

ABELIANIZATION OF SOME GROUPS OF INTERVAL EXCHANGES

OCTAVE LACOURTE

ABSTRACT. Let IET be the group of bijections from $[0, 1[$ to itself that are continuous outside a finite set, right-continuous and piecewise translations. The abelianization homomorphism $f : \text{IET} \rightarrow A$, called SAF-homomorphism, was described by Arnoux-Fathi and Sah. The abelian group A is the second exterior power of the reals over the rationals.

For every subgroup Γ of \mathbb{R}/\mathbb{Z} we define $\text{IET}(\Gamma)$ as the subgroup of IET consisting of all elements f such that f is continuous outside Γ . Let $\tilde{\Gamma}$ be the preimage of Γ in \mathbb{R} . We establish an isomorphism between the abelianization of $\text{IET}(\Gamma)$ and the second skew-symmetric power of $\tilde{\Gamma}$ over \mathbb{Z} denoted by ${}^{\circ}\Lambda_{\mathbb{Z}}^2 \tilde{\Gamma}$. This group often has non-trivial 2-torsion, which is not detected by the SAF-homomorphism.

We then define IET^{\heartsuit} the group of all interval exchange transformations with flips. Arnoux proved that this group is simple thus perfect. However for every subgroup $\text{IET}^{\heartsuit}(\Gamma)$ we establish an isomorphism between its abelianization and $\langle \{a \otimes a \pmod{2} \mid a \in \tilde{\Gamma}\} \rangle \times \langle \{\ell \wedge \ell \pmod{2} \mid \ell \in \tilde{\Gamma}\} \rangle$ which is a 2-elementary abelian subgroup of $\otimes_{\mathbb{Z}}^2 \tilde{\Gamma} / (2 \otimes_{\mathbb{Z}}^2 \tilde{\Gamma}) \times {}^{\circ}\Lambda_{\mathbb{Z}}^2 \tilde{\Gamma} / (2 {}^{\circ}\Lambda_{\mathbb{Z}}^2 \tilde{\Gamma})$.

1. INTRODUCTION

Let X be the right-open and left-closed interval $[0, 1[$. We denote by $\mathfrak{S}(X)$ the group of all permutations of X and let $\mathfrak{S}_{\text{fin}}$ be its subgroup consisting of all finitely supported elements. For every group G we denote by $D(G)$ its derived subgroup and by $G_{\text{ab}} = G/D(G)$ its abelianization.

We define IET (Interval Exchange Transformations) as the subgroup of $\mathfrak{S}(X)$ consisting of all elements that are continuous outside a finite set, right-continuous and that are piecewise a translation. For every subgroup Γ of \mathbb{R}/\mathbb{Z} we define $\text{IET}(\Gamma)$ as the subgroup of IET consisting of all elements f that are continuous outside Γ .

We also consider an overgroup of IET and $\text{IET}(\Gamma)$, where we allow flips, that is, we replace “translation” with “isometry”. There is a difficulty in defining this group, namely that, properly speaking, it cannot be defined as a group acting on $[0, 1[$. For this we define $\widehat{\text{IET}^{\heartsuit}}$ as the subgroup of $\mathfrak{S}(X)$ consisting of all elements that are continuous outside a finite set and piecewise isometric. Let

Date: September 16, 2020.

2010 Mathematics Subject Classification. 37E05, 20F65, 20J06 .

$\widehat{\text{IET}}^{\boxtimes}(\Gamma)$ be its subgroup consisting of all elements f continuous outside Γ . We define the group of interval exchanges with flips, denoted by IET^{\boxtimes} , as the image of $\widehat{\text{IET}}^{\boxtimes}$ in $\mathfrak{S}(X)/\mathfrak{S}_{\text{fin}}(X)$. Similarly let $\text{IET}^{\boxtimes}(\Gamma)$ be the image of $\widehat{\text{IET}}^{\boxtimes}(\Gamma)$ in $\mathfrak{S}(X)/\mathfrak{S}_{\text{fin}}(X)$.

The group IET is a subgroup of $\widehat{\text{IET}}^{\boxtimes}$. Also it is naturally isomorphic to its image in IET^{\boxtimes} thus we freely see IET as a subgroup of IET^{\boxtimes} .

Most of the existing research on interval exchanges since the 70s pertains to classical dynamics, mostly studying the dynamical and ergodic properties of a single element of IET (see the survey of Viana [10]). The study of IET as a group started with the identification of its abelianization by Sah [9] and Arnoux-Fathi [1] (see below), and was much later revived by work of Dahmani-Fujiwara-Guirardel [3, 4] and C. Novak [8] in the late 2000s.

In comparison the group IET^{\boxtimes} is considerably less known. P. Arnoux proved in [2] that it is a simple group. Recently I. Liousse and N. Guelman [6] proved that IET^{\boxtimes} is uniformly perfect.

For every \mathbb{Z} -module A , we denote by ${}^{\circ}\Lambda_{\mathbb{Z}}^2 A$ the second skew-symmetric power. It is the quotient of the second tensor power $\otimes_{\mathbb{Z}}^2 A$ by the \mathbb{Z} -submodule generated by the set $\{x \otimes y + y \otimes x \mid x, y \in A\}$. We denote by $a \wedge b$ the projection of $a \otimes b$ in ${}^{\circ}\Lambda_{\mathbb{Z}}^2 A$; the set of all these elements generates ${}^{\circ}\Lambda_{\mathbb{Z}}^2 A$. There exists a canonical surjective group homomorphism from ${}^{\circ}\Lambda_{\mathbb{Z}}^2 A$ into the second exterior power $\Lambda_{\mathbb{Z}}^2 A$, which is injective if $A = 2A$. In general, its kernel is a 2-elementary abelian group, which is not trivial in general. For instance, if $A \simeq \mathbb{Z}^d$, then $\Lambda_{\mathbb{Z}}^2 A$ is isomorphic to $\mathbb{Z}^{d(d-1)/2}$, while this kernel is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^d$. In the case where $A = \mathbb{R}$ we recall that every map is \mathbb{Z} -bilinear if and only if it is \mathbb{Q} -bilinear, thus $\Lambda_{\mathbb{Q}}^2 \mathbb{R} = \Lambda_{\mathbb{Z}}^2 \mathbb{R}$.

We recall the result of Arnoux-Fathi and Sah [1, 9] about the identification of the abelianization of IET :

Theorem (Arnoux-Fathi-Sah [1, 9]). *The map:*

$$\begin{aligned} \varphi : \text{IET} &\longrightarrow \Lambda_{\mathbb{Z}}^2 \mathbb{R} \\ f &\longmapsto \left[\sum_{a \in \mathbb{R}} a \otimes \lambda((f - \text{Id})^{-1}(\{a\})) \right] \end{aligned}$$

is a surjective group homomorphism whose kernel is the derived subgroup $D(\text{IET})$. It is called the SAF-homomorphism. It induces an isomorphism between IET_{ab} and the second exterior power of the reals over the rationals.

In this article we identify $\text{IET}(\Gamma)_{\text{ab}}$ and $\text{IET}^{\boxtimes}(\Gamma)_{\text{ab}}$ for every subgroup Γ of \mathbb{R}/\mathbb{Z} . We denote by $\tilde{\Gamma}$ the preimage of Γ in \mathbb{R} .

When Γ is finite, both $\text{IET}(\Gamma)$ and $\text{IET}^{\boxtimes}(\Gamma)$ are finite Coxeter groups and their abelianization can be described easily (see Section 2.1). So from now on, we always assume that Γ is a dense subgroup of \mathbb{R}/\mathbb{Z} .

In this case, we show that the derived subgroups $D(\text{IET}(\Gamma))$ and $D(\text{IET}^{\boxtimes}(\Gamma))$ are simple (see Section 2.5). They are respectively the smallest normal subgroup of $\text{IET}(\Gamma)$ and $\text{IET}^{\boxtimes}(\Gamma)$. Then we see $\text{IET}(\Gamma)$ and $\text{IET}^{\boxtimes}(\Gamma)$ as topological full group of the groupoids of germs of their action on a Stone space. Then by the work of Nekrashevych [7], we obtain the simplicity of their smallest normal subgroup.

We denote by φ_{Γ} the restriction of the SAF-homomorphism φ to $\text{IET}(\Gamma)$ for every subgroup Γ of \mathbb{R}/\mathbb{Z} . It turns out that φ_{Γ} may fail to give the abelianization of $\text{IET}(\Gamma)$ because not every element of order 2 of $\text{IET}(\Gamma)$ is necessarily in $D(\text{IET}(\Gamma))$. In fact the restriction of φ_{Γ} is a surjective group homomorphism onto $2^{\odot}\Lambda_{\mathbb{Z}}^2\tilde{\Gamma}$ and here the main theorem is:

Theorem 1.1. *There exists a surjective group homomorphism $\varepsilon_{\Gamma} : \text{IET}(\Gamma) \rightarrow {}^{\odot}\Lambda_{\mathbb{Z}}^2\tilde{\Gamma}$ whose kernel is the derived subgroup $D(\text{IET}(\Gamma))$. It induces an isomorphism between $\text{IET}(\Gamma)_{\text{ab}}$ and the second skew-symmetric power ${}^{\odot}\Lambda_{\mathbb{Z}}^2\tilde{\Gamma}$.*

Remark 1.2. One can notice that if $2\tilde{\Gamma} = \tilde{\Gamma}$ then $\varepsilon_{\Gamma} = \varphi_{\Gamma}$. Actually, in this case, the proof of Arnoux-Fathi-Sah works with immediate changes.

The novelty occurs when $\tilde{\Gamma} \neq 2\tilde{\Gamma}$. In this case 2-torsion appears in ${}^{\odot}\Lambda_{\mathbb{Z}}^2\tilde{\Gamma}$ with in particular the subset $\{a \wedge a \mid a \in \tilde{\Gamma}\}$.

Before constructing ε_{Γ} it is important to understand the kernel of φ_{Γ} . For this we introduce some elements of IET and $\text{IET}(\Gamma)$.

A *restricted rotation of type (a, b)* is an element r of IET such that there exist consecutive intervals I and J of length a and b respectively with $\sup(I) = \inf(J)$, where r is the translation by b on I and the translation by $-a$ on J (and the identity outside $I \cup J$). The two intervals I and J are called the intervals associated with r . A Γ -*restricted rotation* is a restricted rotation in $\text{IET}(\Gamma)$. A tuple of restricted rotations is *balanced* if the number of factors of type (a, b) is equal to the one of type (b, a) . A product of restricted rotations is balanced if it can be written as the product of a balanced tuple of restricted rotations.

Remark 1.3. Let r be a restricted rotation of type (a, b) and I and J be the two consecutive intervals associated with r . Then if we only look at the interval $I \cup J$ and identify its endpoints, we obtain that r is the actual rotation of angle the length of J . This is why we called these elements “restricted rotation”.

Y. Vorobets proves in [11] the following:

Theorem (Y. Vorobets [11]). *The kernel $\text{Ker}(\varphi)$ of the SAF-homomorphism is generated by the set of all balanced products of restricted rotations and it is also generated by the set of all elements of order 2.*

Here we prove that this result also holds in restriction to $\text{IET}(\Gamma)$:

Lemma 1.4. *The set of all balanced products of Γ -restricted rotations is a generating subset of $\text{Ker}(\varphi_{\Gamma})$.*

A corollary can be proved by following the ideas of Y.Vorobets. We introduce some elements of order 2. An IET-*transposition* of type a is an element of IET that swaps two nonoverlapping intervals of length a while fixing the rest of $[0, 1[$ and an IET(Γ)-*transposition* is an IET-transposition in IET(Γ).

Corollary 1.5. *The set of all IET(Γ)-transpositions is a generating subset of the kernel $\text{Ker}(\varphi_\Gamma)$.*

After this, in Section 5, we define the group homomorphism ε_Γ in the same spirit as the signature on finite permutation groups. For every $n \in \mathbb{N}$ and σ in the finite permutation group \mathfrak{S}_n , a pair $(x, y) \in \{1, 2, \dots, n\}$ is an inversion of σ if $x < y$ and $\sigma(x) > \sigma(y)$. The signature of σ is equal to the counting measure of the set of all inversions of σ .

We keep the same definition for an inversion of an element of IET(Γ) and explicit how to measure this set. We define A_Γ as the Boolean algebra of subsets of $[0, 1[$ generated by the set of intervals $\{[a, b[\mid a, b \in \tilde{\Gamma}\}$. Thanks to the Lebesgue measure λ on \mathbb{R} we obtain a Boolean algebra measure w_Γ for $A_\Gamma \otimes A_\Gamma$ in the second tensor power $\otimes_{\mathbb{Z}}^2 \tilde{\Gamma}$. We show that the set of inversions of an element of IET(Γ) is an union of rectangles inside $A_\Gamma \otimes A_\Gamma$. We define ε_Γ as the projection on ${}^\circ\Lambda_{\mathbb{Z}}^2 \tilde{\Gamma}$ of the measure of the set of all inversions. We show that ε_Γ is a surjective group homomorphism. We establish the equality $-2\varepsilon_\Gamma = \varphi_\Gamma$ and manage to prove that $\text{Ker}(\varepsilon_\Gamma)$ is equal to the derived subgroup $D(\text{IET}(\Gamma))$.

In Section 6 we treat the case of $\text{IET}^{\boxtimes}(\Gamma)$. A *reflection map of type a* is an element of $\widehat{\text{IET}^{\boxtimes}}$ that reverses a right-open and left-closed subinterval of $[0, 1[$ of length a . A *reflection of type a* is the image of a reflection map of type a in IET^{\boxtimes} . A Γ -reflection is a reflection in $\text{IET}^{\boxtimes}(\Gamma)$. We know that the group $\text{IET}^{\boxtimes}(\Gamma)$ is generated by the set of reflections (see Proposition 2.9) so its abelianization is a 2-group. We are no longer able to measure the set of inversions but we can measure the union of this set with its symmetric. By projecting on $\otimes_{\mathbb{Z}}^2 \tilde{\Gamma} / 2(\otimes_{\mathbb{Z}}^2 \tilde{\Gamma})$ we obtain a group homomorphism $\varepsilon_\Gamma^{\boxtimes}$ whose image is $\langle \{a \otimes a \pmod{2} \mid a \in \tilde{\Gamma}\} \rangle$. We also prove that its kernel is generated by all reflections of type inside $\tilde{\Gamma} \setminus (2\tilde{\Gamma})$. Thus for $a \in \tilde{\Gamma} \setminus 2\tilde{\Gamma}$ every restricted rotation of type (a, a) is in the kernel and by the work on IET(Γ) we do not expect it to be in $D(\text{IET}^{\boxtimes}(\Gamma))$. We notice that such an element is conjugate to a reflections of type $2a$. In fact we prove :

Proposition 1.6. *Let Ω_Γ be the conjugate closure of the group generated by the set of all Γ -reflections of type 2ℓ with $\ell \in \tilde{\Gamma} \setminus 2\tilde{\Gamma}$. Then :*

$$\text{Ker}(\varepsilon_\Gamma^{\boxtimes}) = D(\text{IET}^{\boxtimes}(\Gamma))\Omega_\Gamma$$

Then we construct a group homomorphism $\psi_\Gamma : \text{IET}^{\boxtimes}(\Gamma) \rightarrow {}^\circ\Lambda_{\mathbb{Z}}^2 \tilde{\Gamma} / 2({}^\circ\Lambda_{\mathbb{Z}}^2 \tilde{\Gamma})$ which uses ε_Γ , called the positive contribution of $\text{IET}^{\boxtimes}(\Gamma)$. Morally we approximate every element $f \in \text{IET}^{\boxtimes}(\Gamma)$ by elements in IET(Γ) and we use the group homomorphism ε_Γ on them. This can also be seen as approximate triangles by rectangles in the set of inversions. The construction is strongly dependent on

a totally ordered set given by the fact that $\tilde{\Gamma}$ is ultrasimplicially ordered (see Section 2.4). For every $a \in \tilde{\Gamma} \setminus 2\tilde{\Gamma}$, this group homomorphism sends restricted rotation of type (a, a) on a nontrivial element. We also prove that the image of its restriction to $\text{Ker}(\varepsilon_{\tilde{\Gamma}}^{\boxtimes})$ is $\langle \{\ell \wedge \ell \pmod{2} \mid \ell \in \tilde{\Gamma}\} \rangle$.

In order to obtain the following theorem we will need to notice that groups of exponent 2 are also \mathbb{F}_2 -vector spaces.

Theorem 1.7. *For every subgroup Γ of \mathbb{R}/\mathbb{Z} we have the following isomorphism:*

$$\text{IET}^{\boxtimes}(\Gamma)_{\text{ab}} \simeq \langle \{a \otimes a \pmod{2} \mid a \in \tilde{\Gamma}\} \rangle \times \langle \{\ell \wedge \ell \pmod{2} \mid \ell \in \tilde{\Gamma}\} \rangle,$$

where the left term of the product is in $\otimes_{\mathbb{Z}}^2 \tilde{\Gamma} / (2 \otimes_{\mathbb{Z}}^2 \tilde{\Gamma})$ and the right one is in $\odot \Lambda_{\mathbb{Z}}^2 \tilde{\Gamma} / (2 \odot \Lambda_{\mathbb{Z}}^2 \tilde{\Gamma})$.

The inclusion of $\text{IET}(\Gamma)$ in $\text{IET}^{\boxtimes}(\Gamma)$ induces a natural group homomorphism ι from $\text{IET}(\Gamma)_{\text{ab}} / (2 \text{IET}(\Gamma)_{\text{ab}})$ to $\text{IET}^{\boxtimes}(\Gamma)_{\text{ab}}$. We show that ι is injective for every dense subgroup Γ of \mathbb{R}/\mathbb{Z} . The image of ι is $\langle \{p \otimes q + q \otimes p \pmod{2} \mid p, q \in \tilde{\Gamma}\} \rangle \times \langle \{\ell \wedge \ell \pmod{2} \mid \ell \in \tilde{\Gamma}\} \rangle$; it is also isomorphic to $\odot \Lambda_{\mathbb{Z}}^2 \tilde{\Gamma} / (2 \odot \Lambda_{\mathbb{Z}}^2 \tilde{\Gamma})$. We deduce that ι is surjective if and only if $\Gamma = 2\Gamma$. In the case where Γ is finitely generated we can precise the dimension of $\text{IET}^{\boxtimes}(\Gamma)_{\text{ab}}$ and $\text{Im}(\iota)$ as \mathbb{F}_2 -vector spaces. The group $\tilde{\Gamma}$ is also finitely generated and we denote by d its rank. Then $\text{IET}^{\boxtimes}(\Gamma)_{\text{ab}}$ has dimension $\frac{d(d+3)}{2}$ and $\text{Im}(\iota)$ has dimension $\frac{d(d+1)}{2}$.

Acknowledgments. I would like to thank Friedrich Wehrung for communicating me references about ultrasimplicial groups and in particular the work of G.A. Elliott in [5]. I thank also Laurent Bartholdi, Mikael de la Salle and Nicolás Matte Bon for pertinent comments. I wish to express my gratitude to my advisor Yves Cornuier for his careful rereading and support.

2. PRELIMINARIES

The aim of this section is to introduce some notation and gather known results used in the next sections. Let Γ be a subgroup of \mathbb{R}/\mathbb{Z} and let $\tilde{\Gamma}$ be its preimage in \mathbb{R} .

We recall that \mathfrak{S}_n is the permutation group of n elements. For every group G we denote by G_{ab} the abelianization of G , which is the quotient of G by its derived subgroup.

For every function f we define the *support of f* by $\text{Supp}(f) := \{x \in [0, 1[: f(x) \neq x\}$.

For any real interval I let I° be its interior in \mathbb{R} . If I is equal to $[0, t[$ we agree that its interior I° is equal to $]0, t[$. We denote by $\text{Itv}(\Gamma)$ the set of all intervals $[a, b[$ with $a, b \in \tilde{\Gamma}$ and $0 \leq a < b \leq 1$. We denote by λ the Lebesgue measure on \mathbb{R} .

2.1. When Γ is finite.

We assume that Γ is a finite subgroup of \mathbb{R}/\mathbb{Z} . Then there exists $n \in \mathbb{N}_{\geq 1}$ such that $\tilde{\Gamma}$ is equal to $\frac{1}{n}\mathbb{Z}$. We deduce that for every $1 \leq i \leq n$, every element of $\widehat{\text{IET}^{\boxtimes}(\Gamma)}$ is continuous on the interval $[\frac{i-1}{n}, \frac{i}{n}[$ up to a finitely supported permutation.

Then the group $\text{IET}(\Gamma)$ is naturally isomorphic to the finite permutation group \mathfrak{S}_n . It is a Coxeter group of type A_{n-1} so its abelianization is $\{1\}$ if $n = 1$ and it is $\mathbb{Z}/2\mathbb{Z}$ if $n > 1$.

The group $\text{IET}^{\boxtimes}(\Gamma)$ is isomorphic to the signed symmetric group $\mathbb{Z}/2\mathbb{Z} \wr \mathfrak{S}_n$. It is a Coxeter group of type B_n so its abelianization is $\mathbb{Z}/2\mathbb{Z}$ for $n = 1$ and it is $(\mathbb{Z}/2\mathbb{Z})^2$ if $n > 1$.

2.2. Descriptions of an element of IET.

In order to describe an element f of IET we use partitions into right-open and left-closed intervals of $[0, 1[$. Let \mathcal{P} be such a partition.

Definition 2.1. If f is continuous on the interior of every interval of \mathcal{P} then \mathcal{P} is called a *partition into intervals associated with f* . The set of all partitions into intervals associated with f is denoted by Π_f . If every interval I of \mathcal{P} is in $\text{Itv}(\Gamma)$ then \mathcal{P} is said to be a Γ -*partition*. We denote by $f(\mathcal{P})$ the partition into intervals of $[0, 1[$ composed of all right-open and left-closed intervals whose interior is the image by f of the interior of an interval in \mathcal{P} . It is called *the arrival partition of f associated with \mathcal{P}* .

Remark 2.2. For every $f \in \text{IET}$, the set Π_f has a minimal element for the refinement. This element is also the unique partition that has a minimal number of interval.

