# Convergence of finite volume schemes for the coupling between the inviscid Burgers equation and a particle

Nina Aguillon[*], Frédéric Lagoutière[†], Nicolas Seguin[‡]

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**Abstract**

The convergence of a class of finite volume schemes for the model of coupling between a Burgers fluid and a pointwise particle is proved. In this model, introduced by Lagoutière, Seguin and Takahashi in 2008, the particle is seen as a moving point through which an interface condition is imposed, which links the velocity of the fluid on the left and on the right of the particle and the velocity of the particle (the three quantities are all not equal in general). The total momentum of the system is conserved through time.

The proposed schemes are consistent with a "large enough" part of the interface conditions. The proof of convergence is an extension of the one of Andreianov and Seguin (2012) to the case where the particle moves under the influence of the fluid (two-way coupling). This extension contains two new main difficulties: first, the fluxes and interface conditions are time-dependent, and second, the coupling between and ODE and a PDE.

**Key phrases:** Fluid-particle interaction; Burgers equation; Non-conservative coupling; moving interface; convergence of finite volume schemes; PDE-ODE coupling

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## 1 Introduction and main results

We study the convergence of finite volume schemes for the Cauchy problem

$$\begin{cases} \partial_t u + \partial_x \frac{u^2}{2} = -\lambda(u - h'(t))\delta_{h(t)}(x), \\ m_p h''(t) = \lambda(u(t,h(t)) - h'(t)), \\ u_{|t=0} = u^0, h(0) = h^0, h'(0) = v^0. \end{cases} \tag{1}$$

It models the behavior of a pointwise particle of position $h$, velocity $h'$ and acceleration $h''$ with mass $m_p$, immersed into a "fluid", whose velocity at time $t$ and point $x$ is $u(t,x)$. The velocity of the fluid is assumed to follow the inviscid Burgers equation. This system is fully coupled: the fluid exerts a drag force $D = \lambda(u(t,h(t)) - h'(t))$ on the particle, where $\lambda$ is a positive friction parameter. In accordance with the action-reaction principle, the particle exerts the force $-D$ on the fluid. The interaction is local: it applies only at the point where the particle is. This friction force tends to bring the velocities of the fluid and the particle closer to each other: as $\lambda$ is positive, the particle accelerates if $u(t,h(t))$ is larger than $h'(t)$, and vice-versa. This toy model was introduced in [LST08] (see also [BCG13] and [Agu15] for related problems). In contrast with the model studied in [VZ03], [Hil05] and [VZ06], the particle and the fluid do not share the same velocity and the fluid is inviscid. In particular the fluid velocity is typically discontinuous through the particle. It yields issues to define correctly the product $(u-h')\delta_h$ and the ODE for the particle in system (1). To do so, the idea is to regularize the Dirac measure in (1), and to remark that the values of the fluid velocity on both sides of this thickened particle are independent of the regularization. It allows to reformulate

[*]nina.aguillon@math.u-psud.fr, Université Paris Sud
[†]frederic.lagoutiere@math.u-psud.fr, Université Paris Sud
[‡]nicolas.seguin@upmc.fr, Université Pierre et Marie Curie

System (1) as an interface problem, where the traces around the particle $u_-(t) = \lim_{x\to h(t)^-} u(t,x)$ and $u_+(t) = \lim_{x\to h(t)^+} u(t,x)$ must belong to a set $\mathcal{G}_\lambda(h'(t))$, which takes into account the interface conditions. Another way to understand $\mathcal{G}_\lambda(v)$ is to say that for a fixed $v$, this set describes the family of piecewise constant equilibrium for System (1) in the case where the particle velocity is fixed equal to $v$. Indeed, for every $(u_-, u_+) \in \mathbb{R}^2$, the function

$$u(t,x) = \begin{cases} u_- & \text{if } x < vt, \\ u_+ & \text{if } x > vt, \end{cases}$$

is the solution of

$$\begin{cases} \partial_t u + \partial_x \frac{u^2}{2} = -\lambda(u-v)\delta_{vt}(x), \\ u^0(x) = u_- \mathbf{1}_{x<0} + u_+ \mathbf{1}_{x>0}, \end{cases}$$

if and only if $(u_-, u_+)$ belongs to $\mathcal{G}_\lambda(v)$. Following [AKR11], we call this family of equilibrium states the germ at speed $v$. The derivation of $\mathcal{G}_\lambda(v)$ via a regularization of the Dirac mass has been done in details in [LST08], and we recall its definition below.

**Definition 1.1.** For any given speed $v \in \mathbb{R}$, the germ at speed $v$, $\mathcal{G}_\lambda(v)$, is the set

$$\mathcal{G}_\lambda(v) = \mathcal{G}_\lambda^1 \cup \mathcal{G}_\lambda^2(v) \cup \mathcal{G}_\lambda^3(v),$$

where

$$\mathcal{G}_\lambda^1 = \left\{(u_-, u_+) \in \mathbb{R}^2 : u_- = u_+ + \lambda\right\},$$
$$\mathcal{G}_\lambda^2(v) = \left\{(u_-, u_+) \in \mathbb{R}^2 : v \le u_- \le v + \lambda,\ v - \lambda \le u_+ \le v \ \text{ and } \ u_- - u_+ < \lambda\right\},$$

and

$$\mathcal{G}_\lambda^3(v) = \left\{(u_-, u_+) \in \mathbb{R}^2 : -\lambda \le u_+ + u_- - 2v \le \lambda \ \text{ and } \ u_- - u_+ > \lambda\right\}.$$

The germ $\mathcal{G}_\lambda(0)$ and its partition are depicted on Figure 1 on the left (note that the germ $\mathcal{G}_\lambda(v)$ is the translation of $\mathcal{G}_\lambda(0)$ by the vector $(v, v)$). Here, we choose a slightly different partition of the germ than in [AS12] and [ALST13], which is depicted on the right of Figure 1. The reason is that we are able to find a class of finite volume schemes which are consistent with $\mathcal{G}_\lambda^1 \cup \mathcal{G}_\lambda^2(0)$ with this choice, but not with the original partition. However, the essential property that $\mathcal{G}_\lambda^1 \cup \mathcal{G}_\lambda^2(0)$ is a *definite* germ still holds true with the partition of Definition 1.1 (more details are given in Definition 1.4 and Proposition 4.8). Once the germ has been defined, System (1) is defined as an interface problem.

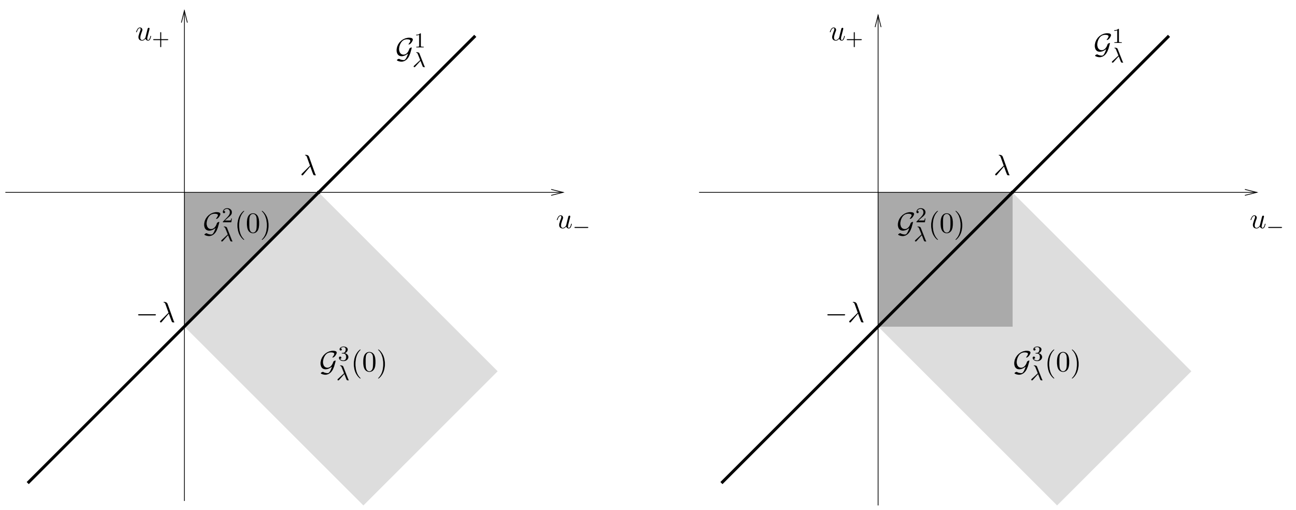



Figure 1: The germ for a motionless particle and its partitions. Left: the partition used in this work. Right: the partition used in [AS12] and [ALST13].

The equation on the particle is reformulated to keep the conservation of total momentum

$$m_p h'(t) + \int_{\mathbb{R}} u(t,x)dx$$

which holds formally in (1). In [LST08], an entropy inequality that takes into account the particle is also derived.

**Definition 1.2.** A pair $(u, h)$ of functions in $L^\infty(\mathbb{R}_+ \times \mathbb{R}) \times W^{2,\infty}(\mathbb{R}_+)$ is a solution of (1) with initial data $u^0$ in $L^\infty(\mathbb{R})$ and $(h^0, v^0) \in \mathbb{R}^2$ if:

- the function $u$ is an entropy weak solution of the Burgers equations on the sets $\{(t,x),\, x < h(t)\}$ and $\{(t,x),\, x > h(t)\}$,
- for almost every positive time $t$,

$$m_p h''(t) = (u_-(t) - u_+(t)) \left( \frac{u_-(t) + u_+(t)}{2} - h'(t) \right) \tag{2}$$

and

$$(u_-(t), u_+(t)) \in \mathcal{G}_\lambda(h'(t)).$$

This definition requires the existence of traces along the particle's trajectory $h$. It follows from the works of Panov [Pan07] and Vasseur [Vas01]. For the one way coupling when the particle is motionless, well-posedness in the $BV$ setting was proved in [AS12], while for the fully coupled system (1), it is proved in [ALST10] and [ALST13].

Remark that Definition 1.2 is not suitable to prove convergence of finite volume schemes in a general framework. Indeed, a scheme can create a numerical boundary layer near the particle, of several cells width. It does not prevent the scheme from converging in, say, $L^\infty_{loc}$ in time and $L^1$ in space; but in that case we cannot expect the numerical traces to converge to their correct values. Nevertheless we will prove the convergence of some schemes that create such boundary layers. The key point is to use, instead of Definition 1.2, an equivalent definition which does not involve the traces of $u$. We begin with some properties of the germ $\mathcal{G}_\lambda(v)$.

In the sequel, we denote by $\Phi_v$ the so-called Kruzhkov entropy flux associated with $f^v(u) = \frac{u^2}{2} - vu$:

$$\Phi_v : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (a,b) & \longmapsto & \operatorname{sgn}(a-b)\left(\left(\frac{a^2}{2} - va\right) - \left(\frac{b^2}{2} - vb\right)\right) \end{array}$$

and we define

$$\Xi_v : \begin{array}{ccc} \mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ ((a_-, a_+), (b_-, b_+)) & \longmapsto & \Phi_v(a_-, b_-) - \Phi_v(a_+, b_+) \end{array}$$

**Proposition 1.3.** *If both $(a_-, a_+)$ and $(b_-, b_+)$ belong to $\mathcal{G}_\lambda(v)$, then*

$$\Xi_v((a_-, a_+), (b_-, b_+)) \geq 0.$$

*Conversely, if $(a_-, a_+)$ is such that*

$$\forall (b_-, b_+) \in \mathcal{G}_\lambda(v), \quad \Xi_v((a_-, a_+), (b_-, b_+)) \geq 0,$$

*then $(a_-, a_+)$ belongs to the germ.*

Adopting once again the vocabulary of [AKR11], Proposition 1.3 means that $\mathcal{G}_\lambda(v)$ is a maximal $L^1$-dissipative germ. The proof can be found in [AS12], see Proposition 2. We now introduce another notion, which will allow us to restrict our attention to a small part of $\mathcal{G}_\lambda(v)$.

**Definition 1.4.** A subset $\mathcal{H}_\lambda(v)$ of $\mathcal{G}_\lambda(v)$ is said to be definite if any $(a_-, a_+)$ that satisfies

$$\forall (b_-, b_+) \in \mathcal{H}_\lambda(v), \quad \Xi_v((a_-, a_+), (b_-, b_+)) \geq 0 \tag{3}$$

belongs to the germ $\mathcal{G}_\lambda(v)$.

**Example 1.5.** The subset $\mathcal{G}^1_\lambda \cup \mathcal{G}^2_\lambda(v)$ is definite. This is proved in Proposition 4.8.

We now focus on alternative traceless characterizations of entropy solutions. In the sequel $\mathcal{H}_\lambda(v)$ always denotes a definite part of $\mathcal{G}_\lambda(v)$. For all $(c_-, c_+)$ we denote by $c$ the piecewise constant function

$$c(t,x) = c_- \mathbf{1}_{x<h(t)} + c_+ \mathbf{1}_{x \geq h(t)},$$

and by $\operatorname{dist}_1(a, X)$ the $L^1$-distance of a point $a := (a_-, a_+)$ of $\mathbb{R}^2$ to a set $X$ included in $\mathbb{R}^2$:

$$\operatorname{dist}_1((a_-, a_+), X) = \inf_{(x_-, x_+) \in X} \Big( |a_- - x_-| + |a_+ - x_+| \Big).$$

**Proposition 1.6.** *Let $h$ be a function of $W^{2,\infty}_{loc}(\mathbb{R}_+)$ and let $u$ be a function of $L^\infty_{loc}(\mathbb{R}_+\times\mathbb{R})$, which is an entropy solution of the Burgers equation on the sets $\{(t,x),\, x<h(t)\}$ and $\{(t,x),\, x>h(t)\}$. The following assertions are equivalent.*

- *For almost every time $t>0$, $(u_-(t),u_+(t))$ belongs to $\mathcal{G}_\lambda(h'(t))$.*
- *For almost every time $t>0$, for all $(c_-,c_+)\in\mathbb{R}^2$, there exist $\delta\in(0,t)$ and a constant $A$ depending only on $||u^0||_\infty$, $\lambda$, $(c_-,c_+)$ and $||h'||_\infty$ such that for every nonnegative function $\varphi$ in $\mathcal{C}^\infty_0((t-\delta,t+\delta)\times\mathbb{R})$,*

$$\begin{aligned}\int_{\mathbb{R}_+}\int_{\mathbb{R}}|u-c|(s,x)\partial_s\varphi(s,x-h(s))+\Phi_{h'(t)}(u,c)(s,x)\partial_x\varphi(s,x-h(s))dx\,ds\\ \geq -A\int_{\mathbb{R}_+}\operatorname{dist}_1((c_-,c_+),\mathcal{H}_\lambda(h'(s)))\varphi(s,0)\,ds.\end{aligned}\tag{4}$$

*Proof.* For the sake of completeness we reproduce here the main ingredients of the proof that can be found in [AS12]. Let $\varphi$ be in $\mathcal{C}^\infty_0((t-\delta,t+\delta)\times\mathbb{R})$, where $\delta$ belongs to $(0,t)$. For positive $\varepsilon$, we introduce the function

$$\zeta_\varepsilon(z)=1-\min(1,|z|/\varepsilon),$$

whose support is $(-\varepsilon,\varepsilon)$. The support of the function

$$\psi_\varepsilon(t,x)=(1-\zeta_\varepsilon)\varphi(t,x-h(t))$$

is included in $\{(t,x),\, t>0,\, x\neq h(t)\}$. The function $u$ is an entropy solution of the Burgers equation on the sets $\{(t,x),\, x<h(t)\}$ and $\{(t,x),\, x>h(t)\}$, thus for all real $\kappa$,

$$\iint_{\mathbb{R}_+\times\mathbb{R}}|u(s,x)-\kappa|\partial_s\psi_\varepsilon(s,x)+\Phi_0(u(s,x),\kappa)\partial_x\psi_\varepsilon dx\,ds\geq 0.$$

But $\partial_s\psi_\varepsilon(s,x)=\partial_s((1-\zeta_\varepsilon)\varphi)(s,x-h(s))-h'(s)\partial_x((1-\zeta_\varepsilon)\varphi)(s,x-h(s))$, and using the fact that

$$\Phi_v(a,b)=\Phi_0(a,b)-v|a-b|,$$

we obtain

$$\iint_{\mathbb{R}_+\times\mathbb{R}}|u-c|(s,x)(\partial_s(1-\zeta_\varepsilon)\varphi)(s,x-h(s))+\Phi_0(u,c)(s,x)\partial_x((1-\zeta_\varepsilon)\varphi)(s,x-h(s))dx\,ds\geq 0.$$

Thus we have

$$\begin{aligned}&\iint_{\mathbb{R}_+\times\mathbb{R}}|u-c|(s,x)\partial_s\varphi(s,x-h(s))+\Phi_{h'(t)}(u,c)(s,x)\partial_x\varphi(s,x-h(s))dx\,ds\\ &\geq\liminf_{\varepsilon\to 0}\iint_{\mathbb{R}_+\times\mathbb{R}}|u-c|(s,x)(\partial_s(\zeta_\varepsilon\varphi))(s,x-h(s))+\Phi_{h'(t)}(u,c)(s,x)(\partial_x(\zeta_\varepsilon\varphi))(s,x-h(s))dx\,ds\\ &=\int_{\mathbb{R}_+}\Phi_{h'(s)}(u_-(s),c_-)-\Phi_{h'(s)}(u_+(s),c_+)\varphi(s,0)ds\\ &=\int_{\mathbb{R}_+}\Xi_{h'(s)}((u_-(s),u_+(s)),(c_-,c_+))\varphi(s,0)ds\end{aligned}$$

For all $s$ for which the pair $(u_-(s),u_+(s))$ exists and belongs to $\mathcal{G}_\lambda(h'(s))$, we denote by $(\tilde{c}_-(s),\tilde{c}_+(s))$ a $L^1$-projection of $(c_-,c_+)$ on $\mathcal{H}_\lambda(h'(s))$, i.e. a point that minimizes the distance $\operatorname{dist}_1((c_-,c_+),\mathcal{H}_\lambda(h'(s)))$. We have

$$\begin{aligned}\Xi_{h'(s)}((u_-(s),u_+(s)),(c_-(s),c_+(s)))\geq\Xi_{h'(s)}((u_-(s),u_+(s)),(\tilde{c}_-(s),\tilde{c}_+(s)))\\ -|\Xi_{h'(s)}((u_-(s),u_+(s)),(c_-(s),c_+(s)))-\Xi_{h'(s)}((u_-(s),u_+(s)),(\tilde{c}_-(s),\tilde{c}_+(s)))|.\end{aligned}$$

Since $(\tilde{c}_-(s),\tilde{c}_+(s))$ belongs to $\mathcal{H}_\lambda(h'(s))$, Proposition 1.3 yields

$$\int_{\mathbb{R}_+}\Xi_{h'(s)}((u_-(s),u_+(s)),(\tilde{c}_-(s),\tilde{c}_+(s)))\varphi(s,0)ds\geq 0.$$

On the other hand

$$
\begin{aligned}
&|\Xi_{h'(s)}((u_-(s),u_+(s)),(c_-(s),c_+(s))) - \Xi_{h'(s)}((u_-(s),u_+(s)),(\tilde c_-(s),\tilde c_+(s)))|\\
&\qquad \le |\Phi_{h'(s)}(u_-(s),c_-(s)) - \Phi_{h'(s)}(u_-(s),\tilde c_-(s))| + |\Phi_{h'(s)}(u_+(s),c_+(s)) - \Phi_{h'(s)}(u_+(s),\tilde c_+(s))|
\end{aligned}
$$

which is smaller than a constant depending only on $||h'||_\infty$, $||u||_\infty$, $c$ and $\lambda$ (since $c \mapsto \tilde c$ depends on $\lambda$), multiplied by the $L^1$-distance between $(c_-, c_+)$ and $(\tilde c_-(s), \tilde c_+(s))$, and we obtain the result.

