bowtie}(\Gamma))$.

Definition 3.2.15. We denote by Ω_Γ the conjugate closure of the group generated by the set of all Γ-reflections of type 2ℓ with $\ell \in \tilde{\Gamma} \smallsetminus 2\tilde{\Gamma}$ (the closure inside $\mathrm{IET}^{\bowtie}(\Gamma)$).

The inclusion $\Omega_\Gamma \subset \mathrm{Ker}(\varepsilon_\Gamma^{\bowtie})$ is immediate. With Figure 3.2.2 we obtain the following:

Proposition 3.2.16. *Every Γ-reflections of type 2ℓ with $\ell \in \tilde{\Gamma} \smallsetminus 2\tilde{\Gamma}$ is conjugated to a Γ-restricted rotation of type (ℓ, ℓ).*
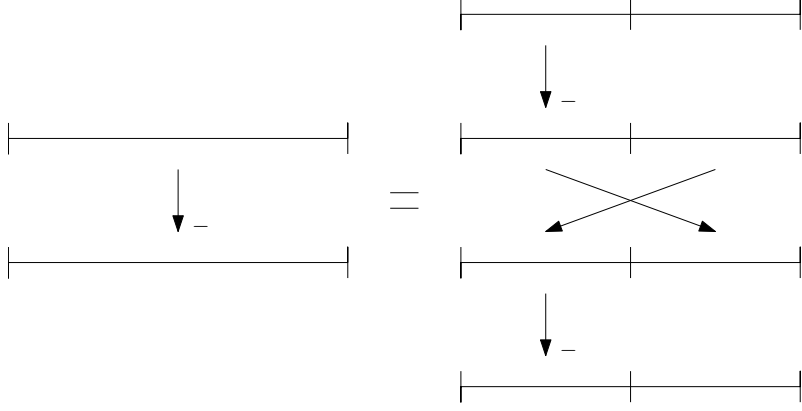
Figure 3.2.2

The aim here is to show the equality $\text{Ker}(\varepsilon_\Gamma^{\bowtie}) = D(\text{IET}^{\bowtie}(\Gamma))\Omega_\Gamma$.

We begin by proving the result in the specific case where Γ has finite rank. We will reduce the general case to this one.

Lemma 3.2.17. *Let Γ be a finitely generated subgroup of $\mathbb{R}/\mathbb{Z}$. Then for every $f \in \text{Ker}(\varepsilon_\Gamma^{\bowtie})$ there exist $\delta \in D(\text{IET}^{\bowtie}(\Gamma))$ and $h \in \Omega_\Gamma$ such that $f = \delta h$.*

Proof. As Γ is finitely generated, we know that $\tilde{\Gamma}$ is finitely generated. Let d be the rank of $\tilde{\Gamma}$.

Let $f \in \text{Ker}(\varepsilon_\Gamma^{\bowtie})$. Let $n \in \mathbb{N}$ and $\mathcal{P} := \{I_1, I_2, \ldots, I_n\}$ be a partition into Γ-intervals associated with f. We denote by L_i the length of I_i for every $1 \leq i \leq n$. By Corollary 2.1.3 there exists $B := \{\ell_1, \ell_2, \ldots, \ell_d\}$ a basis of $\tilde{\Gamma}$ with elements in $\tilde{\Gamma}_+$ such that $L_i \in \text{Vect}_{\mathbb{N}}(B)$ for every $1 \leq i \leq n$. Hence we can cut each I_i into smaller intervals with length in B. This operation gives us a new partition $\mathcal{Q} := \{J_1, J_2, \ldots, J_k\}$, with $k \in \mathbb{N}$, into Γ-intervals associated with f.

For every $1 \leq i \leq k$ we define r_i as the J_i-reflection if f is order-reversing on J_i else we put $r_i = \text{Id}$. Let g be the product $fr_1r_2\ldots r_k$; it is an element of $\text{IET}(\Gamma)$ and $\mathcal{Q}$ is a partition into Γ-intervals associated with g. By Theorem 1.2.8 the element g can be written as a finite product of Γ-restricted rotations with type inside $\{(l_p, l_q) \mid p, q \in \{1, 2, \ldots, d\}\}$. Thanks to an element of $D(\text{IET}^{\bowtie}(\Gamma))$ we can organize this product to put all Γ-restricted rotations of type (l_p, l_p) together (with $1 \leq p \leq d$; they are elements in Ω_Γ): there exist $w_1 \in D(\text{IET}^{\bowtie}(\Gamma))$ and $h \in \Omega_\Gamma$ and $m \in \mathbb{N}$ and $s_1, s_2, \ldots, s_m$ some Γ-restricted rotations with type inside $\{(l_p, l_q) \mid p, q \in \{1, 2, \ldots, d\}, \ p \neq q\}$ such that $g = w_1 h s_1 s_2 \ldots s_m$. Then $f = w_1 h s_1 s_2 \ldots s_m r_k r_{k-1} \ldots r_1$.

We define $u_p := \text{Card}\{i \in \{1, 2, \ldots, k\} \mid r_i \neq \text{Id}, \ \text{type}(r_i) = l_p\}$. Let $v_{p,p} = 0$ for every $1 \leq p \leq d$ and let $v_{p,q} := \text{Card}\{j \in \{1, 2, \ldots, m\} \mid \text{type}(s_j) = (l_p, l_q)\}$ for every $1 \leq p \neq q \leq d$. Then we have:

$$\varepsilon_\Gamma^{\bowtie}(f) = \sum_{i=1}^{k} \varepsilon_\Gamma^{\bowtie}(r_i) + \sum_{j=1}^{m} \varepsilon_\Gamma^{\bowtie}(s_j) = \sum_{p=1}^{d} u_p l_p \otimes l_p + \sum_{p=1}^{d} \sum_{q=1}^{d} v_{p,q}(l_p \otimes l_q + l_q \otimes l_p) \ [\text{mod } 2] = 0$$

We notice that $\sum_{p=1}^{d} \sum_{q=1}^{d} v_{p,q}(l_p \otimes l_q + l_q \otimes l_p) = \sum_{p=1}^{d} \sum_{q=1}^{d} (v_{p,q} + v_{q,p}) l_p \otimes l_q$. Furthermore B is a basis of $\tilde{\Gamma}$ so $\{l_p \otimes l_q\}_{1 \leq p,q \leq d}$ is a basis of $\bigotimes_{\mathbb{Z}}^2 \tilde{\Gamma}$, thus we deduce that 2 divides u_p for every $1 \leq p \leq d$ and 2 divides $v_{p,q} + v_{q,p}$ for every $1 \leq p, q \leq d$.

We obtain that $r_1 r_2 \ldots r_k$ is a balanced product of Γ-reflections hence by Lemma 3.2.2 it is an element of $D(\text{IET}^{\bowtie}(\Gamma))$ denoted w_2. We also deduce that the product $s_1 s_2 \ldots s_m$ is

a balanced product of Γ-restricted rotations with type inside $\{(a,b) \mid a \neq b \in \tilde{\Gamma}_+\}$. Hence by Proposition 3.2.4 we obtain that it is also an element of $D(\mathrm{IET}^{\bowtie}(\Gamma))$, denoted w_3.

Finally we have $f = w_1 h w_3 w_2 = \delta h$ with $\delta \in D(\mathrm{IET}^{\bowtie}(\Gamma))$ and $h \in \Omega_\Gamma$. $\qquad\square$

The next lemma gives an inclusion used to conclude in the general case:

Lemma 3.2.18. *For all Γ, A subgroups of $\mathbb{R}/\mathbb{Z}$ such that $A \subset \Gamma$ we have:*

$$D(\mathrm{IET}^{\bowtie}(A))\Omega_A \subset D(\mathrm{IET}^{\bowtie}(\Gamma))\Omega_\Gamma$$

Proof. The inclusion $D(\mathrm{IET}^{\bowtie}(A)) \subset D(\mathrm{IET}^{\bowtie}(\Gamma))$ is immediate. It is sufficient to show that $\Omega_A \subset D(\mathrm{IET}^{\bowtie}(\Gamma))\Omega_\Gamma$. Let $\tilde{A}$ be the preimage of A in $\mathbb{R}$.

Let f be an element of Ω_A. Then there exist $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in \tilde{A} \smallsetminus 2\tilde{A}$ and $w_1, w_2, \ldots, w_n$ some Γ-reflections such that the type of w_i is $2a_i$ and there exist $g_1, g_2, \ldots g_n \in \mathrm{IET}^{\bowtie}(A)$ such that $f = \prod\limits_{i=1}^{n} g_i w_i g_i^{-1}$.

Let $U := \{i \in \{1, 2, \ldots, n\} \mid a_i \in \tilde{\Gamma} \smallsetminus 2\tilde{\Gamma}\}$. By definition we have $\{g_i w_i g_i^{-1} \mid i \in U\} \subset \Omega_\Gamma$. Take $V := \{1, 2, \ldots, n\} \smallsetminus U$. As A is a subgroup of Γ we deduce that $V = \{j \in \{1, 2, \ldots, n\} \mid a_j \in 2\tilde{\Gamma}\}$. Thus for every $j \in V$ the type of w_j is in $4\tilde{\Gamma}$ so we deduce that w_j and $g_j w_j g_j^{-1}$ belong to $D(\mathrm{IET}^{\bowtie}(\Gamma))$ (see Figure 1.3.1). We know that there exists $h \in D(\mathrm{IET}^{\bowtie}(\Gamma))$ such that:

$$f = h \prod_{j \in V} g_j w_j g_j^{-1} \prod_{i \in U} g_i w_i g_i^{-1}$$

Then $f \in D(\mathrm{IET}^{\bowtie}(\Gamma))\Omega_\Gamma$. $\qquad\square$

We can prove the theorem for the general case:

Theorem 3.2.19. *For any dense subgroup Γ of $\mathbb{R}/\mathbb{Z}$ we have:*

$$\mathrm{Ker}(\varepsilon_\Gamma^{\bowtie}) = D(\mathrm{IET}^{\bowtie}(\Gamma))\Omega_\Gamma$$

Proof. The inclusion from right to left is already proved.

Let $f \in \mathrm{Ker}(\varepsilon_\Gamma^{\bowtie})$, let $n \in \mathbb{N}$ and $\mathcal{P} := \{I_1, I_2, \ldots, I_n\}$ be a partition into Γ-intervals associated with f. We denote by L_i the length of I_i for every $1 \leq i \leq n$. As $\varepsilon_\Gamma^{\bowtie}(f) = 0$ we know that there exist $k \in \mathbb{N}$ and $a_1, a_2 \ldots, a_k, b_1, b_2, \ldots, b_k \in \tilde{\Gamma}$ such that $\omega_\Gamma(\mathcal{E}_f) = 2\sum\limits_{i=1}^{k} a_i \otimes b_i$ inside $\bigotimes_{\mathbb{Z}}^2 \tilde{\Gamma}$.

Let $\tilde{A}$ be the subgroup of $\mathbb{R}$ generated by $\{L_i\}_{i=1\ldots n} \cup \{1\} \cup \{a_i, b_i\}_{i=1\ldots k}$. Then $\tilde{A}$ contains $\mathbb{Z}$, is finitely generated and is a subgroup of $\tilde{\Gamma}$. Let A be the image of $\tilde{A}$ in $\mathbb{R}/\mathbb{Z}$. The partition $\mathcal{P}$ is also a partition into A-intervals associated with f thus f belongs to $\mathrm{IET}^{\bowtie}(A)$. Furthermore we have $\omega_A(\mathcal{E}_f) = \omega_\Gamma(\mathcal{E}_f)$. Hence:

$$\varepsilon_A^{\bowtie}(f) = [\omega_A(\mathcal{E}_f)]_{\bigotimes_{\mathbb{Z}}^2 A/2(\bigotimes_{\mathbb{Z}}^2 A)} = [\omega_\Gamma(\mathcal{E}_f)]_{\bigotimes_{\mathbb{Z}}^2 A/2(\bigotimes_{\mathbb{Z}}^2 A)} = [2\sum_{i=1}^{k} a_i \otimes b_i]_{\bigotimes_{\mathbb{Z}}^2 A/2(\bigotimes_{\mathbb{Z}}^2 A)} = 0$$

By Lemma 3.2.17 we deduce that $f \in D(\mathrm{IET}^{\bowtie}(A))\Omega_A$ and by Lemma 3.2.18 we deduce that $f \in D(\mathrm{IET}^{\bowtie}(\Gamma))\Omega_\Gamma$. $\qquad\square$

3.3 The positive contribution

Thanks to Proposition 3.2.16, we know that every Γ-reflection of type 2ℓ with $\ell \in \tilde{\Gamma} \smallsetminus 2\tilde{\Gamma}$ is conjugated to a Γ-restricted rotation of type (ℓ, ℓ). Also this is an element of $\mathrm{IET}^+(\Gamma)$ which is not send on the trivial element by the morphism ε_Γ. We use the notion of positive substitute in order to use the group homomorphism ε_Γ to send such a reflection on a nontrivial element. For this we need to use the upward directed set M_Γ and the $\mathbb{Z}$-linearly independent finite subset S_a of $\tilde{\Gamma}_+$ for every $a \in M_\Gamma$ given by Corollary 2.1.2. We recall that for every $a, b \in M_\Gamma$ with $a \leq b$ we have $\mathrm{Vect}_\mathbb{N}(S_a) \subset \mathrm{Vect}_\mathbb{N}(S_b)$ and $\tilde{\Gamma}_+$ is equal to the direct limit $\varinjlim \mathrm{Vect}_\mathbb{N}(S_a)$.

For every $a \in M_\Gamma$, we construct a map depending on S_a. They are not group homomorphisms but satisfy the group homomorphism property on some products thanks to Proposition 3.1.7.

3.3.1 Some subsets of $\mathrm{IET}^\bowtie(\Gamma)$

Let S be a finite set of $\tilde{\Gamma}_+$. We denote by G_S the set of all f in $\mathrm{IET}^\bowtie(\Gamma)$ such that there exists a partition $\mathcal{P}$ into S-intervals associated with f. We remark that G_S is not a group in general. We want to know how these sets and $\mathrm{IET}^\bowtie(\Gamma)$ are linked.

Proposition 3.3.1. *Let S and T be two finite subsets of $\tilde{\Gamma}_+$. If $S \subset \mathrm{Vect}_\mathbb{N}(T)$ then $G_S \subset G_T$. More precisely for every partition $\mathcal{P}$ into S-intervals of $[0, 1[$ there exists a refinement $\mathcal{Q}$ of $\mathcal{P}$ which is a partition into T-intervals of $[0, 1[$.*

Proof. Let $f \in G_S$ and let $\mathcal{P}$ be a partition into S-intervals associated with f. As $S \subset \mathrm{Vect}_\mathbb{N}(T)$ each interval $I \in P$ can be subdivided with intervals of length in T. After subdividing this way the intervals of $\mathcal{P}$, we obtain a refinement $\mathcal{Q}$ of $\mathcal{P}$. We notice that $\mathcal{Q}$ is a partition into T-intervals associated with f so $f \in G_T$. $\square$

As $\tilde{\Gamma}_+$ is the direct limit of the $\mathrm{Vect}_\mathbb{N}(S_a)$ we deduce the next proposition:

Proposition 3.3.2. *For every finite subset S of $\tilde{\Gamma}_+$, there exists $a \in M_\Gamma$ such that for every $b \geq a$ we have $S \subset \mathrm{Vect}_\mathbb{N}(S_b)$.*

From Propositions 3.3.1 and 3.3.2 we obtain:

Corollary 3.3.3. *Let $f \in \mathrm{IET}^\bowtie(\Gamma)$. There exists $a \in M_\Gamma$ such that for every $b \geq a$ we have $f \in G_{S_b}$.*

We also want to check that if we take any product of elements in $\mathrm{IET}^\bowtie(\Gamma)$ there will be a moment where we have a partition associated to the tuple of these elements.

Proposition 3.3.4. *Let $k \in \mathbb{N}$ and $f_1, f_2, \ldots, f_k \in \mathrm{IET}^\bowtie(\Gamma)$. There exists $a \in M_\Gamma$ such that for every $b \geq a$ there exists $\mathcal{P}_b$ a partition into S_b-intervals associated with $(f_1, f_2, \ldots, f_k)$.*

Proof. Thanks to Proposition 3.3.1 and Remark 3.1.5 we deduce that it is sufficient to find only one $a \in M_\Gamma$ such that there exists a partition into S_a-intervals associated with $(f_1, f_2, \ldots, f_k)$.

By Corollary 3.3.3 there exist $a_i \in M_\Gamma$ and $\mathcal{P}_i$ a partition into S_{a_i} intervals associated with f_i for every $1 \leq i \leq k$. We denote by V_i the set of all the endpoints of the intervals in $\mathcal{P}_i$. Let $V = V_1 \cup f_1^{-1}(V_2) \cup \ldots \cup (f_{k-1} \ldots f_1)^{-1}(V_k)$. We know that V is finite thus there exist $m \in \mathbb{N}$ and $v_0, v_1, \ldots, v_m \in \tilde{\Gamma}_+$ such that $V = \{v_i\}_{i=0\ldots m}$. Up to change the order we can assume that $v_0 = 0 < v_1 < v_2 < \ldots < v_{m-1} < v_m = 1$. Let I_j be the interval $[v_{j-1}, v_j[$ and ℓ_j be the length of I_j for every $1 \leq j \leq m$. Let $T = \{\ell_j\}_{j=1\ldots m}$ and $\mathcal{P} = \{I_j\}_{j=1\ldots m}$.

We have $V_1 \subset V$ then $\mathcal{P}$ is a refinement of $\mathcal{P}_1$ so $\mathcal{P}$ is a partition into T-intervals associated with f_1. Similarly for every $2 \leq i \leq k$ we know that $V_i \subset f_{i-1} \ldots f_1(V)$ thus $f_{i-1} \ldots f_1(\mathcal{P})$ is a refinement of $\mathcal{P}_i$ so $f_{i-1} \ldots f_1(\mathcal{P})$ is a partition into T-intervals associated with f_i. Hence $\mathcal{P}$ is a partition into T-intervals associated with $(f_1, f_2, \ldots, f_k)$.

Thanks to Proposition 3.3.2 there exists $a \in M_\Gamma$ such that for every $b \geq a$ we have $T \subset \mathrm{Vect}_\mathbb{N}(S_b)$ and by Proposition 3.3.1 and Remark 3.1.5 there exists a refinement $\mathcal{Q}$ of $\mathcal{P}$ which is a partition into S_a-intervals associated with $(f_1, f_2, \ldots, f_k)$.

$\square$

3.3.2 The S-map

Let S be a finite subset of $\tilde{\Gamma}_+$ for all this subsection. We assume that S is free inside $\mathrm{Vect}_\mathbb{Z}(S)$. For every $a, b \in \tilde{\Gamma}$, we use the notation $[a \wedge b]_{2 \odot \bigwedge_\mathbb{Z}^2 \tilde{\Gamma}} = a \wedge b \ [\mathrm{mod}\ 2]$.

Let $f \in G_S$ and $\mathcal{P}$ be a S-partition associated with f. We show that the value $\varepsilon_\Gamma(f_\mathcal{P}^+) \ [\mathrm{mod}\ 2]$ does not depend on $\mathcal{P}$, where ε_Γ is the group homomorphism define in 2.3.9.

Proposition 3.3.5. *Let $f \in \mathrm{IET}^\bowtie(\Gamma)$ and let $\mathcal{P}, \mathcal{Q}$ be two partitions into S-intervals associated with f. Then $\varepsilon_\Gamma(f_\mathcal{P}^+) \ [\mathrm{mod}\ 2] = \varepsilon_\Gamma(f_\mathcal{Q}^+) \ [\mathrm{mod}\ 2]$.*

Proof. For this proof we denote by r_I the I-reflection for every subinterval I of $[0, 1[$ and we recall that λ is the Lebesgue measure. First we reduce the case to a simpler one.

Let $f \in \mathrm{IET}^\bowtie(\Gamma)$ and $\mathcal{P}, \mathcal{Q}$ be two partitions into S-intervals associated with f. Let $n, k \in \mathbb{N}$ such that $\mathcal{P} = \{I_1, I_2, \ldots, I_n\}$ and $\mathcal{Q} = \{J_1, J_2, \ldots J_k\}$. Up to change the index we can assume that the I_i are consecutive intervals and so are the J_i. We denote by $\mathcal{M}$ the unique partition into Γ-intervals associated with f that has the minimal number of intervals. Let m be this number and $\mathcal{M} := \{M_1, M_2, \ldots, M_m\}$ where the M_i are consecutive intervals. As $\mathcal{P}, \mathcal{Q}$ are also partitions into Γ-intervals we know that they are refinements of M.

Let $1 \leq i \leq m$ and let $n_0 = k_0 = 0$. There exist $n_1 < n_2 < \ldots < n_m, k_1 < k_2 < \ldots < k_m \in \mathbb{N}$ such that $M_i = \bigsqcup_{j=n_{i-1}+1}^{n_i} I_j = \bigsqcup_{j=k_{i-1}+1}^{k_i} J_j$. Hence $\lambda(M_i) = \sum_{j=n_{i-1}+1}^{n_i} \lambda(I_i) = \sum_{j=k_{i-1}+1}^{k_i} \lambda(J_j)$. For every $s \in S$ we define $a_s = \mathrm{Card}(\{j \in \{n_{i-1}+1, \ldots, n_i\} \mid \lambda(I_j) = s\}$ and $b_s = \mathrm{Card}(\{j \in \{k_{i-1}+1, \ldots, k_i\} \mid \lambda(J_j) = s\})$. Then $\sum_{s \in S} a_s s = \sum_{s \in S} b_s s$. As S is free in $\mathrm{Vect}_\mathbb{Z}(S)$ we deduce that $a_s = b_s$ for every $s \in S$. This implies that $n_i = k_i$ for every $1 \leq i \leq n$ and $n = k$. We also get the existence of a permutation σ_i of the set $\{n_{i-1}+1, \ldots, n_i\}$ such that $\lambda(I_j) = \lambda(J_{\sigma_i(j)})$ for every $j \in \{n_{i-1}+1, \ldots, n_i\}$.

We deduce that it is sufficient to show the result when only one permutation σ_i is a transposition $(a\ a+1)$ with $n_{i-1}+1 \leq a < n_i$.

Let i_0 in $\{1, 2, \ldots m\}$ and $a \in \{n_{i_0-1}+1, \ldots, n_{i_0}\}$. We assume now that σ_{i_0} is the transposition $(a\ a+1)$ and $\sigma_i = \mathrm{Id}$ for every $i \neq i_0$. Then $I_j = J_j$ for every $j \in \{1, 2, \ldots, n\} \smallsetminus \{a, a+1\}$. We also have $I_a \cup I_{a+1} = J_a \cup J_{a+1}$ and $\lambda(I_{a+1}) = \lambda(J_a)$. In the case where f is order-preserving on M_{i_0} then $f_\mathcal{P}^+ = f_\mathcal{Q}^+$. In the case where f is order-reversing on M_{i_0} we deduce that $(f_\mathcal{Q}^+)^{-1} \circ f_\mathcal{P}^+$ is equal to the square of the restricted rotation whose intervals associated are I_a and I_{a+1} (see Figure 3.3.1).