One can notice that for every $f \in \text{IET}$ there is an equivalence between f belongs to $\text{IET}(\Gamma)$ and the existence of a Γ -partition inside Π_f . From now on every partition is assumed to be finite.

There are descriptions of an element of IET that are more combinatorial. They are useful to understand the SAF-homomorphism and further ε_Γ :

Definition 2.3. Let f be an element of IET and let $\mathcal{P} \in \Pi_f$; let n be the number of intervals of \mathcal{P} . For $1 \leq i \leq n$ let I_i (resp. J_i) be the consecutive intervals composing \mathcal{P} (resp. $f(\mathcal{P})$). Let α be the n -tuple such that α_i is the length of I_i for every $1 \leq i \leq n$. There exists $\sigma \in \mathfrak{S}_n$ such that $f(I_i^\circ) = J_{\sigma(i)}^\circ$. Then we say that (α, τ) is a *combinatorial description of f* . If each component of α is in $\tilde{\Gamma}$ we say that (α, τ) is a Γ -*combinatorial description of f* . The partition \mathcal{P} can also be called the *partition associated with (α, τ)* .

Proposition 2.4. *Let $f \in \text{IET}$ and let (α, τ) be a combinatorial description of f ; let n be the length of α . Then we have:*

$$\varphi(f) = \sum_{j=1}^n \left(\sum_{\substack{i: \\ \tau(i) < \tau(j)}} \alpha_i - \sum_{i < j} \alpha_i \right) \wedge \alpha_j$$

Proof. Let $f \in \text{IET}$ and (α, τ) be a combinatorial description of f . Let n be the length of α and let $\{I_1, I_2, \dots, I_n\}$ be the partition associated with (α, τ) . For each j we denote by $v(j)$ the value of $f - \text{Id}$ on I_j . Thus we deduce that $\varphi(f) = \sum_{j=1}^n v(j) \wedge \lambda(I_j) = \sum_{j=1}^n v(j) \wedge \alpha_j$. Also we know that $v(j) = \sum_{\substack{i \\ \tau(i) < \tau(j)}} \alpha_i - \sum_{i < j} \alpha_i$

(see Figure 1 below), and this gives the conclusion. □

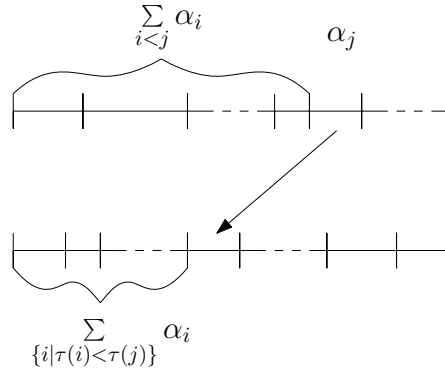


FIGURE 1. Illustration for the value of the SAF-homomorphism in Proposition 2.4

2.3. Three important families.

There are two families of elements of IET appearing in this work and one more for IET^{\boxtimes} .

Definition 2.5. Let $a \in [0, \frac{1}{2}[$ and f be in IET. We call f an *IET-transposition of type a* if it swaps two subintervals of $[0, 1[$ of length a with non-overlapping support while fixing the rest of $[0, 1[$. If in addition f belongs to $\text{IET}(\Gamma)$ then f is called an *IET(Γ)-transposition of type a* .

Also it is natural to consider the identity as an IET(Γ)-transposition. We precise it whenever it is needed.

One can notice that, for f an IET-transposition of type a , asking f to be in $\text{IET}(\Gamma)$ implies that a is in $\tilde{\Gamma} \cap [0, 1[$ but the converse is false.

These elements are called *interval swap maps* by Y. Vorobets in [11]. They are elements of order 2 and generate $D(\text{IET})$. More precisely, we describe below (Figure 2) how such an element can be seen in $D(\text{IET})$. For f an IET(Γ)-transposition of type a , this suggests to ask if a belongs to $2\tilde{\Gamma}$ or not, before concluding that f is in $D(\text{IET}(\Gamma))$ or not.

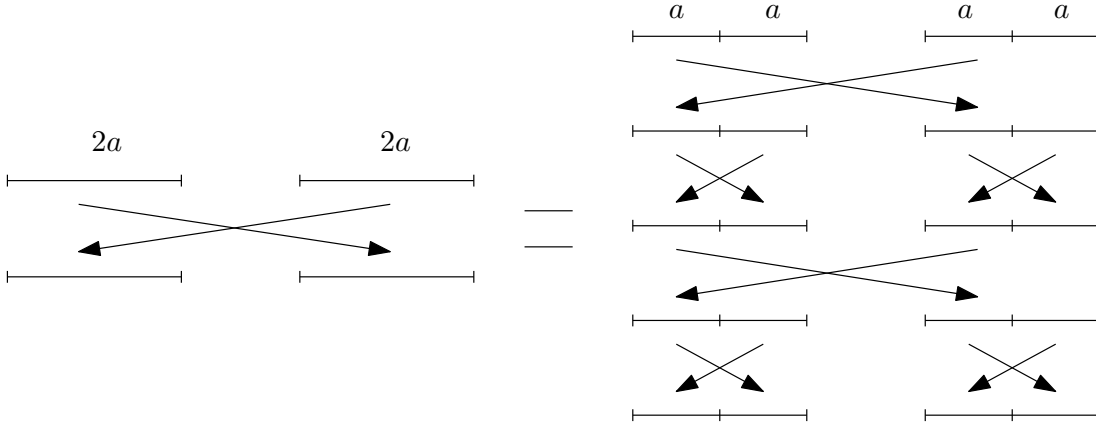


FIGURE 2. Illustration of the property that an IET-transposition of type $2a$ is a commutator of two IET-transpositions of type a .

A *restricted rotation of type (a, b)* is an element r of IET such that there exist consecutive intervals I and J of length a and b respectively with $\sup(I) = \inf(J)$, where r is the translation by b on I and the translation by $-a$ on J (and the identity outside $I \cup J$). The two intervals I and J are called the intervals associated with r .

Definition 2.6. Let $a, b \in [0, 1[$ with $0 \leq a + b \leq 1$ and f be in IET. We call f an *IET-restricted rotation of type (a, b)* if there exist consecutive intervals I and J of length a and b respectively with $\sup(I) = \inf(J)$, where f is the translation by b on I and the translation by $-a$ on J (and the identity outside $I \cup J$). The two intervals I and J are called the intervals associated with f . If in addition f belongs to $\text{IET}(\Gamma)$ then f is called an Γ -*restricted rotation of type (a, b)* .

Again one can notice that for f an IET-restricted rotation of type (a, b) , if f is in $\text{IET}(\Gamma)$ this implies that a and b are in $\tilde{\Gamma} \cap [0, 1[$, but the converse is false.

The case $a = b$ coincide with an IET-transposition of type a .

The family of restricted rotations generates IET and we have more precision on the type of restricted rotations in the decomposition, by the following fact due to Vorobets in [11]:

Lemma 2.7 (Vorobets). *Any transformation f in IET is a product of IET-restricted rotations. Furthermore for every $\mathcal{P} \in \Pi_f$ there exists a decomposition of f into IET-restricted rotations with type inside the set of length of the intervals in \mathcal{P} .*

In particular if we start with an element of $\text{IET}(\Gamma)$ we can find a decomposition into Γ -restricted rotations that satisfies the condition about the type of restricted rotations.

The group IET can be seen as a subgroup of IET^{\boxtimes} ; thus the notion of restricted rotation still makes sense in IET^{\boxtimes} . We now add a family that occurs only in IET^{\boxtimes} .

Definition 2.8. For every right-open and left-closed subinterval I of $[0, 1[$ we define the I -reflection map as the element of $\widehat{\text{IET}}^\boxtimes$ that reverses I° while fixing the rest of $[0, 1[$. The type of an I -reflection map is the length of I . We define the I -reflection as the image of the I -reflection map in IET^\boxtimes . The type of an I -reflection is the length of I . A Γ -reflection is an I -reflection for some $I \in \text{Itv}(\Gamma)$.

The set of all reflections is a generating subset of IET^\boxtimes . This follows from Proposition 6.10 and the fact that the restricted rotation, whose intervals associated are I and J , is the product of the I -reflection, the J -reflection and the $(I \sqcup J)$ -reflection. More precisely, with this proposition and Lemma 2.7 we obtain the following:

Proposition 2.9. *The set of all Γ -reflections is a generating subset of $\text{IET}^\boxtimes(\Gamma)$.*

We make explicit some elements of the conjugacy class of a reflection.

Proposition 2.10. *Two reflections r, s which have the same type are conjugate in $\text{IET}^\boxtimes(\Gamma)$.*

Proposition 2.11. *Let $a \in [0, 1[$. A reflection of type $2a$ is conjugate to a restricted rotation of type (a, a) .*

Proof. Let r be such a reflection and let I be the interval reversed by r . By cutting I into two intervals of same size I_1 and I_2 we obtain that r is conjugate to the restricted rotation that permutes I_1 and I_2 , by the I_1 -reflection. \square

2.4. Ultrasimplicial groups.

Let us introduce some classical terminology from the theory of ordered abelian groups. An ordered abelian group is an abelian group endowed with an invariant partial ordering.

For any subgroup H of \mathbb{R} we denote by $H_+ := \{x \in H \mid x \geq 0\}$ the positive cone of H . A difficulty is that H_+ is not, in general, finitely generated as a subsemigroup. For every subset B of \mathbb{R} we denote by $\text{Vect}_{\mathbb{N}}(B)$ the subsemigroup generated by B .

A subsemigroup of an abelian group is *simplicial* if it is generated, as a subsemigroup, by a finite \mathbb{Z} -independent subset, and it is *ultrasimplicial* if it is the filtered union of simplicial subsemigroups. An ordered abelian group is *simplicially ordered* if its positive cone is simplicial and is *ultrasimplicially ordered* if its positive cone is ultrasimplicial. The next theorem is proved by G.A. Elliott in [5]:

Theorem 2.12. *Every totally ordered abelian group is ultrasimplicially ordered.*

We deduce the following corollary for the totally ordered abelian group $\tilde{\Gamma}$:

Corollary 2.13. *There exists a sequence $(S_n)_n$ of finite \mathbb{Z} -independent subset of $\tilde{\Gamma}$ such that for each n we have $\text{Vect}_{\mathbb{N}}(S_n) \subset \text{Vect}_{\mathbb{N}}(S_{n+1})$ and $\tilde{\Gamma}_+$ is equal to the increasing union $\bigcup_n \text{Vect}_{\mathbb{N}}(S_n)$.*

We will use the Theorem 2.12 in the form of the following corollary which specifies the finite rank case:

Corollary 2.14. *Let H be a subgroup of \mathbb{R} which is free abelian of finite rank d . Then there exists a sequence $(B_n)_{n \in \mathbb{N}}$ of \mathbb{Z} -basis of H such that for each n we have $\text{Vect}_{\mathbb{N}}(B_n) \subset \text{Vect}_{\mathbb{N}}(B_{n+1})$ and H_+ is equal to the increasing union $\bigcup_n \text{Vect}_{\mathbb{N}}(B_n)$. Furthermore for every $k \in \mathbb{N}$ and $L_1, L_2, \dots, L_k \in H_+$ there exists a basis $\{\ell_1, \ell_2, \dots, \ell_d\} \subset H_+$ of H such that for every $1 \leq i \leq k$ the element L_i is a linear combination of $\ell_1, \ell_2, \dots, \ell_d$ with coefficients in \mathbb{N} .*

2.5. About the derived subgroup.

We prove here that the derived subgroups $D(\text{IET}(\Gamma))$ and $D(\text{IET}^{\boxtimes}(\Gamma))$ are simple. There are two steps in the proof. The first one is to give a sufficient condition for a group G to obtain that its derived subgroup $D(G)$ is contained in every nontrivial normal subgroup of G ; this is used by Arnoux in [2] to show that IET^{\boxtimes} is simple.

The second step is to remark that $\text{IET}(\Gamma)$ and $\text{IET}^{\boxtimes}(\Gamma)$ are subgroups of the group of homeomorphism of a Stone space Y ; that is a compact totally disconnected space. We show that they satisfy some conditions that allows us to use the work of Nekrashevych [7] which gives us a normal subgroup of $\text{IET}(\Gamma)$ and one of $\text{IET}^{\boxtimes}(\Gamma)$ which are simple and contained in all normal subgroups.

Definition 2.15. For every $\varepsilon > 0$ we denote by \mathcal{F}_ε the union of the set of all Γ -restricted rotations of type (a, b) with $a + b \leq \varepsilon$, with the set of all $\text{IET}(\Gamma)$ -transpositions of type a with $2a \leq \varepsilon$. We denote by $\mathcal{F}_\varepsilon^{\boxtimes}$ the union of the set \mathcal{F}_ε with the set of all Γ -reflections of type a with $a \leq \varepsilon$.

Lemma 2.16. *For every $\varepsilon > 0$, any element f in $\text{IET}(\Gamma)$ (resp. $\text{IET}^{\boxtimes}(\Gamma)$) can be written as a finite product of element in \mathcal{F}_ε (resp. $\mathcal{F}_\varepsilon^{\boxtimes}$).*

Proof. Let $\varepsilon > 0$ and by density of $\tilde{\Gamma}$ in \mathbb{R} , let $w \in \tilde{\Gamma}_+$ such that $w < \varepsilon$.

If f is an $\text{IET}(\Gamma)$ -transposition of type a let I and J be the two non-overlapping intervals swapped by f . There exist $k \in \mathbb{N}$ and $u \in \tilde{\Gamma}_+$ such that $u < w$ and $a = kw + u$. Then I and J can be cut into $k+1$ consecutive intervals I_1, I_2, \dots, I_{k+1} and J_1, J_2, \dots, J_{k+1} respectively; the first k intervals with length w and the last one with length u . Then f is equal to the product $s_1 s_2 \dots s_{k+1}$ where s_i is the $\text{IET}(\Gamma)$ -transposition that swaps I_i with J_i .

If f is an I -reflection of type a with $a = kw + u$ as in the previous case. Then we consider $\{I_1, I_2, \dots, I_{k+1}\}$ and $\{J_1, J_2, \dots, J_{k+1}\}$ two partitions into consecutive intervals of I such that the length of I_i and J_j is w for every $1 \leq i \leq k$ and $2 \leq j \leq k+1$ and the length of I_{k+1} and J_1 is u . Then f is equal to the product $r_1 r_2 \dots r_{k+1} s_1 s_2 \dots s_{k+1}$ where r_i is the I_i -reflection and s_i is the $\text{IET}(\Gamma)$ -transposition that swaps I_i with J_{k+1-i} .

If f is a Γ -restricted rotation of type (a, b) . We can assume $a \geq b$ as the other case is similar. Then f is the product of an $\text{IET}(\Gamma)$ -transposition of type b with

a restricted rotation of type $(a - b, b)$ (see Figure 3). Then by iterating this operation and thanks to the first case we deduce the result.

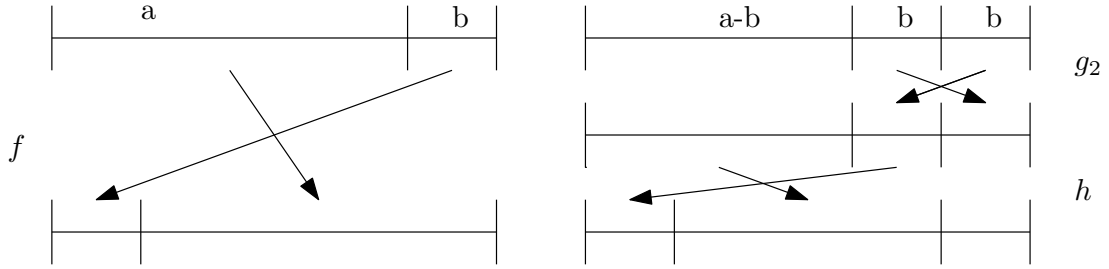


FIGURE 3. Illustration of the decomposition of a restricted rotation in Lemma 2.16.

By Lemma 2.7 and Proposition 2.9 we obtain the result for any f . □

Proposition 2.17. *The derived subgroup $D(\text{IET}(\Gamma))$ (resp. $D(\text{IET}^{\boxtimes}(\Gamma))$) is contained in every nontrivial normal subgroup of $\text{IET}(\Gamma)$ (resp. $\text{IET}^{\boxtimes}(\Gamma)$).*

Proof. We only do the case of $\text{IET}(\Gamma)$. In $\text{IET}^{\boxtimes}(\Gamma)$ one should be careful about having equalities up to finitely supported permutations, but the proof is similar. For every $\varepsilon > 0$, we recall that \mathcal{F}_ε is a generating subset of $\text{IET}(\Gamma)$. Then the set of all commutators of two elements in \mathcal{F}_ε is a generating subset of $D(\text{IET}(\Gamma))$.

Let N be a normal subgroup of $\text{IET}(\Gamma)$ and $f \in N \setminus \{\text{Id}\}$. Then there exists a subinterval I of $[0, 1[$ such that f is continuous on I and $f(I) \cap I = \emptyset$. Let δ be the length of I and let $\varepsilon = \frac{\delta}{3}$. Let $g_1, g_2 \in \mathcal{F}_\varepsilon$ and let $A := \text{Supp}(g_1) \cup \text{Supp}(g_2)$. There exists $h \in \text{IET}(\Gamma)$ such that $h(A) \subset I$ because the measure of A is at most 2ε which is less than δ . Then the element $f' := h^{-1} \circ f \circ h$ is an element of N and $f'(A) \cap A = \emptyset$. This implies that the support of g_1 and the one of $f'g_2f'^{-1}$ do not overlap; thus these elements commute and we obtain that $[g_1, g_2] = [g_1, [g_2, f']]$. Also N is normal in $\text{IET}(\Gamma)$, hence as $f' \in N$ we deduce that $[g_2, f'] \in N$, thus $[g_1, g_2] \in N$. We deduce that every commutator of elements in \mathcal{F}_ε is in N so $D(\text{IET}(\Gamma))$ is a subgroup of N . □

Definition 2.18. Let X be a Stone space and G be a subgroup of $\text{Homeo}(X)$. The group G is a topological full group if for every $n \in \mathbb{N}$ and for all (Y_1, Y_2, \dots, Y_n) and (Z_1, Z_2, \dots, Z_n) finite partitions into clopen subsets of X and every $g_i \in G$ such that $g_i(Y_i) = Z_i$, the element g that satisfies $g = g_i$ on X_i is an element of G .

One can notice that $\text{IET}(\Gamma)$ and $\text{IET}^{\boxtimes}(\Gamma)$ act minimally on the space Y obtained from \mathbb{R}/\mathbb{Z} by replacing every point $x \in \Gamma$ by two points x_- and x_+ . The space Y is a Stone space without isolated points. The fact that $\text{IET}(\Gamma)$ and $\text{IET}^{\boxtimes}(\Gamma)$ are both topological full group is immediate. We can use the following theorem due to V. Nekrashevych in [7]:

Theorem 2.19 (Nekrashevych). *Let X be a infinite Stone space and G be a subgroup of $\text{Homeo}(X)$ such that G is a topological full group and acts minimally on X . Then the subgroup $\mathcal{A}(G)$ of G generated by the set of all elements of order 3, is simple and contained in every nontrivial normal subgroup of G .*

Thanks to this theorem and Proposition 2.17 we deduce that $\mathcal{A}(\text{IET}(\Gamma)) = D(\text{IET}(\Gamma))$ and $\mathcal{A}(\text{IET}^{\boxtimes}(\Gamma)) = D(\text{IET}^{\boxtimes}(\Gamma))$ and both of them are simple.

3. BEHAVIOUR OF THE COMPOSITION OF TWO IET-TRANSPOSITIONS

Understanding compositions of $\text{IET}(\Gamma)$ -transpositions is important to describe the abelianization of $\text{IET}(\Gamma)$. The next lemma relates finite order elements with IET -transpositions (see Definition 2.5). In [11], Y. Vorobets proves it in the case $\Gamma = \mathbb{R}/\mathbb{Z}$. The general case follows from this case:

Lemma 3.1. *Let f be a finite order element of $\text{IET}(\Gamma)$. Let $n \in \mathbb{N}$ and $\mathcal{P} := \{I_1, \dots, I_n\} \in \Pi_f$. Then there exists $\sigma \in \mathfrak{S}_n$ such that $f(I_i) = I_{\sigma(i)}$ thus f is a product of $\text{IET}(\Gamma)$ -transpositions.*

The aim is to show that the product of two IET -transpositions has finite order. It is not a trivial question because we can build such examples of product of order n for every $n \in \mathbb{N}$ (see Proposition 3.5). The result will be used in Section 5.

Lemma 3.2. *Let f and g be two IET -transpositions. Let I and J be the non-overlapping intervals swapped by f with $\text{sup}(I) = \text{inf}(J)$. Let A and B be the non-overlapping intervals swapped by g with $\text{sup}(A) = \text{inf}(B)$. Let α and β be the elements of $[0, \frac{1}{2}[$ such that $J = I + \alpha$ and $B = A + \beta$. If $\alpha > \beta$ then for all x in $[0, 1[$ there exists n in $\mathbb{N}_{\geq 1}$ such that $(gf)^n(x) \notin \text{Supp}(f) \cap \text{Supp}(g)$.*

Proof. We have the conclusion for all x outside $\text{Supp}(f) \cup \text{Supp}(g)$. By contradiction we assume that there exists x in $\text{Supp}(f) \cup \text{Supp}(g)$ such that for every n in $\mathbb{N}_{\geq 1}$, the element $(gf)^n(x)$ belongs to $\text{Supp}(f) \cap \text{Supp}(g)$. Up to consider $(gf)(x)$ instead of x we can assume that x is in $\text{Supp}(f) \cap \text{Supp}(g)$. We distinguish three cases:

- (1) If $I \cap B = \emptyset$ and $J \cap A = \emptyset$. Then $\text{Supp}(f) \cap \text{Supp}(g) = (I \cap A) \sqcup (J \cap B)$. Moreover we check that for every $n \in \mathbb{N}_{\geq 1}$ if $y = (gf)^n(x)$ is in $I \cap A$ then $f(y)$ is in $J \cap B$. Indeed $f(y)$ is clearly in J thus $f(y) \notin A$. If it is not in B then $f(y)$ is fixed by g . Thus $gf(y) = f(y)$ is neither in A nor in B which is in contradiction with the assumption on x . Similarly we get $gf(y)$ in $I \cap A$. Thus in one iteration we oscillate between the two intervals. The same is true if y is in $J \cap B$ at first. Hence for $n > \frac{1}{\alpha - \beta}$ if x is in $I \cap A$ then $(gf)^n(x) = x + n \times (\alpha - \beta) > x + 1 \geq 1$ which is a contradiction. Also if x is in $J \cap B$ then $(gf)^n(x) = x - n \times (\alpha - \beta) < x - 1 < 0$ which is a contradiction.