Conversely, using a sequence of test functions $\varphi$ concentrating at a time $t$ for which $u$ has traces in (4), we obtain that for all $(c_-, c_+)$ in $\mathcal{H}_\lambda(h'(t))$,

$$
\Xi_{h'(t)}((u_-(t),u_+(t)),(c_-,c_+)) \ge 0,
$$

and thus by Proposition 1.3, $(u_-(t), u_+(t))$ belongs to the germ $\mathcal{G}_\lambda(h'(t))$. □

**Proposition 1.7.** *Let $u$ in $L^\infty_{loc}(\mathbb{R}_+\times\mathbb{R})$ be a solution of the Burgers equation on the sets $\{(t,x),\ x < h(t)\}$ and $\{(t,x),\ x > h(t)\}$. A function $h$ in $W^{2,\infty}_{loc}(\mathbb{R}_+)$ satisfies (2) almost everywhere, with initial data $(h(0), h'(0)) = (h^0, v^0)$, if and only if for all $\xi \in \mathcal{C}^\infty_0([0,T])$ and for all $\psi \in \mathcal{C}^\infty_0(\mathbb{R})$ such that $\psi(0) = 1$ the following holds:*

$$
\begin{aligned}
-\int_0^T m_p h'(t)\xi'(t)dt = m_p v^0 \xi(0) &+ \int_{\mathbb{R}}\int_0^T \frac{u^2}{2}(s,x)\xi(s)\psi'(x-h(s))ds\,dx\\
&+ \int_{\mathbb{R}}\int_0^T u(s,x)[\xi'(s) - h'(s)\psi'(x-h(s))]ds\,dx\\
&+ \int_{\mathbb{R}} u^0(x)\psi(x-h^0)\xi(0)dx.
\end{aligned}
\tag{5}
$$

*Proof.* This characterization was proved in [ALST10]. It follows from the application of the Green–Gauss theorem and the fact that $u$ is an entropy solution of the Burgers equation away from the particle:

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{R}} \frac{u^2}{2}(s,x)\xi(s)\psi'(x-h(s)) + u(s,x)[\xi'(s) - h'(s)\psi'(x-h(s))]ds\,dx\\
&\quad = \int_0^T\int_{\mathbb{R}} \frac{u^2}{2}(s,x)\partial_x(\xi\psi(x-h(s))) + u(s,x)\partial_s(\xi\psi(x-h(s)))ds\,dx\\
&\quad = -\int_0^T\int_{x\ne h(t)} \left(\partial_x\frac{u^2}{2} + \partial_t u\right)(\xi\psi)dx\,ds - \int_{\mathbb{R}} u^0(x)\psi(x-h(0))\xi(0)dx\\
&\qquad + \int_0^T \xi(s)\left(\left(\frac{u_-^2(s)}{2} - h'(s)u_-(s)\right) - \left(\frac{u_+^2(s)}{2} - h'(s)u_+(s)\right)\right)ds\\
&\quad = \int_0^T m_p h''(s)\xi(s)ds - \int_{\mathbb{R}} u^0(x)\psi(x-h(0))\xi(0)dx.
\end{aligned}
$$

□

We now present the family of finite volume schemes for which we prove convergence. The proof follows the guidelines of the Lax–Wendroff theorem. In Section 3, we obtain a $BV$ bound on the fluid velocity and a $W^{2,\infty}$ bound on the particle's trajectory that allow us to extract convergent subsequences in $L^1_{loc}(\mathbb{R}_+\times\mathbb{R})$ and $W^{1,\infty}_{loc}(\mathbb{R}_+)$. The difficulties are to treat numerically the interface conditions enclosed in the germ and the coupling between an ODE and a PDE. More precisely:

- First, we have to take into account at the numerical level the interface condition of Definition 1.1. We will use schemes that preserves a "sufficiently large" part of the germ.
- Second, the moving particle must be handle with care. It is crucial that the particle lies at an interface of the mesh at the beginning of the time step. To do so and avoid the problem of the replacement of the particle, we use a mesh that tracks the particle and we update the particle's velocity by conservation of total momentum.

Let us fix a time step $\Delta t$ and a space step $\Delta x$. In the sequel we assume that the time step and the space step are proportional, and we denote by $\mu = \frac{\Delta t}{\Delta x}$ their ratio. We propose to approximate the solution of (1) with a finite volume scheme. We use a mesh that follows the particle, which is placed between the cells numbered 0 and 1. The speed of the particle is approximated by a piecewise constant $(v^n)_{n\in\mathbb{N}}$. Given the solution a time $n\Delta t$: we consider that the particle has constant velocity $v^n$ on the whole time step $(n\Delta t, (n+1)\Delta t)$ to update the fluid velocity, then we update $v^n$ by conservation of the total momentum. The interface $1/2$ where the particle lies is special, and we have to use appropriate fluxes at this interface. Due to the source term, the equation is not conservative around the particle, thus we have two different fluxes $f^{n,-}_{1/2}$ and $f^{n,+}_{1/2}$ on the left and on the right of the particle respectively. Away from the particle, Equation (1) writes as a scalar conservation law, and we can use any standard flux for the Burgers equation. The scheme is initialized with

$$\forall j \in \mathbb{Z},\ u^0_j = \frac{1}{\Delta x}\int_{x^0_{j-1/2}}^{x^0_{j+1/2}} u^0(x)\,dx.$$

From the integration of the first equation of (1) on the space time cell

$$\mathcal{C}^n_j = \{(n\Delta t + s, x^n_{j-1/2} + y + sv^n), 0 \le s < \Delta t, 0 \le y < \Delta x\},$$

we obtain the finite volume scheme

$$\begin{cases} u^{n+1}_j &= u^n_j - \mu(f^n_{j+1/2}(v^n) - f^n_{j-1/2}(v^n)) \text{ for } j \in \mathbb{Z}, j \notin \{0,1\},\\ u^{n+1}_0 &= u^n_0 - \mu(f^n_{1/2,-}(v^n) - f^n_{-1/2}(v^n)),\\ u^{n+1}_1 &= u^n_1 - \mu(f^n_{3/2}(v^n) - f^n_{1/2,+}(v^n)),\\ v^{n+1} &= v^n + \frac{\Delta t}{m_p}(f^n_{1/2,-}(v^n) - f^n_{1/2,+}(v^n)),\\ x^{n+1}_j &= x^n_j + v^n\Delta t. \end{cases} \tag{6}$$

Here we emphasized the dependency of the flux on the particle's velocity. In the sequel we denote by $u_{\Delta t}$ the constant by cell function

$$u_{\Delta t}(t,x) = u^n_j \quad \text{if} \quad (t,x) \in \mathcal{C}^n_j. \tag{7}$$

and by $v_{\Delta t}$ and $h_{\Delta t}$ the constant and linear by cell functions:

$$\begin{cases} v_{\Delta t}(t) &= v^n \quad \text{if } n\Delta t \le t < (n+1)\Delta t,\\ h_{\Delta t}(t) &= h^0 + \Delta t\sum_{m=0}^{n-1} v^m + v^n(t - n\Delta t) \quad \text{if } n\Delta t \le t < (n+1)\Delta t. \end{cases} \tag{8}$$

Another way to proceed is to perform the change of variable

$$\tilde{u}(t,x) = u(t, x + h(t))$$

in (1). This function satisfies the PDE

$$\partial_t \tilde{u} + \partial_x\left(\frac{\tilde{u}^2}{2} - h'(t)\tilde{u}\right) = -\lambda(\tilde{u} - h')\delta_0(x) \tag{9}$$

The particle is now motionless but the flux depends on time. Integrating (9) on $[n\Delta t, (n+1)\Delta t] \times [x^0_{j-1/2}, x^0_{j+1/2}]$, and using special fluxes around the particle (still placed at interface 1/2), we obtain the finite volume scheme

$$\begin{cases} \tilde{u}^{n+1}_j &= \tilde{u}^n_j - \mu(f^{v^n,n}_{j+1/2} - f^{v^n,n}_{j-1/2}) \text{ for } j \in \mathbb{Z}\setminus\{0,1\},\\ \tilde{u}^{n+1}_0 &= \tilde{u}^n_0 - \mu(f^{v^n,n-}_{1/2} - f^{v^n,n}_{-1/2}),\\ \tilde{u}^{n+1}_1 &= \tilde{u}^n_1 - \mu(f^{v^n,n}_{3/2} - f^{v^n,n+}_{1/2}),\\ v^{n+1} &= v^n + \frac{\Delta t}{m_p}(f^{v^n,n-}_{1/2} - f^{v^n,n+}_{1/2}). \end{cases} \tag{10}$$

The two points of view are illustrated on Figure 2.

The fluxes $f^n_{j-1/2}(v^n)$ with $j \neq 1/2$ (or $f^{n\pm}_{1/2}(v^n)$ if $j = 1/2$) are strongly related to the fluxes $f^{v^n,n}_{j-1/2}$: in (6), $f^n_{j+1/2}(v^n)$ is an approximation of

$$\frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} (f^0(u) - v^n u)(t, x^n_{j+1/2} + v^n t)dt$$

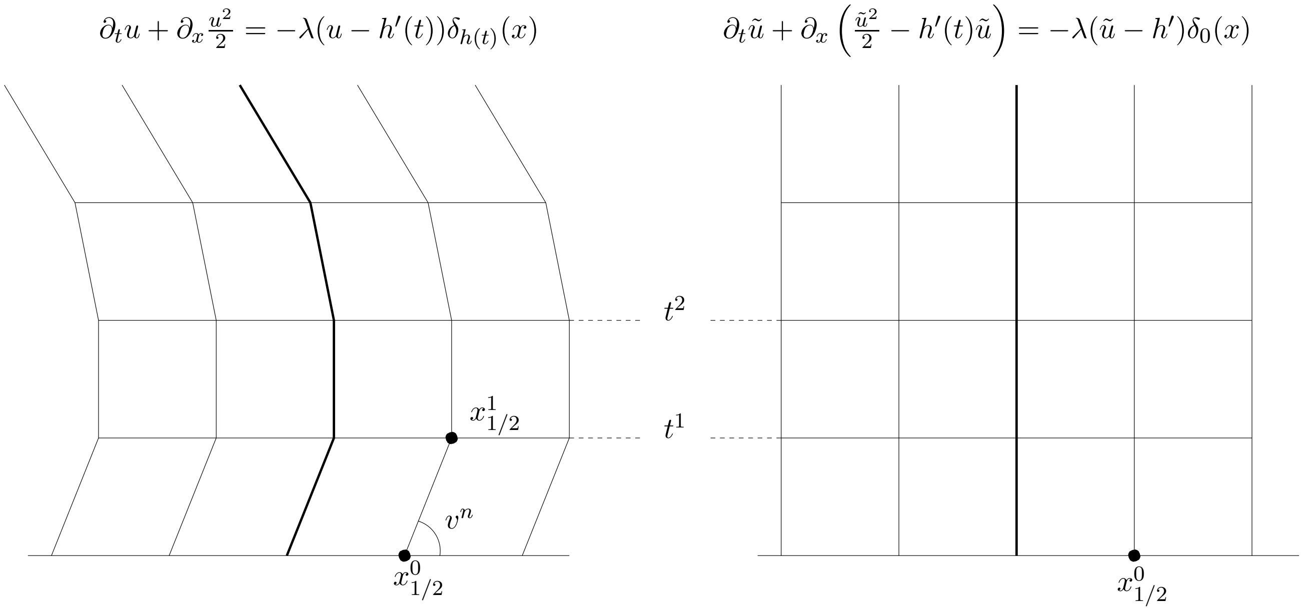



Figure 2: To approximate the solution of (1), we can either use a mesh that follows the particle (on the left) or straighten the particle's trajectory and approximate the solution of (9). In both cases, the particle's trajectory is the bold line.

while in (10), $f^{v^n,n}_{-1/2}$ is an approximation of

$$\frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} f^{v^n}(\tilde{u})(t, x^0_{j+1/2})dt.$$

In the following we prove the convergence of Scheme (6) under a set of assumptions on the fluxes $f^n_{j+1/2}$, $f^n_{1/2,-}$ and $f^n_{j+1/2,+}$ and a Courant–Friedrichs–Lewy condition. We restrict the study to two-points fluxes

$$f^n_{j+1/2} = g(u^n_j, u^n_{j+1}, v^n) \quad \text{and} \quad f^n_{1/2,\pm} = g^{\pm}_\lambda(u^n_j, u^n_{j+1}, v^n).$$

The assumptions on the flux $f^n_{j+1/2}$ away from the particle are the classical ones:

- consistency with the modified Burgers equation:

$$\forall a \in \mathbb{R},\ \forall v \in \mathbb{R},\ g(a, a, v) = \frac{a^2}{2} - va, \tag{11}$$

- monotonicity with respect to the first two arguments:

$$\forall (a, b) \in \mathbb{R}^2,\ \forall v \in \mathbb{R},\quad \partial_1 g(a, b, v) \geq 0 \quad \text{and} \quad \partial_2 g(a, b, v) \leq 0. \tag{12}$$

- local Lipschitz-continuity of $g$; (13)

they ensure convergence of the scheme to an entropy solution of the Burgers equation away from the particle.

The assumptions on the fluxes around the particle are the following. We first have some consistency assumptions, which ensure that some particular solutions corresponding to a large enough part of the germ are exactly preserved by the numerical scheme. More precisely, the hypotheses on the fluxes $g^{\pm}_\lambda$ are:

- consistency the part $\mathcal{G}^1_\lambda$ of the germ:

$$\forall v \in \mathbb{R},\ \forall (a, b) \in \mathcal{G}^1_\lambda,\ g^-_\lambda(a, b, v) = \frac{a^2}{2} - va \quad \text{and} \quad g^+_\lambda(a, b, v) = \frac{b^2}{2} - vb. \tag{14}$$

  In Section 4, we make the stronger assumption that $g$ is consistent with a subset $\mathcal{H}_\lambda$ of $\mathcal{G}_\lambda$ which contains $\mathcal{G}^1_\lambda$ and is *definite* (see Definition 1.4)

$$\forall v \in \mathbb{R},\ \forall (a, b) \in \mathcal{H}_\lambda(v),\ g^-_\lambda(a, b, v) = \frac{a^2}{2} - va \quad \text{and} \quad g^+_\lambda(a, b, v) = \frac{b^2}{2} - vb. \tag{15}$$

Hypothesis (14) will be used to prove $BV$ estimates on the fluid part $(u_j^n)_{j\in\mathbb{Z},n\in\mathbb{N}}$. We also assume that

- if the particle has the same velocity as the fluid, its velocity does not change:

$$\forall v \in \mathbb{R},\ g_\lambda^-(v,v,v) = g_\lambda^+(v,v,v). \tag{16}$$

This hypothesis will be used to prove a $L^\infty$ bound on the particle velocity $(v^n)_{n\in\mathbb{N}}$. We add two classical conditions of regularity and monotonicity, also used to prove the $BV$ bound on $(u_j^n)_{j\in\mathbb{Z},n\in\mathbb{N}}$. We assume that:

- both $g_\lambda^-$ and $g_\lambda^+$ are locally Lipschitz-continuous; (17)
- $g_\lambda^-$ and $g_\lambda^+$ are nondecreasing with respect to their first arguments, and nonincreasing with respect to their second arguments. (18)

Just like in [AS12], we need a dissipativity property to prove discrete entropy inequalities. Moreover, it will also be a key assumption to prove the bounds on the particle's velocity.

- The function $g_\lambda^- - g_\lambda^+$ is nondecreasing with respect to its first two arguments. (19)

For this family of finite volume schemes, we are able to prove the following convergence theorem.

**Theorem 1.8.** *Consider a finite volume scheme of the form* (6) *that satisfies the set of Hypotheses (11–14) and (16–19), and* (15) *in Section 4. Assume that* $u^0$ *belongs to* $BV(\mathbb{R}) \cap L^1(\mathbb{R})$. *Let us denote by* $L$ *the largest Lipschitz constant of* $g$, $g^+$ *and* $g^-$ *on the set* $[m,M]^2 \times [\underline{v},\bar{v}]$, *where*

$$\begin{cases} m = \min\{\operatorname{ess\,inf}_{\mathbb{R}^-} u^0 - \lambda, \operatorname{ess\,inf}_{\mathbb{R}^+} u^0\}, \\ M = \max\{\operatorname{ess\,sup}_{\mathbb{R}^-} u^0, \operatorname{ess\,sup}_{\mathbb{R}^+} u^0 + \lambda\}, \\ \underline{v} = \min(m, v^0), \\ \bar{v} = \max(M, v^0). \end{cases}$$

*Then, under the Courant-Friedrichs-Lewy condition*

$$L\mu \le \frac{1}{2}, \tag{20}$$

*the sequence* $(u_{\Delta t})$ *converges in* $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R})$ *toward* $u$ *and the sequence* $(h_{\Delta t})$ *converges in* $W^{1,\infty}_{loc}(\mathbb{R}_+)$ *toward* $h$ *when* $\Delta t$ *tends to* 0, *where* $(h,u)$ *is the solution of* (1).

The next two Sections are devoted to the proof. In Section 3, we prove bounds on the total variation of the fluid and on the acceleration of the particle, which permit us to extract converging subsequences. Then in Section 4, we prove Theorem 1.8. Hypothesis (15) is sufficient to obtain a discrete version of (4). In Section 5, we drop hypothesis (15) and prove the following theorem, which extends the proof of convergence of [AS12] to the fully coupled case (1).

**Theorem 1.9.** *Consider a finite volume scheme of the form* (6) *with*

$$\begin{cases} g_\lambda^-(a,b,v) &= g(a, b+\lambda, v), \\ g_\lambda^+(a,b,v) &= g(a-\lambda, b, v). \end{cases}$$

*Assume that* $g$ *satisfies Assumptions (11–13). Then* $g_\lambda^-$ *and* $g_\lambda^+$ *satisfies* (14) *and (16–18). Eventually, assume that Hypothesis* (19) *holds.*

*Let the initial data* $u^0$ *belongs to* $BV(\mathbb{R}) \cap L^1(\mathbb{R})$, *and denote by* $L$ *the largest Lipschitz constant of* $g$, $g^+$ *and* $g^-$ *on the set* $[m,M]^2 \times [\underline{v},\bar{v}]$, *where*

$$\begin{cases} m = \min\{\operatorname{ess\,inf}_{\mathbb{R}^-} u^0 - \lambda, \operatorname{ess\,inf}_{\mathbb{R}^+} u^0\}, \\ M = \max\{\operatorname{ess\,sup}_{\mathbb{R}^-} u^0, \operatorname{ess\,sup}_{\mathbb{R}^+} u^0 + \lambda\}, \\ \underline{v} = \min(m, v^0), \\ \bar{v} = \max(M, v^0). \end{cases}$$

*Then, under the Courant-Friedrichs-Lewy condition* $L\mu \le \frac{1}{2}$, *the sequence* $(u_{\Delta t})$ *converges in* $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R})$ *toward* $u$ *and the sequence* $(h_{\Delta t})$ *converges in* $W^{1,\infty}_{loc}(\mathbb{R}_+)$ *toward* $h$ *when* $\Delta t$ *tends to* 0, *where* $(h,u)$ *is the solution of* (1).

This type of schemes was introduced in [AS12]. They *a priori* only preserve the part $\mathcal{G}_\lambda^1$ of the germ, i.e. they do not satisfy (15). As $\mathcal{G}_\lambda^1$ is not a definite subset of $\mathcal{G}_\lambda(v)$. This makes the proof more difficult, because we cannot use a numerical equivalent of Formule (4) anymore. In Section 2, we present several numerical illustrations, illustrating either Theorem 1.8 or Theorem 1.9. We also compare the results given by different numerical fluxes. In particular, we bring out the possible occurrence of numerical boundary layers at the particle position for the less accurate scheme.

# 2 Numerical Results

To begin with, let us give some numerical illustration of solution of Model (1) with Scheme (6). For the fluxes around the particle, we compare the choice

$$\begin{cases} g_\lambda^-(u_-,u_+,v) = g(u_-, \min(u_+ + \lambda, \max(u_-,v)), v) \\ g_\lambda^+(u_-,u_+,v) = g(\max(u_- - \lambda, \min(u_+,v)), u_+, v) \end{cases} \tag{21}$$

which satisfies Hypothesis (15), with the choice

$$\begin{cases} g_\lambda^-(u_-,u_+,v) = g(u_-, u_+ + \lambda, v) \\ g_\lambda^+(u_-,u_+,v) = g(u_- - \lambda, u_+, v) \end{cases} \tag{22}$$

studied in Section 5, which only satisfies (14). In both cases, we compare the Rusanov flux and the Godunov flux. We will prove in Propositions 4.9 and 4.10 that the monotonicity assumption (18) and the dissipativity property (19) hold for these fluxes. Numerical schemes for (1) are proposed in [ALST10] and [Tow15]. They both use a fixed grid and the flux form (22) around the particle in order to preserve the equilibrium in $\mathcal{G}_\lambda$. In [ALST10] the motion of the particle is updated with a random sampling procedure, while in [Tow15], a modified stencil is used when the particle crosses an interface. Below we present several numerical test cases. The initial data is a Riemann problem, in the sense that the fluid's velocity is constant on both sides of the particle. The solutions are classified and described in [LST08].