In both cases we deduce $\varepsilon_\Gamma(f_P^+) \ [\mathrm{mod}\ 2] = \varepsilon_\Gamma(f_Q^+) \ [\mathrm{mod}\ 2]$. $\square$
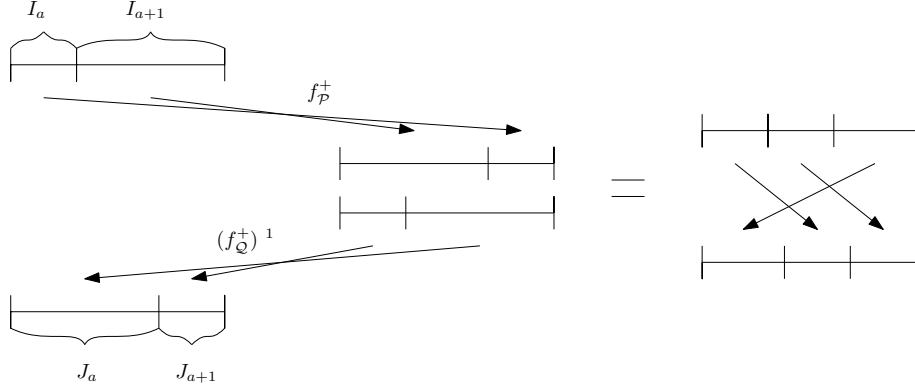
Figure 3.3.1 – Illustration of the fact that $(f_\mathcal{Q}^+)^{-1} \circ f_\mathcal{P}^+$ is equal to the square of a restricted rotation in Proposition 3.3.5

Definition 3.3.6. We define the S-map as:

$$\psi_S : \begin{array}{ccc} \text{IET}^{\bowtie}(\Gamma) & \longrightarrow & {}^{\ominus}\bigwedge_\mathbb{Z}^2 \tilde{\Gamma}/2^{\ominus}\bigwedge_\mathbb{Z}^2 \tilde{\Gamma} \\ f & \longmapsto & \begin{cases} \varepsilon_\Gamma(f_\mathcal{P}^+) \ [\text{mod } 2] & f \in G_S \\ 0 & f \notin G_S \end{cases} \end{array}$$

Where $\mathcal{P}$ is a partition into S-intervals associated with f.

Thanks to Proposition 3.3.5, the map ψ_S is well-defined. We check that ψ_S satisfies the morphism condition when we have the existence of a partition in S-intervals associated with a tuple.

Proposition 3.3.7. *Let $n \in \mathbb{N}$ and $f_1, f_2, \dots, f_n \in \text{IET}^{\bowtie}(\Gamma)$. If there exists $\mathcal{P}$ a partition into S-intervals associated with $(f_1, f_2, \dots, f_n)$ then:*

$$\psi_S(f_n f_{n-1} \dots f_1) = \sum_{i=1}^n \psi_S(f_i)$$

Proof. Let $n \in \mathbb{N}$ and $f_1, f_2, \dots, f_n \in \text{IET}^{\bowtie}(\Gamma)$. The case $n = 1$ is trivial thus we assume $n \geq 2$. Let $\mathcal{P}$ be a partition into S-intervals associated with $(f_1, f_2, \dots, f_n)$. Let $g_1 = (f_1)_\mathcal{P}^+$ and for every $2 \leq i \leq n$ let $g_i = (f_i)_{f_{i-1}\dots f_1(\mathcal{P})}^+$. By Proposition 3.1.7 we know that $(f_n f_{n-1} \dots f_1)_\mathcal{P}^+ = g_n g_{n-1} \dots g_1$, then:

$$\psi_S(f_n f_{n-1} \dots f_1) = \varepsilon_\Gamma((f_n f_{n-1} \dots f_1)_\mathcal{P}^+) \ [\text{mod } 2] = \varepsilon_\Gamma(g_n g_{n-1} \dots g_1) \ [\text{mod } 2]$$

Also ε_Γ is a group homomorphism thus:

$$\psi_S(f_n f_{n-1} \dots f_1) = \sum_{i=1}^n \varepsilon_\Gamma(g_i) \ [\text{mod } 2] = \sum_{i=1}^n \psi_S(f_i)$$

$\square$

3.3.3 The group homomorphism

We recall that M_Γ is the upward directed set introduced by Corollary 2.1.2 in the context of ultrasimplicially ordered groups. We define the group of germs at infinity of functions from M_Γ to ${}^{\ominus}\bigwedge_\mathbb{Z}^2 \tilde{\Gamma}/2^{\ominus}\bigwedge_\mathbb{Z}^2 \tilde{\Gamma}$ as the quotient: $\Xi_\Gamma = ({}^{\ominus}\bigwedge_\mathbb{Z}^2 \tilde{\Gamma}/2^{\ominus}\bigwedge_\mathbb{Z}^2 \tilde{\Gamma})^{M_\Gamma}/\text{Net}_0$, where Net_0 is the normal subgroup of eventually zero functions defined by:

$$\text{Net}_0 := \{(w_a)_{a \in M_\Gamma} \mid \exists a \in M_\Gamma \ \forall b \in M_\Gamma, \ b \geq a \Rightarrow w_b = 0\}$$

We are now able to define a new group homomorphism from $\text{IET}^{\bowtie}(\Gamma)$ to Ξ_Γ.

Definition 3.3.8. We define the *positive contribution of* $\mathrm{IET}^{\bowtie}(\Gamma)$ as the next map:

$$\psi_\Gamma : \quad \mathrm{IET}^{\bowtie}(\Gamma) \quad \longrightarrow \quad \Xi_\Gamma$$
$$f \quad \longmapsto \quad [(\psi_{S_a}(f))_{a \in M_\Gamma}]$$

Proposition 3.3.9. *The map ψ_Γ is a group homomorphism.*

Proof. Let $f, g \in \mathrm{IET}^{\bowtie}(\Gamma)$. By Proposition 3.3.4, there exists $a_0 \in M_\Gamma$ such that for every $b \geq a_0$, there exists a partition $\mathcal{P}_b$ into S_b-intervals associated with (f, g). Then by Proposition 3.3.7 we deduce that for every $b \geq a_0$ we have $\psi_{S_b}(g \circ f) = \psi_{S_b}(g) + \psi_{S_b}(f)$.

We denote by $w = (w_a)_{a \in M_\Gamma}$ the element of $(^\ominus\bigwedge^2_{\mathbb{Z}} \tilde{\Gamma} / 2^\ominus\bigwedge^2_{\mathbb{Z}} \tilde{\Gamma})^{M_\Gamma}$ defined by $w_a = \psi_{S_a}(g \circ f) - \psi_{S_a}(g) - \psi_{S_a}(f)$ for every $a \in M_\Gamma$. We remark that $w_b = 0$ for every $b \geq a_0$ thus $w \in \mathrm{Net}_0$. Then we have $(\psi_{S_a}(g \circ f))_a = (\psi_{S_a}(g))_a + (\psi_{S_a}(f))_a - w$. So $\psi_\Gamma(g \circ f) = \psi_\Gamma(g) + \psi_\Gamma(f)$. $\qquad\square$

The following lemma gives the value of an element in $\mathrm{IET}(\Gamma)$. The proof is immediate from the definition of ψ_{S_a}.

Lemma 3.3.10. *For every $f \in \mathrm{IET}(\Gamma)$, there exists $a_0 \in M_\Gamma$ such that for every $b \in M_\Gamma$ with $b \geq a_0$ we have we have $\psi_{S_b}(f) = \varepsilon_\Gamma(f) \ [\mathrm{mod} \ 2]$. In particular $\psi_\Gamma(f) = [(\varepsilon_\Gamma(f) \ [\mathrm{mod} \ 2])_{a \in M_\Gamma}]$.*

Remark 3.3.11. In fact the definition of Ψ_{S_a} gives us that for every $a \in M_\Gamma$ there exists $a' \in M_\Gamma$ with $a' \geq a$ such that for every $b \geq a'$ we have $\psi_{S_b}(f) = \varepsilon_\Gamma(f) \ [\mathrm{mod} \ 2]$. This is a sufficient condition to obtain the wanted equality in Ξ_Γ. It is also a necessary condition. Indeed, let us assume that there exists $a_0 \in M_\Gamma$ such that for every $b \in M_\Gamma$ with $b \geq a_0$ we have $\psi_{S_b}(f) = \varepsilon_\Gamma(f) \ [\mathrm{mod} \ 2]$. Then thanks to the property of M_Γ to be an upward directed set we deduce that for every $a \in M_\Gamma$ there exists $c_a \in M_\Gamma$ such that $c_a \geq a$ and $c_a \geq a_0$. Thus for every $b \in M_\Gamma$ such that $b \geq c_a$ we have $\psi_{S_b}(f) = \varepsilon_\Gamma(f) \ [\mathrm{mod} \ 2]$.

The next proposition show that the set we used to define Ω_Γ is not sent to the trivial element by ψ_Γ; thus this is not a subset of $D(\mathrm{IET}^{\bowtie}(\Gamma))$.

Proposition 3.3.12. *Let $\ell \in \tilde{\Gamma} \smallsetminus 2\tilde{\Gamma}$ and r be a Γ-reflection of type 2ℓ. Then $\psi_\Gamma(r) = [(\ell \wedge \ell \ [\mathrm{mod} \ 2])_{a \in M_\Gamma}] \neq 0$.*

Proof. We know that such a Γ-reflection is conjugate to a Γ-restricted rotation of type (ℓ, ℓ). We denote by s this Γ-restricted rotation. We have $\varepsilon_\Gamma(s) = \ell \wedge \ell \ [\mathrm{mod} \ 2]$. Then by Lemma 3.3.10 we have $\psi_\Gamma(r) = \psi_\Gamma(s) = [(\ell \wedge \ell \ [\mathrm{mod} \ 2])_a]$. Thus if $\psi_\Gamma(r) = 0$ then there exists $(w_a)_{a \in M_\Gamma} \in \mathrm{Net}_0$ such that $(\ell \wedge \ell \ [\mathrm{mod} \ 2])_a + (w_a)_a = 0$. We deduce that $\ell \wedge \ell \ [\mathrm{mod} \ 2] = 0$. Or this implies $\ell \in 2\tilde{\Gamma}$ and this is a contradiction. Hence $\psi_\Gamma(r) \neq 0$. $\qquad\square$

3.4 Description of $\mathrm{IET}^{\bowtie}(\Gamma)_{\mathrm{ab}}$

With both morphisms $\varepsilon_\Gamma^{\bowtie}$ and ψ_Γ we are now able to describe $D(\mathrm{IET}^{\bowtie}(\Gamma))$. We recall that Ω_Γ is the conjugate closure of the group generated by the set of all Γ-reflections of type 2ℓ with $\ell \in \tilde{\Gamma} \smallsetminus 2\tilde{\Gamma}$.

Lemma 3.4.1. *We have the inclusion:*

$$\Omega_\Gamma \cap \mathrm{Ker}(\psi_\Gamma) \subset D(\mathrm{IET}^{\bowtie}(\Gamma))$$

Proof. Let $w \in \Omega_\Gamma \cap \mathrm{Ker}(\psi_\Gamma)$. There exist $n \in \mathbb{N}$ and $g_1, g_2, \ldots g_n \in \mathrm{IET}^{\bowtie}(\Gamma)$ and $r_1, r_2, \ldots, r_n$ some Γ-reflections with type inside $\tilde{\Gamma} \smallsetminus 2\tilde{\Gamma}$ such that $w = \prod_{i=1}^{n} g_i r_i g_i^{-1}$. We know that every Γ-reflection is conjugate to a Γ-restricted rotation so there exist $h_1, h_2, \ldots h_n \in$

$\text{IET}^{\bowtie}(\Gamma)$ and $s_1, s_2, \ldots, s_n$ some Γ-restricted rotations such that $s_i = h_i r_i h_i^{-1}$ for every $1 \le i \le n$. Then $w = \prod_{i=1}^{n} g_i h_i^{-1} s_i h_i g_i^{-1}$. By Lemma 3.2.2 it is sufficient to show that w is a balanced product of Γ-reflections.

As for every $1 \le i \le n$ we have g_i and g_i^{-1} which appear the same number of time in w (the same is true for h and h^{-1}) we deduce that it is sufficient to show that $w' := \prod_{i=1}^{n} s_i$ is a balanced product of Γ-reflections. By Lemma 3.2.2 it is enough to show that $w' \in D(\text{IET}^{\bowtie}(\Gamma))$.

We notice that $\psi_\Gamma(w) = \psi_\Gamma(w')$ thus by the assumption we deduce that $\psi_\Gamma(w') = 0$. Hence by Lemma 3.3.10 we deduce that $\varepsilon_\Gamma(w')$ [mod 2] $= 0$. This equality stands in $^\ominus \bigwedge_{\mathbb{Z}}^2 \tilde{\Gamma}/2^\ominus\bigwedge_{\mathbb{Z}}^2 \tilde{\Gamma}$. There exists $k \in \mathbb{N}$ and for every $1 \le j \le k$ there exist $a_j, b_j \in \tilde{\Gamma}_+$ with $a_j + b_j < 1$ and $n_j \in \mathbb{N}$ and $\nu_j \in \{-1, 1\}$ such that $\varepsilon_\Gamma(w') + \sum_{j=1}^{k} 2n_j \nu_j a_j \wedge b_j = 0$. For every $1 \le j \le k$ let γ_j be a Γ-restricted rotation of type (a_j, b_j). Then the element $w' \prod_{j=1}^{k} (\gamma_j^{n_j \nu_j})^2$ is in $\text{IET}(\Gamma)$ and satisfies:

$$\varepsilon_\Gamma\left(w' \prod_{j=1}^{k} (\gamma_j^{n_j \nu_j})^2\right) = \varepsilon_\Gamma(w') + \sum_{j=1}^{k} 2n_j \nu_j a_j \wedge b_j = 0$$

By Theorem 2.3.24 the element $w' \prod_{j=1}^{k} (\gamma_j^{n_j \nu_j})^2$ is in $D(\text{IET}(\Gamma)) \subset D(\text{IET}^{\bowtie}(\Gamma))$. By Corollary 3.2.3 we know that $\prod_{j=1}^{k} (\gamma_j^{n_j \nu_j})^2$ is in $D(\text{IET}^{\bowtie}(\Gamma))$. Then we deduce that $w' \in D(\text{IET}^{\bowtie}(\Gamma))$. Hence w is in $D(\text{IET}^{\bowtie}(\Gamma))$. $\qquad\square$

Theorem 3.4.2. *We have $D(\text{IET}^{\bowtie}(\Gamma)) = \text{Ker}(\varepsilon_\Gamma^{\bowtie}) \cap \text{Ker}(\psi_\Gamma)$.*

Proof. The inclusion from left to right is trivial.

Let $f \in \text{Ker}(\varepsilon_\Gamma^{\bowtie}) \cap \text{Ker}(\psi_\Gamma)$. By Theorem 3.2.19 there exist $g \in D(\text{IET}^{\bowtie}(\Gamma))$ and $h \in \Omega_\Gamma$ such that $f = gh$. We deduce that $\psi_\Gamma(h) = \psi_\Gamma(f) = 0$ so $h \in \text{Ker}(\psi_\Gamma) \cap \Omega_\Gamma$. By Lemma 3.4.1 we obtain that $h \in D(\text{IET}^{\bowtie}(\Gamma))$, thus $f \in D(\text{IET}^{\bowtie}(\Gamma))$. $\qquad\square$

Corollary 3.4.3. *We have $D(\text{IET}^{\bowtie}(\Gamma)) = \text{Ker}(\psi_\Gamma|_{\text{Ker}(\varepsilon_\Gamma^{\bowtie})})$.*

Lemma 3.4.4. *The quotient $\text{Ker}(\varepsilon_\Gamma^{\bowtie})/D(\text{IET}^{\bowtie}(\Gamma))$ is isomorphic to the subgroup $\langle\{\ell \wedge \ell \text{ [mod 2]} \mid \ell \in \tilde{\Gamma}\}\rangle$ of $^\ominus\bigwedge_{\mathbb{Z}}^2 \tilde{\Gamma}/2^\ominus\bigwedge_{\mathbb{Z}}^2 \tilde{\Gamma}$.*

Proof. By Corollary 3.4.3 we have:

$$\text{Ker}(\varepsilon_\Gamma^{\bowtie})/D(\text{IET}^{\bowtie}(\Gamma)) \simeq \text{Im}(\psi_\Gamma|_{\text{Ker}(\varepsilon_\Gamma^{\bowtie})}) = \psi_\Gamma(\text{Ker}(\varepsilon_\Gamma^{\bowtie}))$$

By Theorem 3.2.19 and as ψ_Γ is a group homomorphism we have the equality $\psi_\Gamma(\text{Ker}(\varepsilon_\Gamma^{\bowtie})) = \psi_\Gamma(\Omega_\Gamma)$. Furthermore Ω_Γ is the normal closure of the group generated by all Γ-reflections of type 2ℓ with $\ell \in \tilde{\Gamma} \smallsetminus 2\tilde{\Gamma}$. Hence we deduce that:

$$\psi_\Gamma(\Omega_\Gamma) = \langle\{[(\ell \wedge \ell \text{ [mod 2]})_{n \in \mathbb{N}}] \mid \ell \in \tilde{\Gamma} \smallsetminus 2\tilde{\Gamma}\rangle = \langle\{[(\ell \wedge \ell \text{ [mod 2]})_{n \in \mathbb{N}}] \mid \ell \in \tilde{\Gamma}\rangle$$

Thus $\psi_\Gamma(\text{Ker}(\varepsilon_\Gamma^{\bowtie})) \simeq \langle\{\ell \wedge \ell \text{ [mod 2]} \mid \ell \in \tilde{\Gamma}\}\rangle$. $\qquad\square$

Theorem 3.4.5. *We have the following group isomorphisms:*

$$\text{IET}^{\bowtie}(\Gamma)_{\text{ab}} \simeq \text{Im}(\varepsilon_\Gamma^{\bowtie}) \times \text{Ker}(\varepsilon_\Gamma^{\bowtie})/D(\text{IET}^{\bowtie}(\Gamma))$$
$$\simeq \langle\{a \otimes a \text{ [mod 2]} \mid a \in \tilde{\Gamma}\}\rangle \times \langle\{\ell \wedge \ell \text{ [mod 2]} \mid \ell \in \tilde{\Gamma}\}\rangle,$$

where the left term of the product is in $\bigotimes_{\mathbb{Z}}^2 \tilde{\Gamma}/(2\bigotimes_{\mathbb{Z}}^2 \tilde{\Gamma})$ and the right one is in $^\ominus\bigwedge_{\mathbb{Z}}^2 \tilde{\Gamma}/(2^\ominus\bigwedge_{\mathbb{Z}}^2 \tilde{\Gamma})$.

Proof. The second isomorphism is given by Corollary 3.2.14 and Lemma 3.4.4.

For the first isomorphism we recall that we have the following exact sequence:

$$1 \to \operatorname{Ker}(\varepsilon_\Gamma^{\bowtie})/D(\operatorname{IET}^{\bowtie}(\Gamma)) \to \operatorname{IET}^{\bowtie}(\Gamma)_{\mathrm{ab}} \to \operatorname{IET}^{\bowtie}(\Gamma)/\operatorname{Ker}(\varepsilon_\Gamma^{\bowtie}) \to 1$$

Each group in this exact sequence has exponent 2. Then they are also $\mathbb{F}_2$-vectorial spaces. We deduce that this exact sequence is an exact sequence of $\mathbb{F}_2$-vectorial spaces, thus it splits and gives the result. $\square$

Remark 3.4.6. If $\tilde{\Gamma}$ has dimension d then $\langle\{a \otimes a \text{ [mod 2]} \mid a \in \tilde{\Gamma}\}\rangle$ has dimension $\frac{d(d+1)}{2}$, as $\mathbb{F}_2$-vector space, and $\langle\{\ell \wedge \ell \text{ [mod 2]} \mid \ell \in \tilde{\Gamma}\}\rangle$ has dimension d, as $\mathbb{F}_2$-vector space, so $\operatorname{IET}^{\bowtie}(\Gamma)_{\mathrm{ab}}$ has dimension $\frac{d(d+3)}{2}$ over $\mathbb{F}_2$.

Remark 3.4.7. The inclusion of $\operatorname{IET}(\Gamma)$ in $\operatorname{IET}^{\bowtie}(\Gamma)$ induces a group morphism:

$$\eta : \operatorname{IET}(\Gamma)_{\mathrm{ab}}/2(\operatorname{IET}(\Gamma)_{\mathrm{ab}}) \to \operatorname{IET}^{\bowtie}(\Gamma)_{\mathrm{ab}}.$$

By Theorem 1.2.8 we know that $\operatorname{IET}(\Gamma)$ is generated by Γ-restricted rotations thus we deduce that the image of η is the subgroup $\langle\{p \otimes q + q \otimes p \text{ [mod 2]} \mid p, q \in \tilde{\Gamma}\}\rangle \times \langle\{a \wedge a \text{ [mod 2]}\}\rangle$ of $\bigotimes_{\mathbb{Z}}^2 \tilde{\Gamma}/(2\bigotimes_{\mathbb{Z}}^2 \tilde{\Gamma}) \times {}^{\ominus}\bigwedge_{\mathbb{Z}}^2 \tilde{\Gamma}/(2{}^{\ominus}\bigwedge_{\mathbb{Z}}^2 \tilde{\Gamma})$. This is isomorphic to ${}^{\ominus}\bigwedge_{\mathbb{Z}}^2 \tilde{\Gamma}/(2{}^{\ominus}\bigwedge_{\mathbb{Z}}^2 \tilde{\Gamma})$ and if $\tilde{\Gamma}$ has dimension d then its dimension is $\frac{d(d+1)}{2}$ as $\mathbb{F}_2$-vector space. In this case η is not surjective and its cokernel has dimension d over $\mathbb{F}_2$. In the case where $\tilde{\Gamma}$ has infinite dimension over $\mathbb{Z}$ we deduce that $\operatorname{Im}(\eta)$ also has infinite dimension over $\mathbb{F}_2$. By Proposition 2.3.18 we deduce that the group $\langle\{p \otimes q + q \otimes p \text{ [mod 2]} \mid p, q \in \tilde{\Gamma}\}\rangle$ is equal to the group $\langle\{a \otimes a \text{ [mod 2]} \mid a \in \tilde{\Gamma}\}\rangle$ if and only if $\tilde{\Gamma} = 2\tilde{\Gamma}$. Then η is surjective if and only if $\tilde{\Gamma} = 2\tilde{\Gamma}$.