- (2) Now let us assume that $I \cap B \neq \emptyset$ (this gives the strict inequality $\sup(I) > \inf(B)$), then $J \cap A = \emptyset$ because $\inf(J) = \sup(I) > \inf(B) = \sup(A)$. We have two cases:
- (a) If the positive (gf) -orbit of x does not intersect $I \cap B$. Let f' be the IET(Γ)-transposition which swaps the interval $I \setminus B$ with $f(I \setminus B)$. Then for every n in $\mathbb{N}_{\geq 1}$ we have $(gf')^n(x)$ in $\text{Supp}(f') \cap \text{Supp}(g)$. Moreover $(I \setminus B) \cap B = \emptyset$ and $f(I \setminus B) \cap J = \emptyset$. Hence we return in the first case which lead to a contradiction.
 - (b) If the positive (gf) -orbit of x intersect $I \cap B$. Let $k \in \mathbb{Z}$ such that $y := (gf)^k(x)$ belongs to $I \cap B$. Then for every n in $\mathbb{N}_{\geq 1}$ we always have $(gf)^n(y)$ in $\text{Supp}(f) \cap \text{Supp}(g)$. Hence as y is in $I \cap B$ we get $f(y)$ in $J \cap B$. Indeed it is clearly in J and it is in B otherwise $f(y)$ will not be in $A \cup B$ which is equal to $\text{Supp}(g)$ and lead to a contradiction. So $gf(y)$ is in A and $ff(y) = y$ is in B . We deduce that $gf(y) = y + \alpha - \beta \leq \sup(A) = \inf(B) \leq ff(y) = y$ which is a contradiction because $\alpha > \beta$.
- (3) The last case is where $J \cap A \neq \emptyset$ but it is a similar argument as the previous one and it also lead to a contradiction.

In conclusion the lemma holds for every $x \in \text{Supp}(f) \cup \text{Supp}(g)$. \square

Remark 3.3. Let x be an element of $[0, 1[$. If there exists n in $\mathbb{N}_{\geq 1}$ such that $(gf)^n(x)$ is not in $\text{Supp}(f)$ then $gf(gf)^n(x) = g(gf)^n(x) = f(gf)^{n-1}(x)$ so $(gf)^n(gf)^n(x) = f(x)$. If there exists n in $\mathbb{N}_{\geq 1}$ such that $(gf)^n(x)$ is not in $\text{Supp}(g)$ then $(gf)^n(x) = g(gf)^n(x) = f(gf)^{n-1}(x)$ so $(gf)^{n-1}(gf)^n(x) = f(x)$. We deduce that if there exist n and k in $\mathbb{N}_{\geq 1}$ such that $(gf)^n(x)$ and $(gf)^k(f(x))$ are not in $\text{Supp}(f) \cup \text{Supp}(g)$ then x has finite order at most $2n + 2k$.

Lemma 3.4. *Let f and g be two IET-transpositions. Then gf has finite order.*

Proof. As gf is in IET we know that if gf has periodic points, the set of periods is finite. Hence it is sufficient to prove that each point of $[0, 1[$ has a periodic (gf) -orbit to conclude. For every x which is neither in $\text{Supp}(f)$ nor $\text{Supp}(g)$ then $gf(x) = x$. Hence we only need to check for x is in $\text{Supp}(f) \cup \text{Supp}(g)$.

Let I and J be the non-overlapping intervals swapped by f with $\sup(I) = \inf(J)$ and let A and B be the non-overlapping intervals swapped by g with $\sup(A) = \inf(B)$. Let α and β be the elements of $[0, \frac{1}{2}[$ such that $J = I + \alpha$ and $B = A + \beta$.

If $\alpha = \beta$ then for every x in $\text{Supp}(f) \cap \text{Supp}(g)$ then $gf(x) = x$. Moreover for x in $\text{Supp}(f) \setminus \text{Supp}(g)$ we have $f(x)$ not in $\text{Supp}(g)$ thus $gf(x) = f(x)$ and $gfgf(x) = gf(fx) = g(x) = x$. The same stands for x in $\text{Supp}(g) \setminus \text{Supp}(f)$. In conclusion gf has finite order at most 2.

If $\alpha \neq \beta$, we can assume that $\alpha > \beta$ up to change the role of f and g and because fg is conjugate to gf . Let x in $\text{Supp}(f) \cup \text{Supp}(g)$. Then by Lemma 3.2 there exist n and k in $\mathbb{N}_{\geq 1}$ such that $(gf)^n(x)$ is not in $\text{Supp}(f) \cap \text{Supp}(g)$ and

$(gf)^k(f(x))$ is not in $\text{Supp}(f) \cap \text{Supp}(g)$. Thanks to Remark 3.3 we know that x has finite order at most $2n + 2k$. \square

Proposition 3.5. *For every n in $\mathbb{N}_{\geq 1}$ there exist two IET-transpositions f and g such that the product gf has order n .*

Proof. We distinguish the case where n is even or odd. In both cases we illustrate the proof with Figure 4. The case $n = 1$ is given by the equality $f^2 = \text{Id}$ for any IET-transposition f . Let $n \in \mathbb{N}_{\geq 1}$.

Let I and J be two consecutive intervals of the same length $\ell \in [0, \frac{1}{2}]$ and let g be the IET-transposition that swaps I and J . Let A_1, A_2, \dots, A_{n-1} and C be consecutive intervals of length $\frac{\ell}{n}$ such that the left endpoint of A_1 is the left endpoint of I (hence the right endpoint of C is the right endpoint of I). Let D and B_1, B_2, \dots, B_{n-1} be consecutive intervals of length $\frac{\ell}{n}$ such that the right endpoint of B_{n-1} is the right endpoint of J (hence the left endpoint of D is the left endpoint of J). Let f be the IET-transposition that swaps A_i and B_i for every $1 \leq i \leq n-1$. Hence by definition we get $g(A_1) = D$, $g(A_i) = B_{i-1}$ for every $2 \leq i \leq n-1$ and $g(C) = B_{n-1}$. So the composition gf is equal to the permutation $(A_1 A_2 \dots A_{n-1} C B_{n-1} B_{n-2} \dots B_1 D)$. Thus gf has order $2n$.

It remains the case of order $2n-1$. Let I, J and K be three consecutive intervals with I and J of length $\ell \in [0, \frac{1}{3}[$ and K of length $\ell' \in]\frac{\ell}{n}, \ell[$. Let g be the IET-transposition that swaps I and J . We define A_1, A_2, \dots, A_{n-1} consecutive intervals of length $\frac{\ell}{n}$ such that the right endpoint of A_{n-1} is the right endpoint of I . We define also D and B_1, B_2, \dots, B_{n-1} consecutive intervals of length $\frac{\ell}{n}$ such that the left endpoint of B_{n-1} is the left endpoint of K . Let f be the IET-transposition that swaps A_i and B_i for every $1 \leq i \leq n-1$. One can check that the product gf in this case is the permutation $(A_1 A_2 \dots A_{n-1} B_{n-1} B_{n-2} \dots B_1 D)$ so gf has order $2n-1$. \square

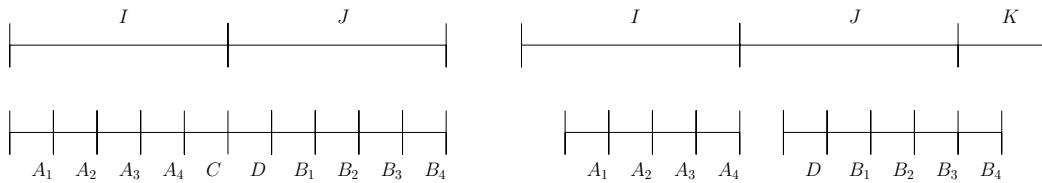


FIGURE 4. Illustration of Proposition 3.5 with $n = 5$. **Left:** For the case “ fg has order $2n$ ”. **Right:** For the case “ fg has order $2n-1$ ”.

4. A GENERATING SUBSET FOR $\text{Ker}(\varphi_\Gamma)$

Let Γ be a dense subgroup of \mathbb{R}/\mathbb{Z} . We follow the idea of Y. Vorobets in [11] and introduce the notion of balanced product of restricted rotations. The aim is to show Lemma 1.4. We recall that φ_Γ is the restriction of the SAF-homomorphism to $\text{IET}(\Gamma)$; and that $\tilde{\Gamma}_+$ is the positive cone of $\tilde{\Gamma}$.

4.1. Balanced product of Γ -restricted rotations.

Definition 4.1. Let $n \in \mathbb{N}$ and let r_i be a restricted rotation for every $1 \leq i \leq n$. For every $a, b \in \tilde{\Gamma}_+$ let $n_{a,b}$ be the number of restricted rotation r_i of type (a, b) . The tuple (r_1, r_2, \dots, r_n) is said to be a *balanced tuple of restricted rotations* if $n_{a,b} = n_{b,a}$ for every $a, b \in \tilde{\Gamma}_+$. We say that a product of restricted rotations is a *balanced product of restricted rotations* if it can be written as a product of a balanced tuple of restricted rotations.

Example 4.2. Let a be an element in $\tilde{\Gamma}_+$ with $a \leq \frac{1}{2}$. Any Γ -restricted rotation of type (a, a) is a balanced product of restricted rotations. These elements are also $\text{IET}(\Gamma)$ -transposition which swapped two consecutive intervals of same length a .

Example 4.3. Let $a, b \in \tilde{\Gamma}_+$ with $a + b \leq 1$. If h is a Γ -restricted rotation of type (a, b) then h^{-1} is a Γ -restricted rotation of type (b, a) . Thus every element of $D(\text{IET}(\Gamma))$ is a balanced product of Γ -restricted rotations.

In order to get the decomposition of the elements of $\text{Ker}(\varphi_\Gamma)$, we need to know the freeness of some family of ${}^\circ\Lambda_{\mathbb{Z}}^2 V$.

Lemma 4.4. *Suppose V is a subgroup of \mathbb{R} (then it is a \mathbb{Z} -module). Let k in $\mathbb{N}_{\geq 1}$ and v_1, v_2, \dots, v_k elements of V which are \mathbb{Z} -linearly independent. Then the wedge products $v_i \wedge v_j$ for $1 \leq i < j \leq k$ are \mathbb{Z} -linearly independent in ${}^\circ\Lambda_{\mathbb{Z}}^2 V$.*

Proof. Let v_1, v_2, \dots, v_k in V which are \mathbb{Z} -linearly independent. It is sufficient to prove the lemma for $V = \mathbb{R}$ because being \mathbb{Z} -linearly independent in ${}^\circ\Lambda_{\mathbb{Z}}^2 \mathbb{R}$ implies being \mathbb{Z} -linearly independent in ${}^\circ\Lambda_{\mathbb{Z}}^2 V$. We know that ${}^\circ\Lambda_{\mathbb{Z}}^2 \mathbb{R}$ is isomorphic to $\Lambda_{\mathbb{Z}}^2 \mathbb{R}$. Let us assume that v_1, v_2, \dots, v_k are in \mathbb{R} . Then being \mathbb{Z} -linearly independent is the same that being \mathbb{Q} -linearly independent. Indeed if there exist p_1, p_2, \dots, p_k in \mathbb{Z} and q_1, q_2, \dots, q_k in $\mathbb{N}_{\geq 1}$ such that $\sum_{i=1}^k \frac{p_i}{q_i} v_i = 0$ then

$\sum_{i=1}^k \left(\prod_{\substack{j=1 \\ j \neq i}}^k q_j p_i \right) v_i = 0$ is an equality in \mathbb{Z} . Thus for each $i \in \{1, 2, \dots, k\}$ we have

$\prod_{\substack{j=1 \\ j \neq i}}^k q_j p_i = 0$. Or $q_j \neq 0$ for every j then $p_i = 0$ for every i .

The \mathbb{Q} -vector space generated by all the v_i for $1 \leq i \leq k$ is isomorphic to \mathbb{Q}^k . We can complete the \mathbb{Q} -linearly independent set $\{v_1, v_2, \dots, v_k\}$ in a basis S of \mathbb{R} seen as a \mathbb{Q} -vector space. Thus in $\Lambda_{\mathbb{Z}}^2 \mathbb{R}$ the elements $v_i \wedge v_j$ for $1 \leq i < j \leq k$ are \mathbb{Q} -linearly independent so they are \mathbb{Z} -linearly independent in ${}^\circ\Lambda_{\mathbb{Z}}^2 \mathbb{R}$ and this gives the conclusion. \square

Lemma 4.5. *Any transformation f in $\text{Ker}(\varphi_\Gamma)$ can be represented as a balanced product of Γ -restricted rotations.*

Proof. Let $f \in \text{Ker}(\varphi_\Gamma)$. This is trivial if $f = \text{id}$; assume otherwise. Let (μ, σ) be a Γ -combinatorial description of f , let $k \in \mathbb{N}$ and $\{I_1, I_2, \dots, I_k\}$ be the partition

into intervals associated to (μ, σ) (we have $k \geq 2$ as f is not the identity). We recall that μ_i is the length of I_i for every $1 \leq i \leq k$.

We treat the case where Γ is finitely generated. Then $\tilde{\Gamma}$ is also finitely generated and we denote by d its rank. By Corollary 2.14 there exist $\ell_1, \ell_2, \dots, \ell_d$ in $\tilde{\Gamma}_+$ such that $\mathcal{L} := \{\ell_1, \ell_2, \dots, \ell_d\}$ is a basis of $\tilde{\Gamma}$ and such that L_i is a linear combination of $\ell_1, \ell_2, \dots, \ell_d$ with non-negative integer coefficients for every $1 \leq i \leq k$. Then I_i can be partitioned into smaller intervals with length in \mathcal{L} for every $1 \leq i \leq k$. We obtain a partition associated with f whose intervals have length in \mathcal{L} . By Lemma 2.7 there exist $n \in \mathbb{N}$ and a restricted rotation f_i of type (a_i, b_i) with $a_i, b_i \in \mathcal{L}$ for $1 \leq i \leq n$ such that $f = f_1 f_2 \dots f_n$. For any $i, j \in \{1, 2, \dots, d\}$ let s_{ij} be the number of Γ -restricted rotation of type (ℓ_i, ℓ_j) in the tuple (f_1, f_2, \dots, f_n) . As $\varphi_\Gamma(f_i) = \ell_j \wedge \ell_i - \ell_i \wedge \ell_j = 2\ell_j \wedge \ell_i$, we obtain that:

$$\varphi_\Gamma(f) = \sum_{i=1}^d \sum_{j=1}^d 2s_{ij}(\ell_j \wedge \ell_i) = \sum_{1 \leq i < j \leq d} 2(s_{ij} - s_{ji})(\ell_j \wedge \ell_i)$$

We know that $\{\ell_1, \ell_2, \dots, \ell_d\}$ is a basis of $\tilde{\Gamma}$ thus by Lemma 4.4 we obtain that $\{\ell_j \wedge \ell_i\}_{1 \leq i < j \leq d}$ is a free family of ${}^\circ\Lambda_{\mathbb{Z}}^2 \tilde{\Gamma}$. Then the assumption $\varphi_\Gamma(f) = 0$ implies $s_{ij} = s_{ji}$ for every $1 \leq i < j \leq d$. This means that the product of Γ -restricted rotations $f_1 f_2 \dots f_n$ is balanced.

We do not assume Γ finitely generated any more. Hence we only know that $\varphi_\Gamma(f) = \sum_{j=1}^k \left(\sum_{\substack{i: \\ \sigma(i) < \sigma(j)}} \mu_i - \sum_{i < j} \mu_i \right) \wedge \mu_j = 0$ in ${}^\circ\Lambda_{\mathbb{Z}}^2 \tilde{\Gamma}$ (see Proposition 2.4). We

denote by $\overline{\varphi_\Gamma(f)} := \sum_{j=1}^k \left(\sum_{\substack{i: \\ \sigma(i) < \sigma(j)}} \mu_i - \sum_{i < j} \mu_i \right) \otimes \mu_j$. It is a representative of $\varphi_\Gamma(f)$

in $\otimes_{\mathbb{Z}}^2 \tilde{\Gamma}$. Then there exist a finite set J and $x_j, y_j \in \tilde{\Gamma}$ for every $j \in J$, such that $\overline{\varphi_\Gamma(f)} = \sum_{j \in J} x_j \otimes y_j + y_j \otimes x_j$. We denote by \tilde{A} the subgroup of $\tilde{\Gamma}$ generated

by $\{\mu_i\}_{1 \leq i \leq k} \cup \{x_j, y_j\}_{j \in J}$. Then \tilde{A} is a finitely generated subgroup of \mathbb{R} which contains \mathbb{Z} . Its image A in \mathbb{R}/\mathbb{Z} is a finitely generated subgroup of \mathbb{R}/\mathbb{Z} . Also we know that f is in $\text{IET}(A)$ and (μ, σ) is also a A -combinatorial description of f and $\overline{\varphi_\Gamma(f)}$ is an element of $\otimes_{\mathbb{Z}}^2 \tilde{A}$. Thus in ${}^\circ\Lambda_{\mathbb{Z}}^2 \tilde{A}$ we have:

$$\varphi_A(f) = \sum_{j=1}^k \left(\sum_{\substack{i: \\ \sigma(i) < \sigma(j)}} \mu_i - \sum_{i < j} \mu_i \right) \wedge \mu_j = [\overline{\varphi_\Gamma(f)}]_{\circ\Lambda_{\mathbb{Z}}^2 \tilde{A}} = \left[\sum_{j \in J} x_j \otimes y_j + y_j \otimes x_j \right]_{\circ\Lambda_{\mathbb{Z}}^2 \tilde{A}} = 0$$

Then we can applied the previous case and conclude that f is a balanced product of A -restricted rotations, thus a balanced product of Γ -restricted rotations. \square

4.2. $\text{Ker}(\varphi_\Gamma)$ is generated by IET(Γ)-transpositions.

The work of Y.Vorobets [11] done for IET can be adapted to show the next two lemmas:

Lemma 4.6. *Let f and g be two Γ -restricted rotations. If they have the same type then $f^{-1}g$ is a product of IET(Γ)-transpositions.*

Lemma 4.7. *Let f be a Γ -restricted rotation and g be any transformation in IET(Γ). Then the commutator $[f, g]$ is a product of IET(Γ)-transpositions.*

These lemmas with Lemma 2.16 give us the next theorem. It is proved by Y.Vorobets in the case $\Gamma = \mathbb{R}/\mathbb{Z}$ in [11], and as the proof is the same we only provide here a sketch.

Theorem 4.8. *Every balanced product of Γ -restricted rotations can be written as a product of IET(Γ)-transpositions.*

Proof of Theorem 4.8 (sketched).

Let (f_1, f_2, \dots, f_n) be a balanced tuple of restricted rotations. The proof is by strong induction on the length n of the tuple. If $n = 1$ then (f_1) is a balanced tuple of Γ -restricted rotations, thus f_1 is a Γ -restricted rotation of type (a, a) with $a \in \tilde{\Gamma}$ so it is also an IET(Γ)-transposition.

For the general case, let (a, b) be the type of f_1 . If $a = b$ then f_1 is an IET(Γ)-transposition and (f_2, f_3, \dots, f_n) is a balanced tuple of restricted rotations. By the induction assumption we obtain the result. If $a \neq b$ then there exists $k \in \{2, \dots, n\}$ such that f_k is a Γ -restricted rotation of type (b, a) . Let $g_1 = f_2 \dots f_{k-1}$ or $g_1 = \text{Id}$ if $k = 2$. Let $g_2 = f_{k+1} \dots f_n$ or $g_2 = \text{Id}$ if $k = n$. Then we can write

$$f_1 f_2 \dots f_n = f_1 g_1 f_k g_2 = (f_1 f_k)(f_k^{-1} g_1 f_k g_1^{-1})(g_1 g_2)$$

Hence, the induction assumption and Lemmas 4.6 and 4.7 give the result. \square

Corollary 4.9. *The kernel $\text{Ker}(\varphi_\Gamma)$ is generated by the set of all IET(Γ)-transpositions.*

5. DESCRIPTION OF THE ABELIANIZATION OF IET(Γ)

In this section we construct a group homomorphism $\varepsilon_\Gamma : \text{IET}(\Gamma) \rightarrow {}^\circ\Lambda_{\mathbb{Z}}^2 \tilde{\Gamma}$ whose kernel is the derived subgroup $D(\text{IET}(\Gamma))$.

5.1. Boolean measures.

In finite permutation groups there is a natural signature. One way to describe the signature is as follows: the signature of a finite permutation f , viewed in $\mathbb{Z}/2\mathbb{Z}$ is the number modulo 2 of pairs (x, y) such that $x < y$ and $f(x) > f(y)$. In our context where $f \in \text{IET}(\Gamma)$, while this set is infinite, the idea is to measure it in a meaningful sense.

Definition 5.1. Let A be a Boolean algebra and G be an abelian group. Let $\mu : A \rightarrow G$ be a finitely additive map: $\forall U, V \in A$ disjoint, $\mu(U \sqcup V) = \mu(U) + \mu(V)$. Such a μ is called a *Boolean algebra measure for A in G* .

Notation 5.2. We recall that $\text{Itv}(\Gamma)$ is the set of all intervals $[a, b[$ with a and b in $\tilde{\Gamma}$ and $0 \leq a < b \leq 1$. Let A_Γ be the Boolean algebra of subsets of $[0, 1[$ generated by $\text{Itv}(\Gamma)$. Then A_Γ is a Boolean subalgebra of $\{0, 1\}^{[0, 1[}$. By noting λ the Lebesgue measure on $[0, 1[$ we get that λ is a Boolean measure for A_Γ in $\tilde{\Gamma}$.