## 2.1 Standing particle in a fluid at rest

One of the aims of this subsection is to show that with Choice (22), boundary layers may appear around the particle. On the contrary, no such boundary layer with Choice (21), which has the additional property (15). The key difference between the two schemes is that (21) exactly preserves the trivial equilibrium $u = 0$ and $v = 0$. This is illustrated on Figure 3. We do not plot the velocity as it is always equal to 0. For this test case, $\lambda = 1$, $m_p = 0.1$, $\Delta x = 0.02$ and $\Delta t = 0.01$. The final time is $T = 1$.

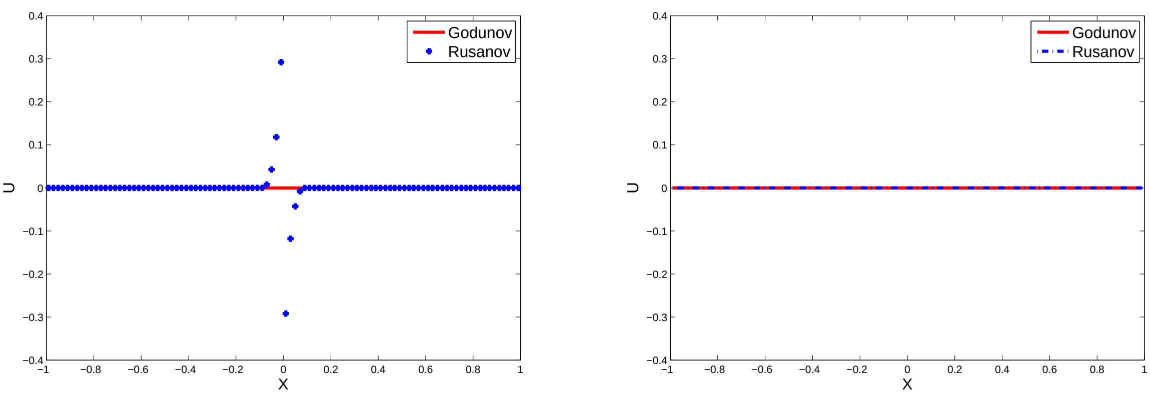



Figure 3: Scheme (21) on the right preserves the solution consisting of a fluid at rest containing a standing particle, while Scheme (22) on the left may create boundary layers.

## 2.2 Particle traveling inside a shock or a rarefaction wave

We now give two other examples of this phenomenon, with $\lambda = 1$, $m_p = 0.1$, $\Delta x = 0.02$ and $\Delta t = 0.01$ as before. The final time is $T = 1$. The first one is the case where

$$u^0(x) = 0.2 * \mathbf{1}_{x<0} - 0.2 * \mathbf{1}_{x>0},\ h^0 = 0, v^0 = 0.1. \tag{23}$$

This is an example of Case (V) of [LST08], in which all the Riemann problems for (1) are solved. The exact solution consists in the initial shock moving at the speed of the particle, whose trajectory is given by

$$h(t) = \frac{m_p v^0}{0.4}\left(1 - e^{-\frac{0.4}{m_p}t}\right).$$

The second one is with the initial data

$$u^0(x) = -0.2 * \mathbf{1}_{x<0} + 0.2 * \mathbf{1}_{x>0},\ h^0 = 0, v^0 = 0.1. \tag{24}$$

This is case $(I)$ in [LST08]. In that case the fluid does not see the particle: the solution is a rarefaction fan for the fluid in the middle of which stands the particle, which velocity is constant equal to $v^0$.

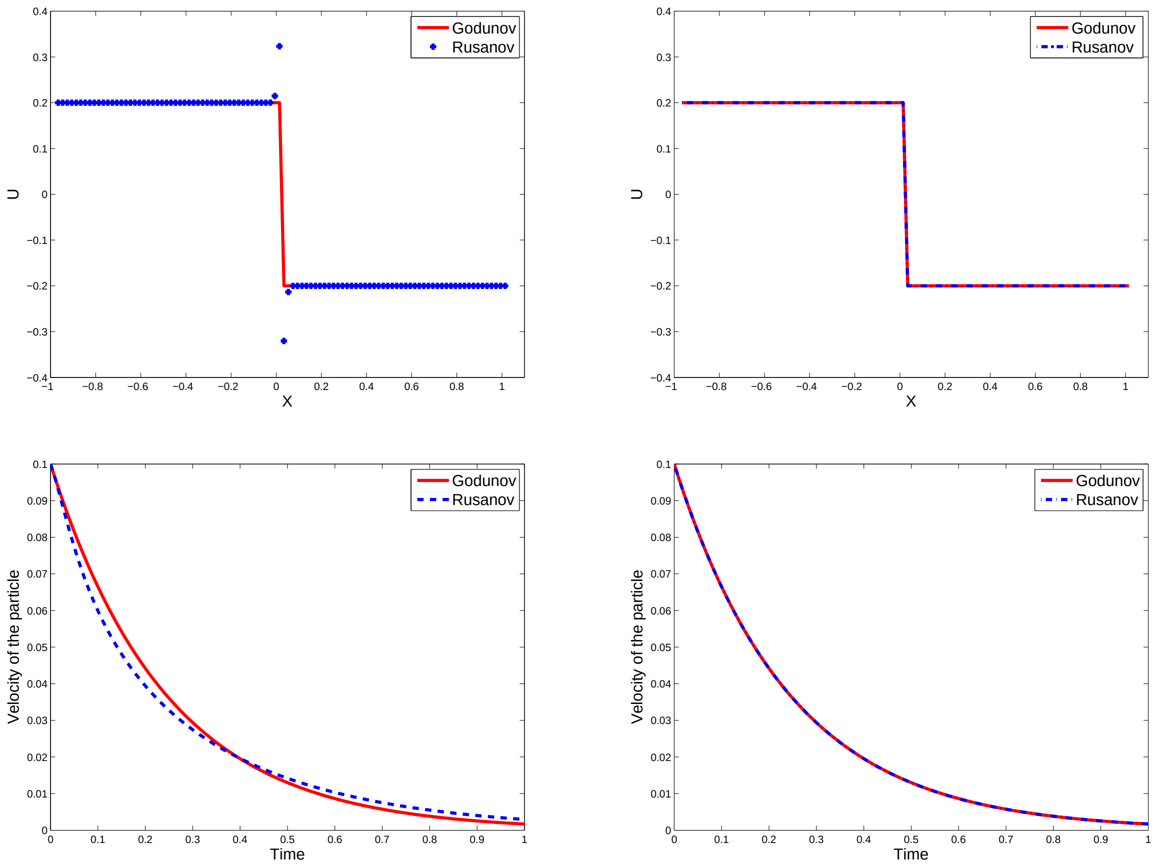



Figure 4: Comparison of Scheme (22) (left) and Scheme (21) for the initial data (23). All schemes except (22) with the Rusanov flux are very close to the exact solution.

The results are displayed on Figures 4 and 5 respectively. Scheme (22) with the Rusanov scheme creates a boundary layer. We also observe that the velocity of the particle is strongly impacted by the boundary layer. In fact the particle's trajectory does not converge in $W^{2,\infty}$: it is easy to check that the numerical acceleration at time 0 is given by

$$m_p \frac{v^1 - v^0}{\Delta t} = \lambda \left(\frac{u_- + u_+}{2} - v^0\right)$$

and not by (2). Let us underline however that this scheme converges toward the exact solution by Theorem 1.8, in $W^{1,\infty}$ for the particle's velocity.

On the other hand, when $g$ is the Godunov flux, no boundary layer appears in Scheme (22). This is because the scheme is more consistent than expected: it satisfies (15) with $\mathcal{H}_\lambda(v) = \mathcal{G}^1_\lambda \cup \mathcal{G}^2_\lambda(v)$.

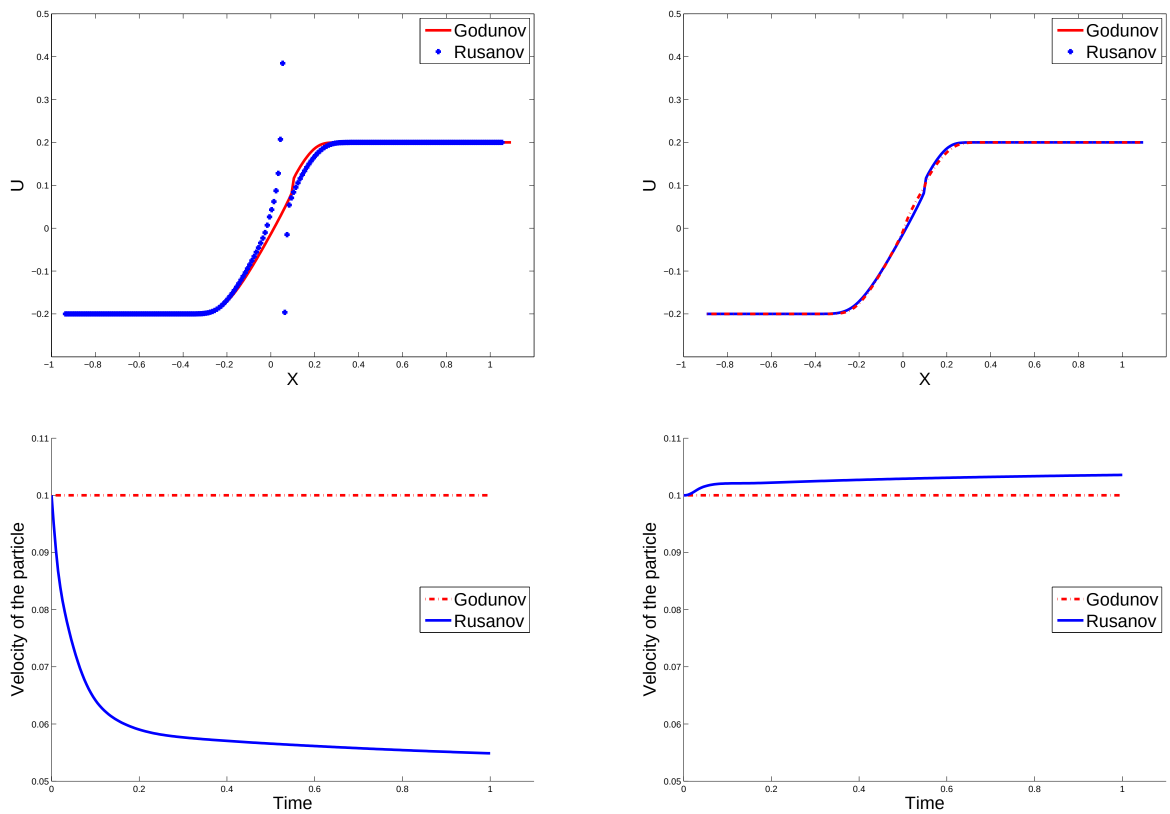



Figure 5: Comparison of Scheme (22) (left) and Scheme (21) for the initial data (23). All the scheme except (22) with the Rusanov flux give solution that are very close to the exact solution.

Once again, a boundary layer is created around the particle if Choice (21) is made.

### 2.3 Particle initially much faster than the fluid, large friction

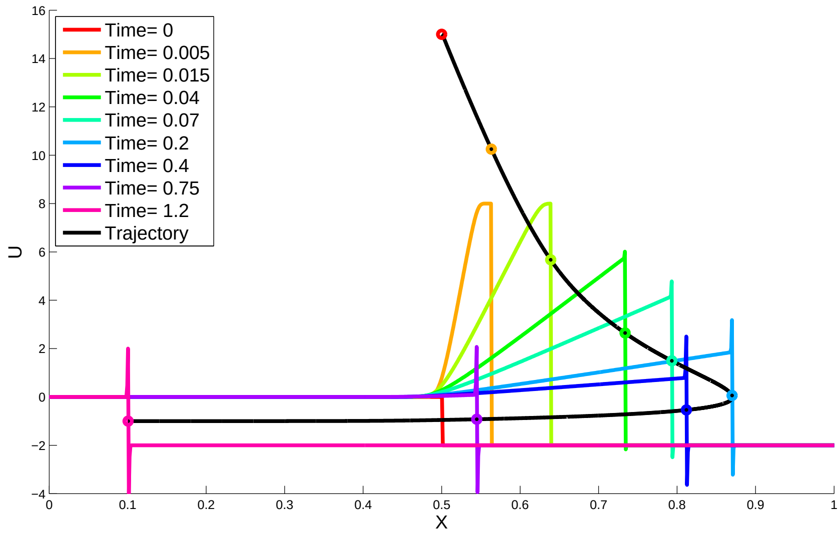
This test case is a Riemann problem in which the particle initially has a velocity much larger than the fluid's. It corresponds to case (III) in [LST08], and can also be compared with the result in [Tow15] (which has the advantage of being written on a fixed grid). At first the particle generates an acceleration of the fluid behind it, creating a rarefaction fan. As $\lambda = 15$ is large (compared to the jump in the fluid's velocity), the particle is quickly slowed down. Eventually, the rarefaction fan disappears and the particle travels inside a shock. The initial data is

$$u^0(x) = -2 * \mathbf{1}_{x>0}, \;\; h^0 = 0.5, v^0 = 10. \tag{25}$$

and we took $\Delta x = 0.001$ and $\Delta t = 0.000025$. The solutions at different times are depicted on Figure 6. Once again, boundary layers appear when the jump in the fluid's velocity through the particle becomes smaller than $\lambda$.

### 2.4 Particle initially much faster than the fluid, small friction

Here again the particle is initially very fast, but this time $\lambda = 1$ is small (compared to $v^0 - u_L$). It corresponds to case (IV) in [LST08]. At first the solution contains two discontinuities: the first one is a shock in the fluid's domain, and the second one is a discontinuity that travels at the speed of the particle. At first the speed of the first shock is smaller than the one of the particle, but it always



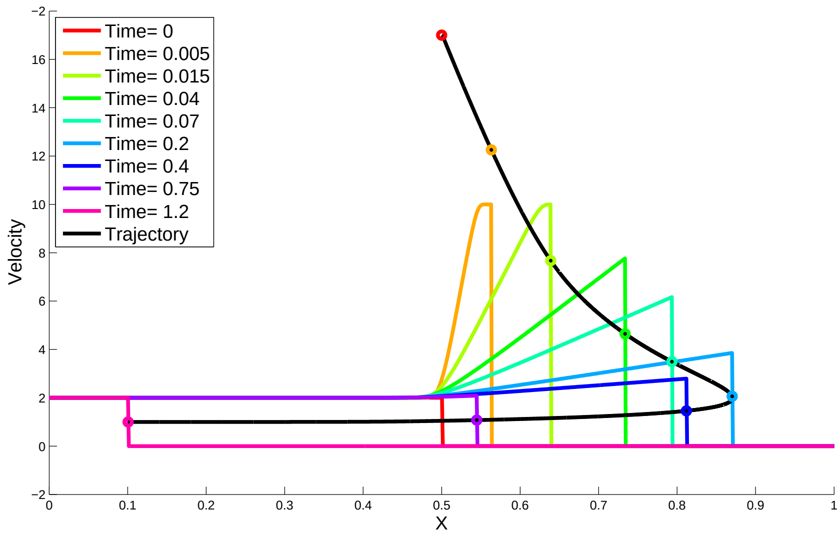




Figure 6: Comparison of Scheme (22) for $g$ the Rusanov flux (top) or the Godunov flux (bottom) for the initial data (25). The particle's trajectory is in black.

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has the same speed, whereas the particle slows down. Thus at some point a new Riemann problem with a single discontinuity appears. In the example below, the pattern with two shocks separating and regrouping occurs three times, after what the particle just travels inside the shock.

The initial data is

$$u^0(x) = 7.2 * \mathbf{1}_{x<0} - 2.2 * \mathbf{1}_{x>0}, \; h^0 = 0, v^0 = 15. \tag{26}$$

and we took $\Delta x = 0.001$ and $\Delta t = 0.000025$ as before. The solution at different times is depicted on Figure 7. Here the fluid's velocity jump through the particle is always larger than $\lambda$ and there is no boundary layers. Thus we only represent the results for Scheme (22). The results for Scheme (21) are almost identical.

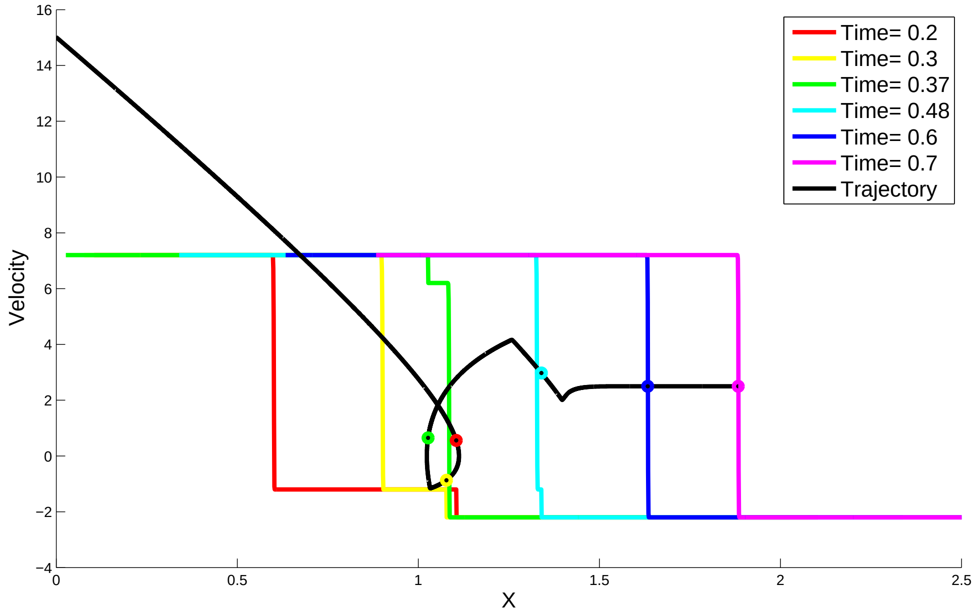


Figure 7: Scheme (22) with the Rusanov flux for the initial data (26). The particle's trajectory is in black.

# 3 A priori bounds

In the sequel we assume that $u^0$ belongs to $L^1(\mathbb{R}) \cap BV(\mathbb{R})$, that Hypotheses (11), (12) and (13) on the flux $g$ are satisfied, and that the monotonicity and regularity assumptions (18) and (17) on $g^\pm$ are satisfied. We will specify the consistency hypotheses on $g^\pm$ along the way. We first consider the uncoupled problem where $(v^n)_{n\in\mathbb{N}}$ is fixed.