Proposition 3.4.8. *The group homomorphism η is injective.*

Proof. For every $f \in \operatorname{IET}(\Gamma)$ we denote by $[f]$ its image in $\operatorname{IET}(\Gamma)_{\mathrm{ab}}$. Thanks to Theorem 2.3.24 we know that $[f] \in 2\operatorname{IET}(\Gamma)_{\mathrm{ab}}$ if and only if $\varepsilon_\Gamma(f) \in 2{}^{\ominus}\bigwedge_{\mathbb{Z}}^2 \Gamma$. Hence to prove the statement it is enough to prove that for every $f \in \operatorname{IET}(\Gamma)$ such that $\varepsilon_\Gamma^{\bowtie}(f) = 0$ and $\operatorname{proj}_{\operatorname{Ker}(\varepsilon_\Gamma^{\bowtie})}(f) = 0$ we have $\varepsilon_\Gamma(f) \in 2{}^{\ominus}\bigwedge_{\mathbb{Z}}^2 \Gamma$. We use notations of inversions defined in Definition 3.2.5. By Corollary 2.1.3 there exists $n \in \mathbb{N}$ and a $\mathbb{Z}$-linearly independent family $\{l_1, l_2, \ldots, l_n\}$ of Γ_+ and $n_{i,j} \in \mathbb{Z}$ such that $\mathcal{E}_{f,1} = \sum\limits_{i,j} n_{i,j} l_i \otimes l_j$. The equality $\operatorname{proj}_{\operatorname{Ker}(\varepsilon_\Gamma^{\bowtie})}(f) = 0$ gives us that $n_{i,i} = 0$. We have $\varepsilon_\Gamma^{\bowtie}(f) = \sum\limits_{i \neq j}(n_{i,j} + n_{j,i}) l_i \otimes l_j \text{ [mod 2]} = 0$. We deduce that 2 divides $(n_{i,j} + n_{j,i})$ for every $1 \leq i \neq j \leq n$. We obtain that :

$$\begin{aligned}
\varepsilon_\Gamma(f) &= \sum_{i \neq j} n_{i,j} l_i \wedge l_j \\
&= \sum_{i < j}(n_{i,j} - n_{j,i}) l_i \wedge l_j \\
&= \sum_{i < j}(n_{i,j} + n_{j,i}) l_i \wedge l_j - 2\sum_{i < j} n_{j,i} l_i \wedge l_j
\end{aligned}$$

We deduce that 2 divides $\varepsilon_\Gamma(f)$ and this gives the result.

$\square$

Signature for piecewise continuous groups

Résumé en français. On montre l'annulation de la classe de Kapoudjian de PC$^{\bowtie}$ et par conséquent l'annulation de la classe de Kapoudjian de tous les sous-groupes de PC$^{\bowtie}$. Cela correspond à ce que le groupe PC$^{\bowtie}$ se relève dans le quotient de $\widehat{\text{PC}^{\bowtie}}$ par son sous-groupe alterné des permutations paires. Pour démontrer ce résultat, on construit un morphisme de groupes non trivial, appelé signature, de $\widehat{\text{PC}^{\bowtie}}$ vers $\mathbb{Z}/2\mathbb{Z}$ qui étend la signature naturelle sur $\mathfrak{S}_{\text{fin}}$. Soit G un sous-groupe de PC$^{\bowtie}$ qui est simple et non-abélien et soit $\widehat{G}$ sa pré-image dans $\widehat{\text{PC}^{\bowtie}}$; grâce au morphisme signature que l'on a construit on peut alors lister l'ensemble des sous-groupes normaux de $\widehat{G}$.

Ce chapitre correspond à l'article [Lac20b].

English abstract. We show that the Kapoudjian class of PC$^{\bowtie}$ vanishes, and by consequence the one of every subgroup of PC$^{\bowtie}$ also vanishes. That is, the quotient map $\widehat{\text{PC}^{\bowtie}} \to \text{PC}^{\bowtie}$ splits modulo the alternating subgroup of even permutations. This is shown by constructing a nonzero group homomorphism, called signature, from $\widehat{\text{PC}^{\bowtie}}$ to $\mathbb{Z}/2\mathbb{Z}$ which extends the natural signature on $\mathfrak{S}_{\text{fin}}$. Let G be a subgroup of PC$^{\bowtie}$ which is simple nonabelian and let $\widehat{G}$ be its preimage in $\widehat{\text{PC}^{\bowtie}}$; thanks to the signature on $\widehat{\text{PC}^{\bowtie}}$ we are able to list normal subgroups of $\widehat{G}$.

This chapter corresponds to the article [Lac20b].

Contents

4.1 Introduction

The beginning of the study of the symmetric group of an infinite set X, denoted by $\mathfrak{S}(X)$, date back at least in 1915 with an article of Vitali [Vit15] and one of Andreoli [And15]. Even if he does not use the language of group theory, Vitali proves that every element of $\mathfrak{S}(\mathbb{Z})$ is a finite product of squares. A direct consequence is the following:

Theorem 4.1.1 (Vitali [Vit15])**.** *The only group homomorphism from $\mathfrak{S}(X)$ into $\mathbb{Z}/2\mathbb{Z}$ is the trivial one.*

A natural subgroup to consider is the one consisting of all finitely supported permutations, denoted by $\mathfrak{S}_{\mathrm{fin}}(X)$. It is a normal subgroup and with the same proof as in finite permutation group, we can define a surjective group homomorphism $\varepsilon_{\mathrm{fin}}(X)$ onto $(\mathbb{Z}/2\mathbb{Z}, +)$ called the classical signature. We denote it by $\mathfrak{A}_{\mathrm{fin}}(X)$ the kernel of the classical signature. It is called the alternating subgroup and it is a subgroup of index two of $\mathfrak{S}_{\mathrm{fin}}(X)$ and also a normal subgroup of $\mathfrak{S}(X)$. In 1929, Onefri [Ono29] proves that $\mathfrak{S}_{\mathrm{fin}}(X)$ and $\mathfrak{A}_{\mathrm{fin}}(X)$ are the only proper normal subgroups of $\mathfrak{S}(X)$ when X is a countable set. This result was extended to every infinite set X by Baer [Bae34]. With these groups we obtain the following exact sequence:

$$0 \to \mathbb{Z}/2\mathbb{Z} = \mathfrak{S}_{\mathrm{fin}}(X)/\mathfrak{A}_{\mathrm{fin}}(X) \to \mathfrak{S}(X)/\mathfrak{A}_{\mathrm{fin}}(X) \to \mathfrak{S}(X)/\mathfrak{S}_{\mathrm{fin}}(X) \to 1$$

It is a central extension, thus it gives a canonical element of the cohomology group $H^2(\mathfrak{S}(X)/\mathfrak{S}_{\mathrm{fin}}(X), \mathbb{Z}/2\mathbb{Z})$. It appears in the work of Kapoudjian and Sergiescu [Kap02, KS05], and also in the work of Cornulier [Cor19a] in the context of near actions.

Here we work with the interval $[0, 1[$ and we recall that $\mathfrak{S}_{\mathrm{fin}} = \mathfrak{S}_{\mathrm{fin}}([0, 1[)$. We denote also $\varepsilon_{\mathrm{fin}} = \varepsilon_{\mathrm{fin}}([0, 1[)$. For every subgroup G of $\mathfrak{S}([0, 1[)/\mathfrak{S}_{\mathrm{fin}}$ we denote by $\widehat{G}$ its preimage in $\mathfrak{S}([0, 1[)$. Then, we have the following exact sequence:

$$0 \to \mathbb{Z}/2\mathbb{Z} = \mathfrak{S}_{\mathrm{fin}}/\mathfrak{A}_{\mathrm{fin}} \to \widehat{G}/\mathfrak{A}_{\mathrm{fin}} \to G \to 1$$

As before, it is a central extension, thus it gives a canonical element of the cohomology group $H^2(G, \mathbb{Z}/2\mathbb{Z})$, called the Kapoudjian class of G. For such a G, Cornulier [Cor19b] asks when its Kapoudjian class vanishes. This is answered positively below. Such a vanishing implies in particular that $\widehat{G}/\mathfrak{A}_{\mathrm{fin}}$ is isomorphic to the direct product $G \times \mathbb{Z}/2\mathbb{Z}$. If there exists a group homomorphism from $\widehat{G}$ onto $\mathbb{Z}/2\mathbb{Z}$ which extends the signature $\varepsilon_{\mathrm{fin}}$, then the above exact sequence splits and this implies the vanishing of the Kapoudjian class of G.

Theorem 4.1.2. *There exists a group homomorphism $\varepsilon : \widehat{\mathrm{PC}^{\bowtie}} \to \mathbb{Z}/2\mathbb{Z}$ that extends the classical signature on $\mathfrak{S}_{\mathrm{fin}}$.*

In contrast, Theorem 4.1.1 implies that the Kapoudjian class of $\mathfrak{S}([0, 1[)/\mathfrak{S}_{\mathrm{fin}}$, does not vanish.

Corollary 4.1.3. *Let G be a subgroup of $\mathrm{PC}^{\bowtie}$. Then the Kapoudjian class of G is zero.*

The idea of the proof of Theorem 4.1.2 is to associate for every $f \in \widehat{\mathrm{PC}^{\bowtie}}$ and every finite partition $\mathcal{P}$ of $[0, 1[$ into intervals associated with f, two numbers. The first is the number of interval of $\mathcal{P}$ where f is order-reversing and the second is the signature of a particular finitely supported permutation. The next step is to prove that the sum modulo 2 of this two numbers is independent from the choice of partition. Then we show that it is enough to prove that $\varepsilon|_{\mathrm{IET}^{\bowtie}}$ is a group homomorphism. For this we show that it is additive when we look at the composition of two elements of $\widehat{\mathrm{IET}^{\bowtie}}$ by calculate the value of the signature with a particular partition.

In Section 4.3, we apply these results to the study of normal subgroups of $\widehat{\mathrm{PC}^{\bowtie}}$ and certain subgroups. More specifically we prove:

Theorem 4.1.4. *Let G be a subgroup of $\mathrm{PC}^{\bowtie}$ which is simple nonabelian and let $\widehat{G}$ be its preimage in $\mathfrak{S}$. Then $\widehat{G}$ has exactly 5 normal subgroups, which constitute the following diagram:*

$$
\begin{array}{ccc}
 & \widehat{G} & \\
\diagup & & \diagdown \\
\mathfrak{S}_{\mathrm{fin}} & & \mathrm{Ker}(\varepsilon|_{\widehat{G}}) \\
\diagdown & & \diagup \\
 & \mathfrak{A}_{\mathrm{fin}} & \\
 & | & \\
 & \{1\} &
\end{array}
$$
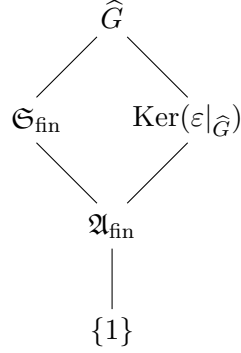
Let $\widehat{\mathrm{AIET}}$ be the subgroup of $\widehat{\mathrm{PC}^{\bowtie}}$ consisting of elements that are piecewise affine, and let $\widehat{\mathrm{AIET}^+}$ be its subgroup of piecewise orientation-preserving elements. N. Guelman and I. Liousse [GL19a] prove that AIET^+ is uniformly perfect and uniformly simple. Then with Theorem 1.3.4 we deduce that $\mathrm{AIET}^{\bowtie}$ is also perfect and so simple (see Corollary 1.3.3). Hence with Theorem 1.3.4 and Corollary 1.3.5 we deduce that $\mathrm{PC}^{\bowtie}, \mathrm{PC}^+, \mathrm{AIET}^{\bowtie}, \mathrm{AIET}^+, \mathrm{IET}^{\bowtie}$ are examples of groups that satisfy conditions of Theorem 4.1.4.

For every real interval I we denote by I° its interior in $\mathbb{R}$ and if $I = [0, t[$ we agree that its interior is $]0, t[$. We use the notion of partition associated defined in Section 1.2.1. We recall that for every $f \in \mathrm{PC}^{\bowtie}$, we denote by Π_f the set of all partitions into intervals associated with f. We recall that $\mathcal{P}_f^{\min}$ is the unique partition in Π_f such that every partition $\mathcal{Q} \in \Pi_f$ is a refinement of $\mathcal{P}_f^{\min}$.

4.2 Construction of the signature homomorphism

4.2.1 Definitions

The main decomposition we use here is the one given by Proposition 3.1.1. We recall the statement:

Proposition 4.2.1. *Let $\widehat{f} \in \widehat{\mathrm{PC}^{\bowtie}}$ (resp. $\widehat{\mathrm{IET}^{\bowtie}}$) and let $\mathcal{P}$ be a partition associated with $\widehat{f}$. Then there exist a unique subset $A(\widehat{f}, \mathcal{P}) \subset \widehat{f}(\mathcal{P})$ and a unique finitely supported permutation $\sigma_{(\widehat{f}, \mathcal{P})}$ such that $\sigma_{(\widehat{f}, \mathcal{P})} \circ (\prod_{I \in A(\widehat{f}, \mathcal{P})} \widehat{r}_I) \circ \widehat{f}$ belongs to $\widehat{\mathrm{PC}^+}$ (resp. $\widehat{\mathrm{IET}^+}$) and is right-continuous.*

Where $\sigma_{(\widehat{f}, \mathcal{P})}$ is the default of pseudo right-continuity of $\widehat{f}$ according to $\mathcal{P}$ and the composition $\sigma_{(\widehat{f}, \mathcal{P})} \circ (\prod_{I \in A(\widehat{f}, \mathcal{P})} \widehat{r}_I) \circ \widehat{f}$ is denoted by $f_{\mathcal{P}}^+$ and it is called the positive $\mathcal{P}$-substitute of $\widehat{f}$ and f.

Definition 4.2.2. The cardinal of $A(\widehat{f}, \mathcal{P})$ is denoted $R(\widehat{f}, \mathcal{P})$ and is called the *flip number of $\widehat{f}$ about $\mathcal{P}$*.

It is important to notice that we will use here partitions associated with an element of $\widehat{\mathrm{PC}^{\bowtie}}$ and not all essential partitions. We also give a definition more dynamical for the default of pseudo right-continuity:

Definition 4.2.3. Let $\widehat{f}$ be an element of $\widehat{\mathrm{PC}^{\bowtie}}$, $n \in \mathbb{N}$ and $\mathcal{P} = \{I_1, I_2, \ldots, I_n\} \in \Pi_{\widehat{f}}$. For every $1 \leq j \leq n$, let α_j be the left endpoint of I_j and β_j be the left endpoint of $\widehat{f}(I_j^\circ)$. We define the *default of pseudo right continuity for $\widehat{f}$ about $\mathcal{P}$* denoted $\sigma_{(\widehat{f}, \mathcal{P})}$ as the finitely supported permutation which sends $\widehat{f}(\alpha_j)$ to β_j for every $1 \leq j \leq n$ (this is well-defined because the set of all $\widehat{f}(\alpha_j)$ is equal to the set of all β_j).

The equivalence between the two definitions is given by the right-continuity of $\widehat{f}_\mathcal{P}^+$.

Definition 4.2.4. For $\widehat{f} \in \widehat{\mathrm{PC}^{\bowtie}}$ and $\mathcal{P} \in \Pi_{\widehat{f}}$, define $\varepsilon(\widehat{f}, \mathcal{P}) \in \mathbb{Z}/2\mathbb{Z}$ as $R(\widehat{f}, \mathcal{P}) + \varepsilon_{\mathrm{fin}}(\sigma_{(\widehat{f}, \mathcal{P})})$ [mod 2]. We define also $\varepsilon(\widehat{f}) = \varepsilon(\widehat{f}, \mathcal{P}_{\widehat{f}}^{\mathrm{fin}})$.

Proposition 4.2.5. *For every $\tau \in \mathfrak{S}_{\mathrm{fin}}$ and every $\mathcal{P} \in \Pi_\tau$ we have $\varepsilon(\tau, \mathcal{P}) = \varepsilon_{\mathrm{fin}}(\tau)$.*

Proof. It is clear that for every $\tau \in \mathfrak{S}_{\mathrm{fin}}$ and every partition $\mathcal{P}$ associated with τ we have $R(\tau, \mathcal{P}) = 0$ and $\sigma_{(\tau, \mathcal{P})} = \tau$. $\qquad\square$

We deduce that ε extends the classical signature $\varepsilon_{\mathrm{fin}}$. Thus we will write ε instead of $\varepsilon_{\mathrm{fin}}$.

Proposition 4.2.6. *Every right-continuous element $\widehat{f}$ of $\widehat{\mathrm{PC}^+}$ satisfies $\varepsilon(\widehat{f}, \mathcal{P}) = 0$ for every $\mathcal{P} \in \Pi_{\widehat{f}}$.*

Proof. In this case, for every partition $\mathcal{P}$ into intervals associated with $\widehat{f}$ we always have $R(\widehat{f}, \mathcal{P}) = 0$ and $\sigma_{(\widehat{f}, \mathcal{P})} = \mathrm{Id}$. $\qquad\square$

4.2.2 Proof of Theorem 4.1.2

In order to prove that ε is a group homomorphism, it is useful to calculate $\varepsilon(\widehat{f})$ thanks to $\varepsilon(\widehat{f}, \mathcal{P})$ for every $\widehat{f} \in \widehat{\mathrm{PC}^{\bowtie}}$ and $\mathcal{P} \in \Pi_{\widehat{f}}$.

Lemma 4.2.7. *For every $\widehat{f} \in \widehat{\mathrm{PC}^{\bowtie}}$ and every $\mathcal{P} \in \Pi_h$ we have $\varepsilon(\widehat{f}) = \varepsilon(\widehat{f}, \mathcal{P})$.*

Proof. Let $\widehat{f}$ and $\mathcal{P}$ be as in the statement. By minimality of $\mathcal{P}_{\widehat{f}}^{\mathrm{min}}$, in term of refinement, we deduce that there exist $n \in \mathbb{N}$ and $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n \in \Pi_{\widehat{f}}$ such that:

1. $\mathcal{P}_1 = \mathcal{P}_{\widehat{f}}^{\mathrm{min}}$;

2. $\mathcal{P}_n = \mathcal{P}$;

3. for every $2 \leq i \leq n$ the partition $\mathcal{P}_i$ is a refinement of the partition $\mathcal{P}_{i-1}$ where only one interval of $\mathcal{P}_{i-1}$ is cut into two.

Hence it is enough to show $\varepsilon(\widehat{f}, \mathcal{Q}) = \varepsilon(\widehat{f}, \mathcal{Q}')$ where $\mathcal{Q}, \mathcal{Q}' \in \Pi_{\widehat{f}}$ such that there exist consecutive intervals $I, J \in \mathcal{Q}$ with $I \cup J \in \mathcal{Q}'$ and $\mathcal{Q}' \smallsetminus \{I \cup J\} = \mathcal{Q} \smallsetminus \{I, J\}$.

Let α be the left endpoint of I and let x be the right endpoint of I (x is also the left endpoint of J). There are only two cases but in both cases, we know that $\sigma_{(\widehat{f}, \mathcal{Q})} = \sigma_{(\widehat{f}, \mathcal{Q}')}$ except maybe on $\widehat{f}(\alpha)$ and $\widehat{f}(x)$:

1. The first case is when $\widehat{f}$ is order-preserving on $(I \cup J)^\circ$. Then as $\mathcal{Q} \smallsetminus \{I, J\} = \mathcal{Q}' \smallsetminus \{I \cup J\}$ we get $R(\widehat{f}, \mathcal{Q}) = R(\widehat{f}, \mathcal{Q}')$. As $\widehat{f}$ is order-preserving on the interior of $I \cup J$ we know that $\sigma_{(\widehat{f}, \mathcal{Q}')}(\widehat{f}(\alpha))$ is the left endpoint of $\widehat{f}((I \cup J)^\circ)$ which is the left endpoint of $\widehat{f}(I^\circ)$ thus equals to $\sigma_{(\widehat{f}, \mathcal{Q})}(\widehat{f}(\alpha))$. With the same reasoning we deduce that $\sigma_{(\widehat{f}, \mathcal{Q}')}(\widehat{f}(x)) = \sigma_{(\widehat{f}, \mathcal{Q})}(\widehat{f}(x))$ hence $\sigma_{(\widehat{f}, \mathcal{Q})} = \sigma_{(\widehat{f}, \mathcal{Q}')}$. Thus in $\mathbb{Z}/2\mathbb{Z}$ we have
$$R(\widehat{f}, \mathcal{Q}') + \varepsilon(\sigma_{(\widehat{f}, \mathcal{Q}')}) = R(\widehat{f}, \mathcal{Q}) + \varepsilon(\sigma_{(\widehat{f}, \mathcal{Q})}).$$

2. The second case is when $\widehat{f}$ is order-reversing on $(I \cup J)^\circ$. Then we get $R(\widehat{f}, \mathcal{Q}) = R(\widehat{f}, \mathcal{Q}') + 1$. This time $\sigma_{(\widehat{f}, \mathcal{Q}')}(\widehat{f}(\alpha))$ is still the left endpoint of $\widehat{f}((I \cup J)^\circ)$ which is the left endpoint of $\widehat{f}(J^\circ)$ thus equals to $\sigma_{(\widehat{f}, \mathcal{Q})}(\widehat{f}(x))$. With the same reasoning we deduce that $\sigma_{(\widehat{f}, \mathcal{Q}')}(\widehat{f}(x)) = \sigma_{(\widehat{f}, \mathcal{Q})}(\widehat{f}(\alpha))$. Then by denoting τ the transposition $(\widehat{f}(x) \; \sigma_{(\widehat{f}, \mathcal{Q}')}(\widehat{f}(\alpha)))$, we obtain $\sigma_{(\widehat{f}, \mathcal{Q})} = \tau \circ \sigma_{(\widehat{f}, \mathcal{Q}')}$. We must notice that the transposition is not the identity because $\widehat{f}^{-1}(\sigma_{(\widehat{f}, \mathcal{Q}')}(\widehat{f}(\alpha)))$ is an endpoint of one of the intervals of $\mathcal{Q}'$ and x is not.

In conclusion in $\mathbb{Z}/2\mathbb{Z}$ we have:

$$R(\widehat{f}, \mathcal{Q}') + \varepsilon(\sigma_{(\widehat{f}, \mathcal{Q}')}) = R(\widehat{f}, \mathcal{Q}') + 1 + 1 + \varepsilon(\sigma_{(\widehat{f}, \mathcal{Q}')}) = R(\widehat{f}, \mathcal{Q}) + \varepsilon(\sigma_{(\widehat{f}, \mathcal{Q})})$$
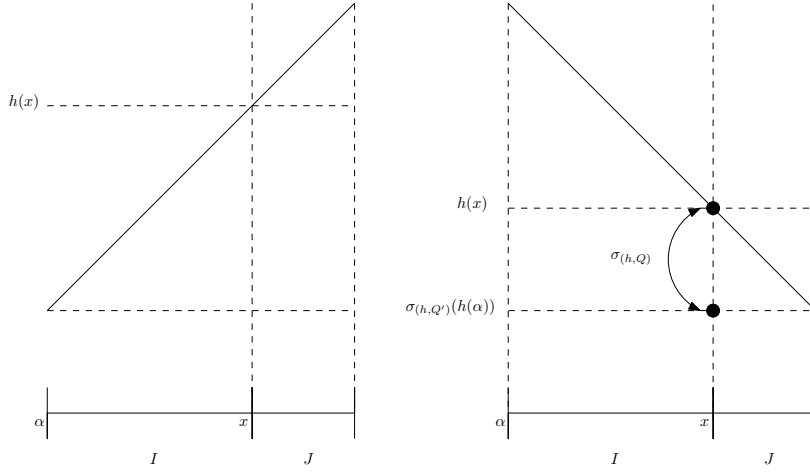
Figure 4.2.1 – Illustrations of the two cases appearing in Lemma 4.2.7. **Left:** Case where $\widehat{f}$ is order-preserving on $(I \cup J)^\circ$ and we see that $\sigma_{(\widehat{f}, \mathcal{Q})}(\widehat{f}(x)) = \sigma_{(\widehat{f}, \mathcal{Q}')}(\widehat{f}(x))$. **Right:** Case where $\widehat{f}$ is order-reversing on $(I \cup J)^\circ$ and we see that $\sigma_{(\widehat{f}, \mathcal{Q})}(\widehat{f}(x)) = (\widehat{f}(x) \; \sigma_{(\widehat{f}, \mathcal{Q}')}(\widehat{f}(\alpha))) \circ \sigma_{(\widehat{f}, \mathcal{Q}')}(\widehat{f}(x))$.