It might be useful to notice that for k in $\mathbb{N}_{\geq 1}$ and every I_1, I_2, \dots, I_k intervals in $\text{Itv}(\Gamma)$, the intersection $\bigcap_i I_i$ is still an element of $\text{Itv}(\Gamma)$. Moreover for every I in $\text{Itv}(\Gamma)$, the complement of I is the disjoint union of two elements of $\text{Itv}(\Gamma)$. Thus any Boolean combination of elements of $\text{Itv}(\Gamma)$ is a finite disjoint union of such elements.

Proposition 5.3. *Let X and Y be two sets, let A be a Boolean subalgebra of $\{0, 1\}^X$ and let B be a Boolean subalgebra of $\{0, 1\}^Y$. Let G and H be two abelian groups, let $\mu : A \rightarrow G$ be a Boolean algebra measure for A in G and $\nu : B \rightarrow H$ be a Boolean algebra measure for B in H . Let $C := A \otimes B$ be the Boolean algebra product (generated by subsets of the form $a \times b$ with a in A and b in B). Then there exists a unique Boolean algebra measure $\omega : C \rightarrow G \otimes H$ for C in $G \otimes H$ such that for every a in A and b in B we have $\omega(a \times b) = \mu(a) \otimes \nu(b)$.*

Proof. Let ω_1 and ω_2 be two such Boolean algebra measures, thus they are equal on every $a \times b$ for $a \in A$ and $b \in B$. Let c be an element of C , then there exist k in \mathbb{N} and a_1, \dots, a_k in A and b_1, \dots, b_k in B such that $c = \bigsqcup_{i=1}^k a_i \times b_i$. So

$$\omega_1(c) = \sum_{i=1}^k \omega_1(a_i \times b_i) = \sum_{i=1}^k \omega_2(a_i \times b_i) = \omega_2(c). \text{ Thus } \omega_1 = \omega_2 \text{ and the unicity is proved.}$$

It is sufficient to prove the existence for every finite Boolean subalgebra of C . Indeed if we assume that for every D finite Boolean subalgebra of C there exists a Boolean algebra measure m_D for D in $G \otimes H$ such that $m_D(a \times b) = \mu(a) \otimes \nu(b)$ for every a in A and b in B with $a \times b$ in D . Let c be an element of C . Then $\{0_C, c, \neg c, 1_C\}$ is a finite Boolean subalgebra of C non-trivial. Moreover if c is in $D_1 \cap D_2$ where D_1 and D_2 are two finite Boolean subalgebras of C then by noting D the Boolean subalgebra generated by D_1 and D_2 we get that D is a finite Boolean subalgebra of C containing c . Thus $m_D|_{D_1}$ is a Boolean measure for D_1 in $G \otimes H$ which satisfies $m_D|_{D_1}(a \times b) = \mu(a) \otimes \nu(b)$ for every a in A and b in B with $a \times b$ in D_1 . By unicity we get $m_D|_{D_1} = m_{D_1}$ and the same argument gives $m_D|_{D_2} = m_{D_2}$ thus $m_{D_1}(c) = m_D(c) = m_{D_2}(c)$. So by putting $\omega(c) = m_D(c)$, the map ω is well-defined. also if we take two disjoint elements c and c' in C . Then by taking any finite Boolean subalgebra D of C which contains c and c' we get $m_D(c + c') = m_D(c) + m_D(c') = \omega(c) + \omega(c')$ and the value does not depend on D . Thus ω is the wanted Boolean algebra measure.

Let now D be a finite Boolean subalgebra of C . Then there exist $k, \ell \in \mathbb{N}$ and $a_1, \dots, a_k \in A$ and $b_1, \dots, b_\ell \in B$ such that every $d \in D$ is a Boolean combination of $a_i \times b_j$. Then let D' be the finite Boolean algebra generated by all the $a_i \times b_j$

with $1 \leq i \leq k$ and $1 \leq j \leq \ell$. Let U be the finite Boolean subalgebra of A generated by all a_i and let V be the finite Boolean subalgebra of B generated by all b_j . Then U and V are atomic. Let u_1, \dots, u_n be the atoms of U and v_1, \dots, v_m be the atoms of V . Hence D' is atomic with atoms given by $u_i \times v_j$ for every $1 \leq i \leq k$ and $1 \leq j \leq \ell$. Then for each element d in D' there exists a unique $J_d \subset \{1, 2, \dots, n\} \times \{1, 2, \dots, m\}$ such that $d = \bigsqcup_{(i,j) \in J_d} u_i \otimes v_j$. Hence the map $m_{D'}$ defined by $m_{D'}(d) = m_{D'}(\bigsqcup_{(i,j) \in J_d} u_i \times v_j) = \sum_{(i,j) \in J_d} \mu(u_i) \otimes \nu(v_j)$ is well-defined, finitely additive and satisfies $m_{D'}(a \times b) = \mu(a) \otimes \nu(b)$ for every $a \in A$, $b \in B$ such that $a \times b \in D'$. \square

Notation 5.4. By applying the previous proposition with $X = Y = [0, 1[$ and $A = B = A_\Gamma$ and $\mu = \nu = \lambda$, there exists a unique Boolean algebra measure $\omega_\Gamma : A_\Gamma \otimes A_\Gamma \rightarrow \otimes_{\mathbb{Z}}^2 \tilde{\Gamma}$ such that for every a, b, c and d in $\tilde{\Gamma}_+$ with $a < b \leq 1$ and $c < d \leq 1$ we have $\omega_\Gamma([a, b[\times [c, d[) = (b - a) \otimes (d - c)$.

We need to check some IET(Γ)-invariance for the measure ω_Γ . For this we define an action of IET(Γ) on $[0, 1[\times [0, 1[$ by $f.(x, y) = (f(x), f(y))$. Hence for every P in A_Γ we have $f.P$ in A_Γ , this gives us a new Boolean algebra measure $f_*\omega_\Gamma$.

Proposition 5.5. *For every f in IET(Γ) and every P in $A_\Gamma \otimes A_\Gamma$ we have $f.P := \{(f(x), f(y)) \mid (x, y) \in P\}$ in $A_\Gamma \otimes A_\Gamma$. Furthermore we have $f_*\omega_\Gamma = \omega_\Gamma$.*

Proof. Let $f \in \text{IET}(\Gamma)$ and (μ, σ) be a Γ -combinatorial description of f and let $\{I_1, \dots, I_n\}$ be the partition into intervals associated. Let P be an element of $A_\Gamma \otimes A_\Gamma$. There exist m in \mathbb{N} and pairwise disjoint elements p_1, p_2, \dots, p_m of $\text{Itv}(\Gamma) \times \text{Itv}(\Gamma)$ such that $P = \bigsqcup_{i=1}^m p_i$. As f is a permutation of $[0, 1[$ we get $f. \bigsqcup_{i=1}^m p_i = \bigsqcup_{i=1}^m f.p_i$, so it is enough to show that $f.p$ belongs to $A_\Gamma \otimes A_\Gamma$. For $i \in \{1, \dots, k\}$, let $a_i, b_i, c_i, d_i \in \tilde{\Gamma}$ such that $p_i = [a_i, b_i[\times [c_i, d_i[$. Then $f.p_i = \bigsqcup_{(k,l)} f([a_i, b_i[\cap I_k) \times f([c_i, d_i[\cap I_l)$ which is a finite disjoint union of elements of $\text{Itv}(\Gamma) \times \text{Itv}(\Gamma)$ because a_i, b_i, c_i, d_i and extremities of I_l are in $\tilde{\Gamma}$ for each $1 \leq l \leq n$. In conclusion $f.p_i$ is in $A_\Gamma \otimes A_\Gamma$ thus $f.P$ is in $A_\Gamma \otimes A_\Gamma$. Also f is piecewise a translation and λ is the Lebesgue measure, so for any J in

$\text{Itv}(\Gamma)$ we have $\lambda(J) = \lambda(f(J))$. Thus:

$$\begin{aligned}
\omega_\Gamma(f.p_i) &= \sum_{(k,l)} \omega_\Gamma(f([a_i, b_i[\cap I_k] \times f([c_i, d_i[\cap I_l])) \\
&= \sum_{(k,l)} \lambda(f([a_i, b_i[\cap I_k]) \otimes \lambda(f([c_i, d_i[\cap I_l])) \\
&= \left(\sum_k \lambda([a_i, b_i[\cap I_k]) \right) \otimes \left(\sum_l \lambda([c_i, d_i[\cap I_l]) \right) \\
&= \lambda([a_i, b_i]) \otimes \lambda([c_i, d_i]) \\
&= \omega_\Gamma(p_i)
\end{aligned}$$

This gives us $\omega_\Gamma(f.P) = \sum_{i=1}^m \omega_\Gamma(f.p_i) = \sum_{i=1}^m \omega_\Gamma(p_i) = \omega_\Gamma(P)$. Hence $\omega_\Gamma = f_*^{-1} \omega_\Gamma$. As f^{-1} is also in $\text{IET}(\Gamma)$ we can do the same to deduce $\omega_\Gamma = f_* \omega_\Gamma$. \square

5.2. Creation of a signature.

Definition 5.6. Let f be a transformation in $\text{IET}(\Gamma)$. Every pair (x, y) in $[0, 1[\times [0, 1[$ such that $x < y$ and $f(x) > f(y)$ is called an *inversion of f* . We denote by \mathcal{E}_f the set of all inversions of f .

Proposition 5.7. Let f be a transformation in $\text{IET}(\Gamma)$ and (μ, τ) be a combinatorial description of f . We have $\mathcal{E}_f = \bigsqcup_{j=1}^n \bigsqcup_{\substack{i < j \\ \tau(i) > \tau(j)}} I_i \times I_j$.

Proof. Let (μ, τ) be a Γ -combinatorial description of f and let $\{I_1, I_2, \dots, I_k\}$ be the partition into intervals associated. Let (x_0, y_0) be an element of \mathcal{E}_f . Then there exist $i, j \in \{1, 2, \dots, k\}$ such that $x_0 \in I_i$ and $y_0 \in I_j$. As $x_0 < y_0$ we have $i \leq j$. Furthermore if $i = j$ then as f is an isometry which preserves the order on I_i we get $f(x_0) < f(y_0)$ which is a contradiction, we deduce that $i < j$. By definition of f we have $f(I_i), f(I_j) \in \text{Itv}(\Gamma)$ and they are disjoint. Thus as $f(x_0) > f(y_0)$ we deduce that for every $x \in I_i$ and $y \in I_j$ we have $x < y$ and $f(x) > f(y)$, so $I_i \times I_j \subset \mathcal{E}_f$. Also, this implies $\sigma(i) > \sigma(j)$ and we deduce that

$$\bigcup_{j=1}^n \bigsqcup_{\substack{i < j \\ \tau(i) > \tau(j)}} I_i \times I_j = \mathcal{E}_f. \quad \square$$

Corollary 5.8. For every $f \in \text{IET}(\Gamma)$ we have $\mathcal{E}_f \in A_\Gamma \otimes A_\Gamma$.

We denote by p the projection from $\otimes_{\mathbb{Z}}^2 \tilde{\Gamma}$ into ${}^\circ\Lambda_{\mathbb{Z}}^2 \tilde{\Gamma}$.

Definition 5.9. The *signature for $\text{IET}(\Gamma)$* is the following map:

$$\begin{aligned}
\varepsilon_\Gamma : \text{IET}(\Gamma) &\longrightarrow {}^\circ\Lambda_{\mathbb{Z}}^2 \tilde{\Gamma} \\
f &\longmapsto p \circ \omega_\Gamma(\mathcal{E}_f)
\end{aligned}$$

Proposition 5.10. For every A and B in A_Γ we have:

$$p \circ \omega_\Gamma(A \times B) = -p \circ \omega_\Gamma(B \times A)$$

Proof. Let $A, B \in A_\Gamma$ then:

$$\begin{aligned} p \circ \omega_\Gamma(A \times B) &= \lambda(A) \wedge \lambda(B) \\ &= -\lambda(B) \wedge \lambda(A) \\ &= -p \circ \omega_\Gamma(B \times A) \end{aligned}$$

□

Theorem 5.11. *The map ε_Γ is a group homomorphism.*

Proof. Let $f, g \in \text{IET}(\Gamma)$. We denote by s the symmetry of axis $y = x$. We remark that every element I of $A_\Gamma \otimes A_\Gamma$ satisfies $s(I) \in A_\Gamma \otimes A_\Gamma$. Then $\{(x, y) \mid x < y, g(x) > g(y), fg(x) < fg(y)\} = \mathcal{E}_g \cap sg^{-1}(\mathcal{E}_f)$ is an element of $A_\Gamma \otimes A_\Gamma$. We also notice that $\{(x, y) \mid x < y, g(x) < g(y), fg(x) > fg(y)\} = \mathcal{E}_{f \circ g} \cap \mathcal{E}_g^c$ and $\{(x, y) \mid x < y, g(x) > g(y), fg(x) > fg(y)\} = \mathcal{E}_{f \circ g} \cap \mathcal{E}_g$ are two elements of $A_\Gamma \otimes A_\Gamma$.

For more clarity we do some calculus first. By Proposition 5.5 and Proposition 5.10 we get:

$$\begin{aligned} & -p \circ \omega_\Gamma(\{(x, y) \mid x < y, g(x) > g(y), fg(x) < fg(y)\}) \\ &= -p \circ \omega_\Gamma(\{(g(x), g(y)) \mid x < y, g(x) > g(y), fg(x) < fg(y)\}) \\ &= p \circ \omega_\Gamma(\{(g(y), g(x)) \mid x < y, g(x) > g(y), fg(x) < fg(y)\}) \\ &= p \circ \omega_\Gamma(\{(u, v) \mid g^{-1}(u) > g^{-1}(v), u < v, f(u) > f(v)\}) \end{aligned}$$

and

$$\begin{aligned} & p \circ \omega_\Gamma(\{(x, y) \mid x < y, g(x) < g(y), fg(x) > fg(y)\}) \\ &= p \circ \omega_\Gamma(\{(g(x), g(y)) \mid x < y, g(x) < g(y), fg(x) > fg(y)\}) \\ &= p \circ \omega_\Gamma(\{(u, v) \mid g^{-1}(u) < g^{-1}(v), u < v, f(u) > f(v)\}) \end{aligned}$$

Then:

$$\begin{aligned} & p \circ \omega_\Gamma(\{(u, v) \mid g^{-1}(u) > g^{-1}(v), u < v, f(u) > f(v)\}) \\ & \quad + p \circ \omega_\Gamma(\{(u, v) \mid g^{-1}(u) < g^{-1}(v), u < v, f(u) > f(v)\}) \\ &= p \circ \omega_\Gamma(\mathcal{E}_f) \\ &= \varepsilon_\Gamma(f) \end{aligned}$$

Hence by adding and remove the same quantity at the fourth equality we obtain:

$$\begin{aligned}
\varepsilon_\Gamma(f \circ g) &= p \circ \omega_\Gamma(\{(x, y) \mid x < y, fg(x) > fg(y)\}) \\
&= p \circ \omega_\Gamma(\{(x, y) \mid x < y, g(x) > g(y), fg(x) > fg(y)\} \\
&\quad \sqcup \{(x, y) \mid x < y, g(x) < g(y), fg(x) > fg(y)\}) \\
&= p \circ \omega_\Gamma(\{(x, y) \mid x < y, g(x) > g(y), fg(x) > fg(y)\}) \\
&\quad + p \circ \omega_\Gamma(\{(x, y) \mid x < y, g(x) < g(y), fg(x) > fg(y)\}) \\
&= p \circ \omega_\Gamma(\{(x, y) \mid x < y, g(x) > g(y), fg(x) > fg(y)\}) \\
&\quad + p \circ \omega_\Gamma(\{(x, y) \mid x < y, g(x) > g(y), fg(x) < fg(y)\}) \\
&\quad - p \circ \omega_\Gamma(\{(x, y) \mid x < y, g(x) > g(y), fg(x) < fg(y)\}) \\
&\quad + p \circ \omega_\Gamma(\{(x, y) \mid x < y, g(x) < g(y), fg(x) > fg(y)\}) \\
&= p \circ \omega_\Gamma(\mathcal{E}_g) + p \circ \omega_\Gamma(\mathcal{E}_f) \\
&= \varepsilon_\Gamma(g) + \varepsilon_\Gamma(f)
\end{aligned}$$

In conclusion, ε_Γ is additive thus it is a group homomorphism. \square

Proposition 5.12. *Let $a \in \tilde{\Gamma}_+$ with $a \leq \frac{1}{2}$ and f be an IET(Γ)-transposition of type a . Then $\varepsilon_\Gamma(f) = a \wedge a$.*

Proof. Let $u, v, b \in \tilde{\Gamma}$ such that $((u, a, b, a, v), (2 \ 4))$ is a Γ -combinatorial description of f (see Figure 5). Let $\{I_1, \dots, I_5\}$ be the partition into intervals associated. We already proved in 5.7 that it is sufficient to check if a pair $(x, y) \in I_i \times I_j$ is in \mathcal{E}_f to know that $I_i \times I_j$ is in \mathcal{E}_f . We also have $I_i \times I_j \notin \mathcal{E}_f$ if $j \leq i$. Thus one can look at the graph of f to find that \mathcal{E}_f is equal to the tiling space on Figure 5. We deduce that $\varepsilon_\Gamma(f) = a \wedge b + a \wedge a + b \wedge a = a \wedge a$. \square

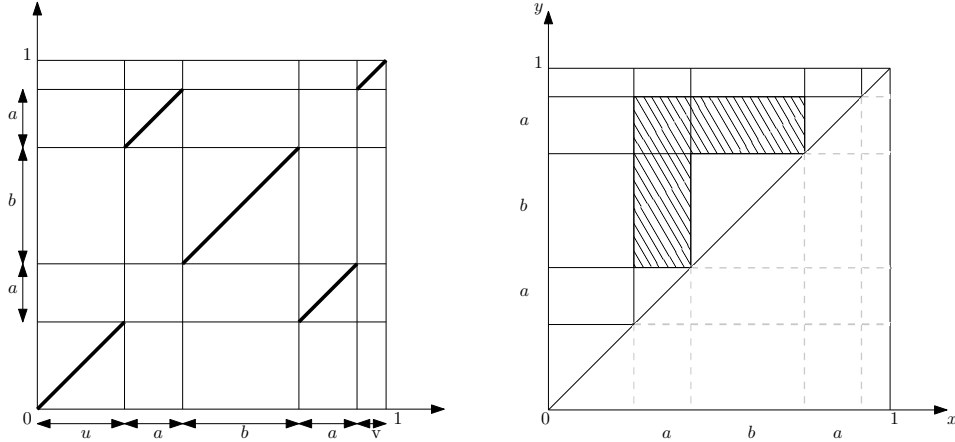


FIGURE 5. Illustration of how to calculate the value of ε_Γ on an IET(Γ)-transposition f in Proposition 5.12. **Left:** The graph of f . **Right:** The set \mathcal{E}_f .

Thanks to Proposition 5.7 we can calculate the value of ε_Γ for every $f \in \text{IET}(\Gamma)$:

Proposition 5.13. *Let $f \in \text{IET}(\Gamma)$ and (α, τ) be a Γ -combinatorial description of f . Let n be the length of α . Then*

$$\varepsilon_\Gamma(f) = \sum_{j=1}^n \sum_{\substack{i < j \\ \tau(i) > \tau(j)}} \alpha_i \wedge \alpha_j$$

Proposition 5.14. *The group homomorphism ε_Γ is surjective.*

Proof. Let $a, b \in \tilde{\Gamma}$. We assume that $0 \leq a, b < 1$ and $0 \leq a + b \leq 1$. Let r be the IET(Γ) restricted rotation of type (a, b) , whose intervals associated are $[0, a[$ and $[a, a + b[$. Then we obtain that $\varepsilon_\Gamma(r) = a \wedge b$.

In the general case, let $w \in \tilde{\Gamma}$ with $0 \leq w \leq \frac{1}{2}$. Then there exist $k, \ell \in \mathbb{Z}$ and $a', b' \in \tilde{\Gamma}$ with $0 \leq a', b' \leq w$ such that $a = kw + a'$ and $b = \ell w + b'$. Then $a \wedge b = k\ell w \wedge w + kw \wedge b' + \ell a' \wedge w + a' \wedge b'$. By the previous case and as ε_Γ is a group homomorphism, we deduce that $a \wedge b$ is in $\text{Im}(\varepsilon_\Gamma)$. \square

5.3. Description of $\text{Ker}(\varepsilon_\Gamma)$.

The aim of this part is to conclude that $\text{Ker}(\varepsilon_\Gamma) = D(\text{IET}(\Gamma))$ and the induced morphism $\text{IET}(\Gamma)_{\text{ab}} \rightarrow {}^\circ\Lambda_{\mathbb{Z}}^2 \tilde{\Gamma}$ is an isomorphism. We recall that φ_Γ is the restriction of the SAF-homomorphism φ to $\text{IET}(\Gamma)$.