**Proposition 3.1.** *Let $u^0$ be in $BV(\mathbb{R}) \cap L^1(\mathbb{R})$. Let $(v^n)_{n\in\mathbb{N}}$ be given and $\underline{v}$ and $\bar{v}$ in $\mathbb{R}$ such that*

$$\forall n \in \mathbb{N}, \; \underline{v} \le v^n \le \bar{v}.$$

*Consider the finite volume scheme*

$$\begin{cases} u_j^{n+1} &= u_j^n - \mu(g(u_j^n, u_{j+1}^n, v^n) - g(u_{j-1}^n, u_j^n, v^n)) \text{ for } j \in \mathbb{Z} \setminus \{0,1\}, \\ u_0^{n+1} &= u_0^n - \mu(g_\lambda^-(u_0^n, u_1^n, v^n) - g(u_{-1}^n, u_0^n, v^n)), \\ u_1^{n+1} &= u_1^n - \mu(g(u_1^n, u_2^n, v^n) - g_\lambda^+(u_0^n, u_1^n, v^n)). \end{cases}$$

*Assume that the fluxes $g^\pm$ satisfy (14) and that the CFL condition (20) holds. Then we have the following $L^\infty$ and $BV$ estimates in space on $u_{\Delta t}$, with $m$ and $M$ the constants of Theorem 1.8:*

$$\forall n \ge 0, \forall j \in \mathbb{Z}, \; m \le u_j^{n+1} \le M \tag{27}$$

*and*

$$\forall n \in \mathbb{N},\ \sum_{j\in\mathbb{Z}} |u_j^n - u_{j-1}^n| \leq \sum_{j\in\mathbb{Z}} |u_j^0 - u_{j-1}^0| + 2\lambda. \tag{28}$$

*Proof.* Due to the presence of the particle, the maximum and the total variation of the exact solution $u$ of (1) can increase through time. For example if $u^0$ is equal to 0 and if $v^0 > \lambda$, then $||u(0^+,\cdot)||_{L^\infty(\mathbb{R})} = ||u^0||_{L^\infty(\mathbb{R})} + \lambda$ and $||u(0^+,\cdot)||_{BV(\mathbb{R})} = ||u^0||_{BV(\mathbb{R})} + 2\lambda$ (see [LST08], Lemma 5.7). This prevents us from applying the LeRoux and Harten lemma (see [Har84] and [LeR77]) directly to $(u_j^n)_{j\in\mathbb{Z},\, n\in\mathbb{N}}$. Yet it can be applied to the sequence $(w_j^n)_{j\in\mathbb{Z},\, n\in\mathbb{N}}$ defined by

$$w_j^n = \begin{cases} u_j^n - \frac{\lambda}{2} & \text{if } j \leq 0, \\ u_j^n + \frac{\lambda}{2} & \text{if } j \geq 1. \end{cases}$$

Let us prove that there exists two families of real numbers $(C_{j+1/2}^n)_{j\in\mathbb{Z},n\in\mathbb{N}}$ and $(D_{j+1/2}^n)_{j\in\mathbb{Z},n\in\mathbb{N}}$ such that for all $j$ in $\mathbb{Z}$, for all $n$ in $\mathbb{N}$,

$$w_j^{n+1} = w_j^n + C_{j+1/2}^n(w_{j+1}^n - w_j^n) - D_{j-1/2}^n(w_j^n - w_{j-1}^n), \tag{29}$$

and

$$0 \leq 1 - C_{j+1/2}^n - D_{j+1/2}^n \leq 1,\ 0 \leq C_{j+1/2}^n \leq 1 \ \text{ and } 0 \leq D_{j+1/2}^n \leq 1.$$

In other words, $w_j^{n+1}$ writes as a convex combination of $w_{j-1}^n$, $w_j^n$ and $w_{j+1}^n$ and therefore,

$$\forall n \geq 0,\ \min_k w_k^n \leq w_j^{n+1} \leq \max_k w_k^n.$$

As a consequence, for all $n \in \mathbb{N}$ and for $j \leq 0$,

$$\min_k w_k^0 + \lambda/2 \leq u_j^n \leq \max_k w_k^0 + \lambda/2,$$

which rewrites

$$\min\{\cdots, u_0^0, u_1^0 + \lambda, \cdots\} \leq u_j^n \leq \max\{\cdots, u_0^0, u_1^0 + \lambda, \cdots\}.$$

Similarly, for all $n \in \mathbb{N}$ and for all $j \geq 1$,

$$\min\{\cdots, u_0^0 - \lambda, u_1^0, \cdots\} \leq u_j^n \leq \max\{\cdots, u_0^0 - \lambda, u_1^0, \cdots\},$$

hence the $L^\infty$ bound (27) is proven. Moreover, the LeRoux and Harten lemma yields

$$\forall n \in \mathbb{N},\ \sum_{j\in\mathbb{Z}} |w_j^{n+1} - w_{j-1}^{n+1}| \leq \sum_{j\in\mathbb{Z}} |w_j^n - w_{j-1}^n|,$$

and thus (28).

Let us go back to the existence of $C_{j+1/2}^n$ and $D_{j-1/2}^n$. In the sequel we denote by $|a,b|$ the interval $[\min(a,b), \max(a,b)]$. Assume first that (29) holds for some $n \in \mathbb{N}$. Then for every $j \leq -1$, there exists $\tilde{w}_{j-1/2}^n \in |w_{j-1}^n, w_j^n|$ and $\bar{w}_{j+1/2}^n \in |w_j^n, w_{j+1}^n|$

$$\begin{aligned} w_j^{n+1} &= w_j^n - \mu\left(g(u_j^n, u_{j+1}^n, v^n) - g(u_{j-1}^n, u_j^n, v^n)\right) \\ &= w_j^n - \mu\left(g_\lambda\left(w_j^n + \frac{\lambda}{2}, w_{j+1}^n + \frac{\lambda}{2}, v^n\right) - g_\lambda\left(w_{j-1}^n + \frac{\lambda}{2}, w_j^n + \frac{\lambda}{2}, v^n\right)\right) \\ &= w_j^n - \mu\left(\partial_1 g_\lambda\left(\tilde{w}_{j-1/2}^n + \frac{\lambda}{2}, w_{j+1}^n + \frac{\lambda}{2}, v^n\right)(w_j^n - w_{j-1}^n)\right. \\ &\qquad \left. + \partial_2 g_\lambda\left(w_{j-1}^n + \frac{\lambda}{2}, \bar{w}_{j+1/2}^n + \frac{\lambda}{2}, v^n\right)(w_{j+1} - w_j)\right). \end{aligned}$$

Both triplets $\left(\tilde{w}_{j-1/2}^n + \frac{\lambda}{2}, w_{j+1}^n + \frac{\lambda}{2}, v^n\right)$ and $\left(w_{j-1}^n + \frac{\lambda}{2}, \bar{w}_{j+1/2}^n + \frac{\lambda}{2}, v^n\right)$ belong to $[m,M]^2 \times [\underline{v}, \bar{v}]$. The CFL condition (20), and the fact that $\partial_1 g \geq 0$ and $\partial_2 g \leq 0$, yield (29) with

$$\begin{cases} D_{j-1/2}^n &= \mu \partial_1 g_\lambda \left(\tilde{w}_{j-1/2}^n + \frac{\lambda}{2}, w_{j+1}^n + \frac{\lambda}{2}, v^n\right), \\ C_{j+1/2}^n &= -\mu \partial_2 g_\lambda \left(w_{j-1}^n + \frac{\lambda}{2}, \bar{w}_{j+1/2}^n + \frac{\lambda}{2}, v^n\right). \end{cases}$$

The case $j \geq 2$ can be treated in the exact same way. We now turn to the trickier case $j = 0$. The facts that $g_\lambda^-$ is consistent with $\mathcal{G}_\lambda^1$ and that $g$ is consistent (Hypothesis (14) and (11)) imply that

$$g_\lambda^-\left(w_0^n + \frac{\lambda}{2}, w_0^n - \frac{\lambda}{2}, v^n\right) = g_\lambda\left(w_0^n + \frac{\lambda}{2}, w_0^n + \frac{\lambda}{2}, v^n\right),$$

which allows us to write

$$\begin{aligned}
w_0^{n+1} &= w_0^n - \mu\left(g_\lambda^-(u_0^n, u_1^n, v^n) - g(u_{-1}^n, u_0^n, v^n)\right)\\
&= w_0^n - \mu\left(g_\lambda^-\left(w_0^n + \frac{\lambda}{2}, w_1^n - \frac{\lambda}{2}, v^n\right) - g_\lambda\left(w_{-1}^n + \frac{\lambda}{2}, w_0^n + \frac{\lambda}{2}, v^n\right)\right)\\
&= w_0^n - \mu\left(g_\lambda^-\left(w_0^n + \frac{\lambda}{2}, w_1^n - \frac{\lambda}{2}, v^n\right) - g_\lambda^-\left(w_0^n + \frac{\lambda}{2}, w_0^n - \frac{\lambda}{2}, v^n\right)\right.\\
&\qquad\left. + g_\lambda\left(w_0^n + \frac{\lambda}{2}, w_0^n + \frac{\lambda}{2}, v^n\right) - g_\lambda\left(w_{-1}^n + \frac{\lambda}{2}, w_0^n + \frac{\lambda}{2}, v^n\right)\right).
\end{aligned}$$

Thus, there exists $\tilde{w}_{-1/2}^n \in |w_{-1}^n, w_0^n|$ and $\bar{w}_{1/2}^n \in |w_0^n, w_1^n|$ such that

$$\begin{aligned}
w_0^{n+1} = w_0^n - \mu&\left(\partial_2 g_\lambda^-\left(w_0^n + \frac{\lambda}{2}, \bar{w}_{1/2}^n - \frac{\lambda}{2}, v^n\right)(w_1^n - w_0^n)\right.\\
&\left. + \partial_1 g_\lambda\left(\tilde{w}_{-1/2}^n + \frac{\lambda}{2}, w_0^n + \frac{\lambda}{2}, v^n\right)(w_0^n - w_{-1}^n)\right).
\end{aligned}$$

Once again, both triplets $\left(w_0^n + \frac{\lambda}{2}, \bar{w}_{1/2}^n - \frac{\lambda}{2}, v^n\right)$ and $\left(w_0^n + \frac{\lambda}{2}, \tilde{w}_{-1/2}^n + \frac{\lambda}{2}, v^n\right)$ belong to $[m, M]^2 \times [\underline{v}, \bar{v}]$. The monotonicity on $g$ and $g_\lambda^-$ allows to conclude with

$$\begin{cases}
D_{-1/2}^n &= \mu \partial_1 g_\lambda\left(\tilde{w}_{-1/2}^n + \frac{\lambda}{2}, w_0^n + \frac{\lambda}{2}, v^n\right),\\
C_{1/2}^n &= -\mu \partial_2 g_\lambda^-\left(w_0^n + \frac{\lambda}{2}, \bar{w}_{1/2}^n - \frac{\lambda}{2}, v^n\right).
\end{cases}$$

The case $j = 1$ can be treated in the exact same way, using the consistency assumption

$$g_\lambda^+\left(w_1^n + \frac{\lambda}{2}, w_1^n - \frac{\lambda}{2}, v^n\right) = g_\lambda\left(w_1^n - \frac{\lambda}{2}, w_1^n - \frac{\lambda}{2}, v^n\right).$$

□

We now turn to the case where the particle's velocity is updated from time to time, and focus on the estimates on the velocity and acceleration of the particle.

**Proposition 3.2.** *Assume that the fluxes $g^\pm$ satisfy* (16) *and* (19)*, that the CFL condition* (20) *holds, and that the time step also satisfies*

$$\frac{4L}{m_p}\Delta t \leq 1. \tag{30}$$

*Then, the sequence $(u_j^n)_{j\in\mathbb{Z}, n\in\mathbb{N}}$ defined by* (6) *satisfies Estimates* (27) *and* (28)*, while for $(v^n)_{n\in\mathbb{N}}$ we have the following estimates:*

$$\forall n \in \mathbb{N}, \quad \underline{v} \leq v^n \leq \bar{v}, \tag{31}$$

*and*

$$\forall n \in \mathbb{N}, \quad \left|\frac{v^{n+1} - v^n}{\Delta t}\right| \leq \frac{2L}{m_p}(||u^0||_\infty + \lambda + ||v||_\infty). \tag{32}$$

*The constants $\bar{v}$ and $\underline{v}$ are defined in Theorem 1.8.*

*Proof.* We proceed by induction. Let us first remark that if the estimate (31) on $v^n$ is satisfied at time $t^n$, the proof of Proposition 3.1 yields the $L^\infty$ and $BV$ estimates on $(u_j^{n+1})_{j\in\mathbb{Z}}$. Therefore, we focus on the estimate on $v^{n+1}$. Using Hypothesis (16), we introduce the null quantity $g_\lambda^-(v^n, v^n, v^n) - g_\lambda^+(v^n, v^n, v^n)$ and write

$$\begin{aligned} v^{n+1} &= v^n + \frac{\Delta t}{m_p}(g_\lambda^-(u_0^n, u_1^n, v^n) - g_\lambda^+(u_0^n, u_1^n, v^n)) \\ &= v^n + \frac{\Delta t}{m_p}\left(\int_0^1 \partial_s(g_\lambda^-(v^n + s(u_0^n - v^n), v^n + s(u_1^n - v^n), v^n))ds \right. \\ &\qquad \left. - \int_0^1 \partial_s(g_\lambda^+(v^n + s(u_0^n - v^n), v^n + s(u_1^n - v^n), v^n))ds\right), \end{aligned}$$

and we obtain

$$\begin{aligned} v^{n+1} = v^n + \frac{\Delta t}{m_p}\bigg(&\int_0^1 (u_0^n - v^n)\,\partial_1(g_\lambda^- - g_\lambda^+)\big(v^n + s(u_0^n - v^n), v^n + s(u_1^n - v^n), v^n\big)ds \\ &+ \int_0^1 (u_1^n - v^n)\,\partial_2(g_\lambda^- - g_\lambda^+)\big(v^n + s(u_0^n - v^n), v^n + s(u_1^n - v^n), v^n\big)ds\bigg). \end{aligned} \tag{33}$$

Assume now that $v^n \leq \min(u_0^n, u_1^n)$. Then both $(u_1^n - v^n)$ and $(u_0^n - v^n)$ are nonnegative. Moreover, the dissipativity assumption (19) implies that $\partial_1(g_\lambda^- - g_\lambda^+)$ and $\partial_2(g_\lambda^- - g_\lambda^+)$ are also nonnegative. Hence we have $v^{n+1} \geq v^n$ and Hypothesis (30) yields

$$\begin{aligned} v^{n+1} &\leq v^n + 2L\frac{\Delta t}{m_p}(u_0^n - v^n + u_1^n - v^n) \\ &\leq \left(1 - \frac{4L\Delta t}{m_p}\right)v^n + \frac{4L\Delta t}{m_p}\max(u_0^n, u_1^n) \\ &\leq \bar{v}. \end{aligned}$$

We now treat the case $u_0^n \leq v^n \leq u_1^n$. The only difference is that $u_0^n - v^n$ is now negative. The integral form (33) of $v^{n+1}$ and Hypothesis (31) yield

$$\underline{v} \leq v^n - 2L\frac{\Delta t}{m_p}(v^n - u_0^n) \leq v^{n+1} \leq v^n + 2L\frac{\Delta t}{m_p}(u_1^n - v^n) \leq \bar{v}.$$

Once the $L^\infty$ bounds on $(u_j^n)_{j\in\mathbb{Z},n\in\mathbb{N}}$ and $(v^n)_{n\in\mathbb{N}}$ are proven, the bound of the particle's acceleration (32) is an easy consequence of the integral form of $v^{n+1}$. □

*Remark* 3.3. Condition (30) is satisfied for small enough $\Delta t$. Thus it is not a restriction to prove the convergence of the scheme. However from the numerical point of view, one has to check Condition (30) in addition to the CFL condition (20). This restriction is severe if the particle is very light. It is possible, at the cost of solving a nonlinear system, to use an implicit version of Scheme (6) for the particle's velocity, i.e.

$$\begin{cases} u_j^{n+1} &= u_j^n - \mu(f_{j+1/2}^n(v^{n+1}) - f_{j-1/2}^n(v^{n+1})) \text{ for } j \in \mathbb{Z}, j \notin \{0,1\}, \\ u_0^{n+1} &= u_0^n - \mu(f_{1/2,-}^n(v^{n+1}) - f_{-1/2}^n(v^{n+1})), \\ u_1^{n+1} &= u_1^n - \mu(f_{3/2}^n(v^{n+1}) - f_{1/2,+}^n(v^{n+1})), \\ v^{n+1} &= v^n + \frac{\Delta t}{m_p}(f_{1/2,-}^n(v^{n+1}) - f_{1/2,+}^n(v^{n+1})), \\ x_j^{n+1} &= x_j^n + v^n\Delta t. \end{cases}$$

In that case, we obtain Bounds (31) and (32) without Constraint (30) on the time step. The proof is exactly the same than the one of Proposition 3.2. For example in the case where $v^{n+1} \leq \min(u_0^n, u_1^n)$, we obtain

$$v^{n+1} \leq v^n + 2L\frac{\Delta t}{m_p}(u_0^n - v^{n+1} + u_1^n - v^{n+1}),$$

and thus, without any constraint on $\Delta t$ other than (20),

$$v^{n+1} \leq \frac{v^n + 2L\frac{\Delta t}{m_p}(u_0^n + u_1^n)}{1 + 4L\frac{\Delta t}{m_p}} \leq \bar{v}. \qquad \triangle$$

We are now in position to extract converging subsequences of $(u_\Delta)$ and $(h_\Delta)$ (defined in (7) and (8)). In Section 4, we will prove that their limits are solutions of the Cauchy problem (1) for the fully coupled problem.

**Proposition 3.4.** *Assume that $u^0$ belongs to $BV(\mathbb{R})\cap L^1(\mathbb{R})$, and that Hypotheses (11-14) and (16-19) are satisfied. Moreover, assume that the CFL condition* (20) *holds. Then there exists $u$ in $BV_{loc}(\mathbb{R}_+\times\mathbb{R})$ and $h$ in $W^{2,\infty}_{loc}(\mathbb{R}_+)$ such that, up to a subsequence, the sequence $(u_{\Delta t})$ converges in $L^1_{loc}(\mathbb{R}_+\times\mathbb{R})$ toward $u$ and the sequence $(h_{\Delta t})$ converges in $W^{1,\infty}_{loc}(\mathbb{R})$ toward $h$ as $\Delta t$ tends to* 0.

*Proof.* Let us first fix a time $T>0$ and a constant $A>0$ and prove the convergence in $L^1([0,T]\times[-A,A])$ and $W^{1,\infty}([0,T])$. By Proposition 3.1, we can use Helly's theorem to prove the convergence in $L^1([0,T]\times[-A,A])$ of $(u_{\Delta t})$, toward a function $u$ in $BV(([0,T]\times[-A,A]))$. Similarly Proposition 3.2 allows us to apply Arzelà-Ascoli's theorem to prove convergence in $W^{1,\infty}([0,T])$ of $(h_{\Delta t})$ to a function $h$ belonging to $W^{2,\infty}([0,T])$. The result is extended to the whole time-space $\mathbb{R}_+\times\mathbb{R}$ thanks to the Cantor diagonal extraction argument. □

*Remark* 3.5. Up to the same subsequence, $(v_{\Delta t})$ converges toward $h'$ in $L^1_{loc}$. Moreover the sequence of functions $(c_{\Delta t})$ defined by

$$c_{\Delta t}(t,x)=\begin{cases}c_- & \text{if } t<h_{\Delta x}(t),\\ c_+ & \text{if } t>h_{\Delta x}(t),\end{cases}$$

converges in $L^1_{loc}$ toward

$$c(t,x)=\begin{cases}c_- & \text{if } t<h(t),\\ c_+ & \text{if } t>h(t).\end{cases}$$

Indeed, we have

$$\int_{-A}^{A}\int_0^T |c_{\Delta t}(t,x)-c(t,x)|dtdx \le |c_+-c_-|\int_0^T |h_{\Delta t}(t)-h(t)|dt \le 2LT\Delta t.$$

△

# 4 Convergence of schemes consistent with a definite part of the germ

From now on, we assume that all the hypotheses of Proposition 3.4 are satisfied, and that both Conditions (20) and (30) are satisfied. The aim of this section is to prove Theorem 1.8. To that purpose, we prove that under Condition (15), which states that the fluxes $g^{\pm}_{\lambda}$ around the particle are consistent with a definite subset $\mathcal{H}_\lambda$ of the germ (see Definition 1.4), the limit $(u,h)$ of the scheme is the solution of (1).

The fact that the Cauchy problem (1) is well-posed in $BV(\mathbb{R})$ is proven in [ALST13]. Once we know that Scheme (6) converges toward a solution of (1), the uniqueness of the solution yields that the whole sequence $(u_{\Delta t},h_{\Delta t})$ converges. Theorem 1.8 gives a different way to prove the existence of a solution (but not the uniqueness).