$\square$

If $\phi \in \mathrm{Homeo}^+([0,1[)$, then it follows from Proposition 4.2.6 that $\varepsilon(\phi) = 0$. We show that ε is invariant by the action of $\mathrm{Homeo}^+([0,1[)$ on $\widehat{\mathrm{PC}^{\bowtie}}$.

Lemma 4.2.8. *For every* $\widehat{f} \in \widehat{\mathrm{PC}^{\bowtie}}$ *and every* $\phi \in \mathrm{Homeo}^+([0,1[)$ *we have* $\varepsilon(\widehat{f}\phi) = \varepsilon(\widehat{f}) = \varepsilon(\phi\widehat{f})$.

Proof. Let $\widehat{f} \in \widehat{\mathrm{PC}^{\bowtie}}$ and $\phi \in \mathrm{Homeo}^+([0,1[)$ be as in the statement. Let $n \in \mathbb{N}$ and $\mathcal{P} := \{I_1, I_2, \ldots, I_n\} \in \Pi_{\widehat{f}}$. Then $\mathcal{Q} := \{\phi^{-1}(I_1), \phi^{-1}(I_2), \ldots, \phi^{-1}(I_n)\}$ is in $\Pi_{\widehat{f}\phi}$. We know that ϕ is order preserving then for every $1, \leq i \leq n$, $\widehat{f}\phi$ preserves (reverses respectively) the order on $\phi^{-1}(I_i^\circ)$ if and only if $\widehat{f}$ preserves (reverses respectively) the order on I_i°, so $R(\widehat{f}, \mathcal{P}) = R(\widehat{f}\phi, \mathcal{Q})$. We can notice that the left endpoint of $\phi^{-1}(I_i^\circ)$ (denoted by α_i) is send on the left endpoint of I_i (denoted by a_i) by ϕ hence $\widehat{f}(a_i) = \widehat{f}\phi(\alpha_i)$ has to be send on $\sigma_{(\widehat{f}, \mathcal{P})}(\widehat{f}(a_i))$ so $\sigma_{(\widehat{f}\phi, \mathcal{Q})} = \sigma_{(\widehat{f}, \mathcal{P})}$. we deduce that $\varepsilon(\widehat{f}\phi) = \varepsilon(\widehat{f})$.

The other equality has a similar proof. We denote by $\widehat{f}(\mathcal{P})$ the arrival partition of $\widehat{f}$ associated with $\mathcal{P}$. We know that ϕ is continuous thus $\widehat{f}(\mathcal{P})$ is in Π_ϕ and we deduce that $\mathcal{P} \in \Pi_{\phi\widehat{f}}$. Also ϕ is order-preserving then $R(\widehat{f}, \mathcal{P}) = R(\phi\widehat{f}, \mathcal{P})$. We know that $\sigma_{(\phi, \widehat{f}(\mathcal{P}))} = \mathrm{Id}$ then we can notice that $\phi \circ \sigma_{(\widehat{f}, \mathcal{P})} \circ \widehat{f}$ sends the left endpoint of I_i to the left endpoint of $\phi\widehat{f}(I_i^\circ)$. Then $\sigma_{(\phi\widehat{f}, \mathcal{P})} = \phi\sigma_{(\widehat{f}, \mathcal{P})}\phi^{-1}$ and we deduce that $\varepsilon(\sigma_{(\phi\widehat{f}, \mathcal{P})}) = \varepsilon(\sigma_{(\widehat{f}, \mathcal{P})})$. Hence $\varepsilon(\phi\widehat{f}) = \varepsilon(\widehat{f})$. $\square$

Thanks to Proposition 1.2.18 it is enough to prove that $\varepsilon|_{\widehat{\mathrm{IET}^{\bowtie}}}$ is a group homomorphism.

Lemma 4.2.9. *The map $\varepsilon|_{\widehat{\mathrm{IET}^{\bowtie}}}$ is a group homomorphism.*

Proof. Let $\widehat{f}, \widehat{g} \in \widehat{\mathrm{IET}^{\bowtie}}$. Let $\mathcal{P} \in \Pi_{\widehat{f}}$ and $\mathcal{Q} \in \Pi_{\widehat{g}}$. For every $I \in \mathcal{Q}$ (resp. $J \in \mathcal{P}$) we denote by α_I (resp. β_J) the left endpoint of I (resp. J). Up to refine $\mathcal{P}$ and $\mathcal{Q}$ we can assume that $\mathcal{P} = \widehat{g}(\mathcal{Q})$ thus $\widehat{g}(\{\alpha_I\}_{I \in \mathcal{Q}}) = \{\beta_J\}_{J \in \mathcal{P}}$. Then $\mathcal{Q} \in \Pi_{\widehat{f} \circ \widehat{g}}$ and for every $K \in \widehat{f} \circ \widehat{g}(\mathcal{Q})$ we denote by γ_K the left endpoint of K.
In $\mathbb{Z}/2\mathbb{Z}$, we get immediately that $R(\widehat{f} \circ \widehat{g}, \mathcal{Q}) = R(\widehat{g}, \mathcal{Q}) + R(\widehat{f}, \widehat{g}(\mathcal{Q}))$. Now we want to describe the default of pseudo right continuity for $\widehat{f} \circ \widehat{g}$ about $\mathcal{Q}$. We recall that $\sigma_{(\widehat{f} \circ \widehat{g}, \mathcal{Q})}$ is the permutation that sends $\widehat{f} \circ \widehat{g}(\alpha_I)$ on $\gamma_{\widehat{f} \circ \widehat{g}(I)}$ for every $I \in \mathcal{Q}$ while fixing the rest of $[0, 1[$. Furthermore $\sigma_{(\widehat{g}, \mathcal{Q})}(\widehat{g}(\alpha_I)) = \beta_{\widehat{g}(I)}$ and $\sigma_{(\widehat{f}, \widehat{g}(\mathcal{Q}))}(\widehat{f}(\beta_{\widehat{g}(I)})) = \gamma_{\widehat{f} \circ \widehat{g}(I)}$. Then $\sigma_{(\widehat{f}, \widehat{g}(\mathcal{Q}))} \circ \widehat{f} \circ \sigma_{(\widehat{g}, \mathcal{Q})} \circ \widehat{g}(\alpha_I) = \gamma_{\widehat{f} \circ \widehat{g}(I)}$ and we deduce that the permutation $\sigma_{(\widehat{f}, \widehat{g}(\mathcal{Q}))} \circ \widehat{f} \circ \sigma_{(\widehat{g}, \mathcal{Q})} \circ \widehat{f}^{-1}$ sends $\widehat{f} \circ \widehat{g}(\alpha_I)$ on $\gamma_{\widehat{f} \circ \widehat{g}(I)}$ for every $I \in \mathcal{Q}$ while fixing the rest of $[0, 1[$. Thus $\sigma_{(\widehat{f} \circ \widehat{g}, \mathcal{Q})} = \sigma_{\widehat{f}, \widehat{g}(\mathcal{Q})} \circ \widehat{f} \circ \sigma_{(\widehat{g}, \mathcal{Q})} \circ \widehat{f}^{-1}$. Then $\varepsilon(\sigma_{(\widehat{f} \circ \widehat{g}, \mathcal{Q})}) = \varepsilon(\sigma_{\widehat{f}, \widehat{g}(\mathcal{Q})}) + \varepsilon(\sigma_{(\widehat{g}, \mathcal{Q})})$ and we conclude that $\varepsilon(\widehat{f} \circ \widehat{g}) = \varepsilon(\widehat{f}) + \varepsilon(\widehat{g})$. $\qquad \square$

Corollary 4.2.10. *The map ε is a group homomorphism.* $\qquad \square$

4.3 Normal subgroups of $\widehat{\mathrm{PC}^{\bowtie}}$ and some subgroups

Here we present some corollaries of Theorem 4.1.2. For every group G we denote by $D(G)$ its derived subgroup.

Definition 4.3.1. For every group H, we define $J_3(H)$ as the subgroup generated by elements of order 3.

Let $\widehat{G}$ be a subgroup of $\widehat{\mathrm{PC}^{\bowtie}}$ containing $\mathfrak{S}_{\mathrm{fin}}$. We denote by G its projection on $\mathrm{PC}^{\bowtie}$. We recall that $\mathfrak{A}_{\mathrm{fin}}$ is a normal subgroup of $\widehat{G}$, and has a trivial centraliser. We deduce that for every nontrivial normal subgroup H of $\widehat{G}$ contains $\mathfrak{A}_{\mathrm{fin}}$.

From the short exact sequence:

$$0 \longrightarrow \mathfrak{S}_{\mathrm{fin}} \longrightarrow \widehat{G} \longrightarrow G \longrightarrow 1$$

we deduce the next short exact sequence which is a central extension:

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \widehat{G}/\mathfrak{A}_{\mathrm{fin}} \longrightarrow G \longrightarrow 1.$$

This short exact sequence splits because the signature $\varepsilon_{|\widehat{G}} : \widehat{G} \to \mathbb{Z}/2\mathbb{Z}$ constructed in § 4.2 is a retraction. Then we deduce that $\widehat{G}/\mathfrak{A}_{\mathrm{fin}}$ is isomorphic to the direct product $\mathbb{Z}/2\mathbb{Z} \times G$.

Corollary 4.3.2. *The projection $\widehat{G}_{\mathrm{ab}} \to G_{\mathrm{ab}}$ extends in an isomorphism $\widehat{G}_{\mathrm{ab}} \sim G_{\mathrm{ab}} \times \mathbb{Z}/2\mathbb{Z}$. Furthermore $D(\widehat{G}) = \mathrm{Ker}(\varepsilon) \cap \widehat{D(G)}$ is a subgroup of index 2 in $\widehat{D(G)}$. In particular, if G is a perfect group then $\widehat{G}_{\mathrm{ab}} = \mathbb{Z}/2\mathbb{Z}$.*

Corollary 4.3.3. *Let G be a subgroup of $\mathrm{PC}^{\bowtie}$ which is simple nonabelian and let $\widehat{G}$ be its preimage in $\mathfrak{S}$. Then we have the following normal subgroups diagram of $\widehat{G}$:*

$$
\begin{array}{ccc}
 & \widehat{G} & \\
\mathfrak{S}_{\mathrm{fin}} & & \mathrm{Ker}(\varepsilon|_{\widehat{G}}) \\
 & \mathfrak{A}_{\mathrm{fin}} & \\
 & | & \\
 & \{1\} &
\end{array}
$$
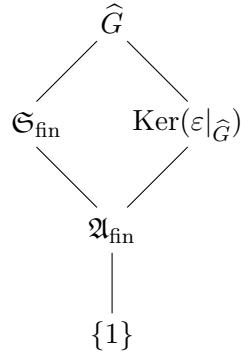
Proof. Let G as in the statement and let $\widehat{G}$ be its preimage in $\mathfrak{S}$. First we immediately check that the subgroups in the diagrams are distinct normal subgroups of $\widehat{G}$. In the case of $\mathrm{Ker}(\varepsilon)$, there exists $g \in \widehat{G} \smallsetminus \mathfrak{S}_{\mathrm{fin}}$ thus either $g \in \mathrm{Ker}(\varepsilon) \smallsetminus \mathfrak{S}_{\mathrm{fin}}$ or $\sigma g \in \mathrm{Ker}(\varepsilon) \smallsetminus \mathfrak{S}_{\mathrm{fin}}$ for any transposition σ.

Second let H be a normal subgroup of $\widehat{G}$ distinct from $\{1\}$. Then it contains $\mathfrak{A}_{\mathrm{fin}}$. Also $H/\mathfrak{A}_{\mathrm{fin}}$ is a normal subgroup of $\widehat{G}/\mathfrak{A}_{\mathrm{fin}} \simeq \mathbb{Z}/2\mathbb{Z} \times G$. Furthermore G is simple then there are only four possibilities for $H/\mathfrak{A}_{\mathrm{fin}}$. As two normal subgroups H, K of $\widehat{G}$ containing $\mathfrak{A}_{\mathrm{fin}}$ such that $H/\mathfrak{A}_{\mathrm{fin}} = K/\mathfrak{A}_{\mathrm{fin}}$ are equal, we deduce that $\widehat{G}$ has at most 5 normal subgroups. $\qquad\square$

Corollary 4.3.4. *Let G be a subgroup of $\mathrm{PC}^{\bowtie}$ which is simple nonabelian and let $\widehat{G}$ be its preimage in $\mathfrak{S}$. If there exists an element of order 3 in $G \smallsetminus \mathfrak{A}_{\mathrm{fin}}$ then $J_3(\widehat{G}) = \mathrm{Ker}(\varepsilon) = D(\widehat{G})$.* $\qquad\square$

Remark 4.3.5. In the context of topological-full groups, the group $J_3(G)$ appears naturally (with some mild assumptions) and is denoted by $\mathsf{A}(G)$ by Nekrashevych in [Nek19]. In some case of topological-full groups of minimal groupoids (see [Mat15]) we have the equality $A(G) = D(G)$ thanks to the simplicity of $D(G)$. In spite of the analogy, it is not clear that the corollary can be obtained as particular case of this result.

Rectangle exchange transformations
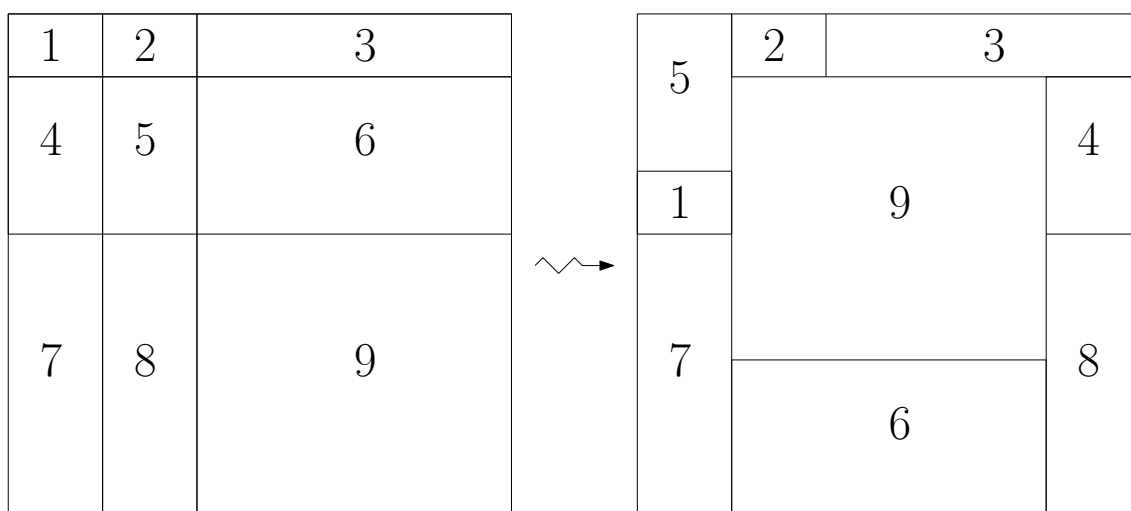
Résumé en français.   On étudie une généralisation du groupe des échanges d'intervalles en dimension $d \geq 1$ quelconque, appelé le groupe des échanges de rectangles (Rec_d). On introduit les battages restreints, qui sont une généralisation des rotations restreintes et on démontre que l'ensemble des battages restreints forme un système générateur de Rec_d. On note $\mathscr{T}_d$ l'ensemble des éléments qui permutent deux rectangles en les translatant, et on prouve qu'il forme un système générateur du sous-groupe dérivé $D(\text{Rec}_d)$. On termine par une généralisation du morphisme SAF qui nous permet d'identifier l'abélianisé de Rec_d.

English abstract.   We study a generalization of the Interval Exchange Transformations group (IET) in every dimension $d \geq 1$ called the Rectangle exchange transformations group (Rec_d). The subset of restricted rotations in IET is a generating subset and we prove that a natural generalization of these elements, called restricted shuffle, form a generating subset of Rec_d. We denote by $\mathscr{T}_d$ the subset of Rec_d made up of those transformations that permute two rectangles by translation. We prove that the derived subgroup is generated

by $\mathscr{T}_d$. We also identify the abelianization of Rec_d.

Contents

5.1 Introduction

We recall that the group IET is the group consisting of all permutations of $[0, 1[$ continuous outside a finite set, right-continuous and piecewise a translation. We study a generalization of IET in higher dimension. Let $d \geq 1$ be an integer. We denote by $X = [0, 1[^d$ the left half-open square of dimension d. Let $\mathcal{B} = \{e_1, e_2, \ldots, e_d\}$ be the canonical basis of $\mathbb{R}^d$ and we denote by λ the Lebesgue measure on $\mathbb{R}$. For $1 \leq i \leq d$ let pr_i be the orthogonal projection on $\mathrm{Vect}(e_i)$ and $\mathrm{pr}_i^\perp$ be the orthogonal projection on the hyperplane $e_i^\perp$. For an element $x \in \mathbb{R}^d$ we use the notation $x_i = \mathrm{pr}_i(x)$. A natural way to generalize left half-open intervals is to consider elements of the form $I_1 \times \ldots \times I_d$ where I_i is a left half-open subinterval of $[0, 1[$. They are called left half-open d-rectangles. In the following, every d-rectangle is supposed to be left half-open.

We define the rectangle exchange transformations group of dimension d, denoted by Rec_d, as the set of all permutations f of $[0, 1[^d$ such that there exists a finite partition of $[0, 1[^d$ into d-rectangles such that f is a translation on each of these d-rectangles. Elements of Rec_d are called *d-rectangle exchange transformations*.

Historically, H. Haller [Hal81] introduced 2-rectangle exchange transformations in 1981 and it is mainly ergodic properties of a single 2-rectangle exchange transformation which are studied. More generally, dynamics of piecewise isometries on polytopes are studied, in particular by A. Goetz [Goe00], however the group itself is rarely considered. In contrast, the group of piecewise affine self-homeomorphisms of some manifolds are recently considered in particular by D. Calegari and D. Rolfsen [CR15].

Definition 5.1.1. Let $f \in \mathrm{Rec}_d$ and $\mathcal{P}$ be a partition of X into rectangles (resp. a grid-pattern). We said that $\mathcal{P}$ *is a partition associated with* f (resp. *a grid-pattern associated with* f) if for every $K \in \mathcal{P}$ the restriction of f to K is a translation. Then the set $f(\mathcal{P}) := \{f(K) \mid K \in \mathcal{P}\}$ is a new partition of X into rectangles called the *arrival partition of* f *with* $\mathcal{P}$. We denote by Π_f the set of all partitions associated with f. From now on every partition is assumed to be finite.

Remark 5.1.2. The fact that Rec_d is a group under composition is immediate. One can see that if $f, g \in \mathrm{Rec}_d$ and $\mathcal{P} \in \Pi_f, \mathcal{Q} \in \Pi_g$, then there exists a partition $\mathcal{R}$ into d-rectangles that refines both $f(\mathcal{P})$ and $\mathcal{Q}$. Thus $f^{-1}(\mathcal{R})$ is a partition into d-rectangles such that $g \circ f$ acts on every d-rectangle of $f^{-1}(\mathcal{R})$ by translation.

In the following, the "d" of d-rectangle may be omitted whenever there is no possible confusion.

We introduce two kinds of special elements in Rec_d.

Definition 5.1.3. A *restricted shuffle in direction* i is an element $\sigma_{R,s,i}$ of Rec_d where R is a $(d-1)$-subrectangle of $e_i^\perp$ and s is a restricted rotation, defined by:

1. if $\mathrm{pr}_i^\perp(x) \notin R$, $\sigma_{R,s,i}(x) = x$;

2. if $\mathrm{pr}_i^\perp(x) \in R$:

   (a) for $j \neq i$, $\sigma_{R,s,i}(x)_j = x_j$;
   (b) $\sigma_{R,s,i}(x)_i = s(x_i)$.

For disjoint translation-isometric rectangles $P, Q \subset [0, 1[^d$, define the Rec_d-*transposition* $\tau_{P,Q}$ as the element of Rec_d defined as the identity outside $P \cup Q$, and as a translation on each of P, Q, exchanging them. The set of all Rec_d-transpositions is denoted by $\mathscr{T}_d$.

Notation 5.1.4. If I and J are the two intervals associated with s then the d-rectangles P_1 and P_2, defined by $\mathrm{pr}_i(P_1) = I$, $\mathrm{pr}_i(P_2) = J$ and $\mathrm{pr}_i^\perp(P_1) = \mathrm{pr}_i^\perp(P_2) = R$, are two rectangles which partitioned the support of f and where f is continuous on both of them. We say that f shuffles this two rectangles.
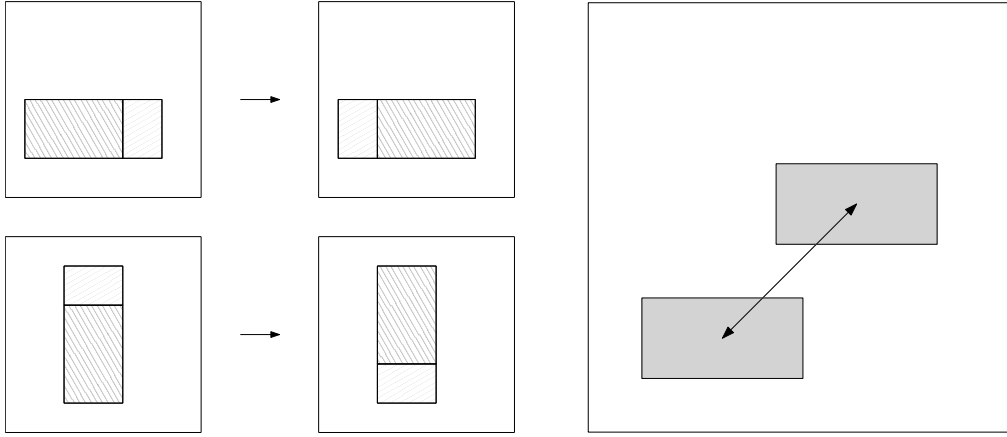
Figure 5.1.1 – **Left**: Examples of restricted shuffles in dimension 2 with different directions. **Right**: Example of a Rec_2-transposition.

We know that the set of all restricted rotation is a generating subset of IET, see Theorem 1.2.8. Restricted shuffles are a natural generalization of these elements in Rec_d. It is natural to ask if they define a generating subset for Rec_d and the next theorem answers it positively. For $d = 1$, it essentially amounts to seeing an IET as a permutation of n intervals, and arguing that the symmetric group S_n is generated by transpositions $(i, i+1)$. For $d \geq 2$ there is no such simple combinatorial argument and much more work is required; it is done in Section 5.4.