Lemma 5.15. *We have $2\varepsilon_\Gamma = -\varphi_\Gamma$.*

Proof. Let $f \in \text{IET}(\Gamma)$ and (α, τ) be a Γ -combinatorial description of f . Let n be the length of α . Thanks to Propositions 2.4 and 5.13 we have:

$$\begin{aligned} \varphi_\Gamma(f) &= \sum_{j=1}^n \left(\sum_{\substack{i \\ \tau(i) < \tau(j)}} \alpha_i - \sum_{i < j} \alpha_i \right) \wedge \alpha_j \\ &= \sum_{j=1}^n \sum_{\substack{i > j \\ \tau(i) < \tau(j)}} \alpha_i \wedge \alpha_j + \sum_{j=1}^n \left(\sum_{\substack{i < j \\ \tau(i) < \tau(j)}} \alpha_i - \sum_{i < j} \alpha_i \right) \wedge \alpha_j \\ &= \sum_{i=1}^n \sum_{\substack{j < i \\ \tau(j) > \tau(i)}} \alpha_i \wedge \alpha_j + \sum_{j=1}^n \sum_{\substack{i < j \\ \tau(i) > \tau(j)}} \alpha_i \wedge \alpha_j \\ &= -2 \sum_{i=1}^n \sum_{\substack{j < i \\ \tau(j) > \tau(i)}} \alpha_j \wedge \alpha_i \\ &= -2\varepsilon_\Gamma(f) \end{aligned}$$

\square

Corollary 5.16. *We have the inclusion $\text{Ker}(\varepsilon_\Gamma) \subset \text{Ker}(\varphi_\Gamma)$.*

By Corollary 4.9 we know that $\text{Ker}(\varphi_\Gamma)$ is generated by the set of all IET(Γ)-transpositions. Thus it is natural to look at these elements who are also in $\text{Ker}(\varepsilon_\Gamma)$. If σ is an IET(Γ)-transposition of type a in $\text{Ker}(\varepsilon_\Gamma)$ then we have the equality $a \wedge a = 0$. We want to prove that σ is in $D(\text{IET}(\Gamma))$ if and only if $a \in 2\tilde{\Gamma}$.

We denote by $S^2\tilde{\Gamma}$ the second symmetric power of $\tilde{\Gamma}$ and we denote by $a \odot a$ image of $a \otimes a$ in $S^2\tilde{\Gamma}$.

For every group G and every $w \in G$ we use the notation $w \pmod{2}$ for the image of w in $G/2G$.

Proposition 5.17. *Let $a \in \tilde{\Gamma}$, if $a \wedge a \pmod{2} = 0$ then a belongs to $2\tilde{\Gamma}$.*

Proof. For every group G , the group $\odot\Lambda_{\mathbb{Z}}^2 G/2(\odot\Lambda_{\mathbb{Z}}^2 G)$ is naturally isomorphic to the second symmetric power $S^2G/2(S^2G)$. This comes from the fact that these groups satisfy the following universal property: for every group G and every abelian elementary 2-group A we denote i the natural inclusion of $G \times G$ into $S^2G/2(S^2G)$. For every bilinear symmetric group homomorphism $b : G \times G \rightarrow A$ there exists a unique group homomorphism $f : S^2G/2(S^2G) \rightarrow A$ such that for every $g, h \in G$ we have $b(g, h) = f(i(g, h))$.

Let $a \in \tilde{\Gamma}$ with $a \neq 0$ (because we already have $0 = 2 \times 0$). We denote by $a \odot a$ the image of $a \otimes a$ in $S^2\tilde{\Gamma}/2(S^2\tilde{\Gamma})$, and we assume that $a \odot a \pmod{2} = 0$. The projection $\tilde{\Gamma} \rightarrow \tilde{\Gamma}/2\tilde{\Gamma}$ gives rise to a morphism $\zeta : S^2\tilde{\Gamma} \rightarrow S^2(\tilde{\Gamma}/2\tilde{\Gamma})$. As $2(S^2\tilde{\Gamma}) \subset \text{Ker}(\zeta)$ we obtain a morphism $\zeta' : S^2\tilde{\Gamma}/2S^2\tilde{\Gamma} \rightarrow S^2(\tilde{\Gamma}/2\tilde{\Gamma})$. Hence if $a \notin 2\tilde{\Gamma}$ then $\zeta(a \odot a) \neq 0$ thus $\zeta'(a \odot a \pmod{2}) \neq 0$ which is a contradiction with the assumption. In conclusion $a \in 2\tilde{\Gamma}$. \square

Corollary 5.18. *Every IET(Γ)-transposition f in $\text{Ker}(\varepsilon_\Gamma)$ is in $D(\text{IET}(\Gamma))$.*

Proof. Let $a \in \tilde{\Gamma}_+$, with $a \leq \frac{1}{2}$, be the type of f and let $u, v \in \tilde{\Gamma}_+$ such that $I_1 = [u, u + a[$ and $I_2 = [v, v + a[$ are the two intervals swapped by f . From $f \in \text{Ker}(\varepsilon)$ we deduce that $\varepsilon_\Gamma(f) = a \wedge a = 0$. Hence $a \wedge a \pmod{2} = 0$ in $\odot\Lambda_{\mathbb{Z}}^2 \tilde{\Gamma}/2(\odot\Lambda_{\mathbb{Z}}^2 \tilde{\Gamma})$. Then by Proposition 5.17 there exists $b \in \tilde{\Gamma}$ such that $a = 2b$. Thus if we define g as the unique IET(Γ)-transposition of type b that swaps $[u, u + b[$ and $[v, v + b[$ and h as the unique IET(Γ) that swaps $[u, u + b[$ with $[u + b, u + a[$ and $[v, v + b[$ with $[v + b, v + a[$. Then $f = ghgh$ (see Figure 2) and $g^2 = h^2 = \text{Id}$, hence $f \in D(\text{IET}(\Gamma))$. \square

In order to show that $\text{Ker}(\varepsilon_\Gamma) = D(\text{IET}(\Gamma))$ we prove that any element $f \in \text{Ker}(\varepsilon_\Gamma)$ can be written as $f = \sigma h$ where $h \in D(\text{IET}(\Gamma))$ and σ is an IET(Γ)-transposition. This concludes because we just show that an IET(Γ)-transposition which is also in $\text{Ker}(\varepsilon_\Gamma)$ is in $D(\text{IET}(\Gamma))$. We begin by a particular case of a product of IET(Γ)-transpositions with pairwise disjoint support. The aim will be to reduce the general case to this one. We recall that the identity is considered as an IET(Γ)-transposition.

Lemma 5.19. *Let $k \in \mathbb{N}$ and $\tau_1, \tau_2, \dots, \tau_k$ be IET(Γ)-transpositions with pairwise disjoint support. Then $\tau_1\tau_2 \dots \tau_k = \sigma h$ where σ is an IET(Γ)-transposition and*

h is an element of $D(\text{IET}(\Gamma))$. Moreover the support of h and σ do not overlap and are included in the union of the supports of the τ_i .

Proof. By induction it is enough to show the result in the case $k = 2$. Let respectively a_1 and a_2 be the type of τ_1 and τ_2 . As their support do not overlap we know that τ_1 and τ_2 commute. Hence we can assume $a_1 \geq a_2$ without loss of generality. If $a_1 = a_2$ then there exists $f \in \text{IET}(\Gamma)$ such that $\tau_1 = f\tau_2f^{-1}$. Thus $\tau_1\tau_2$ is in $D(\text{IET}(\Gamma))$. If $a_1 > a_2$ let $u, v \in \tilde{\Gamma}$ such that $[u, u + a_1[$ and $[v, v + a_1[$ are the intervals swapped by τ_1 . Let g and h be $\text{IET}(\Gamma)$ -transpositions such that g swaps the intervals $[u, u + a_2[$ and $[v, v + a_2[$ and h swaps the intervals $[u + a_2, u + a_1[$ and $[v + a_2, v + a_1[$. Thus $\tau_1 = hg$. Moreover g and τ_2 are two $\text{IET}(\Gamma)$ -transpositions with same type and non-overlapping support. Then by the previous case, the product $g\tau_2$ is in $D(\text{IET}(\Gamma))$ and its support does not intersect the support of h . Then $f = h(g\tau_2)$ is the wanted decomposition. \square

We can now treat the case of finite order elements:

Lemma 5.20. *If $f \in \text{IET}(\Gamma)$ has finite order then there exist an $\text{IET}(\Gamma)$ -transposition σ and $h \in D(\text{IET}(\Gamma))$ such that the support of σ and h are inside the support of f and $f = \sigma h$.*

Proof. Let $f \in \text{IET}(\Gamma)$ as in the statement and let $n \in \mathbb{N}$ and $\mathcal{P} = \{I_1, I_2, \dots, I_n\} \in \Pi_f$. By Lemma 3.1 there exists $\sigma \in \mathfrak{S}_n$ such that $f(I_i) = I_{\sigma(i)}$. Let $k \in \mathbb{N}$ and $\sigma = c_1 c_2 \dots c_k$ be the disjoint cycle decomposition for σ . Let f_i be the element of $\text{IET}(\Gamma)$ that is equal to f on I_j for every $j \in \text{Supp}(c_i)$ while fixing the rest of $[0, 1[$. Then $f = f_1 \dots f_k$ and f_i commutes with f_j for every $1 \leq i \neq j \leq k$. Then if the statement is true for every f_i we can write $f_i = \tau_i h_i$ with τ_i an $\text{IET}(\Gamma)$ -transposition and $h_i \in D(\text{IET}(\Gamma))$, both of them with support inside the support of f_i . Then $f = f_1 \dots f_k = \tau_1 h_1 \dots \tau_k h_k = \tau_1 \dots \tau_k h_1 \dots h_k$ because the support of τ_i does not overlap with the support of τ_j of h_j for every $1 \leq j \leq k$ and $j \neq i$. We conclude with Lemma 5.19 applied to $\tau_1 \dots \tau_k$.

Let c be a cycle of length $n \geq 2$ and let I_1, I_2, \dots, I_n be non-overlapping intervals of $\text{Itv}(\Gamma)$ of same length. Let $f \in \text{IET}(\Gamma)$ be the element that permutes the set $\{I_1, I_2, \dots, I_n\}$ by c . Then if $c \in D(\mathfrak{S}_n) = \mathfrak{A}_n$ we deduce that $f \in D(\text{IET}(\Gamma))$. If $c \notin D(\mathfrak{S}_n)$ then let g be the unique $\text{IET}(\Gamma)$ -transposition that swaps I_1 with $I_{c(1)}$ (we notice that the support of g is included in the support of f). By the previous case, $gf \in D(\text{IET}(\Gamma))$ and we conclude that $f = g(gf)$ is a wanted decomposition. \square

Corollary 5.21. *Let τ and τ' be two $\text{IET}(\Gamma)$ -transpositions. There exist an $\text{IET}(\Gamma)$ -transposition σ and $h \in D(\text{IET}(\Gamma))$ such that $\tau\tau' = \sigma h$ and the support of σ and h are included in the union of the support of τ and the one of τ' .*

Proof. By Proposition 3.4 we deduce that $f := \tau\tau'$ has finite order. Hence by Lemma 5.20 we obtain the result. \square

Lemma 5.22. *Let k in \mathbb{N} and $\tau_1, \tau_2, \dots, \tau_k$ be some IET(Γ)-transpositions. Then there exist an IET(Γ)-transposition σ and $h \in D(\text{IET}(\Gamma))$ such that $\tau_1\tau_2 \dots \tau_k = \sigma h$.*

Proof. The proof is by induction on k . The initialisation $k = 1$ is immediate. The case $k = 2$ is Corollary 5.21. Now if we assume the result for $k \geq 2$ let $\tau_1, \tau_2, \dots, \tau_k, \tau_{k+1}$ be IET(Γ)-transpositions. Then by assumption, applied to $\tau_2\tau_3 \dots \tau_{k+1}$, there exist an IET(Γ)-transposition σ and $h \in D(\text{IET}(\Gamma))$ such that $\tau_2\tau_3 \dots \tau_{k+1} = \sigma h$. Hence $\tau_1\tau_2 \dots \tau_{k+1} = \tau_1\sigma h$. By using the case $k = 2$ we deduce that there exist an IET(Γ)-transposition σ' and $h' \in D(\text{IET}(\Gamma))$ such that $\tau_1\sigma = \sigma'h'$. Thus $\tau_1\tau_2 \dots \tau_{k+1} = \sigma'g$, with $g = h'h \in D(\text{IET}(\Gamma))$, which is a wanted decomposition. \square

Finally we can prove the main theorem of the section:

Theorem 5.23. *We have the equality $\text{Ker}(\varepsilon_\Gamma) = D(\text{IET}(\Gamma))$, and the induced morphism $\text{IET}(\Gamma)_{\text{ab}} \rightarrow {}^\circ\Lambda_{\mathbb{Z}}^2 \tilde{\Gamma}$ is an isomorphism.*

Proof. The inclusion from right to left is immediate. For the other inclusion let $f \in \text{Ker}(\varepsilon_\Gamma)$. By Corollary 5.16 we know that $f \in \text{Ker}(\varphi_\Gamma)$, then by Corollary 4.5 there exists $k \in \mathbb{N}$ such that f is equal to the product $\tau_1\tau_2 \dots \tau_k$ where τ_i is an IET(Γ)-transposition. By Lemma 5.22 there exist an IET(Γ)-transposition σ and $h \in D(\text{IET}(\Gamma))$ such that $f = \sigma h$. Then $\varepsilon(f) = \varepsilon(\sigma)\varepsilon(h) = \varepsilon(\sigma) = 0$. By Corollary 5.18 we deduce $\sigma \in D(\text{IET}(\Gamma))$. Hence $f = \sigma h \in D(\text{IET}(\Gamma))$.

We deduce that the induced group homomorphism $\varepsilon'_\Gamma : \text{IET}(\Gamma)_{\text{ab}} \rightarrow {}^\circ\Lambda_{\mathbb{Z}}^2 \tilde{\Gamma}$ is injective. Furthermore ε_Γ is surjective by Proposition 5.14 thus ε'_Γ is surjective and we conclude that ε'_Γ is an isomorphism. \square

6. ABELIANIZATION OF $\text{IET}^{\bowtie}(\Gamma)$

In order to identify $\text{IET}^{\bowtie}(\Gamma)_{\text{ab}}$ we construct two group homomorphisms. One is an analogue of the signature homomorphism ε_Γ constructed in Section 5. Its kernel will be slightly larger than $D(\text{IET}^{\bowtie}(\Gamma))$; in fact, it will miss some reflections of a certain type. In order to tackle this issue we notice that, in some cases, a reflection is a conjugate of a restricted rotation and thus we will try to use the group homomorphism ε_Γ . To this we need to affect for every element of $\text{IET}^{\bowtie}(\Gamma)$ an element of $\text{IET}(\Gamma)$.

We recall that the group IET can be seen as a subgroup of IET^{\bowtie} . Thus restricted rotations in IET^{\bowtie} are well-defined.

The next proposition implies that a lot of Γ -reflections are in $D(\text{IET}^{\bowtie}(\Gamma))$ with a sufficient condition on their type (see Definition 2.8 for the definition of reflection and type of a reflection). It is a consequence of Proposition 2.11 and Theorem 5.23:

Proposition 6.1. *Every Γ -reflection of type $\ell \in 4\tilde{\Gamma}$ belongs to $D(\text{IET}^{\bowtie}(\Gamma))$*

6.1. Partitions associated and positive substitute.

6.1.1. Partitions.

Partitions of $[0, 1[$ into finite intervals will be really useful there but the definition of $\text{IET}^{\boxtimes}(\Gamma)$ imposes to be more careful and to take them up to a finitely supported permutation.

Definition 6.2. Let \mathcal{P} be a finite partition of $[0, 1[$ into right-open and left-closed intervals and let $\hat{f} \in \widehat{\text{IET}^{\boxtimes}}$. The partition \mathcal{P} is called a *partition associated with \hat{f}* if \hat{f} is continuous on every $I \in \mathcal{P}$. Also \mathcal{P} is an *essential partition associated with \hat{f}* if there exists a finitely supported permutation σ such that \mathcal{P} is a partition associated with $\sigma\hat{f}$. We denote by $\hat{f}(\mathcal{P})$ the partition into intervals of $[0, 1[$ composed of all right-open and left-closed intervals whose interior is the image by \hat{f} of the interior of an interval in \mathcal{P} up to a finite number of points; it is called the *arrival partition of \hat{f} associated to \mathcal{P}* .

We can now define partitions associated with an element of IET^{\boxtimes} :

Definition 6.3. Let \mathcal{P} be a finite partition of $[0, 1[$ into right-open and left-closed intervals. Let $f \in \text{IET}^{\boxtimes}$ and let \hat{f} be a representative of f in $\widehat{\text{IET}^{\boxtimes}}$. The partition \mathcal{P} is called a *partition associated with f* if \mathcal{P} is an essential partition associated to \hat{f} . We denote by $f(\mathcal{P})$ the partition $\hat{f}(\mathcal{P})$; it is called the *arrival partition of f associated to \mathcal{P}* .

Remark 6.4. For every $f \in \text{IET}^{\boxtimes}$, the set Π_f has a minimal element for the refinement. This element is also the unique partition that has a minimal number of interval.

We recall that a partition \mathcal{P} is a Γ -partition if for every $I \in \mathcal{P}$ we have $I \in \text{Itv}(\Gamma)$. Thus for every $f \in \text{IET}^{\boxtimes}$ we have $f \in \text{IET}^{\boxtimes}(\Gamma)$ if and only if there exists a Γ -partition associated with f .

Sometimes we will need to be more precise on where the length of intervals are. We recall that λ denote the Lebesgue measure:

Definition 6.5. Let S be a finite subset of \mathbb{R} , we say that a subinterval I of \mathbb{R} is a *S -interval* if $\lambda(I) \in S$.

In order to have more rigidity we also associate a partition to a tuple:

Definition 6.6. Let $n \in \mathbb{N}$ and $f_1, f_2, \dots, f_n \in \text{IET}^{\boxtimes}(\Gamma)$. Let S be a finite subset of \mathbb{R} and let \mathcal{P} be a partition into intervals of $[0, 1[$, we said that \mathcal{P} is a *partition into S -intervals associated with (f_1, f_2, \dots, f_n)* if:

- (1) \mathcal{P} is a partition into S -intervals associated with f_1 ,
- (2) For every $2 \leq i \leq n - 1$, $f_i f_{i-1} \dots f_1(\mathcal{P})$ is a partition into S -intervals associated with f_{i+1} .

Remark 6.7. Let S and T be two finite subsets of \mathbb{R} and let \mathcal{P} be a partition into S -intervals associated with (f_1, f_2, \dots, f_n) . Then any refinement of \mathcal{P} into T -intervals is a partition into T -intervals associated with (f_1, f_2, \dots, f_n) .

We also want to talk about order-preserving and order-reversing for elements in IET^{\boxtimes} .

Definition 6.8. Let $f \in \text{IET}^{\boxtimes}$ and $\mathcal{P} \in \Pi_f$. Let $I \in \mathcal{P}$, we say that f is *order-preserving on I* (resp. *order-reversing on I*) if there exists a representative of f in $\widehat{\text{IET}^{\boxtimes}}$ that is order-preserving on I (resp. order-reversing on I).

Thanks to this when we have a partition \mathcal{P} associated with an element $f \in \text{IET}^{\boxtimes}$ we can always say that f is either order-preserving or order-reversing on every interval of \mathcal{P} .

6.1.2. Positive substitute.

We introduce the notion of positive substitute to affect an element of IET for every element of IET^{\boxtimes} . This substitute depends on a partition associated with f .

Definition 6.9. Let $\hat{f} \in \widehat{\text{IET}^{\boxtimes}}$ and \mathcal{P} be an essential partition associated with f . Let f be its image in IET^{\boxtimes} . Then there exists a unique $f_{\mathcal{P}}^+ \in \text{IET}$ such that \mathcal{P} is a partition associated with $f_{\mathcal{P}}^+$ and such that for every $I \in \mathcal{P}$ we have $f_{\mathcal{P}}^+(I) = \hat{f}(I)$ up to a finite number of points. This element is called the *positive \mathcal{P} -substitute of \hat{f} and f* .

This element exists because there exists a finitely supported permutation σ such that \mathcal{P} is a partition associated with $\sigma\hat{f}$ and we define $f_{\mathcal{P}}^+$ as the element of IET such that $f_{\mathcal{P}}^+(I) = \sigma\hat{f}(I)$ for every $I \in \mathcal{P}$. The unicity come from the right continuity of $f_{\mathcal{P}}^+$. The dependance on the partition is really important as we can see on Figures 6 and 7.

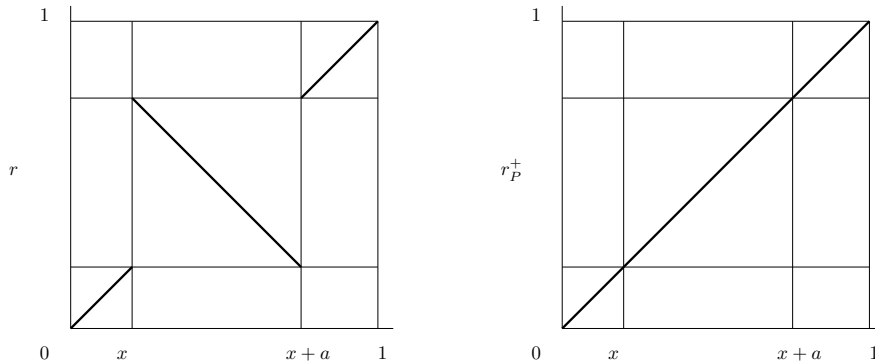


FIGURE 6. Positive substitute for a reflection in the case $S = \{x, a, 1 - (x + a)\}$.

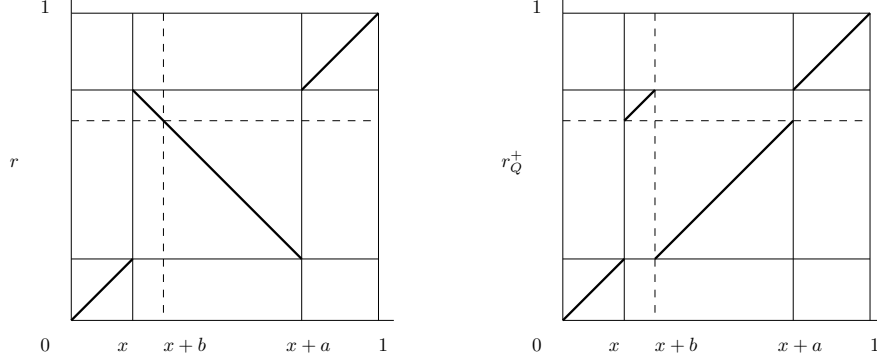


FIGURE 7. Positive substitute for a reflection in the case $S = \{x, b, a - b, 1 - (x + a)\}$.

This gives us a decomposition for every element of $\widehat{\text{IET}}^{\boxtimes}$.