## 4.1 Convergence of the fluid's part

The aim of this subsection is to prove that the limit $u$ of $(u_{\Delta t})$ satisfies (4). We prove in Proposition 4.3 that $(u^n_j)_{j\in\mathbb{Z},n\in\mathbb{N}}$ satisfies a discrete version of (4). In the sequel, for all real numbers $a$ and $b$ we denote by

$$a\top b=\max(a,b) \ \text{ and by } \ a\bot b=\min(a,b).$$

In the following proposition, we establish a discrete entropy inequality.

**Proposition 4.1.** *Assume that Hypotheses (11-19) hold (included* (15)*) and that the CFL condition* (20) *is satisfied. Then for all $(c_-,c_+)$ in $\mathbb{R}^2$, there exists a constant $A$, depending only on $\lambda$, $||u^0||_\infty$, $||v||_\infty$ and $(c_-,c_+)$, such that for all $j\in\mathbb{Z}$, for all $n\in\mathbb{N}$, the following inequality holds:*

$$\frac{|u^{n+1}_j-c_j|-|u^n_j-c_j|}{\Delta t}+\frac{G^n_{j+1/2,-}-G^n_{j-1/2,+}}{\Delta x}\le \varepsilon_j\frac{A}{\Delta x}\,\mathrm{dist}_1((c_-,c_+),\mathcal{H}_\lambda(v^n)), \tag{34}$$

*where*

$$\forall j \neq 0,\ G^n_{j+1/2,-} = G^n_{j+1/2,+} = G^n_{j+1/2},$$

*with*

$$G^n_{j+1/2} = g(u^n_j \top c_j, u^n_{j+1} \top c_{j+1}, v^n) - g(u^n_j \bot c_j, u^n_{j+1} \bot c_{j+1}, v^n),$$
$$G^n_{1/2,\pm} = g^{\pm}_{\lambda}(u^n_0 \top c_0, u^n_1 \top c_1, v^n) - g^{\pm}_{\lambda}(u^n_0 \bot c_0, u^n_1 \bot c_1, v^n),$$
$$c_j = \begin{cases} c_- & \text{if } j \leq 0, \\ c_+ & \text{if } j \geq 1, \end{cases} \quad \text{and} \quad \varepsilon_j = \begin{cases} 1 & \text{if } j \in \{0,1\}, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We follow the guidelines of proofs of classical entropy inequalities. They rely on the identity

$$|u^{n+1}_j - c_j| = u^{n+1}_j \top c_j - u^{n+1}_j \bot c_j.$$

For $j \in \mathbb{Z} \setminus \{0,1\}$, we use the condensed notation $u^{n+1}_j = H(u^n_{j-1}, u^n_j, u^n_{j+1}, v^n)$. Hypothesis (12) on the monotonicity of the fluxes and the CFL condition (20) ensure that for every $v$, $H$ is increasing with respect to its first three arguments. Moreover if $j \in \mathbb{Z} \setminus \{0,1\}$, $c_{j-1} = c_j = c_{j+1}$ and we use the consistency of the flux away from the particle (11) to write $c_j = H(c_{j-1}, c_j, c_{j+1}, v^n)$. It follows that

$$\begin{aligned} u^{n+1}_j \top c_j &= H(u^n_{j-1}, u^n_j, u^n_{j+1}, v^n) \top H(c_{j-1}, c_j, c_{j+1}, v^n) \\ &\leq H(u^n_{j-1} \top c_{j-1}, u^n_j \top c_j, u^n_{j+1} \top c_{j+1}, v^n), \\ u^{n+1}_j \bot c_j &= H(u^n_{j-1}, u^n_j, u^n_{j+1}, v^n) \bot H(c_{j-1}, c_j, c_{j+1}, v^n) \\ &\geq H(u^n_{j-1} \bot c_{j-1}, u^n_j \bot c_j, u^n_{j+1} \bot c_{j+1}, v^n), \end{aligned}$$

and that

$$\begin{aligned} |u^{n+1}_j - c_j| &\leq H(u^n_{j-1} \top c_{j-1}, u^n_j \top c_j, u^n_{j+1} \top c_{j+1}, v^n) - H(u^n_{j-1} \bot c_{j-1}, u^n_j \bot c_j, u^n_{j+1} \bot c_{j+1}, v^n) \\ &\leq u^n_j \top c_j - u^n_j \bot c_j - \mu(G^n_{j+1/2} - G^n_{j-1/2}) \\ &\leq |u^n_j - c_j| - \mu(G^n_{j+1/2} - G^n_{j-1/2}). \end{aligned}$$

Let us now focus on the more complicated case $j = 0$ (the case $j = 1$ can be treated in the exact same way). We denote by $(\tilde{c}^n_0, \tilde{c}^n_1)$ a projection of $(c_-, c_+) = (c^n_0, c^n_1)$ on $\mathcal{H}_\lambda(v^n)$ for the $L^1$-norm, and by $(\tilde{c}^n_j)_{j\in\mathbb{Z}, n\in\mathbb{N}}$ and $(\tilde{G}^n_{j+1/2})_{j\in\mathbb{Z}, n\in\mathbb{N}}$ the analogues of $(c_j)_{j\in\mathbb{Z}}$ and $(G^n_{j+1/2})_{j\in\mathbb{Z}, n\in\mathbb{N}}$ constructed with $\tilde{c}$:

$$\forall j \neq 0,\ \tilde{G}^n_{j+1/2,-} = \tilde{G}^n_{j+1/2,+} = \tilde{G}^n_{j+1/2} = g(u^n_j \top \tilde{c}_j, u^n_{j+1} \top \tilde{c}_{j+1}, v^n) - g(u^n_j \bot \tilde{c}_j, u^n_{j+1} \bot \tilde{c}_{j+1}, v^n),$$

$$\tilde{G}^n_{1/2,\pm} = g^{\pm}_{\lambda}(u^n_0 \top \tilde{c}_0, u^n_1 \top \tilde{c}_1, v^n) - g^{\pm}_{\lambda}(u^n_0 \bot \tilde{c}_0, u^n_1 \bot \tilde{c}_1, v^n).$$

Let us first remark that

$$\begin{aligned} |u^{n+1}_0 - c_0| - |u^n_0 - c_0| &\leq |u^{n+1}_0 - \tilde{c}^n_0| + |\tilde{c}^n_0 - c_0| - \big| |u^n_0 - \tilde{c}^n_-| - |\tilde{c}^n_0 - c_0| \big| \\ &\leq |u^{n+1}_0 - \tilde{c}^n_0| - |u^n_0 - \tilde{c}^n_-| + 2|\tilde{c}^n_0 - c_0|. \end{aligned}$$

Thus we have

$$\begin{aligned} &\frac{|u^{n+1}_0 - c_0| - |u^n_0 - c_0|}{\Delta t} + \frac{G^n_{1/2,-} - G^n_{-1/2}}{\Delta x} \\ &\quad \leq \frac{|u^{n+1}_0 - \tilde{c}^n_0| - |u^n_0 - \tilde{c}^n_0|}{\Delta t} + \frac{G^n_{1/2,-} - G^n_{-1/2}}{\Delta x} + \frac{2}{\Delta t} \operatorname{dist}_1((c_-, c_+), \mathcal{H}_\lambda(v^n)) \\ &\quad \leq \frac{G^n_{1/2,-} - G^n_{-1/2}}{\Delta x} - \frac{\tilde{G}^n_{1/2,-} - \tilde{G}^n_{-1/2}}{\Delta x} + \frac{2}{\Delta t} \operatorname{dist}_1((c_-, c_+), \mathcal{H}_\lambda(v^n)). \end{aligned}$$

Indeed, as $(\tilde{c}^n_0, \tilde{c}^n_1)$ belongs to $\mathcal{H}_\lambda(v^n)$, Hypothesis (15) yields that $\tilde{c}^n_0 = H_\lambda(\tilde{c}^n_{-1}, \tilde{c}^n_0, \tilde{c}^n_1, v^n)$, and we obtain as before

$$|u^{n+1}_0 - \tilde{c}_j| \leq |u^n_j - \tilde{c}_j| - \mu(\tilde{G}^n_{j+1/2} - \tilde{G}^n_{j-1/2}).$$

We now attempt to bound

$$\begin{aligned} G^n_{1/2,-} - \tilde{G}^n_{1/2,-} &= g^-_\lambda(u^n_0 \top c_0, u^n_1 \top c_1, v^n) - g^-_\lambda(u^n_0 \bot c_0, u^n_1 \bot c_1, v^n) \\ &\quad - g^-_\lambda(u^n_0 \top \tilde{c}^n_0, u^n_1 \top \tilde{c}^n_1, v^n) + g^-_\lambda(u^n_0 \bot \tilde{c}^n_0, u^n_1 \bot \tilde{c}^n_1, v^n). \end{aligned}$$

As $(v^n)_{n\in\mathbb{Z}}$ is bounded (Proposition 3.2), the maximum and minimum over $n$ of $\tilde{c}_\pm^n$ is a bounded function of $(c_-, c_+)$ and $||v||_\infty$. Thus the set

$$[\min(m, c_-, c_+, \tilde{c}_-^n, \tilde{c}_+^n), \max(M, c_-, c_+, \tilde{c}_-^n, \tilde{c}_+^n)]^2 \times [\underline{v}, \bar{v}].$$

is compact. Therefore, with $L_c$ the Lipschitz constant of $g_\lambda^-$ over this set, we have

$$\begin{aligned}
&|g_\lambda^-(u_0^n \top c_0, u_1^n \top c_1, v^n) - g_\lambda^-(u_0^n \top \tilde{c}_0^n, u_1^n \top \tilde{c}_1^n, v^n)| \\
&\quad \le |g_\lambda^-(u_0^n \top c_0, u_1^n \top c_1, v^n) - g_\lambda^-(u_0^n \top \tilde{c}_0^n, u_1^n \top c_1, v^n)| \\
&\qquad + |g_\lambda^-(u_0^n \top \tilde{c}_0^n, u_1^n \top c_1, v^n) - g_\lambda^-(u_0^n \top \tilde{c}_0^n, u_1^n \top \tilde{c}_1^n, v^n)| \\
&\quad \le L_c \operatorname{dist}_1((c_-, c_+), \mathcal{H}_\lambda(v^n)),
\end{aligned}$$

and similarly

$$|g_\lambda^-(u_0^n \bot c_0, u_1^n \bot c_1, v^n) - g_\lambda^-(u_0^n \top \tilde{c}_0^n, u_1^n \top \tilde{c}_1^n, v^n)| \le L_c \operatorname{dist}_1((c_-, c_+), \mathcal{H}_\lambda(v^n)),$$

which conclude the proof with $A = 2L_c + \frac{2\Delta x}{\Delta t}$. □

We are now in position to obtain a discrete version of (4). Let us first state a lemma which ensures that the interface is "numerically dissipative".

**Lemma 4.2.** *If $g_\lambda^- - g_\lambda^+$ is nondecreasing with respect to its first two arguments then we have the dissipativity property*

$$G_{1/2,-}^n - G_{1/2,+}^n \ge 0.$$

*Proof of Lemma 4.2.* Let us denote by $a = u_0^n \top c_0$, $\tilde{a} = u_0^n \bot c_0$, $b = u_1^n \top c_1$ and $\tilde{b} = u_1^n \bot c_1$, such that $a \ge \tilde{a}$ and $b \ge \tilde{b}$. The dissipativity property holds if and only if

$$g_\lambda^-(a, b, v^n) - g_\lambda^-(\tilde{a}, \tilde{b}, v^n)) \ge g_\lambda^+(a, b, v^n) - g_\lambda^+(\tilde{a}, \tilde{b}, v^n),$$

which is a straightforward consequence of the monotonicity of $g_\lambda^- - g_\lambda^+$ with respect to its first two variables. □

**Proposition 4.3.** *Let $(\varphi_j^n)_{j\in\mathbb{Z},n\in\mathbb{N}}$ be a compactly supported sequence of nonnegative reals. If* (34) *holds for all $n$ in $\mathbb{N}$ and $j$ in $\mathbb{Z}$, then*

$$\begin{aligned}
&\Delta t \Delta x \sum_{j\in\mathbb{Z},n\in\mathbb{N}} |u_j^{n+1} - c_j| \frac{\varphi_j^{n+1} - \varphi_j^n}{\Delta t} + \Delta x \sum_{i\in\mathbb{Z}} |u_j^0 - c_j| \varphi_j^0 + \Delta t \Delta x \sum_{j\in\mathbb{Z}^*,n\in\mathbb{N}} G_{j+1/2}^n \frac{\varphi_{j+1}^n - \varphi_j^n}{\Delta x} \\
&\quad + \Delta t \Delta x \sum_{n\in\mathbb{N}} G_{j+1/2,+}^n \frac{\varphi_1^n - \varphi_0^n}{\Delta x} \ge -A \Delta t \sum_{n\in\mathbb{N}} \operatorname{dist}_1(c, \mathcal{H}_\lambda(v^n))(\varphi_0^n + \varphi_1^n).
\end{aligned} \tag{35}$$

*Proof.* Classically, the starting point is to multiply Equation (34) by $\varphi_j^n$ and to sum over $j \in \mathbb{Z}$ and $n \in \mathbb{N}$. Then the different terms are rearranged to bring out discrete time and space derivatives of $\varphi$. However, this is not straightforward around the particle, because two different fluxes are used on its left and on its right. The first term of (34) yields

$$\sum_{j\in\mathbb{Z},n\in\mathbb{N}} \frac{|u_j^{n+1} - c_j| - |u_j^n - c_j|}{\Delta t} \varphi_j^n = \sum_{j\in\mathbb{Z},n\in\mathbb{N}} |u_j^{n+1} - c_j| \frac{\varphi_j^n - \varphi_j^{n+1}}{\Delta t} - \frac{1}{\Delta t} \sum_{j\in\mathbb{Z}} |u_j^0 - c_j| \varphi_j^0,$$

and the second term yields

$$\begin{aligned}
\sum_{j\in\mathbb{Z},n\in\mathbb{N}} \frac{G_{j+1/2,-}^n - G_{j-1/2,+}^n}{\Delta x} \varphi_j^n &= \sum_{j\in\mathbb{Z}^*,n\in\mathbb{N}} G_{j+1/2}^n \frac{\varphi_j^n - \varphi_{j+1}^n}{\Delta x} + \sum_{n\in\mathbb{N}} \frac{\varphi_0^n}{\Delta x} G_{1/2,-}^n - \frac{\varphi_1^n}{\Delta x} G_{1/2,+}^n \\
&= \sum_{j\in\mathbb{Z}^*,n\in\mathbb{N}} G_{j+1/2}^n \frac{\varphi_j^n - \varphi_{j+1}^n}{\Delta x} + \sum_{n\in\mathbb{N}} \frac{\varphi_0^n}{\Delta x} (G_{1/2,-}^n - G_{1/2,+}^n) \\
&\quad + \sum_{n\in\mathbb{N}} \frac{\varphi_0^n - \varphi_1^n}{\Delta x} G_{1/2,+}^n.
\end{aligned}$$

We almost have a discrete version of (4). Hypothesis (19) exactly says that $g_\lambda^- - g_\lambda^+$ is nondecreasing with respect to its first two arguments. Thus we can apply Lemma 4.2 to obtain

$$\sum_{j\in\mathbb{Z},n\in\mathbb{N}} \frac{G^n_{j+1/2,-} - G^n_{j-1/2,+}}{\Delta x}\varphi^n_j \geq \sum_{j\in\mathbb{Z}^*,n\in\mathbb{N}} G^n_{j+1/2}\frac{\varphi^n_j - \varphi^n_{j+1}}{\Delta x} + \sum_{n\in\mathbb{N}} \frac{\varphi^n_0 - \varphi^n_1}{\Delta x} G^n_{1/2,+}.$$

Eventually, we have

$$\sum_{j\in\mathbb{Z},n\in\mathbb{N}} \varepsilon_j \frac{A}{\Delta x}\operatorname{dist}_1((c_-,c_+),\mathcal{H}_\lambda(v^n))\varphi^n_j = \frac{A}{\Delta x}\sum_{n\in\mathbb{N}} \operatorname{dist}_1((c_-,c_+),\mathcal{H}_\lambda(v^n))(\varphi^n_0 + \varphi^n_1)$$

and (35) is obtained by regrouping all the terms and changing their signs, and multiplying by $\Delta t\Delta x$. ☐

Passing to the limit $\Delta t \to 0$ in Equation (35), we obtain the following proposition.

**Proposition 4.4.** *If $u^0$ belongs to $BV(\mathbb{R})\cap L^1(\mathbb{R})$, if the CFL condition (20) holds and if Hypotheses (11-19), (*included (15)*), are satisfied, then the limit $u$ of $(u_{\Delta t})$ satisfies Inequality (4) for any nonnegative function $\varphi$ in $\mathcal{C}^\infty_0(\mathbb{R}_+\times\mathbb{R})$.*

*Proof.* For small enough $\Delta t$, Condition (30) is satisfied. We recall that the ratio $\frac{\Delta t}{\Delta x}$ is fixed, so the limit $\Delta t \to 0$ also reads $\Delta x \to 0$ in what follows. Let us fix $(c_-,c_+)$ in $\mathbb{R}^2$, and prove that for every nonnegative $\varphi$ in $\mathcal{C}^\infty_0$, the discrete inequality (35) converges to the continuous entropy inequality (4), where the sequence $(\varphi^n_j)_{j\in\mathbb{Z},n\in\mathbb{N}}$ is defined by $\varphi^n_j = \varphi(n\Delta t, x^n_j - h^n)$. We recall that $\mathcal{C}^n_j$ is the space-time cell

$$\mathcal{C}^n_j = \{(n\Delta t + s, x^n_{j-1/2} + y + sv^n), s\in[0,\Delta t), y\in[0,\Delta x)\},$$

that $h^n$ is the discrete position of the particle's trajectory deduced from its velocity:

$$h^{n+1} = h^n + v^n\Delta t,$$

and that the mesh is moving with the particle: $x^{n+1}_j = x^n_j + v^n\Delta t$. We first treat the first term of (35). The sequence of piecewise constant functions $(\zeta_{\Delta t})$ defined by

$$\zeta_{\Delta t}(t,x) = \frac{\varphi^{n+1}_j - \varphi^n_j}{\Delta t} \quad \text{if } (t,x)\in\mathcal{C}^{n+1}_j$$

converges uniformly to the function $(t,x)\mapsto(\partial_t\varphi)(t,x-h(t))$. Indeed, for every $(t,x)\in\mathcal{C}^{n+1}_j$, there exists $\tilde{t}\in[n\Delta t,(n+1)\Delta t]$ such that

$$\begin{aligned}|\zeta_{\Delta x}(t,x) - (\partial_t\varphi)(t,x-h(t))| &= \left|\frac{\varphi((n+1)\Delta t, x^{n+1}_j - h^{n+1}) - \varphi(n\Delta t, x^n_j - h^n)}{\Delta t} - (\partial_t\varphi)(t,x-h(t))\right| \\ &= |(\partial_t\varphi)(\tilde{t}, x^n_j - h^n) - (\partial_t\varphi)(t,x-h(t))| \\ &\leq C(|\tilde{t}-t| + |x - x^n_j| + |h^n - h(t)|) \\ &\leq C(\Delta t + \Delta x + ||h_{\Delta t} - h||_\infty).\end{aligned}$$

We used the fact that $x^{n+1}_j - h^{n+1} = x^n_j - h^n$. We conclude thanks to Remark 3.5 :

$$\begin{aligned}\Delta t\Delta x\sum_{j\in\mathbb{Z},n\in\mathbb{N}} |u^{n+1}_j - c_j|\frac{\varphi^{n+1}_j - \varphi^n_j}{\Delta t} &= \sum_{j\in\mathbb{Z},n\in\mathbb{N}}\int_{\mathcal{C}^{n+1}_j} |u_{\Delta t} - c_{\Delta t}|\zeta_{\Delta t}\,dt\,dx \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}_+} \mathbf{1}_{t\geq\Delta t}|u_{\Delta t} - c_{\Delta t}|\zeta_{\Delta t}\,dt\,dx \\ &\underset{\Delta t\to 0}{\longrightarrow} \int_{\mathbb{R}}\int_{\mathbb{R}_+} |u-c|(\partial_t\varphi)(t,x-h(t))\,dt\,dx.\end{aligned}$$