Theorem 5.1.5. *The set of all restricted shuffles is a generating subset of Rec_d.*

For $d = 2$ a variant of the proof, providing a combinatorial refinement of Theorem 5.1.5, is performed in Section 5.5

The group IET^d acting coordinate-wise is a subgroup of Rec_d which for $d \geq 2$ is a proper subgroup.

One main difficulty we have to dealt with for $d \geq 2$ is that there does not always exist a grid that is mapped to another grid by translation of all components.

In the case where $d \geq 2$, if we take two disjoint d-rectangles which are translation-isometric such that their orthogonal projections on $\mathrm{Vect}(e_i)$ are not $[0, 1[$ for every $1 \leq i \leq d$, then the Rec_d-transposition which permutes these two rectangles is not in IET^d.

Thanks to Theorem 5.1.5 we obtain that $D(\mathrm{Rec}_d)$ is generated by conjugates of commutators of two restricted shuffles. With this result we prove the following theorem:

Theorem 5.1.6. *The derived subgroup $D(\mathrm{Rec}_d)$ is simple and generated by its subset $\mathscr{T}_d$ of Rec_d-transpositions. It is contained in every in every nontrivial normal subgroup of Rec_d.*

In Section 5.7 we extend Theorem 1.3.6 which identifies the abelianization of IET thanks to the SAF-homomorphism. We denote by $\mathbb{R}^{\otimes k}$ the $k - th$ tensor power of $\mathbb{R}$ over $\mathbb{Q}$. In the case of Rec_d we prove that it is enough to consider an analogue of this group homomorphism in every direction to obtain the following result:

Theorem 5.1.7. *There exists a natural surjective group homomorphism from Rec_d onto $(\mathbb{R}^{\otimes(d-1)} \otimes (\bigwedge_{\mathbb{Q}}^2 \mathbb{R}))^d$, called the generalized SAF-homomorphism, whose kernel is the derived subgroup $D(\mathrm{Rec}_d)$.*

This theorem is done by Arnoux-Fathi-Sah in dimension 1 and thanks to Theorem 5.1.5, the proof is similar in dimension d.

Finally in Section 5.7.4 we consider the subgroup GtG_d of Rec_d generated by the subset $\mathrm{IET}^d \cup \mathscr{T}_d$. While for $d = 1$ it is obviously equal to the whole group $\mathrm{IET} = \mathrm{Rec}_1$, a consequence of Theorem 5.1.7 for $d \geq 2$ is:

Corollary 5.1.8. *The group GtG_d is a proper normal subgroup of Rec_d, which strictly contains $D(\mathrm{Rec}_d)$.*

5.2 Grid-patterns, setwise $\mathbb{Q}$-freeness

We fix $d \geq 1$; in a first reading, one can assume $d = 2$. We introduce the notion of partition associated with an element of Rec_d and also the one of grid-pattern.

Definition 5.2.1. A partition $\mathcal{P}$ of X into rectangles is called a grid-pattern if for every $1 \leq i \leq d$, there exists a partition $\mathcal{Q}_i$ of $[0, 1[$ into half-open intervals such that $\mathcal{P} = \mathcal{Q}_1 \times \mathcal{Q}_2 \times \ldots \times \mathcal{Q}_d$.

Definition 5.2.2. Let $n \in \mathbb{N}$ and $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ be a partition of X into rectangles. For every $1 \leq i \leq d$, let $\mathcal{H}_i = \{\mathrm{pr}_i(P_j)\}_{1 \leq j \leq n}$ and let $E_i := \{a \in [0, 1[\: : \: \exists I \in \mathcal{H}_i$ such that a is an endpoint of $I\}$. Let k_i be the cardinal of E_i and let $a_1 < a_2 < \ldots < a_{k_i}$ be its elements. We define $\mathcal{Q}_i := \{[a_j, a_{j+1}[\}_{1 \leq j \leq k_i - 1}$, it is a finite partition of $[0, 1[$ into intervals. Then $\mathcal{Q} = \mathcal{Q}_1 \times \mathcal{Q}_2 \times \ldots \times \mathcal{Q}_d$ is a grid-pattern called *the grid-pattern refinement of $\mathcal{P}$.*

In fact we will need some rigidity on partitions associated with an element of Rec_d. For this we want to have some objects to be $\mathbb{Q}$-free.

Definition 5.2.3. Let $\mathcal{P}$ be a partition into rectangles of $[0, 1[^d$. For every $1 \leq i \leq d$ we denote by $\mathcal{F}_i$ the set $\{\lambda(\mathrm{pr}_i(K)) \mid K \in \mathcal{P}\}$. If for every $1 \leq i \leq d$ the set $\mathcal{F}_i$ is $\mathbb{Q}$-linearly independent then we say that $\mathcal{P}$ *is a setwise $\mathbb{Q}$-free partition.*

Warning. *The $\mathbb{Q}$-independence required is that of the* **set** $\{\lambda(\mathrm{pr}_i(K)) \mid K \in \mathcal{P}\}$, *and not the family* $(\lambda(\mathrm{pr}_i(K)))_{K \in \mathcal{P}}\}$. *So the set-wise freeness condition says, roughly speaking, that the only $\mathbb{Q}$-linear dependence relations among the $\lambda(\mathrm{pr}_i(K))$, for $K \in \mathcal{P}$ (for each fixed K) are equalities.*

Proposition 5.2.4. *Let $\mathcal{Q}$ be a grid-pattern. There exists a setwise $\mathbb{Q}$-free grid-pattern $\mathcal{Q}'$ that refines $\mathcal{Q}$.*

Proof. Thanks to Corollary 2.1.3 we know that for every finite subset S of positive real numbers there exists a finite $\mathbb{Q}$-free subset of positive real numbers S' such that every element of S is a linear combination of elements in S' with nonnegative integer coefficient. This statement is Lemma 4.1 of Vorobets in [Vor17]. Here let $\mathcal{Q} = \mathcal{Q}_1 \times \ldots \times \mathcal{Q}_d$ where $\mathcal{Q}_i$ is a partition into intervals of $[0, 1[$ and let $\mathcal{F}_i := \{\lambda(I) \mid I \in \mathcal{Q}_i\}$. By applying the previous fact for every $\mathcal{F}_i$ we obtain a new finite set of length $\mathcal{F}_i'$ and we know we can refines every $\mathcal{Q}_i$ into a finite partition $\mathcal{Q}_i'$ into intervals of $[0, 1[$ such that for every $I \in \mathcal{Q}_i'$ we have $\lambda(I) \in \mathcal{F}_i'$. Then $\mathcal{Q}' := \mathcal{Q}_1' \times \ldots \times \mathcal{Q}_d'$ is a setwise $\mathbb{Q}$-free grid-pattern which refines $\mathcal{Q}$. $\qquad\square$
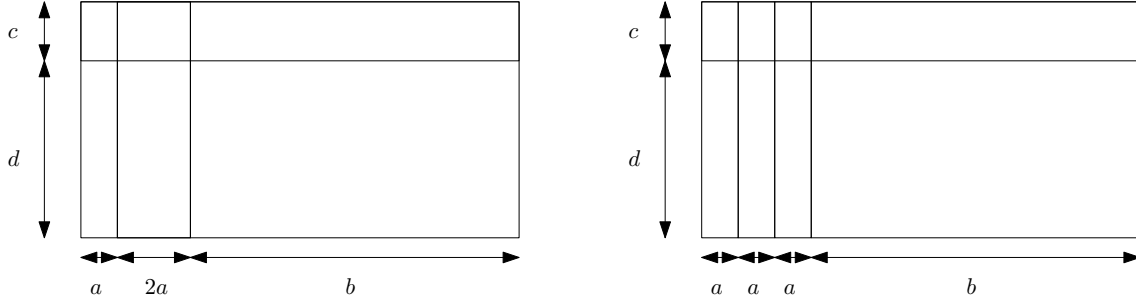
Figure 5.2.1 – We assume that $\{a, b\}$ is a setwise $\mathbb{Q}$-free subset of $\mathbb{R}$ as well as $\{c, d\}$. **Left:** A grid-pattern that is not setwise $\mathbb{Q}$-free. **Right:** A setwise $\mathbb{Q}$-free grid-pattern which refines the left grid-pattern.

It will be important to have some rigidity on the number of pieces which are translation-isometric inside a partition. The next lemma explains how there exists a bijection between two partitions which share the same combinatorial conditions such that every piece and its image by this bijection are translation-isomorphic. The following corollary is the main result to use the induction property.

Lemma 5.2.5. *For every $1 \leq i \leq d$, let F_i be a setwise $\mathbb{Q}$-free subset of $\mathbb{R}^+$. Let $\mathcal{P}$ and $\mathcal{P}'$ be two partitions into d-rectangles of $[0, 1[^d$ such that for every $K \in \mathcal{P} \cup \mathcal{P}'$ we have $\lambda(\mathrm{pr}_i(P)) \in F_i$. Then, there exists a bijection δ between $\mathcal{P}$ and $\mathcal{P}'$ such that for every $K \in \mathcal{P}$ we have K and $\delta(K)$ which are translation-isometric. If $K \in \mathcal{P} \cap \mathcal{P}'$ we can also ask $\delta(K) = K$.*

Proof. Let $\mathcal{A}$ be the Boolean algebra of subsets of $[0, 1[$ generated by the set of all left half-open subintervals of $[0, 1[$. We denote by $\mathcal{A}^{\otimes d}$ the Boolean algebra tensor product. It is generated by subsets of the form $I_1 \times \ldots \times I_d$ with I_i in $\mathcal{A}$ for every $1 \leq i \leq d$. We recall that λ is the Lebesgue measure on $\mathbb{R}$ and we denote by $\mathbb{R}^{\otimes d}$ the $d - th$ tensor power of $\mathbb{R}$ over $\mathbb{Q}$. Thanks to Proposition 2.3.3 we know that there exists a Boolean algebra measure μ_d for $\mathcal{A}^{\otimes d}$ in $\mathbb{R}^{\otimes d}$ such that $\mu_d(I_1 \times \ldots \times I_d) = \lambda(I_1) \otimes \ldots \otimes \lambda(I_d)$.

By setwise $\mathbb{Q}$-freeness of F_i we deduce that $\mathcal{F} = \{a_1 \otimes \ldots \otimes a_d \mid (a_1, \ldots, a_j) \in F_1 \times \ldots \times F_j\}$ is a setwise $\mathbb{Q}$-free subset of $\mathcal{A}^{\otimes d}$. By definition of $\mathcal{P}$ and $\mathcal{P}'$ we have:

$$
\begin{aligned}
\mu_d([0, 1[^d) &= \mu_d(\bigsqcup_{K \in \mathcal{P}} K) = \sum_{\overline{a} \in \mathcal{F}} \mathrm{Card}(\{K \in \mathcal{P} \mid \mu_d(K) = \overline{a}\})\overline{a} \\
&= \mu_d(\bigsqcup_{K \in \mathcal{P}'} K) = \sum_{\overline{a} \in \mathcal{F}} \mathrm{Card}(\{K \in \mathcal{P}' \mid \mu_d(K) = \overline{a}\})\overline{a}
\end{aligned}
$$

Then by setwise $\mathbb{Q}$-freeness of $\mathcal{F}$ we deduce that for every $\overline{a} \in \mathcal{F}$ we have $\mathrm{Card}(\{K \in \mathcal{P} \mid \mu_d(K) = \overline{a}\}) = \mathrm{Card}(\{K \in \mathcal{P}' \mid \mu_d(K) = \overline{a}\})$. From this we deduce that there exists a bijection δ between $\mathcal{P}$ and $\mathcal{P}'$ such that for every $K \in \mathcal{P}$ we have K and $\delta(K)$ which are translation-isometric and for every $K \in \mathcal{P} \cap \mathcal{P}'$ we can have $\delta(K) = K$. $\qquad\square$

Corollary 5.2.6. *In particular, there exists $f \in \mathrm{Rec}_d$ such that $\mathcal{P} \in \Pi_f$ and $f(\mathcal{P}) = \mathcal{P}'$ and for every $K \in \mathcal{P} \cap \mathcal{P}'$ we have $f(K) = K$. More precisely the support of f is partitioned by $\mathcal{P} \smallsetminus (\mathcal{P} \cap \mathcal{P}')$ and for every $K \in \mathcal{P}$, the restriction of f to K is a translation.*

5.3 First observations

We establish some easy particular cases of Theorem 5.1.5, which asserts that Rec_d is generated by restricted shuffles.

A direct consequence of the definition of a restricted shuffle, Definition 5.1.3, and Theorem 1.2.8 is the following proposition, which is a first easy particular case of Theorem 5.1.5, and a step in its proof.

Proposition 5.3.1. *Every element of* IET^d *is a finite product of restricted shuffles.* $\quad\square$

Here is a second elementary particular case of Theorem 5.1.5, which will also be needed.

Proposition 5.3.2. *For all disjoint translation-isometric* P, Q *rectangles, the* Rec_d-*transposition* $\tau_{P,Q}$ *is a product of restricted shuffles.*

Proof. We first prove this in the special case when there exists $1 \leq i \leq d$ such that $\mathrm{pr}_i(P) \cap \mathrm{pr}_i(Q) = \emptyset$ and $\mathrm{pr}_i^\perp(P) = \mathrm{pr}_i^\perp(Q)$. In this case we obtain it is a product of two restricted shuffles. Indeed, this is a consequence of the fact that this lemma is true when $d = 1$. Let $a, b, a', b' \in [0, 1[$ such that $\mathrm{pr}_i(P) = [a, b[$ and $\mathrm{pr}_i(Q) = [a', b'[$. Up to change the role of P and Q we can assume that $b < a'$. Let R and S be the two rectangles such that $\mathrm{pr}_i^\perp(R) = \mathrm{pr}_i^\perp(S) = \mathrm{pr}_i^\perp(P)$ and $\mathrm{pr}_i(R) = [b, b'[$ and $\mathrm{pr}_i(S) = [b, a'[$. Let r_1 be the restricted shuffle in direction i that shuffles P with R (this one send P on Q) and r_2 be the restricted shuffle in direction i that permutes P with S. Then the composition $r_2^{-1} r_1$ is equals to the rectangle transposition that permutes P with Q.

Now let us prove the general case. Let P and Q be two rectangles which are translation-isometric such that $P \cap Q = \emptyset$. Let $P_i := \mathrm{pr}_i(P)$ and $Q_i := \mathrm{pr}_i(Q)$ for every $1 \leq i \leq d$. Thus $P = P_1 \times P_2 \times \ldots \times P_d$ and $Q = Q_1 \times Q_2 \times \ldots \times Q_d$. For every $1 \leq i \leq d - 1$ let R_i be the rectangle $Q_1 \times \ldots \times Q_i \times P_{i+1} \times \ldots \times P_d$. We put $R_0 = P$ and $R_d = Q$. Let t_i be the rectangle transposition that permutes R_{i-1} with R_i for every $1 \leq i \leq d$. Then $\tau_{P,Q} = t_1 \ldots t_{d-1} t_d t_{d-1} \ldots t_1$ and by the special case above, we know that t_i is a product of two restricted shuffles in direction i. Then s is a finite product of restricted shuffles. $\quad\square$

We now consider another special case, of an element of Rec_d mapping grid to grid by translating pieces. Recall from Remark 5.3.4 that not every element of Rec_d has this form.

Proposition 5.3.3. *Every element* $f \in \mathrm{Rec}_d$ *such that there exists a setwise* $\mathbb{Q}$-*free grid-pattern* $\mathcal{Q}$ *such that* $f(\mathcal{Q})$ *is a grid-pattern can be written as a finite product of restricted shuffles.*

Proof. Let $\mathcal{Q} = \mathcal{Q}_1 \times \ldots \times \mathcal{Q}_d$ and $f(\mathcal{Q}) = \mathcal{Q}'_1 \times \ldots \times \mathcal{Q}'_d$, where $\mathcal{Q}_i$ and $\mathcal{Q}'_i$ is a partition into intervals of $[0, 1[$. Thanks to the setwise $\mathbb{Q}$-freeness of $\mathcal{Q}$ we know that $f(\mathcal{Q})$ is setwise $\mathbb{Q}$-free, also for every $1 \leq i \leq d$ and every $a \in [0, 1[$ we have:

$$\mathrm{Card}(\{I \in \mathcal{Q}_i \mid \lambda(I) = a\}) = \mathrm{Card}(\{I \in \mathcal{Q}'_i \mid \lambda(I) = a\})$$

Hence there exists an element g of IET^d such that $g(f(\mathcal{Q})) = \mathcal{Q}$. By Proposition 5.3.1 we know that g is a finite product of restricted shuffles. Also as $g \circ f$ send $\mathcal{Q}$ on itself we deduce that $g \circ f$ is a permutation on every maximal subset of translation-isometric rectangles of $\mathcal{Q}$. Hence it is a product of Rec_d-transpositions and by Proposition 5.3.2 we deduce that f is a finite product of restricted shuffles. $\quad\square$

Remark 5.3.4. For an element of Rec_d there does not always exist a grid-pattern associated which is sent to another grid-pattern. For example this is the case for every restricted shuffle $\sigma_{R,s,i}$ of infinite order such that $R \neq [0, 1[^{d-1}$.

5.4 Generation by restricted shuffles

The proof of Theorem 5.1.5 is by induction on the dimension d and the case of the dimension 1 is already known to be true.

Let $f \in \mathrm{Rec}_d$ and $\mathcal{Q}$ be a grid-pattern associated with f. Thanks to Proposition 5.2.4 we can assume that $\mathcal{Q}$ is a setwise $\mathbb{Q}$-free grid-pattern.

Let $d \geq 2$ be the ambient dimension and assume Theorem 5.1.5 true for Rec_{d-1}. Here the height is represented by the d-th coordinate. For every illustration in dimension 2 we use the following element f_{test} of Rec_2. The following partition $\mathcal{P}_{\mathrm{test}}$ (on the left of the following picture) is associated with f_{test}, and is understood to be setwise $\mathbb{Q}$-free. We denote by $\mathcal{P}'_{\mathrm{test}} = f_{\mathrm{test}}(\mathcal{P}_{\mathrm{test}})$ (on the right of the following picture).
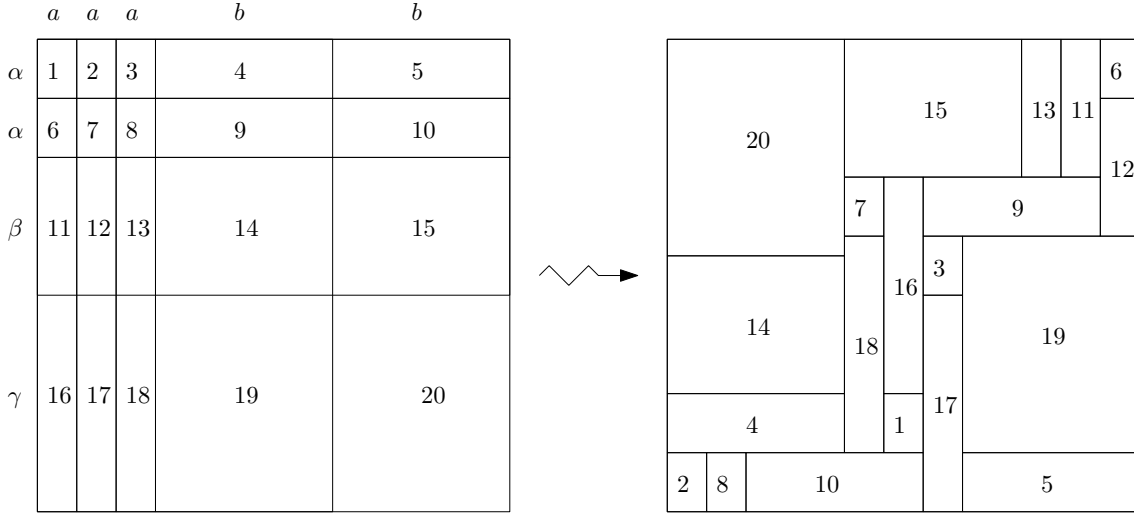
Figure 5.4.1 – Definition of $f_{\mathrm{test}}, \mathcal{P}_{\mathrm{test}}$ and $\mathcal{P}'_{\mathrm{test}}$.

We now introduce a number of simple definitions in this setting, which for the above test example $\mathcal{P}'_{\mathrm{test}}$ are illustrated in Figures 5.4.2, 5.4.5, 5.4.6

Definition 5.4.1. Let $\mathcal{P}$ be a setwise $\mathbb{Q}$-free rectangle partition of $[0, 1[^d$. The *ground of* $\mathcal{P}$ is the following subset of $\mathcal{P}$:

$$\mathrm{Grd}(\mathcal{P}) = \{K \in \mathcal{P} \mid 0 \in \mathrm{pr}_d(K)\}.$$

Let K_0 be an element of $\mathrm{Grd}(\mathcal{P})$. A *tower above* K_0 is a subset T of $\mathcal{P}$ such that :

1. $K_0 \in T$;

2. $\forall K \in T, \ \mathrm{pr}_d^{\perp}(K) = \mathrm{pr}_d^{\perp}(K_0)$;

3. The set $\bigcup_{K \in T} \mathrm{pr}_d(K)$ is a subinterval of $[0, 1[$.

The element K of T which satisfies $\sup(\mathrm{pr}_d(K)) = \sup \left(\bigcup_{K \in T} \mathrm{pr}_d(K) \right)$ is called the *top of the tower* T, denoted by $\mathrm{Top}(T)$. The *highest tower above* K_0, denoted by $T(K_0)$, is the maximal tower above K_0 according to the inclusion order.

Definition 5.4.2. A *city of* $\mathcal{P}$ is a subset of $\mathcal{P}$ containing $\mathrm{Grd}(\mathcal{P})$, and which is a union of towers. The *highest city of* $\mathcal{P}$, denoted by $\mathrm{City}(\mathcal{P})$, is the union of all highest towers above an element of the ground $\mathrm{Grd}(\mathcal{P})$. The top of a city $\mathcal{V} \subset \mathcal{P}$ is the set of $\mathrm{Top}(T)$ when T ranges over maximal towers in $\mathcal{V}$. The *sky of* $\mathcal{P}$, denoted by $\mathrm{Sky}(\mathcal{P})$, is the complement of $\mathrm{City}(\mathcal{P})$ in $\mathcal{P}$.
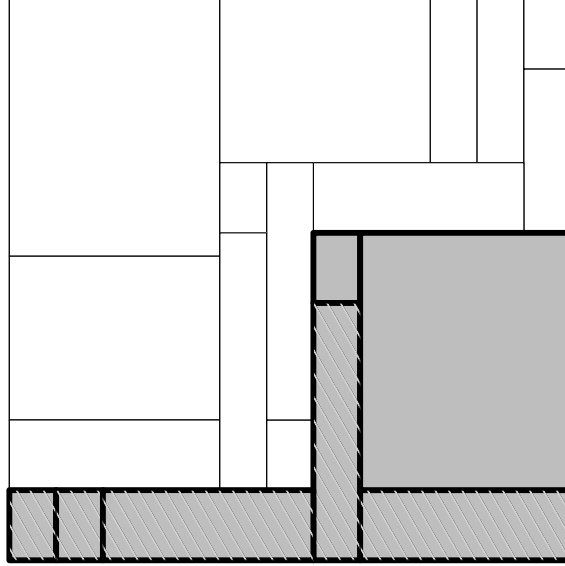
Figure 5.4.2 – Hatched pieces compose the Ground of $\mathcal{P}'_{\text{test}}$, it is also a city of $\mathcal{P}'_{\text{test}}$. All grey pieces (hatched or not) compose City($\mathcal{P}'_{\text{test}}$). Full white pieces represent the sky of $\mathcal{P}'_{\text{test}}$.