Proposition 6.10. *Let $\hat{f} \in \widehat{\text{IET}}^{\boxtimes}$ and \mathcal{P} be an essential partition associated with \hat{f} . Then there exist a unique finitely supported permutation σ and a product r of I -reflection maps with $I \in \hat{f}(\mathcal{P})$ such that $\hat{f} = \sigma r f_{\mathcal{P}}^+$.*

With some conditions we deduce how positive substitute behaves with composition.

Proposition 6.11. *Let $n \in \mathbb{N}$ and $f_1, f_2, \dots, f_n \in \text{IET}^{\boxtimes}(\Gamma)$. Let \mathcal{P} be a partition into S -intervals associated with (f_1, f_2, \dots, f_n) . Let $g_1 = (f_1)_{\mathcal{P}}^+$ and $g_i = (f_i f_{i-1} \dots f_1)_{f_{i-1} \dots f_1(\mathcal{P})}^+$ for every $2 \leq i \leq n$. Then $(f_n f_{n-1} \dots f_1)_{\mathcal{P}}^+ = g_n g_{n-1} \dots g_1$.*

Proof. For this proof we denote by r_I the I -reflection for every subinterval I of $[0, 1[$.

By iteration it is sufficient to show the result for $n = 2$. Let $f, g \in \text{IET}^{\boxtimes}(\Gamma)$ and \mathcal{P} be a partition into S -intervals associated with (f, g) . Let $n \in \mathbb{N}$ such that $\mathcal{P} = \{I_1, I_2, \dots, I_n\}$ and $f(\mathcal{P}) = \{J_1, J_2, \dots, J_n\}$ and $f_{\mathcal{P}}^+(I_i) = J_i$ for $1 \leq i \leq n$. Let $1 \leq i \leq n$, we notice that $r_{J_i} \circ f_{\mathcal{P}}^+|_{I_i} \circ r_{I_i} = f_{\mathcal{P}}^+|_{I_i}$. There are 4 cases:

- (1) If f is order-preserving on I_i and g is order-preserving on J_i then $g \circ f$ is order-preserving on I_i so :

$$(g \circ f)_{\mathcal{P}}^+|_{I_i} = (g \circ f)|_{I_i} = g|_{J_i} \circ f|_{I_i} = g_{f(\mathcal{P})}^+|_{J_i} \circ f_{\mathcal{P}}^+|_{I_i}$$

- (2) If f is order-preserving on I_i and g is order-reversing on J_i then $g \circ f$ is order-reversing on I_i so :

$$(g \circ f)_{\mathcal{P}}^+|_{I_i} = (g \circ f)_{I_i} \circ r_{I_i} = g|_{J_i} \circ f|_{I_i} \circ r_{I_i} = g_{f(\mathcal{P})}^+|_{J_i} \circ r_{J_i} \circ f_{\mathcal{P}}^+|_{I_i} \circ r_{I_i} = g_{f(\mathcal{P})}^+|_{J_i} \circ f_{\mathcal{P}}^+|_{I_i}$$

- (3) If f is order-reversing on I_i and g is order-preserving on J_i then $g \circ f$ is order-reversing on I_i so :

$$(g \circ f)_{\mathcal{P}}^+|_{I_i} = (g \circ f)_{I_i} \circ r_{I_i} = g|_{J_i} \circ f|_{I_i} \circ r_{I_i} = g_{f(\mathcal{P})}^+|_{J_i} \circ f_{\mathcal{P}}^+|_{I_i} \circ r_{I_i} \circ r_{I_i} = g_{f(\mathcal{P})}^+|_{J_i} \circ f_{\mathcal{P}}^+|_{I_i}$$

(4) If f is order-reversing on I_i and g is order-reversing on J_i then $g \circ f$ is order-preserving on I_i so :

$$(g \circ f)_{\mathcal{P}}^{\dagger}|_{I_i} = (g \circ f)|_{I_i} = g|_{J_i} \circ f|_{I_i} = g_{f(\mathcal{P})}^{\dagger}|_{J_i} \circ r_{J_i} \circ f_{\mathcal{P}}^{\dagger}|_{I_i} \circ r_{I_i} = g_{f(\mathcal{P})}^{\dagger}|_{J_i} \circ f_{\mathcal{P}}^{\dagger}|_{I_i}$$

□

6.2. Analogue of the signature.

6.2.1. Balanced product of reflections.

We give here a first description of $D(\text{IET}^{\boxtimes})$. It is inspired of the work done for IET with balanced product restricted rotations.

Definition 6.12. Let $n \in \mathbb{N}$ and r_1, r_2, \dots, r_n be some Γ -reflections. For every $\ell \in \tilde{\Gamma}_+$ let n_{ℓ} be the number of Γ -reflections of type ℓ among these elements. The tuple (r_1, r_2, \dots, r_n) is a *balanced tuple of Γ -reflections* if 2 divides n_{ℓ} for every $\ell \in \tilde{\Gamma}_+$. We say that a product of Γ -reflections is a *balanced product of Γ -reflections* if it can be written as a product of a balanced tuple of Γ -reflections.

Lemma 6.13. *The set of all balanced products of Γ -reflections is a generating subset of $D(\text{IET}^{\boxtimes}(\Gamma))$.*

Proof. As any element of $\text{IET}^{\boxtimes}(\Gamma)$ is a finite product of Γ -reflections (see Proposition 2.9) and as a reflection has order 2 we deduce that every element of $D(\text{IET}^{\boxtimes}(\Gamma))$ is a balanced product of reflections.

Let r and s be two reflections with the same type. By Proposition 2.10 we deduce that they are conjugate. As a reflection has order 2, the product rs is a commutator.

Let $n \in \mathbb{N}$ and r_1, r_2, \dots, r_n be Γ -reflections such that $r_1 r_2 \dots r_n$ is a balanced product of reflections. Then n is even and up to compose with an element of $D(\text{IET}^{\boxtimes})$ we can assume that r_{2i-1} and r_{2i} have the same type for every $1 \leq i \leq \frac{n}{2}$. Thus by the previous case we deduce that $r_1 r_2 \dots r_n$ is in $D(\text{IET}^{\boxtimes})$. □

By Proposition 2.9 and Lemma 6.13 we deduce the following:

Corollary 6.14. *Let $f \in \text{IET}^{\boxtimes}(\Gamma)$ then $f^2 \in D(\text{IET}^{\boxtimes}(\Gamma))$.*

Proposition 6.15. *Any balanced product of Γ -restricted rotations with type in $\{(a, b) \mid a \neq b \in \tilde{\Gamma}_+\}$ is a balanced product of Γ -reflections.*

Proof. Let $a, b \in \tilde{\Gamma}_+$. Let r be a restricted rotation of type (a, b) and s be a restricted rotation of type (b, a) . Let I and J be the two consecutive intervals associated with r . Then r is the composition of the I -reflection, the J -reflection and the $I \cup J$ -reflection thus a product of a reflections of type a with one of type b and one of type $a + b$. The same is true for s thus we obtain that rs is the product of two reflections of type a , two of type b and two of type $a + b$; so rs is a balanced product of Γ -reflections. □

6.2.2. The group homomorphism.

Here we start with the work done in Section 5. We remark that if we denote $A'_{\mathbb{R}}$ the Boolean algebra of subsets of $[0, 1[$ generated by the set of all intervals $[a, b]$, $A_{\mathbb{R}}$ the one generated by the set of all intervals $[a, b[$ and A_{fin} the one generated by all the singletons $\{x\}$ then $A_{\mathbb{R}}$ is isomorphic to $A_{\mathbb{R}}^* := A'_{\mathbb{R}}/A_{\text{fin}}$. This is why we do not make a difference between A_{Γ} and its image in $A_{\mathbb{R}}^*$.

The notion of inversions as defined in 5.6 is no longer relevant because for every reflection r we have $\mathcal{E}_r \notin A_{\Gamma} \otimes A_{\Gamma}$; indeed \mathcal{E}_r is a triangle. We need to be more precise:

Definition 6.16. For every element $f \in \widehat{\text{IET}^{\boxtimes}(\Gamma)}$ we define:

- (1) $\mathcal{E}_{f,1} := \{(x, y) \mid x < y, f(x) > f(y)\}$, the set of all inversions of type 1 of f ,
- (2) $\mathcal{E}_{f,2} := \{(x, y) \mid y < x, f(y) > f(x)\}$, the set of all inversions of type 2 of f ,
- (3) $\mathcal{E}_f^{\boxtimes} := \mathcal{E}_{f,1} \cup \mathcal{E}_{f,2}$, the set of all inversions of f .

From now on we will write $\mathcal{E}_f^{\boxtimes} = \mathcal{E}_f$.

- Example 6.17.** (1) If we consider a Γ -restricted rotation r of type (a, b) (with $a, b \in \tilde{\Gamma}_+$), we obtain $w_{\Gamma}(\mathcal{E}_r) = a \otimes b + b \otimes a$ (see Figure 8).
 (2) Let $I \in \text{Itv}(\Gamma)$ be a subinterval of $[0, 1[$ of length a . Let r be the I -reflection. We have $w_{\Gamma}(\mathcal{E}_r) = a \otimes a$ (see Figure 8).
 (3) Let τ a finitely supported permutation. Then \mathcal{E}_{τ} is a union of singletons thus $\mathcal{E}_{\tau} = \emptyset \in A_{\Gamma} \otimes A_{\Gamma}$ and $w_{\Gamma}(\mathcal{E}_{\tau}) = w_{\Gamma}(\emptyset) = 0$.

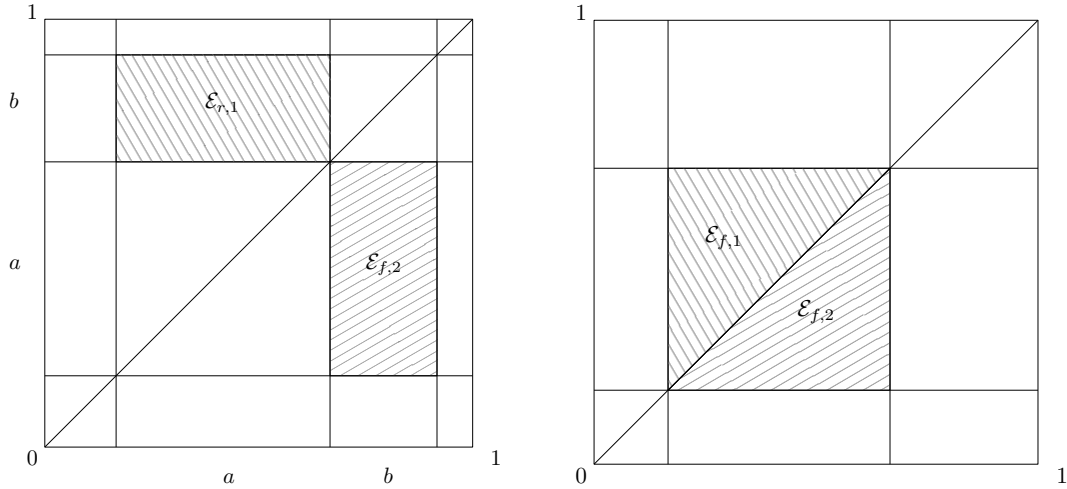


FIGURE 8. **Left:** Set of inversions for a restricted rotation. **Right:** Set of inversions for a I -reflection

Proposition 6.18. For every $f \in \widehat{\text{IET}^{\boxtimes}(\Gamma)}$ we have $\mathcal{E}_f \in A_{\Gamma} \otimes A_{\Gamma}$.

Proof. Let $f \in \widehat{\text{IET}}^{\boxtimes}$ and let \mathcal{P} be a partition associated with f . Let $f_{\mathcal{P}}^+$ be the positive \mathcal{P} -substitute of f . Let $\mathcal{J} \subset \mathcal{P}$ be the subset of intervals where f is order-reversing. By seeing A_{Γ} as its image in $A_{\mathbb{R}}^*$ then we deduce that $\mathcal{E}_f = \mathcal{E}_{f_{\mathcal{P}}^+} \cup \bigcup_{I \in \mathcal{J}} I \times I$ is an element of $A_{\Gamma} \otimes A_{\Gamma}$. \square

Then this set can be measured with the same measure ω_{Γ} used in Section 5. Here we expect a 2-group for the image of our group homomorphism. We denote by π the projection of $\otimes_{\mathbb{Z}}^2 \tilde{\Gamma}$ onto $\otimes_{\mathbb{Z}}^2 \tilde{\Gamma} / 2(\otimes_{\mathbb{Z}}^2 \tilde{\Gamma})$ and for every $a \in \tilde{\Gamma}$ we denote $\pi(a \otimes a)$ by $a \otimes a \pmod{2}$.

Definition 6.19. We define the signature for $\widehat{\text{IET}}^{\boxtimes}(\Gamma)$ as the map:

$$\begin{aligned} \widehat{\varepsilon}_{\Gamma}^{\boxtimes} : \widehat{\text{IET}}^{\boxtimes}(\Gamma) &\longrightarrow \otimes_{\mathbb{Z}}^2 \tilde{\Gamma} / 2(\otimes_{\mathbb{Z}}^2 \tilde{\Gamma}) \\ f &\longrightarrow w_{\Gamma}(\mathcal{E}_f) \pmod{2} \end{aligned}$$

For more clarity we explicit some equalities used to show that $\widehat{\varepsilon}_{\Gamma}^{\boxtimes}$ is a group homomorphism.

Lemma 6.20. Let $f, g \in \widehat{\text{IET}}^{\boxtimes}(\Gamma)$. We have the following equalities:

- (1) $\mathcal{E}_{f \circ g} \cup (\mathcal{E}_g \cap g^{-1}(\mathcal{E}_f)) = \mathcal{E}_g \cup g^{-1}(\mathcal{E}_f)$,
- (2) $\mathcal{E}_{f \circ g} \cap \mathcal{E}_g \cap g^{-1}(\mathcal{E}_f) = \emptyset$.

Remark 6.21. We notice that $\mathcal{E}_g \cap g^{-1}(\mathcal{E}_f)$ is an element of $A_{\Gamma} \otimes A_{\Gamma}$.

Proof. (1) We proceed by double inclusions.

From left to right, we know that $\mathcal{E}_g \cap g^{-1}(\mathcal{E}_f) \subset \mathcal{E}_g \cup g^{-1}(\mathcal{E}_f)$ hence it is sufficient to show the inclusion $\mathcal{E}_{f \circ g} \subset \mathcal{E}_g \cup g^{-1}(\mathcal{E}_f)$. Let (x, y) be in $\mathcal{E}_{f \circ g}$. We can assume that $x < y$, the case $x > y$ is similar. Then we deduce that $f(g(x)) > f(g(y))$. We have two cases, if $g(x) > g(y)$ then $(x, y) \in \mathcal{E}_g$ else we have $g(x) < g(y)$ and $f(g(x)) > f(g(y))$ thus $(x, y) = g^{-1}(g(x), g(y)) \in g^{-1}\mathcal{E}_f$.

From right to left, let $(x, y) \in \mathcal{E}_g \cup g^{-1}(\mathcal{E}_f)$. We can assume that $x < y$ the case $x > y$ is similar. If $(x, y) \in \mathcal{E}_g \cap g^{-1}(\mathcal{E}_f)$ then it is done. We have two cases, if $(x, y) \in \mathcal{E}_g$ and $(x, y) \notin g^{-1}(\mathcal{E}_f)$ then as $x < y$ we have $g(x) > g(y)$ and $f(g(x)) > f(g(y))$ thus $(x, y) \in \mathcal{E}_{f \circ g}$. If $(x, y) \in g^{-1}(\mathcal{E}_f)$ and $(x, y) \notin \mathcal{E}_g$ then as $x < y$ we have $g(x) < g(y)$ and $f(g(x)) > f(g(y))$ thus $(x, y) \in \mathcal{E}_{f \circ g}$.

- (2) By contradiction let us assume that there exists $(x, y) \in \mathcal{E}_{f \circ g} \cap \mathcal{E}_g \cap g^{-1}(\mathcal{E}_f)$. We can assume that $x < y$, the case $x > y$ is similar. As $x < y$ and $(x, y) \in \mathcal{E}_{f \circ g} \cap \mathcal{E}_g$ we know that $g(x) > g(y)$ and $f(g(x)) > f(g(y))$. However $g.(x, y) = (g(x), g(y)) \in \mathcal{E}_f$ thus as $g(x) > g(y)$ we have $f(g(x)) < f(g(y))$ which is a contradiction. \square

Theorem 6.22. The signature $\widehat{\varepsilon}_{\Gamma}^{\boxtimes}$ is a group homomorphism.

Proof. Let $f, g \in \widehat{\text{IET}}^\boxtimes(\Gamma)$. Thanks to Lemma 6.20 we have:

$$\begin{aligned}
\widehat{\varepsilon}_\Gamma^\boxtimes}(f \circ g) &= \omega_\Gamma(\mathcal{E}_{f \circ g}) \pmod{2} \\
&= \omega_\Gamma(\mathcal{E}_{f \circ g}) + \omega_\Gamma(\mathcal{E}_g \cap g^{-1}(\mathcal{E}_f)) - \omega_\Gamma(\mathcal{E}_g \cap g^{-1}(\mathcal{E}_f)) \pmod{2} \\
&= \omega_\Gamma(\mathcal{E}_{f \circ g} \sqcup (\mathcal{E}_g \cap g^{-1}(\mathcal{E}_f))) - \omega_\Gamma(\mathcal{E}_g \cap g^{-1}(\mathcal{E}_f)) \pmod{2} \\
&= \omega_\Gamma(\mathcal{E}_g \cup g^{-1}(\mathcal{E}_f)) - \omega_\Gamma(\mathcal{E}_g \cap g^{-1}(\mathcal{E}_f)) \pmod{2} \\
&= \omega_\Gamma(\mathcal{E}_g) + \omega_\Gamma(g^{-1}(\mathcal{E}_f)) - 2\omega_\Gamma(\mathcal{E}_g \cap g^{-1}(\mathcal{E}_f)) \pmod{2} \\
&= \omega_\Gamma(\mathcal{E}_g) + \omega_\Gamma(g^{-1}(\mathcal{E}_f)) \pmod{2} \\
&= \widehat{\varepsilon}_\Gamma^\boxtimes}(g) + \widehat{\varepsilon}_\Gamma^\boxtimes}(f)
\end{aligned}$$

□

We notice that every finitely supported permutation is in $\text{Ker}(\widehat{\varepsilon}_\Gamma^\boxtimes)$.

Corollary 6.23. *There exists a surjective group homomorphism $\varepsilon_\Gamma^\boxtimes : \text{IET}^\boxtimes(\Gamma) \rightarrow \otimes_{\mathbb{Z}}^2 \tilde{\Gamma} / 2(\otimes_{\mathbb{Z}}^2 \tilde{\Gamma})$.*

Example 6.24. We give the value of $\varepsilon_\Gamma^\boxtimes$ for two kinds of elements:

- (1) Let r be a Γ -reflection of type a then $\varepsilon_\Gamma^\boxtimes}(r) = a \otimes a \pmod{2}$,
- (2) Let s be a Γ -restricted rotation of type (p, q) then:

$$\varepsilon_\Gamma^\boxtimes}(s) = p \otimes q + q \otimes p \pmod{2}$$

As IET^\boxtimes is generated by reflections, we deduce the image of $\varepsilon_\Gamma^\boxtimes$:

Corollary 6.25. *We have the following isomorphism:*

$$\text{IET}^\boxtimes(\Gamma) / \text{Ker}(\varepsilon_\Gamma^\boxtimes) \simeq \text{Im}(\varepsilon_\Gamma^\boxtimes) = \langle \{a \otimes a \pmod{2} \mid a \in \tilde{\Gamma}\} \rangle,$$

where $\langle \{a \otimes a \pmod{2} \mid a \in \tilde{\Gamma}\} \rangle$ is a subgroup of $\otimes_{\mathbb{Z}}^2 \tilde{\Gamma} / 2(\otimes_{\mathbb{Z}}^2 \tilde{\Gamma})$.

6.2.3. Description of $\text{Ker}(\varepsilon_\Gamma^\boxtimes)$.

As $\otimes_{\mathbb{Z}}^2 \tilde{\Gamma} / 2(\otimes_{\mathbb{Z}}^2 \tilde{\Gamma})$ is an abelian group we know that $D(\text{IET}^\boxtimes(\Gamma))$ is included in $\text{Ker}(\varepsilon_\Gamma^\boxtimes)$. We will see later that the other inclusion is false in general.

Notation 6.26. We denote by Ω_Γ the conjugate closure of the group generated by the set of all Γ -reflections of type 2ℓ with $\ell \in \tilde{\Gamma} \setminus 2\tilde{\Gamma}$ (the closure inside $\text{IET}^\boxtimes(\Gamma)$).

The inclusion $\Omega_\Gamma \subset \text{Ker}(\varepsilon_\Gamma^\boxtimes)$ is immediate. By 2.11 we know that every restricted rotation of type (ℓ, ℓ) , with $\ell \in \tilde{\Gamma}$, is in Ω_Γ .

The aim here is to show the equality $\text{Ker}(\varepsilon_\Gamma^\boxtimes) = D(\text{IET}^\boxtimes(\Gamma))\Omega_\Gamma$.

We begin by proving the result in the specific case where Γ has finite rank. We will reduce the general case to this one.

Lemma 6.27. *Let Γ be a finitely generated subgroup of \mathbb{R}/\mathbb{Z} . Then for every $f \in \text{Ker}(\varepsilon_\Gamma^\boxtimes)$ there exist $\delta \in D(\text{IET}^\boxtimes(\Gamma))$ and $h \in \Omega_\Gamma$ such that $f = \delta h$.*

Proof. As Γ is finitely generated, we know that $\tilde{\Gamma}$ is finitely generated. Let d be the rank of $\tilde{\Gamma}$.

Let $f \in \text{Ker}(\varepsilon_\Gamma^\boxtimes)$. Let $n \in \mathbb{N}$ and $\mathcal{P} := \{I_1, I_2, \dots, I_n\}$ be a partition into Γ -intervals associated with f . We denote by L_i the length of I_i for every $1 \leq i \leq n$. By Corollary 2.14 there exists $B := \{\ell_1, \ell_2, \dots, \ell_d\}$ a basis of $\tilde{\Gamma}$ with elements in $\tilde{\Gamma}_+$ such that $L_i \in \text{Vect}_{\mathbb{N}}(B)$ for every $1 \leq i \leq n$. Hence we can cut each I_i into smaller intervals with length in B . This operation gives us a new partition $\mathcal{Q} := \{J_1, J_2, \dots, J_k\}$, with $k \in \mathbb{N}$, into Γ -intervals associated with f .