On the other hand,

$$\Delta t\Delta x\sum_{j<0,n\in\mathbb{N}} G^n_{j+1/2}\frac{\varphi^n_{j+1} - \varphi^n_j}{\Delta x} = \int_{x<-\frac{\Delta x}{2}}\int_{\mathbb{R}_+} G_{\Delta t}\xi_{\Delta t}\,dt\,dx$$

where for every $(t,x)$ in $\mathcal{C}^n_{j+1/2} = \{(n\Delta t + s, x_j + y + v^n s), 0 \le s < \Delta t, 0 \le y < \Delta x\}$,

$$\begin{aligned} G_{\Delta t}(t,x) = G_{j+1/2} =& g_\lambda\left(u_{\Delta t}\left(t, x - \frac{\Delta x}{2}\right)\top c_-, u_{\Delta t}\left(t, x + \frac{\Delta x}{2}\right)\top c_-, v_{\Delta t}(t)\right) \\ &- g_\lambda\left(u_{\Delta t}\left(t, x - \frac{\Delta x}{2}\right)\perp c_-, u_{\Delta t}\left(t, x + \frac{\Delta x}{2}\right)\perp c_-, v_{\Delta t}(t)\right) \end{aligned}$$

and for every $(t,x)$ in $\mathcal{C}^n_{j+1/2}$,

$$\xi_{\Delta t}(t,x) = \frac{\varphi^n_{j+1} - \varphi^n_j}{\Delta x}.$$

The sequence $(\xi_{\Delta t})$ converges uniformly to $(t,x) \mapsto \partial_x \varphi(t, x - h(t))$. By continuity of translations in $L^1$, the sequences $(u_{\Delta t}(t, \cdot + \frac{\Delta x}{2}))_{\Delta t}$ and $(u_{\Delta t}(t, \cdot - \frac{\Delta x}{2}))_{\Delta t}$ converge in $L^1_{loc}$ and therefore, up to extraction almost everywhere, toward $u$. On the other hand, $(v_{\Delta t})$ converges almost everywhere toward $h'$. The consistency of the germ implies that $G_{\Delta t}$ converges almost everywhere to

$$g(u\top c_-, u\top c_-, h') - g(u\perp c_-, u\perp c_-, h') = \operatorname{sgn}(u - c_-)\left(\left(\frac{u^2}{2} - h'u\right) - \left(\frac{c_-^2}{2} - h'c_-\right)\right).$$

As $(u_{\Delta t})$ and $(v_{\Delta t})$ are uniformly bounded in $L^\infty$, the dominated convergence theorem yields

$$\Delta t \Delta x \sum_{j<0, n\in\mathbb{N}} G^n_{j+1/2} \frac{\varphi^n_{j+1} - \varphi^n_j}{\Delta x} \underset{\Delta t\to 0}{\longrightarrow} \int_{\mathbb{R}_-}\int_{\mathbb{R}_+} \Phi_{h'(t)}(u(t,x), c_-)\partial_x\varphi(t, x - h(t))dt\,dx.$$

The second and fourth terms of (35) are easily treated:

$$\Delta x \sum_{i\in\mathbb{Z}} |u^0_j - c_j|\varphi^0_j \underset{\Delta x\to 0}{\longrightarrow} \int_{\mathbb{R}} |u^0 - c|\varphi(0,x)dx$$

and

$$\Delta t \Delta x \sum_{n\in\mathbb{N}} G^n_{j+1/2,+} \frac{\varphi^n_1 - \varphi^n_0}{\Delta x} \underset{\Delta t\to 0}{\longrightarrow} 0.$$

Eventually, we study the convergence of

$$\Delta t \sum_{n\in\mathbb{N}} \operatorname{dist}_1(c, \mathcal{H}_\lambda(v^n))(\varphi^n_0 + \varphi^n_1) = 2\int_{\mathbb{R}_+} \operatorname{dist}_1(c, \mathcal{H}_\lambda(v_{\Delta t}))\frac{\varphi_{\Delta t}(t, -\frac{\Delta x}{2}) + \varphi_{\Delta t}(t, \frac{\Delta x}{2})}{2}dt.$$

Clearly, $\frac{\varphi_{\Delta x}(t, -\frac{\Delta x}{2}) + \varphi_{\Delta x}(t, \frac{\Delta x}{2})}{2}$ converges uniformly to $\varphi(\cdot, 0)$. Moreover,

$$\begin{aligned} |\operatorname{dist}_1(c, \mathcal{H}_\lambda(v_{\Delta t})) - \operatorname{dist}_1(c, \mathcal{H}_\lambda(h'))| &= |\operatorname{dist}_1(c, (v_{\Delta t} - h', v_{\Delta t} - h') + \mathcal{H}_\lambda(h')) - \operatorname{dist}_1(c, \mathcal{H}_\lambda(h'))| \\ &= |\operatorname{dist}_1(c - (v_{\Delta t} - h', v_{\Delta t} - h'), \mathcal{H}_\lambda(h')) - \operatorname{dist}_1(c, \mathcal{H}_\lambda(h'))| \\ &\le |v_{\Delta t} - h'| \end{aligned}$$

and

$$\Delta t \sum_{n\in\mathbb{N}} \operatorname{dist}_1(c, \mathcal{H}_\lambda(v^n))(\varphi^n_0 + \varphi^n_1) \underset{\Delta t\to 0}{\longrightarrow} 2\int_{\mathbb{R}_+} \operatorname{dist}_1(c, \mathcal{H}_\lambda(h'))\varphi(t,0)dt,$$

which concludes the proof. □

*Remark* 4.5. In [CS12], the authors are able to derive error estimates for the Godunov scheme adapted to a conservation law with a discontinuous flux (with respect to the space variable). The jump in such a flux can be related to the presence of the particle in our case, and a treatment partially consistent with the interface is also proposed in this paper. A careful investigation of the interface enables the authors to prove adapted $BV$ bounds, which are one of the main difficulties for obtaining error estimates. Due to the particular fluxes we use around the particle, we can also prove here $BV$ bounds, see Proposition 3.1, and one may expect to adapt the proof of [CS12] and thus obtain error estimates for our numerical methods. △

## 4.2 Convergence of the particle's part

We now prove that the limit $h$ of $(h_{\Delta t})$ satisfies (5). To begin with, we prove that a discrete version of (5) holds.

**Proposition 4.6.** *Let $(u_j^n)_{n\in\mathbb{N},j\in\mathbb{Z}}$ and $(v^n)_{n\in\mathbb{N}}$ be given by Scheme (6). Then, for every compactly supported sequences $(\xi^n)_{n\in\mathbb{N}}$ and $(\psi_j^n)_{n\in\mathbb{N},j\in\mathbb{Z}}$ such that $\psi_0^n = \psi_1^n = 1$ for all integrer $n$,*

$$\begin{aligned} -m\Delta t \sum_{n\in\mathbb{N}^*} v^n \frac{\xi^n - \xi^{n-1}}{\Delta t} &= mv^0\xi^0 + \Delta x\Delta t \sum_{n\in\mathbb{N}^*,j\in\mathbb{Z}} u_j^n \frac{\psi_j^n\xi^n - \psi_j^{n-1}\xi^{n-1}}{\Delta t} \\ &\quad + \Delta x \sum_{j\in\mathbb{Z}} u_j^0 \xi^0 \psi_j + \Delta t\Delta x \sum_{n\in\mathbb{N},j\neq 0} f_{j+1/2}^n \xi^n \frac{\psi_{j+1}^n - \psi_j^n}{\Delta x}. \end{aligned} \tag{36}$$

*Proof.* We write

$$\begin{aligned} m\sum_{n\in\mathbb{N}} v^{n+1}\xi^n &= m\sum_{n\in\mathbb{N}} v^n\xi^n + \Delta t\sum_{n\in\mathbb{N}} (f_{1/2,-}^n - f_{1/2,+}^n)\xi^n \\ &\quad + \Delta x \sum_{n\in\mathbb{N}} \sum_{j\notin\{0,1\}} \left[(u_j^n - u_j^{n+1}) - \mu(f_{j+1/2}^n - f_{j-1/2}^n)\right] \xi^n\psi_j^n \\ &\quad + \Delta x \sum_{n\in\mathbb{N}} \left[(u_0^n - u_0^{n+1}) - \mu(f_{1/2,-}^n - f_{-1/2}^n)\right] \xi^n \\ &\quad + \Delta x \sum_{n\in\mathbb{N}} \left[(u_1^n - u_1^{n+1}) - \mu(f_{3/2}^n - f_{1/2,+}^n)\right] \xi^n. \end{aligned}$$

This comes from the fact that the sum of the last three lines is zero. We now rearrange the different terms. On the one hand we have:

$$\sum_{n\in\mathbb{N},j\leq -1} (f_{j+1/2}^n - f_{j-1/2}^n)\xi^n\psi_j^n = \sum_{n\in\mathbb{N},j\leq -1} f_{j+1/2}^n \xi^n(\psi_j^n - \psi_{j+1}^n) + \sum_{n\in\mathbb{N}} \xi^n f_{-1/2}^n,$$

and on the other hand we have:

$$\sum_{n\in\mathbb{N},j\geq 2} (f_{j+1/2}^n - f_{j-1/2}^n)\xi^n\psi_j^n = \sum_{n\in\mathbb{N},j\geq 1} f_{j+1/2}^n \xi^n(\psi_j^n - \psi_{j+1}^n) - \sum_{n\in\mathbb{N}} \xi^n f_{3/2}^n.$$

It follows

$$m\sum_{n\in\mathbb{N}} v^{n+1}\xi^n = m\sum_{n\in\mathbb{N}} v^n\xi^n + \Delta x \sum_{n\in\mathbb{N},j\in\mathbb{Z}} (u_j^n - u_j^{n+1})\xi^n\psi_j^n - \Delta t \sum_{n\in\mathbb{N},j\neq 0} f_{j+1/2}^n\xi^n(\psi_j^n - \psi_{j+1}^n).$$

To conclude, we just have to rearrange the sum over $n$. Being careful with $n = 0$ we obtain

$$\sum_{n\in\mathbb{N}} (v^{n+1} - v^n)\xi^n = \sum_{n\in\mathbb{N}^*} v^n(\xi^{n-1} - \xi^n) - v^0\xi^0$$

and

$$\sum_{n\in\mathbb{N},j\in\mathbb{Z}} (u_j^n - u_j^{n+1})\xi^n\psi_j^n = \sum_{n\in\mathbb{N}^*,j\in\mathbb{Z}} u_j^n(\psi_j^n\xi^n - \psi_j^{n-1}\xi^{n-1}) + \sum_{j\in\mathbb{Z}} u_j^0\xi^0\psi_j^0,$$

and the result follows by regrouping all the terms. □

We can now pass to the limit $\Delta t \to 0$ in Proposition 4.6 to prove that $h$ satisfies (5).

**Proposition 4.7.** *Assume that Hypotheses (11-19) hold, and that the CFL condition* (20) *is satisfied. For all test functions $\xi$ and $\psi$ such that $\psi(0) = 1$, the limit $h$ of $(h_{\Delta t})$ satisfies Inequality* (5).

*Proof.* Define

$$\psi_j^n = \psi(x_j^n - h^n) \quad \text{and} \quad \xi^n = \xi(n\Delta t).$$

Proposition 4.6 applies if $\psi_0^n = \psi_1^n = 1$. Here, we only have

$$\forall j \in \{0,1\}, \quad \left|\psi_j^n - 1\right| \leq C\Delta x.$$

The equality (36) holds up to the following corrections appearing in the left hand side:

$$\Delta x \Delta t \sum_{n\in\mathbb{N}^*, j\in\{0,1\}} u_j^n \frac{(1-\psi_j^n)\xi^n - (1-\psi_j^{n-1})\xi^{n-1}}{\Delta t} + \Delta x \sum_{j\in\{0,1\}} u_j^0 \xi^0 (1-\psi_j^0)$$
$$+ \Delta x \Delta t \sum_{n\in\mathbb{N}} \left( f_{-1/2}^n \xi^n \frac{(1-\psi_0^n)}{\Delta x} - f_{1/2}^n \xi^n \frac{(1-\psi_1^n)}{\Delta x} \right),$$

which all tend to zero since $\psi_0^n - 1 = O(\Delta x)$ and $\psi_1^n - 1 = O(\Delta x)$. The sequence

$$\zeta_{\Delta t}(t,x) = \frac{\psi_j^n \xi^n - \psi_j^{n-1}\xi^{n-1}}{\Delta t} \quad \text{if } (t,x) \in \mathcal{C}_j^n$$

converges uniformly to the function $(t,x) \mapsto \psi \xi'$. Indeed, by definition of the moving mesh, $x_j^n - h^n = x_j^{n-1} - h^{n-1}$. Therefore, $\psi_j^n = \psi_j^{n-1}$ and

$$\frac{\psi_j^n \xi^n - \psi_j^{n-1}\xi^{n-1}}{\Delta t} = \psi_j^n \frac{\xi^n - \xi^{n-1}}{\Delta t}$$

which converges uniformly toward the expected function. Now, define $F_{\Delta t}$ by

$$F_{\Delta t}(t,x) = g_\lambda \left( u_{\Delta t}\left(t, x - \frac{\Delta x}{2}\right), u_{\Delta t}\left(t, x + \frac{\Delta x}{2}\right), v_{\Delta t}(t) \right)$$

in such a way that for all $(t,x)$ in $\mathcal{C}_{j+1/2}^n$,

$$F_{\Delta t}(t,x) = f_{j+1/2}^n.$$

By continuity of translations in $L^1$, the sequences $(u_{\Delta t}(t, \cdot + \frac{\Delta x}{2}))_{\Delta t}$ and $(u_{\Delta t}(t, \cdot - \frac{\Delta x}{2}))_{\Delta t}$ converge in $L^1_{loc}$, and therefore, up to extraction, almost everywhere, toward $u$. On the other hand, $(v_{\Delta t})$ converges almost everywhere toward $h'$. The consistency of the flux (11) implies that $F_{\Delta t}$ converges almost everywhere to

$$g(u,u,h') = \frac{u^2}{2} - h'u.$$

□

### 4.3 A family of schemes consistent with a definite part of the germ

In this section we exhibit a family of schemes that satisfies the set of Assumptions (11-19). Let us clarify which definite subset of $\mathcal{G}_\lambda$ is used.

**Proposition 4.8.** *The part $\mathcal{H}_\lambda(v) = \mathcal{G}_\lambda^1 \cup \mathcal{G}_\lambda^2(v)$ is a definite subset of the germ.*

*Proof.* Following [AS12] (see Equations (13) and (14) in this reference), it suffices to show that if

$$\Xi_v((u_-,u_+),(v_-,v_+)) \geq 0 \quad \text{for any } (v_-,v_+) \in \mathcal{G}_\lambda^2(v), \tag{37}$$

then the stronger following property holds

$$\Xi_v((u_-,u_+),(v_-,v_+)) \geq 0 \quad \text{for any } (v_-,v_+) \in \mathcal{G}_\lambda^1 \cup \mathcal{G}_\lambda^2(v).$$

In the sequel we assume that $v = 0$. The general case follows by translation. The two main arguments are, first, that Proposition 1.4 implies that this is automatically satisfied if $(u_-,u_+)$ belongs to the germ, and, second, that for all $(v_-,v_+)$ in $\mathcal{G}_\lambda^2$, $\left|\frac{v_-^2 - v_+^2}{2}\right| \leq \frac{\lambda^2}{2}$. In the sequel, $(v_-,v_+)$ always denotes an element of $\mathcal{G}_\lambda^2$. We proceed by a tedious, but not difficult, disjunction of cases.

- If $u_- \geq \lambda$ and $u_+ \geq 0$, then we want to prove that

  $$\frac{u_-^2 - v_-^2}{2} - \frac{u_+^2 - v_+^2}{2} \geq 0.$$

  If we apply Equation (37) to $(\lambda, 0)$, we obtain that

  $$\frac{u_-^2 - u_+^2}{2} \geq \frac{\lambda^2}{2}$$

  and the result follows.

- If $0 \leq u_- \leq \lambda$ and $u_+ \geq 0$, then $(u_-, u_+)$ belongs to the germ. Indeed, Equation (37) applied to $(u_-, 0)$ yields $-\frac{u_+^2}{2} \geq 0$ and therefore, $u_+ = 0$.
- If $u_- \leq 0$ and $u_+ \geq 0$, then $(u_-, u_+)$ belongs to the germ. Indeed, Equation (37) applied to $(0, 0)$ yields
$$-\frac{u_-^2}{2} - \frac{u_+^2}{2} \geq 0$$
and therefore, $u_- = u_+ = 0$.
- If $u_- \leq 0$ and $-\lambda \leq u_+ \leq 0$, then $(u_-, u_+)$ belongs to the germ. Indeed, Equation (37) applied to $(0, u_+)$ yields $-\frac{u_-^2}{2} \geq 0$ and therefore, $u_- = 0$.
- If $u_- \leq 0$ and $\leq u_+ \leq -\lambda$, then we want to prove that
$$-\frac{u_-^2 - v_-^2}{2} + \frac{u_+^2 - v_+^2}{2} \geq 0.$$
If we apply Equation (37) to $(0, -\lambda)$, we obtain
$$-\frac{u_-^2}{2} + \frac{u_+^2 - \lambda^2}{2} \geq 0.$$
and the result follows.
- If $0 \leq u_- \leq \lambda$ and $u_+ \leq -\lambda$, let us first assume that $u_- \geq v_-$. We have to prove that
$$\frac{u_-^2 - v_-^2}{2} + \frac{u_+^2 - v_+^2}{2} \geq 0.$$
But $0 \leq v_- \leq u_-$ and $0 \geq v_+ \geq u_+$, and we have the result:
$$\frac{v_-^2 + v_+^2}{2} \leq \frac{u_-^2 + v_+^2}{2} \leq \frac{u_-^2 + u_+^2}{2}.$$
We now assume that $u_- \leq v_-$. We want to prove that
$$-\frac{u_-^2 - v_-^2}{2} + \frac{u_+^2 - v_+^2}{2} \geq 0.$$
Moreover, $(u_-, u_+)$ does not belong to the germ $\mathcal{G}_\lambda$, and therefore $u_+ \leq -u_- - \lambda$ and
$$\frac{u_+^2 - u_-^2}{2} \geq \frac{2u_-\lambda + \lambda^2}{2} \geq \frac{\lambda^2}{2} \geq \frac{v_+^2 - v_-^2}{2}.$$
- If $\lambda \leq u_-$ and $u_+ \leq -\lambda$, the result
$$\frac{u_-^2 - v_-^2}{2} + \frac{u_+^2 - v_+^2}{2} \geq 0$$
is a straightforward consequence of
$$\frac{u_-^2 + u_+^2}{2} \geq \lambda^2 \geq \frac{v_-^2 + v_+^2}{2}.$$
- Eventually, if $\lambda \leq u_-$ and $-\lambda \leq u_+ \leq 0$, let us first assume that $u_+ \leq v_+$ and prove
$$\frac{u_-^2 - v_-^2}{2} + \frac{u_+^2 - v_+^2}{2} \geq 0.$$
It follows from
$$\frac{v_+^2 + v_-^2}{2} \leq \frac{u_+^2 + v_-^2}{2} \leq \frac{u_+^2 + u_-^2}{2}.$$
Assume now that $u_+ > v_+$ and $u_+ \geq -u_- + \lambda$. The result
$$\frac{u_-^2 - v_-^2}{2} - \frac{u_+^2 - v_+^2}{2} \geq 0$$
comes from
$$\frac{u_-^2 - u_+^2}{2} \geq \frac{-2\lambda u_+ + \lambda^2}{2} \geq \frac{\lambda^2}{2} \geq \frac{v_-^2 - v_+^2}{2}.$$

□

It is possible to find fluxes that satisfy (15) with $\mathcal{H}_\lambda = \mathcal{G}^1_\lambda \cup \mathcal{G}^2_\lambda$ and (18).