Definition 5.4.3. The complexity of $\mathcal{P}$ is the following subset of $]0, 1[$:

$$\mathscr{C}(\mathcal{P}) = \{\min(\mathrm{pr}_d(K)) \mid K \in \mathrm{Sky}(\mathcal{P})\}.$$

The set $\mathscr{C}(\mathcal{P})$ is empty if and only if $\mathcal{P} = \mathrm{City}(\mathcal{P})$. Otherwise, the minimum of the set $\mathscr{C}(\mathcal{P})$ is called the *working height of* $\mathcal{P}$ denoted by $\mathscr{C}_{\min}(\mathcal{P})$.

The idea is to move pieces of City($\mathcal{P}$) with restricted shuffles in direction in $\{1, 2, \ldots, d-1\}$ such that the new partition $\mathcal{P}'$ obtained satisfies $\mathscr{C}(\mathcal{P}') \subset \mathscr{C}(\mathcal{P}) \smallsetminus \{\mathscr{C}_{\min}(\mathcal{P})\}$. For this we describe more precisely how and where we move pieces.

Definition 5.4.4. We define the *building worksite of* $\mathcal{P}$, denoted by $\mathrm{Work}^-(\mathcal{P})$, as the following subset of Top(City($\mathcal{P}$)):

$$\mathrm{Work}^-(\mathcal{P}) = \{K \in \mathrm{Top}(\mathrm{City}(\mathcal{P})) \mid \sup(\mathrm{pr}_d(K)) = \mathscr{C}_{\min}(\mathcal{P})\}.$$

Similarly we define the *upper building worksite of* $\mathcal{P}$, denoted by $\mathrm{Work}^+(\mathcal{P})$, as the following subset of Sky($\mathcal{P}$):

$$\mathrm{Work}^+(\mathcal{P}) = \{P \in \mathrm{Sky}(\mathcal{P}) \mid \min(\mathrm{pr}_d(P)) = \mathscr{C}_{\min}(\mathcal{P})\}.$$

We define the *site of* $\mathcal{P}$ as the subset of $e_d^\perp$ define as the following:

$$\mathrm{Site}(\mathcal{P}) = \bigcup_{K \in \mathrm{Work}^-(\mathcal{P})} \mathrm{pr}_d^\perp(K).$$
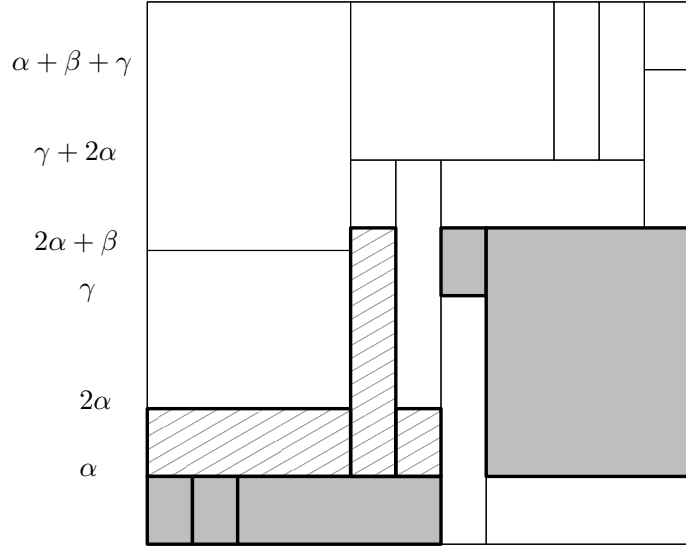
Figure 5.4.3 – The set of all grey pieces represents $\mathrm{Top}(\mathrm{City}(\mathcal{P}'_{\text{test}}))$ and the set of all hatched pieces represents $\mathrm{Work}^+(\mathcal{P})$.
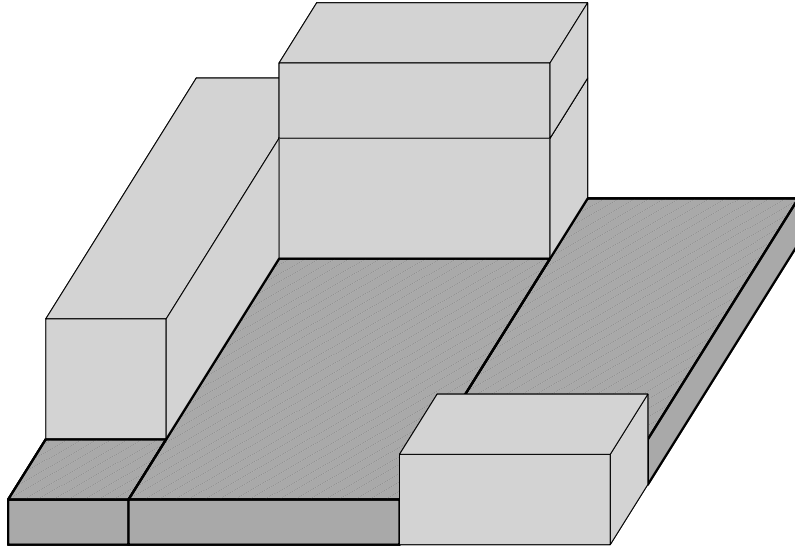
Figure 5.4.4 – In dimension 3, illustration of a city of a partition where the hatched space represents the site of the partition.

The proof of Theorem 5.1.5 is done by induction on the cardinal c of $\mathscr{C}_{\min}(\mathcal{P})$; the core of the arguments being the following two lemmas, which treat $c = 0$ and the induction step for $c > 0$.

Lemma 5.4.5. *Let $\mathcal{P}$ be a setwise $\mathbb{Q}$-free partition such that $\mathscr{C}(\mathcal{P}) = \emptyset$. Then there exists a product r of restricted shuffles in direction d such that $\mathcal{P}$ is associated with r and $r(\mathcal{P})$ is a grid-pattern. (See illustration in Figure 5.4.6.)*

Proof. A consequence of $\mathscr{C}_{\min}(\mathcal{P}) = \emptyset$ is that $\mathrm{City}(\mathcal{P}) = \mathcal{P}$, that is, highest towers partition $[0, 1[^d$. In particular, we have a partition D of $e_d^\perp$ such that for every $x \in [0, 1[$ we have $\{\mathrm{pr}_d^\perp(K) \mid K \in \mathcal{P}, \text{ and } x \in \mathrm{pr}_d(P)\} = D$. Also the set $\{\mathrm{pr}_d(K) \mid K \in \mathcal{P}\}$ is setwise $\mathbb{Q}$-free, thus for every $a \in [0, 1[$, the number of rectangles K such that $\lambda(\mathrm{pr}_d(K)) = a$ is the same in every tower $T \subset \mathrm{City}(\mathcal{P})$. Then by using only restricted shuffles in direction d, we can

move pieces inside every tower $T \subset \mathrm{City}(\mathcal{P})$ to order them according to the length of their projection on $\mathrm{Vect}(e_d)$. The image of $\mathcal{P}$ by the product of these restricted shuffles is a grid-pattern. $\qquad \square$

Lemma 5.4.6. *Let $\mathcal{P}$ be a setwise $\mathbb{Q}$-free partition such that $\mathscr{C}(\mathcal{P}) \neq \emptyset$. There exists a product g of restricted shuffles in direction inside $\{1, 2, \ldots, d-1\}$ such that $\mathcal{P} \in \pi_g$ and:*

$$\mathscr{C}(g(\mathcal{P})) \subset \mathscr{C}(\mathcal{P}) \smallsetminus \{\mathscr{C}_{\min}(\mathcal{P})\}.$$

Proof. For every $1 \leq i \leq d$ let $F_i = \{\lambda(\mathrm{pr}_i(K)) \mid K \in \mathcal{P}\}$; it is a setwise $\mathbb{Q}$-free subset of $\mathbb{R}^+$. By definition we know that $\Omega^- = \{\mathrm{pr}_d^\perp(K) \mid K \in \mathrm{Top}(\mathrm{City})(\mathcal{P})\}$ and $\Omega^+ = \{\mathrm{pr}_2(K) \mid K \in \mathrm{Work}^+ \cup \mathrm{Top}(\mathrm{City}(\mathcal{P})) \smallsetminus \mathrm{Work}^-\}$ are two partitions of $[0,1[^{d-1}$ such that for every $K \in \Omega^- \cup \Omega^+$ and every $1 \leq i \leq d$ we have $\lambda(\mathrm{pr}_i(K)) \in F_i$. Then, by Corollary 5.2.6 we deduce that there exists $\delta \in \mathrm{Rec}_{d-1}$ such that $\Omega^- \in \Pi_\delta$ (for every element K of Ω^-, the restriction of δ to K is a translation) and $\delta(\Omega^-) = \Omega^+$ and for every $K \in \Omega^- \cap \Omega^+$ we have $\delta(K) = K$. As we assumed Theorem 5.1.5 in dimension $d-1$, we know that δ can be written as the product of restricted shuffles of Rec_{d-1}. Then we define $g \in \mathrm{Rec}_d$ such that:

$$g(x) = \left\{ \begin{array}{cl} (\delta \times \mathrm{Id})(x) & \text{if } \mathrm{pr}_2(x) < \mathscr{C}_{\min}(\mathcal{P}) \\ x & \text{else.} \end{array} \right.$$

From this definition we obtain that g is the product of restricted shuffles in Rec_d with direction in $\{1, 2 \ldots, d-1\}$. Also by definition of δ we obtain that for every $K \in \mathrm{Grd}(\mathcal{P})$ we have $g(T(K)) \subset T(g(K))$ and $g(\mathrm{Sky}(\mathcal{P})) = \mathrm{Sky}(\mathcal{P})$. This implies $\mathscr{C}(g(\mathcal{P})) \subset \mathscr{C}(\mathcal{P})$. Also as $\delta(\Omega^-) = \Omega^+$ we deduce that for every $K \in \mathrm{Sky}(\mathcal{P})$ such that $\min(\mathrm{pr}_d(K)) = \mathscr{C}_{\min}(\mathcal{P})$ there exists $Q_K \in \mathrm{Grd}(\mathcal{P})$ such that $\delta(\mathrm{pr}_d^\perp(Q_K) = \mathrm{pr}_d^\perp(K)$. Hence we have $K \in T(g(Q_K))$ and this implies that $\mathscr{C}_{\min}(\mathcal{P}) \notin \mathscr{C}(g(\mathcal{P}))$. $\qquad \square$

Then by induction on the cardinal of the complexity we deduce the following proposition:

Proposition 5.4.7. *Let $\mathcal{Q}$ be a setwise $\mathbb{Q}$-free grid-pattern of $[0,1[^d$. For every $f \in \mathrm{Rec}_d$ such that $\mathcal{Q} \in \Pi_f$, there exists a finite product r_f of restricted shuffles such that $f(\mathcal{Q}) \in \Pi_{r_f}$ and $\mathscr{C}(r_f(f(\mathcal{Q}))) = \emptyset$.*

Thanks to Proposition 5.4.7 and Proposition 5.4.5 we deduce Theorem 5.1.5.
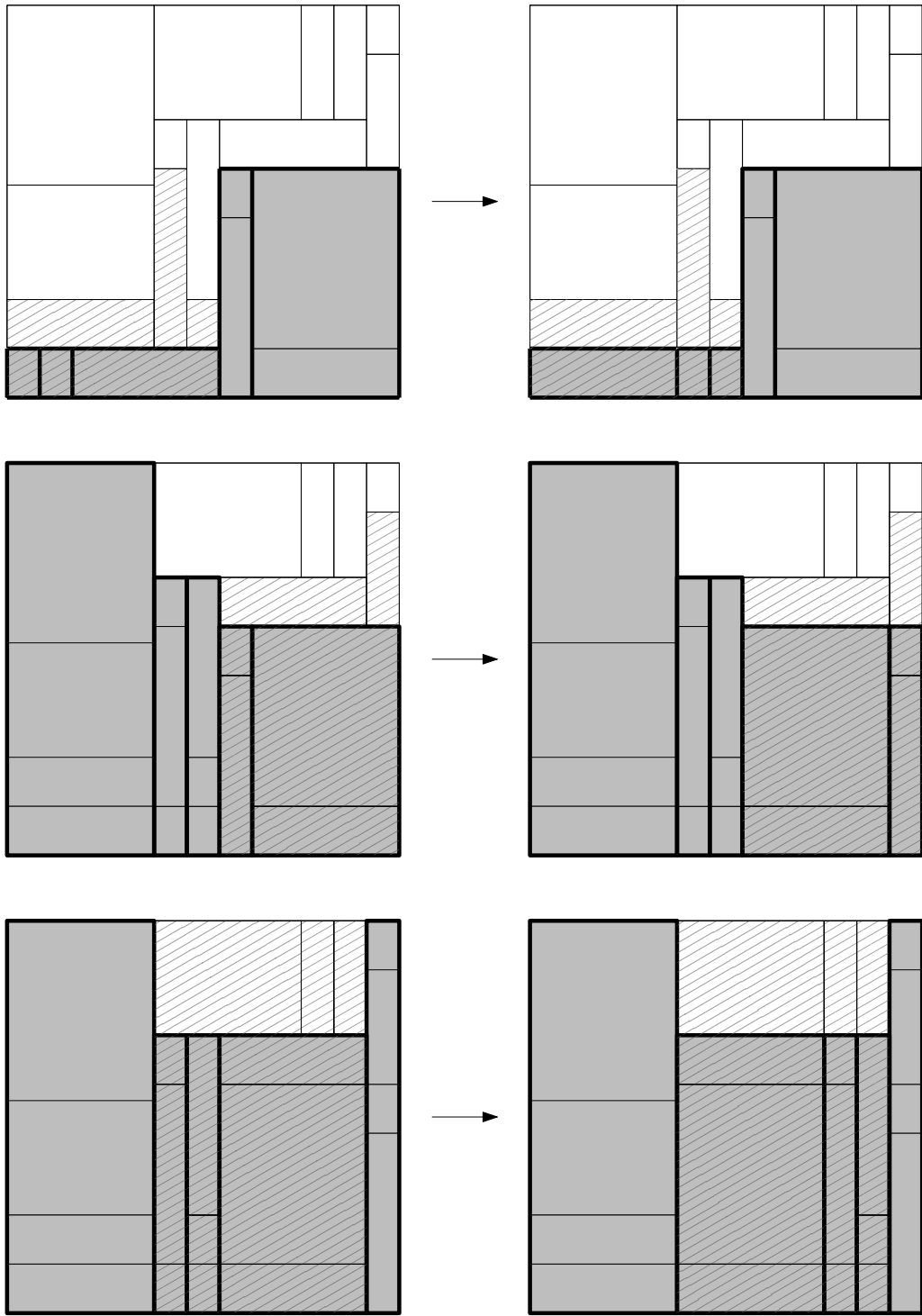
Figure 5.4.5 – Iterations to find a product r of restricted shuffles such that $\mathcal{P}'_{\text{test}} \in \Pi_r$ and $\text{Sky}(r(\mathcal{P}'_{\text{test}})) = \emptyset$. On every left pictures, all grey pieces represent the highest city, all grey hatched pieces represent towers whose top's height is the complexity of the partition and all white hatched pieces represent pieces of sky of the partition which are also in the upper work.
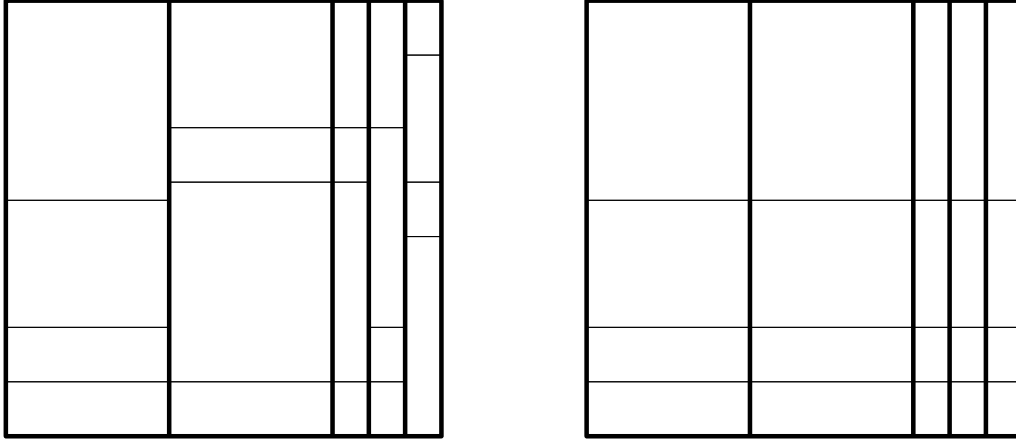
Figure 5.4.6 – Illustration of what looks like a setwise $\mathbb{Q}$-free partition with an empty sky and how moving pieces inside each tower can lead to a setwise $\mathbb{Q}$-free grid-pattern.

5.5 A refinement for Rec_2

Here we establish a more precise and concrete statement in dimension 2. Theorem 5.1.5 says every element f in Rec_d can be obtained as a composition of restricted shuffles. It is tempting to improve this statement by fixing a setwise $\mathbb{Q}$-free partition $\mathcal{P} \in \Pi_f$, and then shuffling rectangles in $f(\mathcal{P})$ without changing the partition. The proof seems at first sight to provide this, but the induction step forces to change the partition. In dimension 2, we can avoid this, see Theorem 5.5.2 below.

In this case we can be more precise than Theorem 5.1.5.

Definition 5.5.1. Let $\mathcal{P}$ be a partition into rectangles of $[0,1[^d$. A *restricted shuffle on* $\mathcal{P}$ is a restricted shuffle which shuffles two rectangles of $\mathcal{P}$. For $n \in \mathbb{N}^*$, a *n-sequence of restricted shuffles on* $\mathcal{P}$ is a a sequence $(r_1, \ldots, r_n)$ of restricted shuffles such that for every $1 \leq i \leq n$ the element r_i is a restricted shuffle on $r_{i-1} \circ \ldots \circ r_1(\mathcal{P})$. The partition $r_n \circ \ldots \circ r_1(\mathcal{P})$ is called the image of $\mathcal{P}$ by this sequence.

Here is the refined version of Theorem 5.1.5, in dimension 2

Theorem 5.5.2. *Suppose $d = 2$. For every $f \in \mathrm{Rec}_d$ and for every setwise $\mathbb{Q}$-free partition $\mathcal{P} \in \Pi_f$, there exists a sequence of restricted shuffles $(r_1, \ldots, r_n)$ on $\mathcal{P}$ such that $f = r_n \circ \ldots \circ r_1$.*

Remark 5.5.3. To motivate the setwise $\mathbb{Q}$-free property, we illustrate with a partition which is a in the image by Rec_d of a grid-pattern $\mathcal{Q}$ which is not setwise $\mathbb{Q}$-free. Indeed if we do not allowed to cut pieces of $\mathcal{Q}$ then for every sequence of restricted shuffles on $\mathcal{Q}$, the image of $\mathcal{Q}$ by this sequence is always $\mathcal{Q}$.

Figure 5.5.1 – **Left:** A grid-pattern $\mathcal{Q}$ which is not setwise $\mathbb{Q}$-free. **Right:** A rearrangement of $\mathcal{Q}$ which is not the image of $\mathcal{Q}$ by a sequence of restricted shuffles on $\mathcal{Q}$.

Theorem 5.5.2 in dimension 1 is Theorem 1.2.8. In dimension 2, we begin by two refinements of Proposition 5.3.3 and Lemma 5.4.5 obtained with immediate changes.

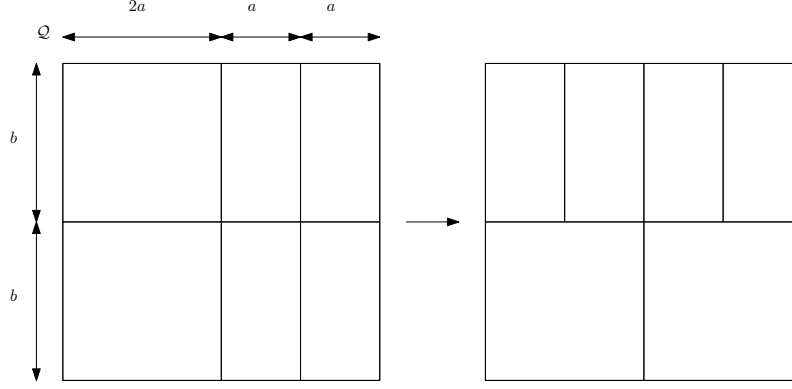**Lemma 5.5.4.** *Let $d \in \mathbb{N}^*$, let $f \in \mathrm{Rec}_d$ such that there exists a setwise $\mathbb{Q}$-free grid-pattern $\mathcal{Q}$ such that $f(\mathcal{Q})$ is a grid-pattern. Then there exists a sequence $(r_1, \ldots, r_n)$ of restricted shuffles on $\mathcal{Q}$ such that the image of $\mathcal{Q}$ by this sequence is $f(\mathcal{Q})$, in particular we have $f = r_n \circ \ldots \circ r_1$.* $\square$

Lemma 5.5.5. *Let $\mathcal{P}$ be a setwise $\mathbb{Q}$-free partition such that $\mathscr{C}(\mathcal{P}) = \emptyset$. Then there exists a sequence of restricted shuffles on $\mathcal{P}$ such that the image of $\mathcal{P}$ by this sequence is a grid-pattern.* $\square$

With these two results, the proof of Theorem 5.5.2 is the same as the one of Theorem 5.1.5 until Lemma 5.4.6, where we proved the following refinement in dimension 2:

Lemma 5.5.6. *Suppose $d = 2$. Let $\mathcal{P}$ be a setwise $\mathbb{Q}$-free partition. There exists a product g of restricted shuffles in direction inside $\{1, 2, \ldots, d-1\}$ such that $\mathcal{P} \in \pi_g$ and:*

$$\mathscr{C}(g(\mathcal{P})) \subset \mathscr{C}(\mathcal{P}) \smallsetminus \{\mathscr{C}_{\min}(\mathcal{P})\}.$$

Also there exists a sequence $(r_1, \ldots, r_n)$ of restricted shuffles on $\mathcal{P}$ such that $g = r_n \circ \ldots \circ r_1$.

Proof. First we rearrange every tower of $\mathrm{City}(\mathcal{P})$ such that pieces of every tower is ordered by increasing order about their length of their 2-projection.