For every $1 \leq i \leq k$ we define r_i as the J_i -reflection if f is order-reversing on J_i else we put $r_i = \text{Id}$. Let g be the product $fr_1r_2 \dots r_k$; it is an element of $\text{IET}(\Gamma)$ and \mathcal{Q} is a partition into Γ -intervals associated with g . By Lemma 2.7 g can be written as a finite product of Γ -restricted rotations with type inside $\{(l_p, l_q) \mid p, q \in \{1, 2, \dots, d\}\}$. Thanks to an element of $D(\text{IET}^\boxtimes(\Gamma))$ we can organize this product to put all Γ -restricted rotations of type (l_p, l_p) together (with $1 \leq p \leq d$; they are elements in Ω_Γ): there exist $w_1 \in D(\text{IET}^\boxtimes(\Gamma))$ and $h \in \Omega_\Gamma$ and $m \in \mathbb{N}$ and s_1, s_2, \dots, s_m some Γ -restricted rotations with type inside $\{(l_p, l_q) \mid p, q \in \{1, 2, \dots, d\}, p \neq q\}$ such that $g = w_1 h s_1 s_2 \dots s_m$. Then $f = w_1 h s_1 s_2 \dots s_m r_k r_{k-1} \dots r_1$.

We define $u_p := \text{Card}\{i \in \{1, 2, \dots, k\} \mid r_i \neq \text{Id}, \text{type}(r_i) = l_p\}$. Let $v_{p,p} = 0$ for every $1 \leq p \leq d$ and let $v_{p,q} := \text{Card}\{j \in \{1, 2, \dots, m\} \mid \text{type}(s_j) = (l_p, l_q)\}$ for every $1 \leq p \neq q \leq d$. Then we have:

$$\varepsilon_\Gamma^\boxtimes(f) = \sum_{i=1}^k \varepsilon_\Gamma^\boxtimes(r_i) + \sum_{j=1}^m \varepsilon_\Gamma^\boxtimes(s_j) = \sum_{p=1}^d u_p l_p \otimes l_p + \sum_{p=1}^d \sum_{q=1}^d v_{p,q} (l_p \otimes l_q + l_q \otimes l_p) \pmod{2} = 0$$

We notice that $\sum_{p=1}^d \sum_{q=1}^d v_{p,q} (l_p \otimes l_q + l_q \otimes l_p) = \sum_{p=1}^d \sum_{q=1}^d (v_{p,q} + v_{q,p}) l_p \otimes l_q$. Furthermore B is a basis of $\tilde{\Gamma}$ so $\{l_p \otimes l_q\}_{1 \leq p, q \leq d}$ is a basis of $\otimes_{\mathbb{Z}}^2 \tilde{\Gamma}$, thus we deduce that 2 divides u_p for every $1 \leq p \leq d$ and 2 divides $v_{p,q} + v_{q,p}$ for every $1 \leq p, q \leq d$.

We obtain that $r_1 r_2 \dots r_k$ is a balanced product of Γ -reflections hence by Lemma 6.13 it is an element of $D(\text{IET}^\boxtimes(\Gamma))$ denoted w_2 . We also deduce that the product $s_1 s_2 \dots s_m$ is a balanced product of Γ -restricted rotations with type inside $\{(a, b) \mid a \neq b \in \tilde{\Gamma}_+\}$. Hence by Proposition 6.15 we obtain that it is also an element of $D(\text{IET}^\boxtimes(\Gamma))$, denoted w_3 .

Finally we have $f = w_1 h w_3 w_2 = \delta h$ with $\delta \in D(\text{IET}^\boxtimes(\Gamma))$ and $h \in \Omega_\Gamma$. \square

The next lemma gives an inclusion used to conclude in the general case:

Lemma 6.28. *For all Γ , A subgroups of \mathbb{R}/\mathbb{Z} such that $A \subset \Gamma$ we have:*

$$D(\text{IET}^\boxtimes(A))\Omega_A \subset D(\text{IET}^\boxtimes(\Gamma))\Omega_\Gamma$$

Proof. The inclusion $D(\text{IET}^\boxtimes(A)) \subset D(\text{IET}^\boxtimes(\Gamma))$ is immediate. It is sufficient to show that $\Omega_A \subset D(\text{IET}^\boxtimes(\Gamma))\Omega_\Gamma$. Let \tilde{A} be the preimage of A in \mathbb{R} .

Let f be an element of Ω_A . Then there exist $n \in \mathbb{N}$ and $a_1, a_2, \dots, a_n \in \tilde{A} \setminus 2\tilde{A}$ and w_1, w_2, \dots, w_n some Γ -reflections such that the type of w_i is $2a_i$ and there exist $g_1, g_2, \dots, g_n \in \text{IET}^{\boxtimes}(A)$ such that $f = \prod_{i=1}^n g_i w_i g_i^{-1}$.

Let $U := \{i \in \{1, 2, \dots, n\} \mid a_i \in \tilde{\Gamma} \setminus 2\tilde{\Gamma}\}$. By definition we have $\{g_i w_i g_i^{-1} \mid i \in U\} \subset \Omega_\Gamma$. Take $V := \{1, 2, \dots, n\} \setminus U$. As A is a subgroup of Γ we deduce that $V = \{j \in \{1, 2, \dots, n\} \mid a_j \in 2\tilde{\Gamma}\}$. Thus for every $j \in V$ the type of w_j is in $4\tilde{\Gamma}$, then by Proposition 6.1 we deduce that w_j and $g_j w_j g_j^{-1}$ belong to $D(\text{IET}^{\boxtimes}(\Gamma))$. We know that there exists $h \in D(\text{IET}^{\boxtimes}(\Gamma))$ such that:

$$f = h \prod_{j \in V} g_j w_j g_j^{-1} \prod_{i \in U} g_i w_i g_i^{-1}$$

Then $f \in D(\text{IET}^{\boxtimes}(\Gamma))\Omega_\Gamma$. \square

We can prove the theorem for the general case:

Theorem 6.29. *For any dense subgroup Γ of \mathbb{R}/\mathbb{Z} we have:*

$$\text{Ker}(\varepsilon_\Gamma^{\boxtimes}) = D(\text{IET}^{\boxtimes}(\Gamma))\Omega_\Gamma$$

Proof. The inclusion from right to left is already proved.

Let $f \in \text{Ker}(\varepsilon_\Gamma^{\boxtimes})$, let $n \in \mathbb{N}$ and $\mathcal{P} := \{I_1, I_2, \dots, I_n\}$ be a partition into Γ -intervals associated with f . We denote by L_i the length of I_i for every $1 \leq i \leq n$. As $\varepsilon_\Gamma^{\boxtimes}(f) = 0$ we know that there exist $k \in \mathbb{N}$ and $a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k \in \tilde{\Gamma}$ such that $\omega_\Gamma(\mathcal{E}_f) = 2 \sum_{i=1}^k a_i \otimes b_i$ inside $\otimes_{\mathbb{Z}}^2 \tilde{\Gamma}$.

Let \tilde{A} be the subgroup of \mathbb{R} generated by $\{L_i\}_{i=1 \dots n} \cup \{1\} \cup \{a_i, b_i\}_{i=1 \dots k}$. Then \tilde{A} contains \mathbb{Z} , is finitely generated and is a subgroup of $\tilde{\Gamma}$. Let A be the image of \tilde{A} in \mathbb{R}/\mathbb{Z} . The partition \mathcal{P} is also a partition into A -intervals associated with f thus f belongs to $\text{IET}^{\boxtimes}(A)$. Furthermore we have $\omega_A(\mathcal{E}_f) = \omega_\Gamma(\mathcal{E}_f)$. Hence:

$$\varepsilon_A^{\boxtimes}(f) = [\omega_A(\mathcal{E}_f)]_{\otimes_{\mathbb{Z}}^2 A/2(\otimes_{\mathbb{Z}}^2 A)} = [\omega_\Gamma(\mathcal{E}_f)]_{\otimes_{\mathbb{Z}}^2 A/2(\otimes_{\mathbb{Z}}^2 A)} = [2 \sum_{i=1}^k a_i \otimes b_i]_{\otimes_{\mathbb{Z}}^2 A/2(\otimes_{\mathbb{Z}}^2 A)} = 0$$

By Lemma 6.27 we deduce that $f \in D(\text{IET}^{\boxtimes}(A))\Omega_A$ and by Lemma 6.28 we deduce that $f \in D(\text{IET}^{\boxtimes}(\Gamma))\Omega_\Gamma$. \square

6.3. The positive contribution.

We know that every Γ -reflection of type 2ℓ with $\ell \in \tilde{\Gamma} \setminus 2\tilde{\Gamma}$ is conjugated to a Γ -restricted rotation of type (ℓ, ℓ) . Also this is an element of $\text{IET}^+(\Gamma)$ which is not send on the trivial element by the morphism ε_Γ . We use the notion of positive substitute in order to use the group homomorphism ε_Γ to send such a reflection on a nontrivial element. For this we need to use the sequence $(S_n)_{n \in \mathbb{N}}$ of \mathbb{Z} -independent subset of $\tilde{\Gamma}_+$ given by Corollary 2.13. We recall that for each $n \in \mathbb{N}$ we have $\text{Vect}_{\mathbb{N}}(S_n) \subset \text{Vect}_{\mathbb{N}}(S_{n+1})$ and $\tilde{\Gamma}_+$ is equal to the direct limit $\varinjlim \text{Vect}_{\mathbb{N}}(S_n)$. Also if Γ is finitely generated we can choose S_n to be a basis of $\tilde{\Gamma}$.

For every $n \in \mathbb{N}$ we construct a map that depends on S_n . They are not group homomorphisms but satisfies the group homomorphism property on some products thanks to Proposition 6.11.

6.3.1. Some subsets of $\text{IET}^{\boxtimes}(\Gamma)$.

Let S be a finite set of $\tilde{\Gamma}_+$. We denote by G_S the set of all f in $\text{IET}^{\boxtimes}(\Gamma)$ such that there exists a partition \mathcal{P} into S -intervals associated with f . We remark that G_S is not a group in general. We want to know how these sets and $\text{IET}^{\boxtimes}(\Gamma)$ are linked.

Proposition 6.30. *Let S and T be two finite subsets of $\tilde{\Gamma}_+$. If $S \subset \text{Vect}_{\mathbb{N}}(T)$ then $G_S \subset G_T$. More precisely for every partition \mathcal{P} into S -intervals of $[0, 1[$ there exists a refinement \mathcal{Q} of \mathcal{P} which is a partition into T -intervals of $[0, 1[$.*

Proof. Let $f \in G_S$ and let \mathcal{P} be a partition into S -intervals associated with f . As $S \subset \text{Vect}_{\mathbb{N}}(T)$ each interval $I \in \mathcal{P}$ can be subdivided with intervals of length in T . After subdividing this way the intervals of \mathcal{P} , we obtain a refinement \mathcal{Q} of \mathcal{P} . We notice that \mathcal{Q} is a partition into T -intervals associated with f so $f \in G_T$. \square

As $\tilde{\Gamma}_+$ is the filtered union of the $\text{Vect}_{\mathbb{N}}(S_n)$ we deduce the next proposition:

Proposition 6.31. *For every finite subset S of $\tilde{\Gamma}_+$ there exists $N \in \mathbb{N}$ such that for every $n \geq N$ we have $S \subset \text{Vect}_{\mathbb{N}}(S_n)$.*

From Propositions 6.30 and 6.31 we obtain:

Corollary 6.32. *Let $f \in \text{IET}^{\boxtimes}(\Gamma)$. There exists $N \in \mathbb{N}$ such that for every $n \geq N$ we have $f \in G_{S_n}$. Furthermore $\text{IET}^{\boxtimes}(\Gamma) = \bigcup_{n \in \mathbb{N}} G_{S_n}$.*

We also want to check that if we take any product of elements in $\text{IET}^{\boxtimes}(\Gamma)$ there will be a moment where we have a partition associated to the tuple of these elements.

Proposition 6.33. *Let $k \in \mathbb{N}$ and $f_1, f_2, \dots, f_k \in \text{IET}^{\boxtimes}(\Gamma)$. There exists $N \in \mathbb{N}$ such that for every $n \geq N$ there exists \mathcal{P}_n a partition into S_n -intervals associated with (f_1, f_2, \dots, f_k) .*

Proof. Thanks to Proposition 6.30 and Remark 6.7 we deduce that it is sufficient to find only one $n \in \mathbb{N}$ where the statement is satisfied.

By Proposition 6.32 there exist $n_i \in \mathbb{N}$ and \mathcal{P}_i a partition into S_{n_i} intervals associated with f_i for every $1 \leq i \leq k$. We denote by V_i the set of all the endpoints of the intervals in \mathcal{P}_i . Let $V = V_1 \cup f_1^{-1}(V_2) \cup \dots \cup (f_{k-1} \dots f_1)^{-1}(V_k)$. We know that V is finite thus there exist $m \in \mathbb{N}$ and $v_0, v_1, \dots, v_m \in \tilde{\Gamma}_+$ such that $V = \{v_i\}_{i=0 \dots m}$. Up to change the order we can assume that $v_0 = 0 < v_1 < v_2 < \dots < v_{m-1} < v_m = 1$. Let I_i be the interval $[v_{i-1}, v_i[$ and ℓ_i be the length of I_i for every $1 \leq i \leq m$. Let $T = \{\ell_i\}_{i=1 \dots m}$ and $\mathcal{P} = \{I_i\}_{i=1 \dots m}$.

We have $V_1 \subset V$ then \mathcal{P} is a refinement of \mathcal{P}_1 so \mathcal{P} is a partition into T -intervals

associated with f_1 . Similarly for every $2 \leq i \leq k$ we know that $V_i \subset f_{i-1} \dots f_1(V)$ thus $f_{i-1} \dots f_1(\mathcal{P})$ is a refinement of \mathcal{P}_i so $f_{i-1} \dots f_1(\mathcal{P})$ is a partition into T -intervals associated with f_i . Hence \mathcal{P} is a partition into T -intervals associated with (f_1, f_2, \dots, f_k) .

Thanks to Proposition 6.31 there exists $n \in \mathbb{N}$ such that $T \subset \text{Vect}_{\mathbb{N}}(S_n)$ and by Proposition 6.30 and Remark 6.7 there exists a refinement \mathcal{Q} of \mathcal{P} which is a partition into S_n -intervals associated with (f_1, f_2, \dots, f_k) . □

6.3.2. The S -map.

Let S be a finite subset of $\tilde{\Gamma}_+$ for all this subsection. We assume that S is free inside $\text{Vect}_{\mathbb{Z}}(S)$. For every $a, b \in \tilde{\Gamma}$, we use the notation $[a \wedge b]_{2 \circ \wedge_{\mathbb{Z}}^2 \tilde{\Gamma}} = a \wedge b \pmod{2}$.

Let $f \in G_S$ and \mathcal{P} be a S -partition associated with f . We show that the value $\varepsilon_{\Gamma}(f_{\mathcal{P}}^+) \pmod{2}$ does not depend on \mathcal{P} , where ε_{Γ} is the group homomorphism define in 5.9.

Proposition 6.34. *Let $f \in \text{IET}^{\boxtimes}(\Gamma)$ and let \mathcal{P}, \mathcal{Q} be two partitions into S -intervals associated with f . Then $\varepsilon_{\Gamma}(f_{\mathcal{P}}^+) \pmod{2} = \varepsilon_{\Gamma}(f_{\mathcal{Q}}^+) \pmod{2}$.*

Proof. For this proof we denote by r_I the I -reflection for every subinterval I of $[0, 1[$ and we recall that λ is the Lebesgue measure. First we reduce the case to a simpler one.

Let $f \in \text{IET}^{\boxtimes}(\Gamma)$ and \mathcal{P}, \mathcal{Q} be two partitions into S -intervals associated with f . Let $n, k \in \mathbb{N}$ such that $\mathcal{P} = \{I_1, I_2, \dots, I_n\}$ and $\mathcal{Q} = \{J_1, J_2, \dots, J_k\}$. Up to change the index we can assume that the I_i are consecutive intervals and so are the J_j . We denote by \mathcal{M} the unique partition into Γ -intervals associated with f that has the minimal number of intervals. Let m be this number and $\mathcal{M} := \{M_1, M_2, \dots, M_m\}$ where the M_i are consecutive intervals. As \mathcal{P}, \mathcal{Q} are also partitions into Γ -intervals we know that they are refinements of \mathcal{M} .

Let $1 \leq i \leq n$ and let $n_0 = k_0 = 0$. There exist $n_1 < n_2 < \dots < n_m, k_1 < k_2 < \dots < k_m \in \mathbb{N}$ such that $M_i = \bigsqcup_{j=n_{i-1}+1}^{n_i} I_j = \bigsqcup_{j=k_{i-1}+1}^{k_i} J_j$. Hence $\lambda(M_i) =$

$\sum_{j=n_{i-1}+1}^{n_i} \lambda(I_j) = \sum_{j=k_{i-1}+1}^{k_i} \lambda(J_j)$. For every $s \in S$ we define $a_s = \text{Card}(\{j \in \{n_{i-1} + 1, \dots, n_i\} \mid \lambda(I_j) = s\})$ and $b_s = \text{Card}(\{j \in \{k_{i-1} + 1, \dots, k_i\} \mid \lambda(J_j) = s\})$. Then $\sum_{s \in S} a_s s = \sum_{s \in S} b_s s$. As S is free in $\text{Vect}_{\mathbb{Z}}(S)$ we deduce that $a_s = b_s$ for every $s \in S$.

This implies that $n_i = k_i$ for every $1 \leq i \leq n$ and $n = k$. We also get the existence of a permutation σ_i of the set $\{n_{i-1} + 1, \dots, n_i\}$ such that $\lambda(I_j) = \lambda(J_{\sigma_i(j)})$ for every $j \in \{n_{i-1} + 1, \dots, n_i\}$.

We deduce that it is sufficient to show the result when only one permutation σ_i is a transposition $(a \ a + 1)$ with $n_{i-1} + 1 \leq a < n_i$.

Let i_0 in $\{1, 2, \dots, m\}$ and $a \in \{n_{i_0-1} + 1, \dots, n_{i_0}\}$. We assume now that σ_{i_0}

is the transposition $(a \ a + 1)$ and $\sigma_i = \text{Id}$ for every $i \neq i_0$. Then $I_j = J_j$ for every $j \in \{1, 2, \dots, n\} \setminus \{a, a + 1\}$. We also have $I_a \cup I_{a+1} = J_a \cup J_{a+1}$ and $\lambda(I_{a+1}) = \lambda(J_a)$. In the case where f is order-preserving on M_{i_0} then $f_{\mathcal{P}}^+ = f_{\mathcal{Q}}^+$. In the case where f is order-reversing on M_{i_0} we deduce that $(f_{\mathcal{Q}}^+)^{-1} \circ f_{\mathcal{P}}^+$ is equal to the square of the restricted rotation whose intervals associated are I_a and I_{a+1} (see Figure 9).

In both cases we deduce $\varepsilon_{\Gamma}(f_{\mathcal{P}}^+) \pmod{2} = \varepsilon_{\Gamma}(f_{\mathcal{Q}}^+) \pmod{2}$. \square

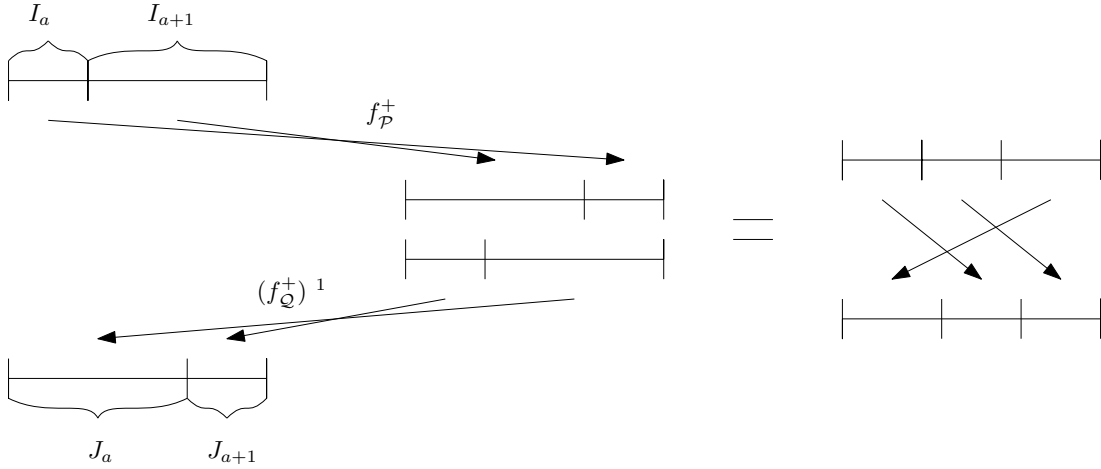


FIGURE 9. Illustration of the fact that $(f_{\mathcal{Q}}^+)^{-1} \circ f_{\mathcal{P}}^+$ is equal to the square of a restricted rotation in Proposition 6.34

Definition 6.35. We define the S -map as:

$$\begin{aligned} \psi_S : \text{IET}^{\boxtimes}(\Gamma) &\longrightarrow \ominus \Lambda_{\mathbb{Z}}^2 \tilde{\Gamma} / 2 \ominus \Lambda_{\mathbb{Z}}^2 \tilde{\Gamma} \\ f &\longmapsto \begin{cases} \varepsilon_{\Gamma}(f_{\mathcal{P}}^+) \pmod{2} & f \in G_S \\ 0 & f \notin G_S \end{cases} \end{aligned}$$

Where \mathcal{P} is a partition into S -intervals associated with f .

Thanks to Proposition 6.34, the map ψ_{Γ} is well-defined. We check that ψ_S satisfies the morphism condition when we have the existence of a partition in S -intervals associated with a tuple.