**Proposition 4.9.** *If $g$ verify the monotonicity assumption* (12)*, the family of finite volume schemes defined by*

$$\begin{cases} g^-_\lambda(u_-, u_+, v) = g(u_-, \min(u_+ + \lambda, \max(u_-, v)), v), \\ g^+_\lambda(u_-, u_+, v) = g(\max(u_- - \lambda, \min(u_+, v)), u_+, v), \end{cases} \tag{38}$$

*is consistent with $\mathcal{G}^1_\lambda \cup \mathcal{G}^2_\lambda(v)$ and satisfies the monotonicity assumptions $\partial_1 g^\pm_\lambda \geq 0$ and $\partial_2 g^\pm_\lambda \leq 0$.*

*Proof.* The proof consists in a simple verification. We first check that for all $u_-$ and $u_+$ in $\mathbb{R}$,

$$g^-_\lambda(u_-, u_- - \lambda, v) = g(u_-, \min(u_-, \max(u_-, v)), v) = g(u_-, u_-, v)$$

and

$$g^+_\lambda(u_+ + \lambda, u_+, v) = g(\max(u_+, \min(u_+, v)), u_+, v) = g(u_+, u_+, v).$$

Then, we verify that for all $u_+$ in $[v - \lambda, v]$,

$$g^-_\lambda(v, u_+, v) = g(u_-, \min(u_+ + \lambda, \max(v, v)), v) = g(v, v, v)$$

and

$$g^+_\lambda(v, u_+, v) = g(\max(v - \lambda, \min(u_+, v)), u_+, v) = g(u_+, u_+, v)$$

while for every $u_-$ in $[v, v + \lambda]$,

$$g^-_\lambda(u_-, v, v) = g(u_-, \min(v + \lambda, \max(u_-, v)), v) = g(u_-, u_-, v)$$

and

$$g^+_\lambda(u_-, v, v) = g(\max(u_- - \lambda, \min(v, v)), v, v) = g(v, v, v).$$

Eventually, the monotonicity properties are implied by those on $g$ as soon as the first component is not $u_+$ and the second is not $u_-$. But if the first component is $u_+$, then $u_+ < v$ and $\partial_2 g^+_\lambda = u_+ - v \leq 0$, while if the second component is $u_-$, then $u_- > v$ and $\partial_2 g^-_\lambda = u_- - v \geq 0$. □

It remains to prove that Assumption (19) holds. This is not the case for every choice of flux $g$ (a counterexample can be found in [AS12]), but we can check it for three classical fluxes.

**Proposition 4.10.** *The family of finite volume schemes* (38) *satisfies that $g^-_\lambda - g^+_\lambda$ is nondecreasing with respect to its first two variables if $g$ is the Godunov, the Rusanov or the Engquist–Osher numerical flux.*

*Proof.* Let us divide the phase space $(u_-, u_+)$ in six zones, depending on which values are taken by $g^-$ and $g^+$:

$$g^-_\lambda(u_-, u_+, v) = \begin{cases} g(u_-, u_-, v) & \text{if } v \leq u_- \leq u_+ + \lambda & \text{zone I,} \\ g(u_-, v, v) & \text{if } u_- \leq v \leq u_+ + \lambda & \text{zone II,} \\ g(u_-, u_+ + \lambda, v) & \text{if } u_+ + \lambda \leq \max(u_-, v) & \text{zone III,} \end{cases}$$

while

$$g^+_\lambda(u_-, u_+, v) = \begin{cases} g(u_+, u_+, v) & \text{if } u_- - \lambda \leq u_+ \leq v & \text{zone 1,} \\ g(v, u_+, v) & \text{if } u_- - \lambda \leq v \leq u_+ & \text{zone 2,} \\ g(u_- - \lambda, u_+, v) & \text{if } \min(u_+, v) \leq u_- - \lambda & \text{zone 3.} \end{cases}$$

These zones are depicted on Figure 8. If $u_+$ belongs to zones 1 or 2, $g^+$ does not depend on $u_-$ and $g^-_\lambda - g^+_\lambda$ is nondecreasing with respect to its first argument. Similarly, if $u_-$ belongs to zones $I$ or $II$, $g^-_\lambda - g^+_\lambda$ is nondecreasing with respect to its second argument. We focus on the case where $u_-$ belongs to zone $III$ or $u_+$ belongs to zone 3. Let us first remark that the case where $u_-$ belongs to zone $III$ and $u_+$ is in zone 3 reduces to the choice of flux studied in [AS12], where the monotonicity property has been proven for the Godunov, Rusanov and Engquist–Osher scheme. Assume that case $u_-$ is in zone $I$ and $u_+$ is in zone 3. Then we have

$$(g^-_\lambda - g^+_\lambda)(u_-, u_+, v) = g(u_-, u_-, v) - g(u_- - \lambda, u_+, v).$$

For the sake of simplicity we assume that $v = 0$.

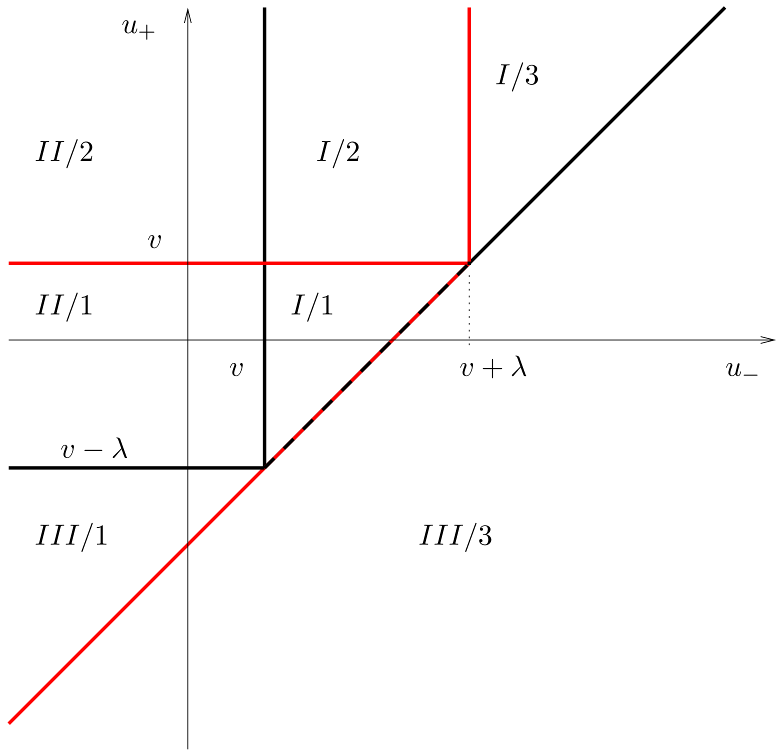


Figure 8: Choice of the fluxes in the family of finite volume schemes (38).

- If $g$ is the Godunov flux, as $u_+ + \lambda \geq u_- \geq \lambda$, the Riemann problem between $u_- - \lambda$ and $u_+$ is a shock traveling faster than $v$. It follows that

$$(g_\lambda^- - g_\lambda^+)(u_-, u_+, 0) = \frac{(u_-)^2}{2} - \frac{(u_- - \lambda)^2}{2} = \lambda u_- - \frac{\lambda^2}{2}$$

  is nondecreasing toward its first two arguments.
- If $g$ is the Rusanov flux,

$$(g_\lambda^- - g_\lambda^+)(u_-, u_+, 0) = \frac{(u_-)^2}{2} - \left(\frac{(u_- - \lambda)^2 + u_+^2}{4} - (u_- - \lambda)\frac{u_+ - (u_- - \lambda)}{2}\right)$$

  and we have

$$\begin{aligned} \partial_1(g_\lambda^- - g_\lambda^+)(u_-, u_+, 0) &= u_- - \left(\frac{u_- - \lambda}{2} - \frac{u_+ - (u_- - \lambda)}{2} + \frac{u_- - \lambda}{2}\right) \\ &= \frac{-u_- + 3\lambda + u_+}{2}. \end{aligned}$$

  As $u_-$ belongs to zone $I$, $u_+ + \lambda \geq u_-$, and the last quantity is larger than $\lambda$. On the other hand,

$$\partial_2(g_\lambda^- - g_\lambda^+)(u_-, u_+, 0) = -\frac{u_+ - (u_- - \lambda)}{2}$$

  and this last quantity is nonnegative because $u_+$ belongs to zone 3.
- Eventually, if $g$ is the Engquist–Osher scheme, the fact that $0 \leq u_- - \lambda \leq u_+$ implies

$$(g_\lambda^- - g_\lambda^+)(u_-, u_+, 0) = \frac{(u_-)^2}{2} - \frac{(u_- - \lambda)^2}{2} = \lambda u_- - \frac{\lambda^2}{2}$$

  is once again nondecreasing with respect to its first two arguments. The case where $u_-$ is in zone $III$ while $u_+$ is in zone 1 can be treated in a symmetrical way.

□

# 5 Convergence of schemes only consistent with $\mathcal{G}^1_\lambda$

In this section, we no longer require Hypothesis (15) to be satisfied, and prove convergence of a family of finite volume schemes that satisfies only (14). The difficulty is that $\mathcal{G}^1_\lambda$ is not a definite part of the germ, and we cannot prove a discrete version of (4) directly. The key point is to study the convergence of the solution of Scheme (6) for initial data in the definite subset of the germ $\mathcal{G}^1_\lambda \cup \mathcal{G}^2_\lambda$. We then extend the comparison argument of [AS12] to prove convergence for arbitrary initial data. Our aim is to prove the following theorem (we do not provide a rigorous statement, the details of the convergence results are stated in Theorem 1.9).

**Theorem 5.1.** *If the numerical fluxes around the particle are given by*

$$\begin{cases} f^n_{1/2,-}(u^n_0, u^n_1, v^n) = g(u^n_0, u^n_1 + \lambda, v^n), \\ f^n_{1/2,+}(u^n_0, u^n_1, v^n) = g(u^n_0 - \lambda, u^n_1, v^n), \end{cases}$$

*where $g$ is a numerical flux satisfying (11-14) and (16-19), and if the CFL condition* (20) *holds, Scheme* (6) *converges toward the solution of* (1).

*Proof.* Let us first remark that Proposition 3.1 and Proposition 3.2 did not use Hypothesis (15), thus we can extract converging subsequences as we did in the previous Section. Now, consider a test function $\varphi$ supported in $\{x < 0\}$ or $\{x > 0\}$, we have $\varphi^n_0 = \varphi^n_1 = 0$ for small enough $\Delta x$. We easily obtain, as in Proposition 4.1, that for all $c$ in $\mathbb{R}$, for all $j \leq -1$,

$$\frac{|u^{n-1}_j - c| - |u^n_j - c|}{\Delta t} + \frac{G^n_{j+1/2} - G^n_{j-1/2}}{\Delta x} \leq 0.$$

Multiplying by $\Delta t \Delta x \varphi^n_j$ and summing over $n \in \mathbb{N}$ and $j \leq -1$, we obtain as in Proposition 4.3

$$\Delta t \Delta x \sum_{j \in \mathbb{Z}, n \leq -1} |u^{n+1}_j - c| \frac{\varphi^{n+1}_j - \varphi^n_j}{\Delta t} + \Delta x \sum_{i \in \mathbb{Z}} |u^0_j - c| \varphi^0_j + \Delta t \Delta x \sum_{j \in \mathbb{Z}^*, n \leq -1} G^n_{j+1/2} \frac{\varphi^n_{j+1} - \varphi^n_j}{\Delta x} \geq 0$$

and we straightforwardly obtain that the limit $u$ of the scheme is an entropy solution of the Burgers equation on the sets $\{x < h\}$ (and similarly on $\{x > h\}$). It remains to prove that the traces around the particle belong to the germ for almost every time. Let us fix a time $t_0$ such that $h'$ and the traces $u_-(t_0)$ and $u_+(t_0)$ exist. Fix $(c_-, c_+)$ in $\mathcal{H}_\lambda(h'(t_0))$. Our aim is to prove a discrete version of (4). Let us first assume that $(c_-, c_+)$ belongs to the straight line $\mathcal{G}^1_\lambda$ but not to the closed square $\overline{\mathcal{G}^2_\lambda(h'(t_0))}$. By continuity of $h'$, there exists $\delta > 0$ such that,

$$\forall t \in (t_0 - \delta, t_0 + \delta),\ \operatorname{dist}_1((c_-, c_+), \mathcal{G}^1_\lambda) = \operatorname{dist}_1((c_-, c_+), \mathcal{H}^1_\lambda(h'(t)))$$

(see Figure 1). Up to taking a smaller $\delta$, this equality is also true at the numerical level for small enough $\Delta t$, since from Proposition 3.4, $(v_n)_{n \in \mathbb{N}}$ converges. Therefore, passing to the limit in (35) with $\varphi$ supported in time in $(t_0 - \delta, t_0 + \delta)$, we directly obtain (4).

We now treat the case where $(c_-, c_+)$ belongs to the interior of $\mathcal{G}^2_\lambda(h'(t_0))$. The principle of the proof is to compare the numerical solution with another one, for which the initial data is much simpler as it corresponds to an element of $\mathcal{G}^2_\lambda(h'(t_0))$. Since $h'$ is continuous, there exists $\delta$ such that

$$\forall t \in (t_0 - \delta, t_0 + \delta),\ (c_-, c_+) \in \mathcal{G}^2_\lambda(h'(t))$$

and on the time interval $(t_0 - \delta, t_0 + \delta)$, (4) becomes

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} |u - c|(s, x) \partial_t \varphi(s, x - h(s)) + \Phi_{h'(t)}(u, c)(s, x) \partial_x \varphi(s, x - h(s)) dx\, ds \geq 0. \tag{39}$$

Up to reducing $\delta$ and for small enough $\Delta t$, this is also true at the numerical level. Now, for $(u^n_j)_{j \in \mathbb{Z}, n \in \mathbb{N}}$ and $(v^n)_{n \in \mathbb{N}}$ given by the fully coupled scheme (6), consider $(c^n_j)_{j \in \mathbb{Z}, n \in \mathbb{N}^*}$ the sequence given by the scheme

$$\begin{cases} c^{n+1}_j &= c^n_j - \mu(g(c^n_j, c^n_{j+1}, v^n) - g(c^n_{j-1}, c^n_j, v^n)) \text{ for } j \notin \{0, 1\}, \\ c^{n+1}_0 &= c^n_0 - \mu(g(c^n_0, c^n_1 + \lambda, v^n) - g(c^n_{-1}, c^n_0, v^n)), \\ c^{n+1}_1 &= c^n_1 - \mu(g(c^n_1, c^n_2, v^n) - g(c^n_0 - \lambda, c^n_1, v^n)), \end{cases} \tag{40}$$

with initial data

$$c_j^0 = \begin{cases} c_- & \text{if } j \le 0, \\ c_+ & \text{if } j > 0. \end{cases} \tag{41}$$

We recall that $(c_-, c_+)$ belongs to $\mathcal{G}^2_\lambda(h'(t_0))$. Simple modifications of Propositions 4.1 and 4.3 yield

$$\Delta t \Delta x \sum_{j\in\mathbb{Z}, n\in\mathbb{N}} |u_j^{n+1} - c_j^{n+1}| \frac{\varphi_j^{n+1} - \varphi_j^n}{\Delta t} + \Delta x \sum_{i\in\mathbb{Z}} |u_j^0 - c_j^0|\varphi_j^0$$
$$+ \Delta t \Delta x \sum_{j\in\mathbb{Z}^*, n\in\mathbb{N}} G_{j+1/2}^n \frac{\varphi_{j+1}^n - \varphi_j^n}{\Delta x} + \Delta t \Delta x \sum_{n\in\mathbb{N}} G_{j+1/2,+}^n \frac{\varphi_1^n - \varphi_0^n}{\Delta x} \ge 0.$$

Assume that $(c_j^n)_{j\in\mathbb{Z},n\in\mathbb{N}}$ converges to $c(t,x) = c_- \mathbf{1}_{x<h(t)} + c_+ \mathbf{1}_{x>h(t)}$ on the interval $(t_0 - \delta, t_0 + \delta)$. Then with $\varphi_j^n = \varphi(t^n, x_j^n)$ where $\varphi$ is a test function supported in $(t_0 - \delta, t_0 + \delta)$, we obtain (39) by passing to the limit. We now study this convergence.

**Lemma 5.2.** *Assume that at iteration $n$, the sequence $(c_j^n)_{j\in\mathbb{Z}}$ given by the scheme (40) is nondecreasing on $j \le 0$ and on $j \ge 1$, and such that*

$$\forall j \le 0,\ c_- \le c_j^n \le c_- + \lambda \quad \text{and} \quad \forall j \ge 1,\ c_+ - \lambda \le c_j^n \le c_+$$

*and*

$$c_0^n - c_1^n \le \lambda,$$

*then the same holds at iteration $n+1$.*

*Proof.* The monotonicity of $(c_j^{n+1})_{j\le 0}$ follows from the monotonicity of $H_\lambda$ under the CFL condition (20). For $j \le -2$, we have

$$c_j^{n+1} = H_\lambda(c_{j-1}^n, c_j^n, c_{j+1}^n) \le H_\lambda(c_j^n, c_{j+1}^n, c_{j+2}^n) = c_{j+1}^{n+1}.$$

As $c_0^n \le c_1^n + \lambda$, we also have

$$c_{-1}^{n+1} = H_\lambda(c_{-2}^n, c_{-1}^n, c_0^n) \le H_\lambda(c_{-1}^n, c_0^n, c_1^n + \lambda) = c_0^{n+1}.$$

Moreover, for $j \le -1$, both $c_{j-1}^n$, $c_j^n$ and $c_{j+1}^n$ are between $c_-$ and $c_- + \lambda$, thus the same holds at iteration $n+1$. For $j = 0$, as $c_+ \le c_-$ (because $(c_-, c_+)$ belongs to $\mathcal{G}^2_\lambda(h'(t_0))$), we conclude by remarking that

$$c_- \le c_0^n \le c_1^n + \lambda \le c_+ + \lambda \le c_- + \lambda.$$

The results for positive integers $j$ are obtained in a similar way. Let us now prove that $u_0^{n+1} - u_1^{n+1} \le \lambda$. We have

$$\begin{aligned} c_0^{n+1} - c_1^{n+1} &= H_\lambda(c_{-1}^n, c_0^n, c_1^n + \lambda) - H_\lambda(c_0^n - \lambda, c_1^n, c_2^n) \\ &\le H_\lambda(c_0^n, c_0^n, c_1^n + \lambda) - H_\lambda(c_0^n - \lambda, c_1^n, c_1^n) \\ &\le c_0^n + \mu L |c_0^n - (c_1^n + \lambda)| - c_1^n + \mu L |(c_0^n - \lambda) - c_1^n| \\ &\le c_0^n - c_1^n + (c_1^n + \lambda - c_0^n) \\ &\le \lambda. \end{aligned}$$

□

For $(c_-, c_+)$ in the *open* subset $\mathcal{G}^2_\lambda(h'(t_0))$, there exists a positive $\delta$ such that $h'(t)$ stays in the interval $(c_+, c_-)$ on the time interval $(t_0 - \delta, t_0 + \delta)$. For small enough $\Delta t$, it is also true at the numerical level. Up to reducing slightly $\delta$, $(c_-, c_+)$ belongs to $\mathcal{G}^2_\lambda(v^n)$ for small enough $\Delta t$ and for all iteration in time such that $t^n$ belongs to $(t_0 - \delta, t_0 + \delta)$, and in particular $c_+ \ge v^n \ge c_-$.