We recall that $\Omega^- = \{\mathrm{pr}_d^\perp(K) \mid K \in \mathrm{Top}(\mathrm{City}(\mathcal{P}))\}$ and $\Omega^+ = \{\mathrm{pr}_d^\perp(K) \mid K \in \mathrm{Work}^+ \cup \mathrm{Top}(\mathrm{City}(\mathcal{P})) \smallsetminus \mathrm{Work}^-\}$. by Corollary 5.2.6 we deduce that there exists $\delta \in \mathrm{Rec}_{d-1}$ such that $\Omega^- \in \Pi_\delta$ (for every element K of Ω^-, the restriction of δ to K is a translation) and $\delta(\Omega^-) = \Omega^+$ and for every $K \in \Omega^- \cap \Omega^+$ we have $\delta(K) = K$. The main argument is that every connected component C of $\mathrm{Site}(\mathcal{P}) = \bigsqcup_{K \in \Omega^-} K$ is a left half-open interval and there exists $\Omega_C^- \subset \Omega^-$ which partitions C. Similarly we can define the subset Ω_C^+ of Ω^+ which partitions C. Then by $\mathbb{Q}$-freeness we can also ask δ to send Ω_C^- on Ω_C^+. Then we define $g_C \in \mathrm{Rec}_2$ such that:

$$g_C(x) = \begin{cases} (\delta \times \mathrm{Id})(x) & \text{if } \mathrm{pr}_2(x) < \mathscr{C}_{\min}(\mathcal{P}) \text{ and } \mathrm{pr}_1(x) \in C \\ x & \text{else} \end{cases}$$

We can see that g_C only moves towers of $\mathrm{City}(\mathcal{P})$. And as these towers are rearrange such that pieces of every tower is ordered by increasing order about their length of their

2-projection. We deduce that there exists a sequence $(r_1, \ldots, r_n)$ of restricted shuffle on $\mathcal{P}$ such that $g_C = r_n \circ \ldots \circ r_1$. Let g be the product of every g_C where C ranges over the set of all connected components of Site($\mathcal{P}$). It satisfies the statement of the lemma. $\qquad \square$

At this point we are unable to prove Theorem 5.5.2 for arbitrary d. Here are some possible step towards a proof.

Definition 5.5.7. Define (S_δ) as the following statement. For every R be finite union of rectangles in $[0, 1[^\delta$. Let $\mathcal{P}, \mathcal{Q}$ be rectangle partitions of R. Suppose that for each i there is a $\mathbb{Q}$-free subset F_i of $]0, 1[$ such that for every $K \in \mathcal{P} \cup \mathcal{Q}$, we have $\lambda(\mathrm{pr}_i(K)) \in F_i$. Then one can change $\mathcal{Q}$ into $\mathcal{P}$ by a finite sequence of shuffles.

Then the statement S_{d-1} implies Theorem 5.5.2 in dimension d, the argument being an immediate adaptation of the above one.

Indeed, we know that S_1 holds. Here R is just a disjoint union of intervals, and the difficulty is that components of R can have complicated shapes in general. Note that proving (S_δ) immediately reduces to the case when R is connected; however it sounds convenient not to assume R connected in order to set up a proof (e.g., by induction on the number of the rectangles).

5.6 The derived subgroup

We recall that $\mathscr{T}_d$ is the subset of all Rec_d-transpositions where $d \geq 1$ is the ambient dimension. In this subsection, we prove Theorem 5.1.6, namely that $\mathscr{T}_d$ is a generating subset of $D(\mathrm{Rec}_d)$. We start with some preliminary observations.

Lemma 5.6.1. *Every element of order 2 in Rec_d is a product of Rec_d-transpositions with pairwise disjoint support.*

Proof. Let $f \in \mathrm{Rec}_d$ have order 2. For $v \in \mathbb{R}^d$, define $X_v = \{x : f(x) - x = v\}$. Note that $X_v \cup X_{-v}$ is f-invariant. Choose a subset V_+ of $\mathbb{R}^d$ of elements called "positive elements", such that $\mathbb{R}^d$ is the disjoint union $V_+ \sqcup -V_+ \sqcup \{0\}$. For v positive, choose a finite partition $\mathcal{W}_v$ of X_v into rectangles, and let $\mathcal{W}$ be the union, for v positive, of all $\mathcal{W}_v$. Then f is the (disjoint support) product of all $\tau_{K, f(K)}$ for K ranging over $\mathcal{W}$. $\qquad \square$

Proposition 5.6.2. *(a) $\mathscr{T}_d \subset D(\mathrm{Rec}_d)$.*

(b) If $D(\mathrm{Rec}_d)$ is simple then it is generated by $\mathscr{T}_d$.

Proof. (a) Let $f \in \mathscr{T}_d$ and P and R be the two rectangles switched by f. We can decompose $P = P_1 \sqcup P_2$ such that P_1 and P_2 are translation-isometric. Let f_1 be the element that switches P_1 with $f(P_1)$ and let f_2 be the element that switches P_1 with P_2 and $f(P_1)$ with $f(P_2)$. Then we have $f = [f_1, f_2]$.

(b) From Lemma 5.6.1, it follows that the subgroup N generated by $\mathscr{T}_d$ coincides with the subgroup generated by elements of order 2. By (a), $N \subset D(\mathrm{Rec}_d)$. Hence, if $D(\mathrm{Rec}_d)$ is simple, it follows that $N = D(\mathrm{Rec}_d)$.

$\qquad \square$

For d=1, simplicity of $D(\mathrm{Rec}_d)$ was proved by Sah [Sah81] and it follows that $D(\mathrm{Rec}_d)$ is generated by Rec_d-transpositions. Vorobets [Vor17] more recently reproved simplicity of $D(\mathrm{Rec}_1)$, by first proving that it is generated by transpositions. Our approach for arbitrary $d \geq 1$ is inspired by the latter.

Definition 5.6.3. For every $\varepsilon > 0$ we define $\mathscr{T}_d^\varepsilon$ as the set of all rectangle transpositions $\tau_{K,L}$ such that $K \cup L$ is contained in a square of length ε.

Proposition 5.6.4. *(a) The subset $\mathscr{T}_d$ generates $D(\mathrm{Rec}_d)$.*

(b) For every $\varepsilon > 0$, the subset $\mathscr{T}_d^\varepsilon$ generates $D(\mathrm{Rec}_d)$.

(c) For every subset $U \subset [0,1[^d$ with nonempty interior, the group $D(\mathrm{Rec}_d)$ is normally generated by those rectangle transpositions $\tau_{K,L}$ such that $K \cup L$ is contained in U.

Proof. (a) From usual commutator formulas it follows that in a group, every commutator $[a_1a_2\ldots, b_1b_2\ldots]$ is a product of conjugates of the $[a_i, b_j]$. Then thanks to Theorem 5.1.5 we deduce that every commutator of elements in Rec_d can be written as the product of conjugates of commutators of restricted shuffles. Hence thanks to Lemma 5.6.1 we deduce that it is enough to prove that every commutator of restricted shuffles is a product of elements of order 2. We already saw that this statement is true in dimension 1. Let $i, j \in \{1, 2 \ldots, d\}$ and s, s' be two restricted rotations and R, R' be two $(d-1)$-subrectangles of $[0,1[^{d-1}$. We have different cases:

   (1) If $i = j$ then for every $x \in [0,1[^d$ and for every $k \in \{1, \ldots, d\}$ with $k \neq i$, we have $[\sigma_{R,s,i}, \sigma_{R',s',i}](x)_k = x_k$. Also $\mathrm{pr}_i^\perp(x) \notin R \cap R'$ we have $[\sigma_{R,s,i}, \sigma_{R',s',i}](x_i) = x_i$ and if $\mathrm{pr}_i^\perp(x) \in R \cap R'$ we have $[\sigma_{R,s,i}, \sigma_{R',s',i}](x)_i = [s, s'](x_i)$. Then by using the result in dimension 1 we deduce that $[\sigma_{R,s,i}, \sigma_{R',s',i}]$ is a product of elements of order 2.

   (2) Let assume $i \neq j$. We remark that if $R = R_1 \sqcup R_2$ then $\sigma_{R,s,i} = \sigma_{R_1,s,i} \circ \sigma_{R_2,s,i}$. Then by using again the equality between commutators we deduce that it is enough to show that the commutator $[\sigma_{R,s,i}, \sigma_{R',s',i}]$ is a product of elements of order 2, where R and R' are as small as we want. In particular as $i \neq j$ we can assume that R and R' are small enough such that for every $x \in \mathrm{Supp}(\sigma_{R,s,i}) \cap \mathrm{Supp}(\sigma_{R',s',i})$ we have both $\sigma_{R,s,i}(x) \notin \mathrm{Supp}(\sigma_{R',s',i})$ and $\sigma_{R',s',i}(x) \notin \mathrm{Supp}(\sigma_{R,s,i})$. Then in this case the commutator $[\sigma_{R,s,i}, \sigma_{R',s',i}]$ permutes cyclically three disjoint rectangles by translations. Hence it is a product of two rectangle transpositions.

 (b) Let f be a rectangle transposition. It can be written as disjoint support product of rectangle transpositions, each exchanging two rectangles K, L each of lengths $\leq \frac{\varepsilon}{3}$. We can then construct disjoint rectangles $K = K_0, K_1 \ldots, K_n = L$, each congruent to K, such that $K_i \cup K_{i+1}$ is contained in a square of size $\leq \varepsilon$ for each i. Since $\tau_{K,L}$ is contained in the subgroup generated by the $\tau_{K_i, K_{i+1}}$, this completes the proof.

 (c) We can assume that U is a square of size ε. Then each generator as in (b) is conjugate, by a rectangle transposition, to an element supported by this square, and the result follows.

$\square$

We deduce the simplicity of the derived subgroup $D(\mathrm{Rec}_d)$:

Theorem 5.6.5. *Every nontrivial subgroup of Rec_d normalized by $D(\mathrm{Rec}_d)$ contains $D(\mathrm{Rec}_d)$. In particular:*

   a) *The group $D(\mathrm{Rec}_d)$ is simple.*

   b) *The group $D(\mathrm{Rec}_d)$ is contained in every nontrivial normal subgroup of Rec_d.*

Proof. Let N be a nontrivial normal subgroup of Rec_d normalized by $D(\mathrm{Rec}_d)$. Let f be a non-identity element of N. For some ε, there exists a square K of length ε, such that f is a translation on K and such that K and $f(K)$ are disjoint.

Let us prove that every rectangle transposition $\tau_{P,Q}$ with $P \cup Q \subset K$ belongs to N. By Proposition 5.6.4 (c) this yields the conclusion.

Cut P and Q in two equal halves according to the d-coordinate: let P_1 and Q_1 be their lower halves, and P_2, Q_2 their upper halves. Then $[f, \tau_{P_1,Q_1}]$ permutes P_1 and Q_1 by translations, permutes $f(P_1)$ and $f(Q_1)$ by translations, and is identity elsewhere. Let s permute P_2 and $f(P_1)$ by translations, Q_2 and $f(Q_1)$ by translations, and be identity elsewhere. Then $s[f, \tau_{P_1,Q_1}]s^{-1} = \tau_{P,Q}$. Hence $\tau_{P,Q} \in N$. $\qquad\square$

5.7 Abelianization of Rec_d

In this section we generalize the abelianization group homomorphism of IET given by Arnoux-Fathi and Sah in [Arn81a, Sah81] called the SAF-homomorphism. The idea is to define, in each direction, a homomorphism, analogous to the 1-dimensional SAF-homomorphism.

For every group G, we denote by $D(G)$ its derived subgroup and by $G_{\mathrm{ab}} := G/D(G)$ its abelianization. For every $f \in \mathrm{Rec}_d$ we denote by $\widetilde{f}$ its image in $(\mathrm{Rec}_d)_{\mathrm{ab}}$. For every $\alpha \in \mathbb{R}$ we denote by $\{\alpha\}$ its fractional part. We recall that $\mathbb{R}^{\otimes d}$ is the d-th tensor power of $\mathbb{R}$ over $\mathbb{Q}$.

5.7.1 Tensor-valued measures

With notation in the proof of Lemma 5.2.5, we recall that $\mathcal{A}$ is the Boolean algebra of subset of $[0,1[$ generated by the set of all left half-open subinterval of $[0,1[$, and $\mu_d : \mathcal{A}^{\otimes d} \to \mathbb{R}^{\otimes d}$ is the Boolean algebra measure such that

$$\mu_d(I_1 \times \ldots \times I_d) = \lambda(I_1) \otimes \ldots \otimes \lambda(I_d)$$

where λ is the Lebesgue measure on $\mathbb{R}$. In order to have more homogeneous results for every direction we need to use variants of the μ_d that place some coordinates at the end:

Definition 5.7.1. For every $1 \le i \le d$ we define a Boolean algebra measure on $\mathcal{A}^{\otimes d}$ by $\mu_{d,i} = \sigma_i \circ \mu_i$: where σ_i is the linear automorphism of $\mathbb{R}^{\otimes d}$ defined by:

$$\sigma_i(x_1 \otimes \cdots \otimes x_d) = (x_1 \otimes \cdots \otimes x_{i-1} \otimes x_{i+1} \otimes \cdots \otimes x_d \otimes x_i).$$

Thus:

$$\mu_{d,i} : \quad \mathcal{A}^{\otimes d} \quad \longrightarrow \quad \mathbb{R}^{\otimes d}$$
$$I_1 \times \ldots \times I_d \quad \longmapsto \quad \mu_d(I_1 \times \ldots \times I_{i-1} \times I_{i+1} \times \ldots \times I_d \times I_i).$$

For every restricted rotation we exhibit a useful representative of its conjugacy class.

Definition 5.7.2. Let $1 \le i \le d$ and let $a_j \in\ ]0,1]$ for every $j \in \{1, \ldots, d\} \smallsetminus \{i\}$; let $a = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_d)$. Let $\alpha \in\ ]0,1]$ and $\beta \in [0,\alpha[$. Let P and Q be the two rectangles such that $\mathrm{pr}_j(P) = \mathrm{pr}_j(Q) = [0, a_j[$ for every $j \ne i$ and $\mathrm{pr}_i(P) = [0, \alpha - \beta[$ and $\mathrm{pr}_i(Q) = [\alpha - \beta, \alpha[$. We call the *standard shuffle of parameters* a, i, α, β the restricted shuffle in direction i that shuffles P and Q; we denote it by $R_{a,i,\alpha,\beta}$.

5.7.2 The generalized SAF-homomorphism

Every $f \in \mathrm{Rec}_d$ is a piecewise translation; thus we directly see that the pushforward measure $f_*(\mu_d)$ is equal to μ_d. Hence for every $1 \le k \le d$ the measure $\mu_{d,i}$ (see Definition 5.7.1) also satisfies $f_*(\mu_{d,i}) = \mu_{d,i}$.

For every $f \in \mathrm{Rec}_d$ and $1 \le i \le d$, we define $\nu_i(f) : [0,1[^d \to \mathbb{R}$ by $\nu_i(f)(x) = \mathrm{pr}_i(f(x)) - \mathrm{pr}_i(x)$. One can notice that for every $g \in \mathrm{Rec}_d$ we have $\nu_i(f \circ g) = \nu_i(f) \circ g + \nu_i(g)$. Also by definition of f it is immediate that for every $x \in \mathbb{R}$, the inverse image $\nu_i(f)^{-1}(x)$ belongs to $\mathcal{A}^{\otimes d}$ and $\nu_i : [0,1[^d \to \mathbb{R}$ takes finitely many values.

Definition 5.7.3. For every $1 \leq i \leq d$ we define the map ξ_i' as the following:

$$
\begin{array}{rcl}
\xi_i' : \quad \mathrm{Rec}_d & \longrightarrow & \mathbb{R}^{\otimes (d+1)} \\
f & \longmapsto & \displaystyle\sum_{\alpha \in \mathbb{R}} \mu_{d,i}((\nu_i(f))^{-1}(\{\alpha\})) \otimes \alpha
\end{array}
$$

We define ξ_i as the projection of ξ_i' in $\mathbb{R}^{\otimes (d-1)} \otimes (\bigwedge_{\mathbb{Q}}^2 \mathbb{R})$.

Example 5.7.4. Let $1 \leq i \leq d$ and $a = (a_1, \ldots, a_{d-1}) \in \,]0,1]^{d-1}$. For every $\alpha \in \,]0,1]$ and every $\beta \in [0, \alpha[$ we have:

$$
\xi_i'(R_{a,i,\alpha,\beta}) = a_1 \otimes \ldots \otimes a_{d-1} \otimes (\alpha - \beta) \otimes \beta + a_1 \otimes \ldots \otimes a_{d-1} \otimes \beta \otimes (-(\alpha - \beta)).
$$

Then we deduce that:

$$
\xi_i(R_{a,i,\alpha,\beta}) = 2(a_1 \otimes \ldots \otimes a_{d-1} \otimes (\alpha \wedge \beta)) = a_1 \otimes \ldots \otimes a_{d-1} \otimes (\alpha \wedge 2\beta).
$$

For every $1 \leq j \neq i \leq d$ we have $\xi_j'(R_{a,i,\alpha,\beta}) = 0$ because $\nu_j(R_{a,i,\alpha,\beta})(x) = 0$ for every $x \in [0,1[^d$.

Definition 5.7.5. The map $\xi : \mathrm{Rec}_d \to \left(\mathbb{R}^{\otimes (d-1)} \otimes (\bigwedge_{\mathbb{Q}}^2 \mathbb{R}) \right)^d$ defined by:

$$
\xi(f) = (\xi_1(f), \xi_2(f), \ldots, \xi_d(f))
$$

for every $f \in \mathrm{Rec}_d$, is called the *generalized* SAF-*homomorphism*.

The map ξ_i is an analogue of the SAF-homomorphism given by Arnoux-Fathi and Sah in [Arn81a, Sah81]. Then the proof that the SAF-homomorphism is a group homomorphism works for ξ_i with immediate changes; we give the proof for the sake of completeness.

Proposition 5.7.6. *The map* $\xi = (\xi_1, \ldots, \xi_d)$ *is a surjective group homomorphism.*

Proof. For every $f, g \in \mathrm{Rec}_d$ we have the following equalities:

$$
\begin{aligned}
\xi_i'(f \circ g) &= \sum_{\alpha \in \mathbb{R}} \mu_{d,i}((\nu_i(f \circ g))^{-1}(\{\alpha\})) \otimes \alpha \\
&= \sum_{\alpha \in \mathbb{R}} \mu_{d,i}((\nu_i(f) \circ g + \nu_i(g))^{-1}(\{\alpha\})) \otimes \alpha \\
&= \sum_{\beta, \gamma \in \mathbb{R}} \mu_{d,i}\big((\nu_i(f) \circ g)^{-1}(\{\beta\}) \cap (\nu_i(g))^{-1}(\{\gamma\})\big) \otimes (\beta + \gamma) \\
&= \sum_{\beta \in \mathbb{R}} \mu_{d,i}((\nu_i(f) \circ g)^{-1}(\{\beta\})) \otimes \beta + \sum_{\gamma \in \mathbb{R}^d} \mu_{d,i}((\nu_i(g))^{-1}(\{\gamma\})) \otimes \gamma \\
&= \xi_i'(f) + \xi_i'(g)
\end{aligned}
$$

The last equality is given by the fact that λ_i is invariant by an element of Rec_d, so that $g_* \mu_{d,i} = \mu_{d,i}$. We deduce that ξ_i' is a group homomorphism and this implies that ξ_i is also a group homomorphism.

To show surjectivity, it is enough to show that $\{0\}^{i-1} \oplus (\mathbb{R}^{\otimes (d-1)} \otimes \Lambda_{\mathbb{Q}}^2 \mathbb{R}) \oplus \{0\}^{d-i}$ is contained in the image. One can notice that the group $\mathbb{R}^{\otimes (d-1)} \otimes (\bigwedge_{\mathbb{Q}}^2 \mathbb{R})$ is generated by the subset $\{a_1 \otimes \ldots \otimes a_{d-1} \otimes (\alpha \wedge \beta) \mid a_j, \alpha, \beta \in [0, \frac{1}{2}]\}$. Then it is sufficient to remark that for every $a_1, \ldots, a_{d-1}, \alpha, \beta \in [0, \frac{1}{2}]$ the element $a_1 \otimes \ldots \otimes a_{d-1} \otimes (\alpha \wedge \beta)$ is the image by ξ_i of the standard shuffle of parameters $(a_1, \ldots, a_{d-1}), i, \alpha, \beta/2$ (see Example 5.7.4). $\qquad \square$

5.7.3   An inverse of the generalized SAF-homomorphism

The group $(\mathbb{R}^{\otimes(d-1)} \otimes (\bigwedge_{\mathbb{Q}}^2 \mathbb{R}))^d$ is abelian, therefore ξ induces a surjective group homomorphism $\overline{\xi}$ from $(\mathrm{Rec}_d)_{\mathrm{ab}}$ onto $(\mathbb{R}^{\otimes(d-1)} \otimes (\bigwedge_{\mathbb{Q}}^2 \mathbb{R}))^d$. In order to show that the kernel $\mathrm{Ker}(\xi)$ is equal to the derived subgroup $D(\mathrm{Rec}_d)$, we construct an inverse of $\overline{\xi}$ in a similar way Arnoux-Fathi and Sah did in [Arn81a, Sah81].

In the following we use additive notations for $(\mathrm{Rec}_d)_{\mathrm{ab}}$. We begin by giving a generating subset of $(\mathrm{Rec}_d)_{\mathrm{ab}}$, for that, we see that we can relate every restricted shuffle to a standard shuffle by conjugating with a rectangle transposition:

Proposition 5.7.7. *Let $1 \leq i \leq d$ and let $a_j \in\ ]0,1]$ for every $j \in \{1, \ldots, d\} \smallsetminus \{i\}$ and let $\alpha, \beta \in\ ]0,1]$. Let r be a restricted shuffle in direction i that shuffles two rectangles P and Q such that $\lambda(\mathrm{pr}_j(P)) = \lambda(\mathrm{pr}_j(Q)) = a_j$ for every $j \neq i$ and $\lambda(\mathrm{pr}_i(P)) = \alpha$ and $\lambda(\mathrm{pr}_i(Q)) = \beta$. We assume that $\sup(\mathrm{pr}_i(P)) = \inf(\mathrm{pr}_i(Q))$. Then r is conjugate to the standard shuffle of parameters $(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_d), i, \alpha + \beta, \beta$.* $\qquad\square$

Proof. All maps considered move only in direction i, and hence this is an immediate consequence of the case $d = 1$ observed in [Arn81a] $\qquad\square$

From Theorem 5.1.5 and Proposition 5.7.7 we deduce the following corollary:

Corollary 5.7.8. *The group $(\mathrm{Rec}_d)_{\mathrm{ab}}$ is generated by the subset*

$$\left\{ \widetilde{R_{a,i,\alpha,\beta}} \mid a \in\ ]0,1]^{d-1},\ 1 \leq i \leq d,\ \alpha \in\ ]0,1],\ \beta \in [0,\alpha[\right\}.$$

Thanks to Theorem 5.1.6 we know that for every $1 \leq i \leq d$, every $a \in [0,1[^{d-1}$ and every $\alpha \in [0,1[$, the standard shuffle $R_{a,i,\alpha,\alpha/2}$ is in the derived subgroup $D(\mathrm{Rec}_d)$. The next lemma gives a relation between some standard shuffles, it is a direct consequence of the case in dimension 1 proved by Arnoux-Fathi in [Arn81a, Theorem IV.2]:

Lemma 5.7.9. *Let $\alpha, \alpha' \in\ ]0,1]$ such that $\alpha + \alpha' \in\ ]0,1]$. Let $\beta \in [0, \min(\alpha, \alpha')[$. Let $a \in\ ]0,1]^{d-1}$ and $1 \leq i \leq d$. Then:*

1. *$R_{a,i,\alpha',\beta}^{-1} \circ R_{a,i,\alpha+\alpha',\beta}$ is conjugate to $R_{a,i,\alpha+\beta,\beta}$;*

2. *$R_{a,i,\alpha+\beta,\beta}^{-1} \circ R_{a,i,\alpha,\beta}$ is conjugate to $R_{a,i,2\beta,\beta}$.*

In the following, we use a lot of parameters. For the sake of clarity, parameters will be inside brackets and not in index.