Proposition 6.36. Let $n \in \mathbb{N}$ and $f_1, f_2, \dots, f_n \in \text{IET}^{\boxtimes}(\Gamma)$. If there exists \mathcal{P} a partition into S -intervals associated with (f_1, f_2, \dots, f_n) then:

$$\psi_S(f_n f_{n-1} \dots f_1) = \sum_{i=1}^n \psi_S(f_i)$$

Proof. Let $n \in \mathbb{N}$ and $f_1, f_2, \dots, f_n \in \text{IET}^{\boxtimes}(\Gamma)$. The case $n = 1$ is trivial thus we assume $n \geq 2$. Let \mathcal{P} be a partition into S -intervals associated with

(f_1, f_2, \dots, f_n) . Let $g_1 = (f_1)_{\mathcal{P}}^+$ and for every $2 \leq i \leq n$ let $g_i = (f_i)_{f_{i-1} \dots f_1(\mathcal{P})}^+$. By Proposition 6.11 we know that $(f_n f_{n-1} \dots f_1)_{\mathcal{P}}^+ = g_n g_{n-1} \dots g_1$, then:

$$\psi_S(f_n f_{n-1} \dots f_1) = \varepsilon_\Gamma((f_n f_{n-1} \dots f_1)_{\mathcal{P}}^+) \pmod{2} = \varepsilon_\Gamma(g_n g_{n-1} \dots g_1) \pmod{2}$$

Also ε_Γ is a group homomorphism thus:

$$\psi_S(f_n f_{n-1} \dots f_1) = \sum_{i=1}^n \varepsilon_\Gamma(g_i) \pmod{2} = \sum_{i=1}^n \psi_S(f_i)$$

□

6.3.3. The group homomorphism.

We are now able to define a new group homomorphism from $\text{IET}^{\boxtimes}(\Gamma)$.

Definition 6.37. We define the *positive contribution* of $\text{IET}^{\boxtimes}(\Gamma)$ as the next map:

$$\begin{aligned} \psi_\Gamma : \text{IET}^{\boxtimes}(\Gamma) &\longrightarrow (\ominus \Lambda_{\mathbb{Z}}^2 \tilde{\Gamma} / 2 \ominus \Lambda_{\mathbb{Z}}^2 \tilde{\Gamma})^{\mathbb{N}} / \{(w_n)_{n \in \mathbb{N}} \mid \exists N \in \mathbb{N}, \forall n \geq N, w_n = 0\} \\ f &\longmapsto [(\psi_{S_n}(f))_{n \in \mathbb{N}}] \end{aligned}$$

Proposition 6.38. *The map ψ_Γ is a group homomorphism.*

Proof. Let $f, g \in \text{IET}^{\boxtimes}(\Gamma)$. By Proposition 6.33 there exists $N \in \mathbb{N}$ such that for every $n \geq N$ there exists a partition \mathcal{P}_n into S_n -intervals associated with (f, g) . Then by Proposition 6.36 we deduce that for every $n \geq N$ we have $\psi_{S_n}(g \circ f) = \psi_{S_n}(g) + \psi_{S_n}(f)$. We denote by $(w_n)_{n \in \mathbb{N}}$ the element of $(\ominus \Lambda_{\mathbb{Z}}^2 \tilde{\Gamma} / 2 \ominus \Lambda_{\mathbb{Z}}^2 \tilde{\Gamma})^{\mathbb{N}}$ defined by $w_n = \psi_{S_n}(g \circ f) - \psi_{S_n}(g) - \psi_{S_n}(f)$ for every $n < N$ and 0 elsewhere. We notice that $(\psi_{S_n}(g \circ f))_n = (\psi_{S_n}(g))_n + (\psi_{S_n}(f))_n - w_n$. So $\psi_\Gamma(g \circ f) = \psi_\Gamma(g) + \psi_\Gamma(f)$. □

The following lemma gives the value of an element in $\text{IET}(\Gamma)$. The proof is immediate from the definition of ψ_{S_n} .

Lemma 6.39. *For every $f \in \text{IET}(\Gamma)$ there exists $N \in \mathbb{N}$ such that for $n < N$ we have $\psi_{S_n}(f) = 0$ and for $n \geq N$ we have $\psi_{S_n}(f) = \varepsilon_\Gamma(f) \pmod{2}$.*

The next proposition show that the set we used to define Ω_Γ is not sent to the trivial element by ψ_Γ ; thus this is not a subset of $D(\text{IET}^{\boxtimes}(\Gamma))$.

Proposition 6.40. *Let $\ell \in \tilde{\Gamma} \setminus 2\tilde{\Gamma}$ and r be a Γ -reflection of type 2ℓ . Then $\psi_\Gamma(r) = [(\ell \wedge \ell \pmod{2})_n] \neq 0$.*

Proof. We know that such a Γ -reflection is conjugate to a Γ -restricted rotation of type (ℓ, ℓ) . We denote by s this Γ -restricted rotation. We have $\varepsilon_\Gamma(s) = \ell \wedge \ell \pmod{2}$. Then by Lemma 6.39 we have $\psi_\Gamma(r) = \psi_\Gamma(s) = [(\ell \wedge \ell \pmod{2})_n]$. Thus if $\psi_\Gamma(r) = 0$ then there exists $(w_n)_n \in (\ominus \Lambda_{\mathbb{Z}}^2 \tilde{\Gamma} / 2 \ominus \Lambda_{\mathbb{Z}}^2 \tilde{\Gamma})^{\mathbb{N}}$ which is 0 above a certain rank such that $(\ell \wedge \ell \pmod{2})_n + (w_n)_n = 0$. As w_n is 0 above a certain rank, we deduce that $\ell \wedge \ell \pmod{2} = 0$. Or this implies $\ell \in 2\tilde{\Gamma}$ and this is a contradiction. Hence $\psi_\Gamma(r) \neq 0$. □

6.4. Description of $\text{IET}^{\boxtimes}(\Gamma)_{\text{ab}}$.

With both morphisms $\varepsilon_{\Gamma}^{\boxtimes}$ and ψ_{Γ} we are now able to describe $D(\text{IET}^{\boxtimes}(\Gamma))$. We recall that Ω_{Γ} is the conjugate closure of the group generated by the set of all Γ -reflections of type 2ℓ with $\ell \in \tilde{\Gamma} \setminus 2\tilde{\Gamma}$.

Lemma 6.41. *We have the inclusion:*

$$\Omega_{\Gamma} \cap \text{Ker}(\psi_{\Gamma}) \subset D(\text{IET}^{\boxtimes}(\Gamma))$$

Proof. Let $w \in \Omega_{\Gamma} \cap \text{Ker}(\psi_{\Gamma})$. There exist $n \in \mathbb{N}$ and $g_1, g_2, \dots, g_n \in \text{IET}^{\boxtimes}(\Gamma)$ and r_1, r_2, \dots, r_n some Γ -reflections with type inside $\tilde{\Gamma} \setminus 2\tilde{\Gamma}$ such that $w = \prod_{i=1}^n g_i r_i g_i^{-1}$. We know that every Γ -reflection is conjugate to a Γ -restricted rotation so there exist $h_1, h_2, \dots, h_n \in \text{IET}^{\boxtimes}(\Gamma)$ and s_1, s_2, \dots, s_n some Γ -restricted rotations such that $s_i = h_i r_i h_i^{-1}$ for every $1 \leq i \leq n$. Then $w = \prod_{i=1}^n g_i h_i^{-1} s_i h_i g_i^{-1}$. By Lemma 6.13 it is sufficient to show that w is a balanced product of Γ -reflections.

As for every $1 \leq i \leq n$ we have g_i and g_i^{-1} which appear the same number of time in w (the same is true for h and h^{-1}) we deduce that it is sufficient to show that $w' := \prod_{i=1}^n s_i$ is a balanced product of Γ -reflections. By Lemma 6.13 it is enough to show that $w' \in D(\text{IET}^{\boxtimes}(\Gamma))$.

We notice that $\psi_{\Gamma}(w) = \psi_{\Gamma}(w')$ thus by the assumption we deduce that $\psi_{\Gamma}(w') = 0$. Hence by Lemma 6.39 we deduce that $\varepsilon_{\Gamma}(w') \pmod{2} = 0$. This equality stands in ${}^{\circ}\Lambda_{\mathbb{Z}}^2 \tilde{\Gamma} / 2 {}^{\circ}\Lambda_{\mathbb{Z}}^2 \tilde{\Gamma}$. There exists $k \in \mathbb{N}$ and for every $1 \leq j \leq k$ there exist $a_j, b_j \in \tilde{\Gamma}_+$ with $a_j + b_j < 1$ and $n_j \in \mathbb{N}$ and $\nu_j \in \{-1, 1\}$ such that $\varepsilon_{\Gamma}(w') + \sum_{j=1}^k 2n_j \nu_j a_j \wedge b_j = 0$. For every $1 \leq j \leq k$ let γ_j be a Γ -restricted rotation of type (a_j, b_j) . Then the element $w' \prod_{j=1}^k (\gamma_j^{n_j \nu_j})^2$ is in $\text{IET}(\Gamma)$ and satisfies:

$$\varepsilon_{\Gamma}(w' \prod_{j=1}^k (\gamma_j^{n_j \nu_j})^2) = \varepsilon_{\Gamma}(w') + \sum_{j=1}^k 2n_j \nu_j a_j \wedge b_j = 0$$

By Theorem 5.23 the element $w' \prod_{j=1}^k (\gamma_j^{n_j \nu_j})^2$ is in $D(\text{IET}(\Gamma)) \subset D(\text{IET}^{\boxtimes}(\Gamma))$.

By Corollary 6.14 we know that $\prod_{j=1}^k (\gamma_j^{n_j \nu_j})^2$ is in $D(\text{IET}^{\boxtimes}(\Gamma))$. Then we deduce that $w' \in D(\text{IET}^{\boxtimes}(\Gamma))$. Hence w is in $D(\text{IET}^{\boxtimes}(\Gamma))$. \square

Theorem 6.42. *We have $D(\text{IET}^{\boxtimes}(\Gamma)) = \text{Ker}(\varepsilon_{\Gamma}^{\boxtimes}) \cap \text{Ker}(\psi_{\Gamma})$.*

Proof. The inclusion from left to right is trivial.

Let $f \in \text{Ker}(\varepsilon_{\Gamma}^{\boxtimes}) \cap \text{Ker}(\psi_{\Gamma})$. By Theorem 6.29 there exist $g \in D(\text{IET}^{\boxtimes}(\Gamma))$ and $h \in \Omega_{\Gamma}$ such that $f = gh$. We deduce that $\psi_{\Gamma}(h) = \psi_{\Gamma}(f) = 0$ so $h \in \text{Ker}(\psi_{\Gamma}) \cap \Omega_{\Gamma}$. By Lemma 6.41 we obtain that $h \in D(\text{IET}^{\boxtimes}(\Gamma))$, thus $f \in D(\text{IET}^{\boxtimes}(\Gamma))$. \square

Corollary 6.43. *We have $D(\text{IET}^{\boxtimes}(\Gamma)) = \text{Ker}(\psi_{\Gamma}|_{\text{Ker}(\varepsilon_{\Gamma}^{\boxtimes})})$.*

Lemma 6.44. *The quotient $\text{Ker}(\varepsilon_{\Gamma}^{\boxtimes})/D(\text{IET}^{\boxtimes}(\Gamma))$ is isomorphic to the subgroup $\langle \{\ell \wedge \ell \pmod{2} \mid \ell \in \tilde{\Gamma}\} \rangle$ of ${}^{\circ}\Lambda_{\mathbb{Z}}^2 \tilde{\Gamma}/2{}^{\circ}\Lambda_{\mathbb{Z}}^2 \tilde{\Gamma}$.*

Proof. By Corollary 6.43 we have:

$$\text{Ker}(\varepsilon_{\Gamma}^{\boxtimes})/D(\text{IET}^{\boxtimes}(\Gamma)) \simeq \text{Im}(\psi_{\Gamma}|_{\text{Ker}(\varepsilon_{\Gamma}^{\boxtimes})}) = \psi_{\Gamma}(\text{Ker}(\varepsilon_{\Gamma}^{\boxtimes}))$$

By Theorem 6.29 and as ψ_{Γ} is a group homomorphism we have the equality $\psi_{\Gamma}(\text{Ker}(\varepsilon_{\Gamma}^{\boxtimes})) = \psi_{\Gamma}(\Omega_{\Gamma})$. Furthermore Ω_{Γ} is the normal closure of the group generated by all Γ -reflections of type 2ℓ with $\ell \in \tilde{\Gamma} \setminus 2\tilde{\Gamma}$. Hence we deduce that:

$$\psi_{\Gamma}(\Omega_{\Gamma}) = \langle \{[(\ell \wedge \ell \pmod{2})]_{n \in \mathbb{N}} \mid \ell \in \tilde{\Gamma} \setminus 2\tilde{\Gamma}\} \rangle = \langle \{[(\ell \wedge \ell \pmod{2})]_{n \in \mathbb{N}} \mid \ell \in \tilde{\Gamma}\} \rangle$$

Thus $\psi_{\Gamma}(\text{Ker}(\varepsilon_{\Gamma}^{\boxtimes})) \simeq \langle \{\ell \wedge \ell \pmod{2} \mid \ell \in \tilde{\Gamma}\} \rangle$. \square

Theorem 6.45. *We have the following group isomorphisms:*

$$\begin{aligned} \text{IET}^{\boxtimes}(\Gamma)_{\text{ab}} &\simeq \text{Im}(\varepsilon_{\Gamma}^{\boxtimes}) \times \text{Ker}(\varepsilon_{\Gamma}^{\boxtimes})/D(\text{IET}^{\boxtimes}(\Gamma)) \\ &\simeq \langle \{a \otimes a \pmod{2} \mid a \in \tilde{\Gamma}\} \rangle \times \langle \{\ell \wedge \ell \pmod{2} \mid \ell \in \tilde{\Gamma}\} \rangle, \end{aligned}$$

where the left term of the product is in $\otimes_{\mathbb{Z}}^2 \tilde{\Gamma}/(2 \otimes_{\mathbb{Z}}^2 \tilde{\Gamma})$ and the right one is in ${}^{\circ}\Lambda_{\mathbb{Z}}^2 \tilde{\Gamma}/(2{}^{\circ}\Lambda_{\mathbb{Z}}^2 \tilde{\Gamma})$.

Proof. The second isomorphism is given by Corollary 6.25 and Lemma 6.44.

For the first isomorphism we recall that we have the following exact sequence:

$$1 \rightarrow \text{Ker}(\varepsilon_{\Gamma}^{\boxtimes})/D(\text{IET}^{\boxtimes}(\Gamma)) \rightarrow \text{IET}^{\boxtimes}(\Gamma)_{\text{ab}} \rightarrow \text{IET}^{\boxtimes}(\Gamma)/\text{Ker}(\varepsilon_{\Gamma}^{\boxtimes}) \rightarrow 1$$

Each group in this exact sequence has exponent 2. Then they are also \mathbb{F}_2 -vectorial spaces. We deduce that this exact sequence is an exact sequence of \mathbb{F}_2 -vectorial spaces, thus it splits and gives the result. \square

Remark 6.46. If $\tilde{\Gamma}$ has dimension d then $\langle \{a \otimes a \pmod{2} \mid a \in \tilde{\Gamma}\} \rangle$ has dimension $\frac{d(d+1)}{2}$, as \mathbb{F}_2 -vector space, and $\langle \{\ell \wedge \ell \pmod{2} \mid \ell \in \tilde{\Gamma}\} \rangle$ has dimension d , as \mathbb{F}_2 -vector space, so $\text{IET}^{\boxtimes}(\Gamma)_{\text{ab}}$ has dimension $\frac{d(d+3)}{2}$ over \mathbb{F}_2 .

Remark 6.47. The inclusion of $\text{IET}(\Gamma)$ in $\text{IET}^{\boxtimes}(\Gamma)$ induces a group morphism ι from $\text{IET}(\Gamma)_{\text{ab}}/2(\text{IET}(\Gamma)_{\text{ab}})$ to $\text{IET}^{\boxtimes}(\Gamma)_{\text{ab}}$. By Lemma 2.7 we know that $\text{IET}(\Gamma)$ is generated by Γ -restricted rotations thus we deduce that the image of ι is the subgroup $\langle \{p \otimes q + q \otimes p \pmod{2} \mid p, q \in \tilde{\Gamma}\} \rangle \times \langle \{a \wedge a \pmod{2}\} \rangle$ of $\otimes_{\mathbb{Z}}^2 \tilde{\Gamma}/(2 \otimes_{\mathbb{Z}}^2 \tilde{\Gamma}) \times {}^{\circ}\Lambda_{\mathbb{Z}}^2 \tilde{\Gamma}/(2{}^{\circ}\Lambda_{\mathbb{Z}}^2 \tilde{\Gamma})$. This is isomorphic to ${}^{\circ}\Lambda_{\mathbb{Z}}^2 \tilde{\Gamma}/(2{}^{\circ}\Lambda_{\mathbb{Z}}^2 \tilde{\Gamma})$ and if $\tilde{\Gamma}$ has dimension d then its dimension is $\frac{d(d+1)}{2}$ as \mathbb{F}_2 -vector space. In this case ι is not surjective and its cokernel has dimension d over \mathbb{F}_2 . In the case where $\tilde{\Gamma}$ has infinite dimension over \mathbb{Z} we deduce that $\text{Im}(\iota)$ also has infinite dimension over \mathbb{F}_2 . By Proposition 5.17 we deduce that the group $\langle \{p \otimes q + q \otimes p \pmod{2} \mid p, q \in \tilde{\Gamma}\} \rangle$ is equal to the group $\langle \{a \otimes a \pmod{2} \mid a \in \tilde{\Gamma}\} \rangle$ if and only if $\tilde{\Gamma} = 2\tilde{\Gamma}$. Then ι is surjective if and only if $\tilde{\Gamma} = 2\tilde{\Gamma}$.

Proposition 6.48. *The group homomorphism ι is injective.*

Proof. For every $f \in \text{IET}(\Gamma)$ we denote by $[f]$ its image in $\text{IET}(\Gamma)_{\text{ab}}$. Thanks to Theorem 5.23 we know that $[f] \in 2\text{IET}(\Gamma)_{\text{ab}}$ if and only if $\varepsilon_{\Gamma}(f) \in 2^{\circ}\Lambda_{\mathbb{Z}}^2\Gamma$. Hence to prove the statement it is enough to prove that for every $f \in \text{IET}(\Gamma)$ such that $\varepsilon_{\Gamma}^{\boxtimes}(f) = 0$ and $\text{proj}_{\text{Ker}(\varepsilon_{\Gamma}^{\boxtimes})}(f) = 0$ we have $\varepsilon_{\Gamma}(f) \in 2^{\circ}\Lambda_{\mathbb{Z}}^2\Gamma$. We use notations of inversions defined in Definition 6.16. By Corollary 2.14 there exists $n \in \mathbb{N}$ and a \mathbb{Z} -linearly independent family $\{l_1, l_2, \dots, l_n\}$ of Γ_+ and $n_{i,j} \in \mathbb{Z}$ such that $\mathcal{E}_{f,1} = \sum_{i,j} n_{i,j} l_i \otimes l_j$. The equality $\text{proj}_{\text{Ker}(\varepsilon_{\Gamma}^{\boxtimes})}(f) = 0$ gives us that $n_{i,i} = 0$. We have $\varepsilon_{\Gamma}^{\boxtimes}(f) = \sum_{i \neq j} (n_{i,j} + n_{j,i}) l_i \otimes l_j \pmod{2} = 0$. We deduce that 2 divides $(n_{i,j} + n_{j,i})$ for every $1 \leq i \neq j \leq n$. We obtain that :

$$\begin{aligned} \varepsilon_{\Gamma}(f) &= \sum_{i \neq j} n_{i,j} l_i \wedge l_j \\ &= \sum_{i < j} (n_{i,j} - n_{j,i}) l_i \wedge l_j \\ &= \sum_{i < j} (n_{i,j} + n_{j,i}) l_i \wedge l_j - 2 \sum_{i < j} n_{j,i} l_i \wedge l_j \end{aligned}$$

We deduce that 2 divides $\varepsilon_{\Gamma}(f)$ and this gives the result. □

REFERENCES

- [1] P. Arnoux. Échanges d'intervalles et flots sur les surfaces. *Ergodic theory (Sem., Les Plans-sur-Bex, 1980)*, pages 5–38, 1981.
- [2] P. Arnoux. *Un invariant pour les échanges d'intervalles et les flots sur les surfaces*. PhD thesis, Université de Reims, 1981.
- [3] F. Dahmani, K. Fujiwara, and V. Guirardel. Free groups of interval exchange transformations are rare. *Groups Geom. Dyn.*, 7(4):883–910, 2013.
- [4] F. Dahmani, K. Fujiwara, and V. Guirardel. Solvable groups of interval exchange transformations. *arXiv: 1701.00377, to appear in Ann. Fac. Sci. Toulouse.*, 2017.
- [5] G. A. Elliott. On totally ordered groups, and K_0 . In *Ring Theory Waterloo 1978 Proceedings, University of Waterloo, Canada, 12–16 June, 1978*, pages 1–49. Springer, 1979.
- [6] N. Guelman and I. Liousse. Bounded simplicity of affine interval exchange transformations and interval exchange transformations. *arXiv:1910.08923*, 2019.
- [7] V. Nekrashevych. Simple groups of dynamical origin. *Ergodic Theory Dynam. Systems*, 39(3):707–732, 2019.
- [8] C. F. Novak. Discontinuity-growth of interval-exchange maps. *J. Mod. Dyn.*, 3(3):379–405, 2009.
- [9] C.-H. Sah. Scissors congruences of the interval. *preprint*, 1981.
- [10] M. Viana. Ergodic theory of interval exchange maps. *Rev. Mat. Complut.*, 19(1):7–100, 2006.
- [11] Y. B. Vorobets. On the commutator of the group of interval exchanges. *Tr. Mat. Inst. Steklova*, 297 (Poryadok i Khaos v Dinamicheskikh Sistemakh):313–325, 2017. English version: Proc. Steklov Inst. Math. 297 (2017), no. 1, 285–296.