Thus the limit $c$ of the scheme (40) with initial data (41) at time $t_0 - \delta$ is such that $c$ is larger than $h'$ on $x < h$ and smaller on $x > h$. It allows to prove that $c$ is, on $\{(t,x) : x < h(t)\}$, the solution of

$$\begin{cases} \partial_t u + \partial_x \frac{u^2}{2} = 0 & \forall t \in (t_0 - \delta, t_0 + \delta), \forall x < h(t), \\ u(t_0 - \delta, x) = c_- & \forall x < h(0), \\ u(t, h(t)) = h'(t) & \forall t \in (t_0 - \delta, t_0 + \delta). \end{cases} \tag{42}$$

As $c_-$ is larger than $h'$ on the whole time interval, the boundary condition is inactive and the solution is $u = c_-$. Let us recall the definition given by Bardos, LeRoux and Nedelec in [BLN79] of this conservation law on a bounded domain. A function $u$ in $L^\infty$ is a solution of

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & \forall t > 0, \forall x < h(t), \\ u(t=0,x) = u^0(x) & \forall x < h(0), \\ u(t,h(t)) = u_b(t) & \forall t > 0, \end{cases}$$

if for all real $\kappa$ and for all nonnegative function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}_+ \times \mathbb{R})$, the following inequality holds:

$$\begin{aligned} &\int_{t>0}\int_{x<h(t)} |u(t,x)-\kappa|\partial_t\varphi(t,x-h(t)) + \Phi_{h'(t)}(u(t,x),\kappa)\partial_x\varphi(t,x-h(t))dx\,dt \\ &\quad + \int_{x<h(0)} |u^0(x)-\kappa|\varphi(0,x)dx + \int_{t>0} \operatorname{sgn}(\kappa - u_b(t))\{f(u(t,h(t)^-)) - f(\kappa)\}\varphi(t,0) \geq 0. \end{aligned} \tag{43}$$

The convergence of finite volume schemes for scalar conservation laws in a bounded domain has been proven in [Vov02] for instance. We are here in a favorable case: we can obtain a discrete version of (43) by summing (34) multiplied by $\Delta t\, \Delta x\, \varphi_j^n$ over $n \geq 0$ and $j \leq -1$. We obtain

$$\begin{aligned} &\Delta t \Delta x \sum_{n\geq 0, j\leq -1} |c_j^{n+1} - \kappa| \frac{\varphi_j^{n+1} - \varphi_j^n}{\Delta t} + \Delta x \sum_{j \leq -1} |c_j^0 - \kappa|\varphi_j^0 \\ &+ \Delta t \Delta x \sum_{n\geq 0, j \leq -1} G_{j+1/2}^n \frac{\varphi_{j+1}^n - \varphi_j^n}{\Delta x} - \Delta t \sum_{n \geq 0} G_{-1/2}^n \varphi_0^n \leq 0. \end{aligned}$$

Passing to the limit yields

$$\begin{aligned} &\int_{t>0}\int_{x<h(t)} |c(t,x)-\kappa|\partial_t\varphi(t,x-h(t)) + \Phi_{h'(t)}(c(t,x),\kappa)\partial_x\varphi(t,x-h(t))dx\,dt \\ &\quad + \int_{x<h(0)} |c_- - \kappa|\varphi(0,x)dx + \int_{t>0} \operatorname{sgn}(\kappa - c(t,h(t)))\{f(c(t,h(t)^-)) - f(\kappa)\}\varphi(t,0) \geq 0. \end{aligned}$$

To conclude we check that

$$\operatorname{sgn}(\kappa - h'(t))\{f(c(t,h(t)^-)) - f(\kappa)\} \geq -\operatorname{sgn}(c(t,h(t)-\kappa))\{f(c(t,h(t)^-)) - f(\kappa)\}.$$

This relies strongly on the fact that $c$ remains larger than $h'$.

- If $h' \leq \kappa \leq c$, the inequality reduces to

$$\{f(c(t,h(t)^-)) - f(\kappa)\} \geq -\{f(c(t,h(t)^-)) - f(\kappa)\}$$

  which holds because $f$ is increasing on $(0,+\infty)$.
- If $h' \leq c \leq \kappa$ or $\kappa \leq h' \leq c$ the inequality reduces to

$$\{f(c(t,h(t)^-)) - f(\kappa)\} \geq \{f(c(t,h(t)^-)) - f(\kappa)\}$$

  or

$$-\{f(c(t,h(t)^-)) - f(\kappa)\} \geq -\{f(c(t,h(t)^-)) - f(\kappa)\},$$

  which are both trivial.

This ends the proof. □

*Remark* 5.3. Of course, Theorem 5.1 applies when the initial data is

$$u^0(x) = c_- \mathbf{1}_{x<0} + c_+ \mathbf{1}_{x \geq 0},$$

with $(c_-, c_+) \in \mathcal{G}_\lambda^2(v^0)$. In Appendix A, we prove the convergence for this specific initial data directly, without using the local in time comparison with the one-way scheme (40) in which the velocity of the particle is fixed.

△

# Appendices

## A Detailed analysis when the initial data belongs to $\mathcal{G}_\lambda^2(v^0)$

Our aim in this section is to prove directly that if

$$u^0(x) = u_- \mathbf{1}_{x<0} + u_+ \mathbf{1}_{x\geq 0} \quad \text{and} \quad h^0 = 0, \tag{44}$$

with $(u_-, u_+)$ in $\mathcal{G}_\lambda^2(v^0)$, Scheme (6) converges toward the exact solution, which in that case is given by

$$\begin{cases} h(t) = \frac{u_-+u_+}{2} t + \left(v^0 - \frac{u_-+u_+}{2}\right) \frac{m_p}{u_- - u_+} \left(1 - e^{-\frac{u_- - u_+}{m_p} t}\right), \\ u(t,x) = u_- \mathbf{1}_{x<h(t)} + u_+ \mathbf{1}_{x \geq h(t)}. \end{cases}$$

In this section only and for technical reasons, we consider a finite volume scheme on a bounded space domain $[-a, a]$, subdivided with $2M_c$ cells and with periodic boundary conditions. The scheme under consideration writes

$$\begin{cases} u_j^{n+1} &= u_j^n - \mu(g(u_j^n, u_{j+1}^n, v^n) - f(u_{j-1}^n, u_j^n, v^n)) \text{ for } j \in \{-M_c+1, \cdots, M_c\} \setminus \{0, 1\}, \\ u_0^{n+1} &= u_0^n - \mu(g(u_0^n, u_1^n + \lambda, v^n) - f(u_{-1}^n, u_0^n, v^n)), \\ u_1^{n+1} &= u_1^n - \mu(g(u_1^n, u_2^n, v^n) - g(u_0^n - \lambda, u_1^n, v^n)), \\ u_{-M_c}^n &= u_{M_c}^n \quad \text{and} \quad u_{M_c+1}^n = u_{-M_c+1}^n, \\ v^{n+1} &= v^n + \frac{\Delta t}{m_p}(g(u_0^n, u_1^n + \lambda, v^n) - g(u_0^n - \lambda, u_1^n, v^n), \\ x_j^{n+1} &= x_j^n + v^n \Delta t. \end{cases} \tag{45}$$

We recall that the ratio of the time step $\Delta t$ and the cell size $\Delta x$ is equal to $\mu$. We fix the final time $T$. At each time step, four new cells (one of both part of the particle and one of each extremities of the interval because of the periodic boundary conditions) are influenced by Scheme (45), in the sense that their values were constant equal to $u_-$ or $u_+$ before. We take $a$ large enough so that the influence of the particle does not interact with the influence of the boundary condition, and stays in the interval $[-a/3, a/3]$ during the time interval $[0, T]$ (see Figure 9 below). This is achieved by taking $a$ larger than $\frac{3T}{\mu}$. The next proposition states that Scheme (45) converges toward the solution

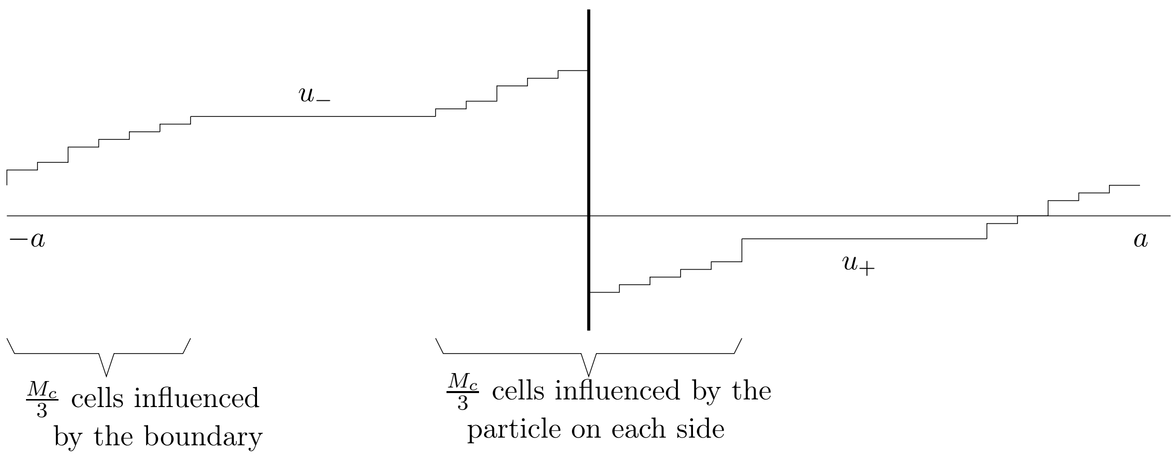


Figure 9: Shape of the numerical solution at time $T$. If $a$ is large enough, the contribution of the particle and of the boundary conditions remain separated.

of the fully coupled problem (1).

**Proposition A.1.** *Assume that the numerical flux $g$ satisfies (11-14) and (16-19), that*

$$\forall A \in \mathbb{R}, \forall B \in \mathbb{R}, \quad g(v - A, v - B, v) = g(v + B, v + A, v), \tag{46}$$

*and that $\partial_3 g$ is decreasing with respect to its first two arguments. Under Condition* (20) *and for the initial data* (44)*, Scheme* (45) *converges toward the solution of* (1) *on*

$$\{(t, x) : t < T \quad \textit{and} \quad -a/3 + h(t) < x < a/3 + h(t)\}.$$

*Proof.* We prove, as we did in Section 5, that $(u_j^n)_{-M_c/3\leq j\leq 0}$ converges toward the solution of (42). The key point is to prove that $v^n$ remains smaller than $u_-$ on the whole time interval $[0,T]$, in which case the boundary condition is inactive and we obtain the result. Similarly on the right of the particle, the boundary condition is inactive if $v^n$ remains larger than $u_+$. It can be seen on Figure 9: if the solution has the shape depicted on this Figure, the part of the solution on the left of the particle has a speed larger than $u_-$, thus if the particle's velocity is smaller than $u_-$, this "wave" is entering inside the particle and is indeed a boundary layer. This intuition was made rigorous in Section 5.

To prove that $u_+ \leq v^n \leq u_-$, we apply the Crandall–Tartar lemma [CT80] to the application

$$T: \begin{array}{ccc} \mathcal{S} & \longrightarrow & \mathcal{S} \\ ((u_j^0)_{-M_c+1\leq j\leq M_c}, v^0) & \longmapsto & ((u_j^n)_{-M_c+1\leq j\leq M_c}, v^n). \end{array}$$

where

$$\mathcal{S} = \{((b_j)_{j\in\{-M_c+1,\cdots,M_c\}}, v) : v\in\mathbb{R},\ b_1\leq b_2\leq\cdots\leq b_{M_c}\leq b_{-M_c+1}\leq b_{-M_c+2}\leq\cdots\leq b_{-1}\leq b_0\leq b_1+\lambda\}.$$

**Lemma A.2** (Crandall–Tartar)**.** *Let $(\Omega,\mu)$ be a measured space, and let $\mathcal{S}$ be a subset of $L^1(\Omega)$ stable by sup:*

$$\forall (u,v)\in\mathcal{S}^2,\quad \max(u,v)\in\mathcal{S}.$$

*Consider a function $T:\mathcal{S}\to\mathcal{S}$ such that the integral is preserved*

$$\forall u\in\mathcal{S}, \int_\Omega T(u) = \int_\Omega u.$$

*Then, if $T$ is order preserving,*

$$||T(u)-T(v)||_{L^1}\leq ||u-v||_{L^1}.$$

In our case, $\Omega = \mathbb{R}^{2M_c}\times\mathbb{R}$,

$$\int_\Omega ((b_j)_{j\in\{-M_c+1,\cdots,M_c\}}, v) = \Delta x\sum_{j=-M_c+1}^{M_c} b_j + mv$$

and

$$||(b_j)_{j\in\{-M_c+1,\cdots,M_c\}}, v||_{L^1} = \Delta x\sum_{j=-M_c+1}^{M_c} |b_j| + m|v|.$$

It is straightforward to verify that Scheme (45) preserves the norm $||\cdot||_{L^1}$. The fact that $T$ takes its values in $\mathcal{S}$ is proven exactly as in the proof of Lemma 5.2. We prove in Lemma A.4 that $T$ is order preserving. Applying the Crandall–Tartar lemma to $((u_j^0), v^0)$ and $(\bar{u}_j^0,\bar{v}) = ((u_j^0), \frac{u_-+u_+}{2})$, we obtain

$$\Delta x\sum_{j=-M_c+1}^{M_c} |\bar{u}_j^{n+1} - u_j^{n+1}| + m|\bar{v}^{n+1} - v^{n+1}| \leq m\left|v^0 - \frac{u_-+u_+}{2}\right|.$$

The result follows since $\bar{v}^{n+1} = \frac{u_-+u_+}{2}$ (see Lemma A.3 below).

□

**Lemma A.3.** *If $g$ satisfies (46) and if the initial data is*

$$\begin{cases} u_j^0 = u_- & \text{for } j\leq 0,\\ u_j^0 = u_+ & \text{for } j\geq 1,\\ v^0 = \frac{u_-+u_+}{2}, & \end{cases}$$

*then Scheme (45) satisfies $v^n = v^0$ for all integer $n$.*

*Proof.* We prove by induction the following stronger result:

$$\forall n \in \mathbb{N}, \forall j \leq 0,\ v^n = \frac{u_- + u_+}{2} \quad \text{and} \quad u^n_{-j} - v^n = v^n - u^n_{j+1}.$$

The symmetry of the initial data ensures that this is satisfied for $n = 0$. Assume that this holds true for some $n \geq 0$. Hypothesis (46) on the flux and the induction hypothesis yield

$$\begin{aligned} g(u^n_0, u^n_1 + \lambda, v^n) &= g(v^n - (u^n_1 + \lambda - v^n), v^n - (u^n_0 - v^n), v^n) \\ &= g(u^n_0 - \lambda, u^n_1, v^n). \end{aligned}$$

Hence, the velocity remains constant. A similar reasoning can be applied to the fluid velocity. Let us give some details for $j \leq -1$:

$$\begin{aligned} u^{n+1}_{-j} &= u^n_{-j} - \mu(g(u^n_{-j}, u^n_{-(j-1)}, v^n) - g(u^n_{-(j+1)}, u^n_{-j}, v^n)) \\ &= 2v^n - u^n_{j+1} - \mu \Big[ g(v^n - (v^n - u^n_{-j}), (v^n - (v^n - u^n_{-(j-1)}), v^n)) \\ &\qquad\qquad -g(v^n - (v^n - u^n_{-(j+1)}), v^n - (v^n - u^n_{-j}), v^n) \Big] \\ &= 2v^n - \Big[ u^n_{j+1} + \mu[g(2v^n - u^n_{-(j-1)}, 2v^n - u^n_{-j}, v^n) \\ &\qquad\qquad -g(2v^n - u^n_{-j}, 2v^n - u^n_{-(j+1)}, v^n) \Big] \\ &= 2v^n - \left( u^n_{j+1} - \mu \left[ g(u^n_{j+1}, u^n_{j+2}, v^n) - g(u^n_j, u^n_{j+1}, v^n) \right] \right) \\ &= 2v^{n+1} - u^{n+1}_{j+1}, \end{aligned}$$

and for $j = 0$:

$$\begin{aligned} u^{n+1}_0 &= u^n_0 - \mu(g(u^n_0, u^n_0 - \lambda, v^n) - g(u^n_{-1}, u^n_0, v^n)) \\ &= 2v^n - u^n_1 - \mu\left[ g(v^n - (v^n - u^n_0), (v^n - (v^n - u^n_0 + \lambda), v^n)) \right. \\ &\qquad\qquad \left. -g(v^n - (v^n - u^n_{-1}), v^n - (v^n - u^n_0), v^n) \right] \\ &= 2v^n - [u^n_1 + \mu[g(2v^n - u^n_0 + \lambda, 2v^n - u^n_0, v^n) \\ &\qquad\qquad -g(2v^n - u^n_0, 2v^n - u^n_{-1}, v^n)] \\ &= 2v^n - (u^n_1 - \mu\left[ g(u^n_1, u^n_2, v^n) - g(u^n_1 + \lambda, u^n_1, v^n) \right]) \\ &= 2v^{n+1} - u^{n+1}_1. \end{aligned}$$

□

We now prove that if two initial data of $\mathcal{S}$ are ordered, this order is conserved after one iteration of the scheme.

**Lemma A.4.** *Let $[(u^n_j)_{j\in\mathbb{Z}}, v^n]$ and $[(\bar{u}^n_j)_{j\in\mathbb{Z}}, \bar{v}^n]$ be two elements of $\mathcal{S}$ such that*

$$\forall j \in \mathbb{Z},\ u^n_j \leq \bar{u}^n_j \quad \text{and} \quad v^n \leq \bar{v}^n.$$

*If the time step satisfy the additional condition*

$$\frac{2\Delta t}{m_p} \max |\partial_3 g| < 1, \tag{47}$$

*then*

$$\forall j \in \mathbb{Z},\ u^{n+1}_j \leq \bar{u}^{n+1}_j \quad \text{and} \quad v^{n+1} \leq \bar{v}^{n+1}.$$

*Proof.* The case where $v^n$ is equal to $\bar{v}^n$ is a straightforward. On the one hand the monotonicity assumption (12) on $g$ and the CFL condition (20) yield as usual

$$u^{n+1}_j = H_\lambda(u^n_{j-1}, u^n_j, u^n_{j+1}, v^n) \leq H_\lambda(\bar{u}^n_{j-1}, \bar{u}^n_j, \bar{u}^n_{j+1}, v^n) = \bar{u}^{n+1}_j.$$

On the other hand,

$$\bar{v}^{n+1} - v^{n+1} = \frac{\Delta t}{m_p}\big((g^-_\lambda - g^+_\lambda)(\bar{u}^n_0, \bar{u}^n_1, v^n) - (g^-_\lambda - g^+_\lambda)(u^n_0, u^n_1, v^n)$$

is nonnegative by Hypothesis (19).

We now focus on the case where $(u_j^n)_{j\in\mathbb{Z}}$ is equal to $(\bar{u}_j^n)_{j\in\mathbb{Z}}$ and $v^n \leq \bar{v}^n$. For $j \leq -1$ and $j \geq 2$, a straightforward computation gives that there exists $a_{j+1/2}^n \in [u_j^n, u_{j+1}^n]$ and $b_{j-1/2}^n \in [u_{j-1}^n, u_j^n]$ such that

$$\begin{aligned} u_j^{n+1} - \bar{u}_j^{n+1} &= \mu \int_0^1 \partial_t g(u_j^n, u_{j+1}^n, v^n + t(\bar{v}^n - v^n)) - \partial_t g(u_{j-1}^n, u_j^n, v^n + t(\bar{v}^n - v^n)) dt \\ &= \mu \int_0^1 (\bar{v}^n - v^n) \big[\partial_3 g(u_j^n, u_{j+1}^n, v^n + t(\bar{v}^n - v^n)) \\ &\qquad\qquad - \partial_3 g(u_{j-1}^n, u_j^n, v^n + t(\bar{v}^n - v^n))\big] \, dt \\ &= \mu \int_0^1 (\bar{v}^n - v^n) \Big[\partial_{23} g(u_j^n, a_{j+1/2}^n, v^n + t(\bar{v}^n - v^n))(u_{j+1}^n - u_j^n) \\ &\qquad\qquad + \partial_{13} g(b_{j-1/2}^n, u_j^n, v^n + t(\bar{v}^n - v^n))(u_j^n - u_{j-1}^n)\Big] . \end{aligned}$$

Moreover, $u_{j-1}^n \leq u_j^n \leq u_{j+1}^n$ because we are considering elements of $\mathcal{S}$, thus if $\partial_3 g$ is decreasing with respect to its first two variables, $u_j^{n+1} \leq \bar{u}_j^{n+1}$. The same reasoning extends to $j \in \{0, 1\}$ because $u_0^n - u_1^n \leq \lambda$. Eventually,

$$\begin{aligned} \bar{v}^{n+1} - v^{n+1} &= \bar{v}^n - v^n + \frac{\Delta t}{m_p}(g(u_0^n, u_1^n + \lambda, \bar{v}^n) - g(u_0^n, u_1^n + \lambda, v^n)) \\ &\quad - \frac{\Delta t}{m_p}(g(u_0^n - \lambda, u_1^n, \bar{v}^n) - g(u_0^n - \lambda, u_1^n, v^n)) \\ &\geq \left(1 - \frac{2\Delta t}{m_p} \max |\partial_3 g|\right) (\bar{v}^n - v^n), \end{aligned}$$

which is nonnegative if (47) holds. □

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