Definition 5.7.10. *Let $a \in\ ]0,1]^{d-1}$ and $1 \leq i \leq d$. For every $\alpha \in\ ]0,1]$ we define:*

$$\Psi[i,a,\alpha] : \begin{array}{ccc} \mathbb{R} & \longrightarrow & (\mathrm{Rec}_d)_{\mathrm{ab}} \\ \beta & \longmapsto & \widetilde{R_{a,i,\alpha,\alpha\{\beta/\alpha\}}} \end{array}.$$

We define $\Psi[i,a,\alpha] = 0$ whenever $\alpha = 0$ or $a_j = 0$ for at least one $1 \leq j \leq d-1$.

Lemma 5.7.11. *Let $a = (a_1, \ldots, a_{d-1}) \in [0,1]^{d-1}$ and $1 \leq i \leq d$. For every $\alpha \in [0,1]$ the map $\Psi[i,a,\alpha]$ is $\mathbb{Z}$-linear.*

Proof. One can see that $R_{a,i,\alpha,\beta}$ is the rotation of angle β on $P \cup Q$ when we look at the projection on $\mathrm{Vect}(e_i)$. We recall that for every $b \in \mathbb{R}$ we have that $\alpha\{b/\alpha\}$ is equal to b modulo α. Hence for every $c \in \mathbb{R}$ we have $\alpha\{(b+c)/\alpha\} = \alpha\{b/\alpha\} + \alpha\{c/\alpha\}$ [mod α]. We deduce that for every $b, c \in \mathbb{R}$ we have $R_{a,i,\alpha,\alpha\{(b+c)/\alpha\}} = R_{a,i,\alpha,\alpha\{b/\alpha\}} \circ R_{a,i,\alpha,\alpha\{c/\alpha\}}$, and this concludes. $\qquad\square$

In the following we prove that Ψ is linear on some other variable for small values.

Lemma 5.7.12. *Let $a = (a_1, \ldots, a_{d-1}) \in [0, 1]^{d-1}$ and $1 \leq i \leq d$. For every $\alpha, \alpha' \in [0, 1]$ such that $\alpha + \alpha' \leq 1$ we have $\Psi[i, a, \alpha + \alpha] = \Psi[i, a, \alpha] + \Psi[i, a, \alpha']$.*

Proof. The case where $\alpha = 0$ or $\alpha' = 0$ is immediate. We assume $\alpha \neq 0$ and $\alpha' \neq 0$. Let $\beta \in \mathbb{R}$. Up to change β in $\beta + k\alpha$ with $k \in \mathbb{Z}$ we can assume that $\beta \geq 0$. Let $n \in \mathbb{N}$ such that $\beta' := \beta/n \in [0, \min(\alpha, \alpha')[$. We have $(\alpha + \alpha')\{\beta'/(\alpha + \alpha')\} = \beta'$. Then $\Psi[i, a, \alpha + \alpha'](\beta') = R_{a,i,\alpha+\alpha',\beta'}$. By Lemma 5.7.9 there exist $g, h \in \text{Rec}$ such that:

$$
\begin{aligned}
R_{a,i,\alpha+\alpha',\beta'} &= R_{a,i,\alpha',\beta'} \circ g \circ R_{a,i,\alpha+\beta',\beta'} \circ g^{-1} \\
&= R_{a,i,\alpha',\beta'} \circ g \circ h \circ R_{a,i,2\beta',\beta'} \circ h^{-1} \circ R_{a,i,\alpha,\beta'} \circ g^{-1}.
\end{aligned}
$$

We know that $R_{a,i,2\beta',\beta'}$ is in the derived subgroup $D(\text{Rec})$ thanks to Theorem 5.1.6. Hence $\widetilde{R_{a,i,\alpha+\alpha',\beta'}} = \widetilde{R_{a,i,\alpha,\beta'}} + \widetilde{R_{a,i,\alpha',\beta'}}$.

We remark that $\alpha\{\beta'/\alpha\} = \alpha'\{\beta'/\alpha'\} = \beta'$ so we have:

$$
\Psi[i, a, \alpha + \alpha'](\beta') = \Psi[i, a, \alpha](\beta') + \Psi[i, a, \alpha'](\beta').
$$

Then we can multiply by n this equality and by linearity on the variable β we deduce $\Psi[i, a_1, \ldots, a_{d-1}, \alpha + \alpha'](\beta) = \Psi[i, a, \alpha](\beta) + \Psi[i, a, \alpha'](\beta)$. $\qquad\square$

Lemma 5.7.13. *Let $a = (a_1, \ldots, a_{d-1}) \in [0, 1]^{d-1}$ and $1 \leq i \leq d$. Let $\alpha \in [0, 1]$. For every $1 \leq j \leq d-1$ and every $u \in [0, 1]$ such that $a_j + u \in [0, 1]$, we have:*

$$
\begin{aligned}
\Psi[i, a + ue_j, \alpha] = \quad &\Psi[i, a, \alpha] \\
&+ \Psi[i, a + (u - a_j)e_j, \alpha].
\end{aligned}
$$

Proof. Let $1_{\mathbb{R}^{d-1}} = (1, \ldots, 1) \in \mathbb{R}^{d-1}$. Let $b = a + ue_j$ and $c = a + (u - a_j)e_j$. One can notice that for every $\beta \in [0, \alpha[$ we have:

$$
R_{b,i,\alpha,\beta} = R_{a,i,\alpha,\beta} \circ R_{1_{\mathbb{R}^{d-1}},j,1,a_j} \circ R_{c,i,\alpha,\beta} \circ R_{1_{\mathbb{R}^{d-1}},j,1,a_j}^{-1}.
$$

Thus $\widetilde{R_{b,i,\alpha,\beta}} = \widetilde{R_{a,i,\alpha,\beta}} + \widetilde{R_{c,i,\alpha,\beta}}$ and this concludes. $\qquad\square$

We extend the definition of $\Psi[i, a, \alpha]$ for $a \in]-1, 1[^{d-1}$ and $\alpha \in]-1, 1[$:

Definition 5.7.14. *Let $a \in [0, 1[^{d-1}$ and $\alpha \in [0, 1[$. For every $1 \leq j \leq d-1$ and for every $\beta \in \mathbb{R}$ we define:*

$$
\begin{aligned}
\Psi[i, a, -\alpha](\beta) \quad &:= \Psi[i, a - 2a_j e_j, \alpha](\beta) \\
&:= \Psi[i, a, \alpha](-\beta).
\end{aligned}
$$

Lemma 5.7.15. *Let $a = (a_1, \ldots, a_{d-1}) \in [0, 1]^{d-1}$ and $1 \leq i \leq d$. Let $\alpha \in [0, 1]$ and $\beta \in \mathbb{R}$. For every $q \in \mathbb{Q}$ such that $|q\alpha| \leq 1$ we have:*

$$
\Psi[i, a, q\alpha](\beta) = \Psi[i, a, \alpha](q\beta).
$$

Also for every $1 \leq j \leq d-1$ such that $|qa_j| \leq 1$, let $b = (a_1, \ldots, a_{j-1}, qa_j, a_{j+1}, \ldots, a_{d-1})$, we have:

$$
\Psi[i, b, \alpha](\beta) = \Psi[i, a, \alpha](q\beta).
$$

Proof. Let $n \in \mathbb{Z}$ and $m \in \mathbb{N}^*$ such that $q = \frac{n}{m}$. As $|q\alpha| \leq 1$ we have $|\frac{1}{m}|a \leq 1$. By Lemma 5.7.11 and Lemma 5.7.12 we deduce that:

$$
\begin{aligned}
\Psi[i, a, \tfrac{n}{m}\alpha](\beta) \quad &= n\Psi[i, a, \tfrac{\alpha}{m}](\beta) \\
&= \Psi[i, a, \tfrac{\alpha}{m}](n\beta).
\end{aligned}
$$

Also:
$$\begin{aligned} \Psi[i,a,\alpha](n\beta) &= \Psi[i,a,m\tfrac{\alpha}{m}](n\beta) \\ &= m\Psi[i,a,\tfrac{\alpha}{m}](n\beta). \end{aligned}$$

Hence:
$$\begin{aligned} \Psi[i,a,q\alpha](\beta) &= \Psi[i,a,\tfrac{\alpha}{m}](n\beta) \\ &= \tfrac{1}{m}\Psi[i,a,\alpha](n\beta) \\ &= \Psi[i,a,\alpha](qb). \end{aligned}$$

The second equality is the same proof by using Lemma 5.7.13 instead of Lemma 5.7.12.
□

For every $u \in \mathbb{R}$ we denote by $\lceil u \rceil$ the ceiling of u. For every $a \in \mathbb{R}^{d-1}$ we denote by $\overline{a} = (\frac{a_1}{\lceil|a_1|\rceil}, \ldots, \frac{a_{d-1}}{\lceil|a_{d-1}|\rceil})$.

Definition 5.7.16. For every $1 \le i \le d$ we define :
$$\Psi_i : \qquad \mathbb{R}^{d+1} \qquad \longrightarrow \qquad \overline{(\mathrm{Rec}_d)_{\mathrm{ab}}}$$
$$(a_1, \ldots, a_{d-1}, \alpha, \beta) \quad \longmapsto \quad \Psi[i, \overline{(a_1, \ldots, a_{d-1})}, \tfrac{\alpha}{\lceil|\alpha|\rceil}](\beta\lceil|\alpha|\rceil \prod_{j=1}^{d-1} \lceil|a_i|\rceil).$$

Proposition 5.7.17. *For every $1 \le i \le d$ the map Ψ_i is $(d+1)$-multilinear over $\mathbb{Z}$.*

Proof. Let $a = (a_1, \ldots, a_{d-1}) \in \mathbb{R}^{d-1}$. The linearity in the variable β is given by Lemma 5.7.11. Let $\alpha, \alpha' \in \mathbb{R}^+$. We have $\frac{\alpha+\alpha'}{\lceil|\alpha+\alpha'|\rceil} \le 1$ so by Lemma 5.7.12 we obtain:

$$\Psi[i, \overline{a}, \tfrac{\alpha+\alpha'}{\lceil|\alpha+\alpha'|\rceil}](\beta\lceil|\alpha+\alpha'|\rceil \prod_{j=1}^{d-1} \lceil|a_i|\rceil) =$$
$$\Psi[i, \overline{a}, \tfrac{\alpha}{\lceil|\alpha+\alpha'|\rceil}](\beta\lceil|\alpha+\alpha'|\rceil \prod_{j=1}^{d-1} \lceil|a_i|\rceil)$$
$$+\Psi[i, \overline{a}, \tfrac{\alpha'}{\lceil|\alpha+\alpha'|\rceil}](\beta\lceil|\alpha+\alpha'|\rceil \prod_{j=1}^{d-1} \lceil|a_i|\rceil).$$

Also $\frac{\alpha}{\lceil|\alpha+\alpha'|\rceil} = \frac{\alpha}{\lceil|\alpha|\rceil} \times \frac{\lceil|\alpha|\rceil}{\lceil|\alpha+\alpha'|\rceil}$. Then by Lemma 5.7.15 we deduce that:

$$\Psi[i, \overline{a}, \tfrac{\alpha}{\lceil|\alpha+\alpha'|\rceil}](\beta\lceil|\alpha+\alpha'|\rceil \prod_{j=1}^{d-1} \lceil|a_i|\rceil)$$
$$= \Psi[i, \overline{a}, \tfrac{\alpha}{\lceil|\alpha|\rceil}](\beta\lceil|\alpha+\alpha'| \times \tfrac{\lceil|\alpha|\rceil}{\lceil|\alpha+\alpha'|\rceil}\rceil \prod_{j=1}^{d-1} \lceil|a_i|\rceil)$$
$$= \Psi_i(a_1, \ldots, a_{d-1}, \alpha, \beta).$$

A similar computation can be done to obtain:

$$\Psi\left[i, \overline{a}, \tfrac{\alpha'}{\lceil|\alpha+\alpha'|\rceil}\right]\left(\beta\lceil|\alpha+\alpha'|\rceil \prod_{j=1}^{d-1} \lceil|a_i|\rceil\right) = \Psi_i(a_1, \ldots, a_{d-1}, \alpha', \beta).$$

Hence we have:

$$\Psi_i(a_1, \ldots, a_{d-1}, \alpha+\alpha', \beta) = \Psi_i(a_1, \ldots, a_{d-1}, \alpha, \beta) + \Psi_i(a_1, \ldots, a_{d-1}, \alpha', \beta).$$

Now let assume $\alpha + \alpha' \in \mathbb{R}$. If $\alpha + \alpha'$ is negative then by Definition 5.7.14 we have:

$$\Psi_i(a, \alpha+\alpha', \beta) = \Psi_i(a, -(\alpha+\alpha'), -\beta).$$

Also for $\alpha, \alpha' \in \mathbb{R}^+$ such that $\alpha - \alpha' \in \mathbb{R}^+$ we have:

$$\Psi_i(a, \alpha, \beta) = \Psi_i(a, \alpha-\alpha', \beta) + \Psi_i(a, \alpha', \beta).$$

In conclusion we obtain that Ψ_i is linear in the variable α.

The linearity of Ψ_i in the variable a_j for $1 \le j \le d$ is the same proof as the linearity in the variable α using Lemma 5.7.13 instead of Lemma 5.7.12.
□

We deduce that Ψ_i can be seen as a group homomorphism from $\mathbb{R}^{\otimes(d+1)}$ onto $(\mathrm{Rec}_d)_{\mathrm{ab}}$.

Lemma 5.7.18. *For every $1 \leq i \leq d$ and every $a_1, \ldots, a_{d-1}, \alpha, \beta \in \mathbb{R}$ we have $\Psi_i(a_1 \otimes \ldots \otimes a_{d-1} \otimes \alpha \otimes \beta + a_1 \otimes \ldots \otimes a_{d-1} \otimes \beta \otimes \alpha) = 0$.*

Proof. Let $a = (a_1, \ldots, a_{d-1})$. By $\mathbb{Q}$-linearity of Ψ it is enough to show the lemma for the case where $a_1, \ldots, a_{d-1}, \alpha, \beta \in [0, \frac{1}{2}[$ and such that $\alpha > \beta > \alpha - \beta$. Then we remark that $\beta\{\alpha/\beta\} = \alpha - \beta$. Then by Lemma 5.7.9 we have:

$$\Psi_i(a_1 \otimes \ldots \otimes a_{d-1} \otimes \alpha \otimes \beta + a_1 \otimes \ldots \otimes a_{d-1} \otimes \beta \otimes \alpha)$$
$$= \Psi[i, a, \alpha](\beta) + \Psi[i, a, \beta](\alpha)$$
$$= \widetilde{R_{a,i,\alpha,\beta}} + \widetilde{R_{a,i,\beta,\alpha-\beta}}$$
$$= \widetilde{R_{a,i,\alpha,\beta}} + \widetilde{R_{a,i,\alpha,\alpha-\beta}} + \widetilde{R_{a,i,2(\alpha-\beta),\alpha-\beta}} = 0. \qquad \square$$

Thanks to Lemma 5.7.18 and Corollary 5.7.8, we deduce that Ψ_i induces a group homomorphism from $\mathbb{R}^{\otimes(d-1)} \otimes (\bigwedge_{\mathbb{Q}}^2 \mathbb{R})$ onto $(\mathrm{Rec}_d)_{\mathrm{ab}}$. It is also denoted by Ψ_i.

Definition 5.7.19. We define the *generalized* SAF-*inverse* as the following group homomorphism:

$$\Psi : \qquad (\mathbb{R}^{\otimes(d-1)} \otimes (\textstyle\bigwedge_{\mathbb{Q}}^2 \mathbb{R}))^d \qquad \longrightarrow \qquad (\mathrm{Rec}_d)_{\mathrm{ab}}$$
$$(a_{1,i} \otimes \ldots \otimes a_{d-1,i} \otimes (\alpha_i \wedge \beta_i))_{1 \leq i \leq d} \;\longmapsto\; \sum_{i=1}^d \Psi_i(a_{1,i} \otimes \ldots \otimes a_{d-1,i} \otimes (\alpha_i \wedge \tfrac{\beta_i}{2})).$$

Lemma 5.7.20. *We have $\Psi \circ \xi = \mathrm{Id}_{(\mathrm{Rec}_d)_{\mathrm{ab}}}$.*

Proof. Let $1 \leq i \leq d$ and let $a = (a_1, \ldots, a_{d-1}) \in (]0, 1])^{d-1}$. For every $\alpha \in]0, 1]$ and every $\beta \in [0, \alpha[$. It is enough to show the result for $\widetilde{R_{a,i,\alpha,\beta}}$. By Example 5.7.4 we have $\xi(\widetilde{R_{a,i,\alpha,\beta}}) = (u_j)_{1 \leq j \leq d}$ where $u_i = 2(a_1 \otimes \ldots \otimes a_{d-1}(\alpha \wedge \beta)) = a_1 \otimes \ldots \otimes a_{d-1}(\alpha \wedge (2\beta))$ and $u_j = 0$ for $j \neq i$. Then by definition of Ψ and Ψ_i we have $\Psi((u_j)_{1 \leq j \leq d}) = \Psi_i(a_1 \otimes \ldots \otimes a_{d-1}(\alpha \wedge \beta)) = \widetilde{R_{a,i,\alpha,\beta}}$. $\qquad \square$

We deduce that the generalized SAF-invariant $\overline{\xi} : (\mathrm{Rec}_d)_{\mathrm{ab}} \to (\mathbb{R}^{\otimes(d-1)} \otimes (\bigwedge^2 \mathbb{R}_{\mathbb{Q}}))^d$ is injective. Thus it gives the following theorem:

Theorem 5.7.21. *The generalized SAF-invariant ξ induces an isomorphism from $(\mathrm{Rec}_d)_{\mathrm{ab}}$ onto $(\mathbb{R}^{\otimes(d-1)} \otimes (\bigwedge_{\mathbb{Q}}^2 \mathbb{R}))^d$.*

5.7.4 A normal subgroup larger than the derived subgroup

We denote by GtG_d the subgroup of Rec_d generated by $\mathrm{IET}^d \cup \mathscr{T}_d$.

Proposition 5.7.22. *Suppose $d \geq 2$. The group GtG_d is a normal subgroup of Rec_d and strictly contains $D(\mathrm{Rec}_d)$.*

Proof. We recall that the group IET is generated by its subset of restricted rotations, see Theorem 1.2.8. For every $1 \leq i \leq d$ and every $\varepsilon > 0$, we define:

$$F_i^\varepsilon = \{\sigma_{[0,1[^{d-1}, r, i} \mid r \text{ is a restricted rotation whose support has measure} < \varepsilon\}.$$

Then the set $\bigcup_{i=1}^d F_i^\varepsilon$ generates IET^d. We obtain that the image of GtG_d by the group homomorphism ξ is equal to $(\{1\}^{d-1} \otimes (\bigwedge^2 \mathbb{R}))^d$ which is not the trivial subgroup. Hence, we deduce that GtG_d strictly contains $D(\mathrm{Rec}_d)$ thanks to Theorem 5.7.21. This gives also that GtG_d is a proper subgroup of Rec_d because the image of Rec_d by ξ is $(\mathbb{R}^{d-1} \otimes (\bigwedge^2 \mathbb{R}))^d$. The fact that GtG_d is a normal subgroup of Rec_d is immediate from the inclusion $D(\mathrm{Rec}_d) \subset \mathrm{GtG}_d$. $\qquad \square$

Remark 5.7.23. The notation GtG_d is for "Grid-to-Grid". We denote by S the subset of Rec_d consisting of elements f such that there exists a grid-pattern associated $\mathcal{Q}$ such that $f(\mathcal{Q})$ is still a grid-pattern. Then S contains $\mathrm{IET}^d \cup \mathscr{T}_d$ but is not equal to GtG_d. However the normal closure in GtG_d of S is GtG_d.

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Invariants de certains groupes d'origine dynamique

Résumé. Le but de cette thèse est d'étudier des groupes agissant par isométries par morceaux sur un intervalle en s'intéressant à leur abélianisé. Le cas du groupe des échanges d'intervalles IET a été traité par Arnoux-Fathi et Sah. Tout d'abord, pour tous sous-groupe Γ de $\mathbb{R}/\mathbb{Z}$, on identifie l'abélianisé de tous les sous-groupes IET(Γ) du groupe IET ; ainsi que celui de tous les sous-groupes IET$^{\bowtie}(\Gamma)$ du groupe des échanges d'intervalles avec renversements IET$^{\bowtie}$. L'idée est adapter le morphisme signature sur les groupes de permutations finis en essayant de mesurer l'ensemble des inversions d'un élément.

Par ailleurs on démontre aussi que la signature des groupes des permutations finis s'étend au groupe des permutations de de $[0, 1[$ qui sont continus en dehors d'un nombre fini de points. Cela a pour conséquence que la classe de Kapoudjian, un élément du second groupe de cohomologie, s'annule.

Ensuite, on se place en dimension $d \geq 1$ et on considère le groupe Rec_d des permutations du rectangle $[0, 1[^d$ qui bougent un nombre fini de sous-rectangles par translations et qui sont l'identité ailleurs. On démontre que la généralisation naturel des rotations restreintes (qui forment un système de générateur du groupe IET) forme un système générateur de Rec_d. Puis on identifie son abélianisé en généralisant le travail fait pour le groupe IET par Arnoux-Fathi et Sah.

Mots-clés : abélianisé, échanges d'intervalles, échanges de rectangles, groupes continus par morceaux, signature.

Invariants of some groups with dynamical origin

Abstract. In this thesis we study groups piecewise acting by isometries on an interval by identifying their abelianization. The case of the Interval Exchange Transformations (IET) group has been done by Arnoux-Fathi and Sah. First, for every subgroup Γ of $\mathbb{R}/\mathbb{Z}$, we identify the abelianization of every subgroup IET(Γ) of the group IET and also every subgroup IET$^{\bowtie}(\Gamma)$ of the Interval Exchange Transformations group with flips IET$^{\bowtie}$. The idea is to adapt the group homomorphism signature on finite permutation groups by measuring the set of inversions of an element.

Also we prove that the group homomorphism signature on finite permutation groups can be extended to the group consisting of permutations of $[0, 1[$ which are continuous outside a finite number of points. A consequence is the vanishing of an element of the second cohomology group called the Kapoudjian class.

Then, we deal with higher dimension. Let $d \geq 1$ and let Rec_d be the group of all permutations of $[0, 1[^d$ which are a translation on a finite number of subrectangles and fix the rest of $[0, 1[^d$. We prove that the natural generalization of restricted rotations (which define a generating subset of IET) define a generating subset of Rec_d. Next we identify the abelianization of Rec_d by extending the work of Arnoux-Fathi and Sah on IET.

Keywords: abelianization, interval exchanges, rectangle exchanges, piecewise continuous groups, signature.

Image de couverture : Tableau réalisé au fusain par mon grand-père André Roddier.