Generalized Harten Formalism and Longitudinal Variation Diminishing schemes for Linear Advection on Arbitrary Grids

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Abstract

We study a family of non linear schemes for the numerical solution of linear advection on arbitrary grids in several space dimension. A proof of weak convergence of the family of schemes is given, based on a new Longitudinal Variation Diminishing (LVD) estimate. This estimate is to be a multidimensional equivalent to the well-known TVD estimate in one dimension. The proof uses a corollary of the Perron-Frobenius theorem applied to a generalized Harten formalism.

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1 Introduction

We address the numerical solution of linear advection in several space dimensions on triangular or quadrangular arbitrary grids, with either linear or non linear schemes. This problem relies on the general theory of numerical approximation of scalar linear and non linear hyperbolic equations by means of finite volume methods. In his seminal work in 1D ([12], [11]), Harten introduced what is commonly referred to as the Harten formalism. TVD (Total Variation Diminishing) schemes derived from the Harten formalism for the numerical solution in 1D of various linear and non linear hyperbolic problems are now very popular: see also [13], [14], [15], [28], [21], [20], [7], [34] and the references therein.

We refer to [18] and [19] for a presentation of major issues about the development of non dissipative schemes for linear advection in 2D for discontinuous flows: see also [29], [20], [7], [30], [31], [2], [32], [33], [35]. We particularly agree with Roe and Sidilkover ([18]): *"Genuinely multidimensional algorithms are only just beginning to be understood"*; see also LeVeque ([7] page 207). To our opinion this is strongly related to the lack of a general multidimensional VD (Variation Diminishing) estimate in 2D on arbitrary grids. Deriving general VD estimates for genuinely multidimensional schemes for various linear and non linear problems on arbitrary grids is challenging. This has been reported for instance by Lax (in Systems of conservation laws and related topics: a conference celebrating Burt Wendroff's birthday): *"Much efforts were spent on trying to devise TVD schemes for multidimensional conservation laws. The search is, of course, doomed to failure, since TV does not D in more than one dimension"*.

The actual theory of approximation ([1], [5], [36], [3]), also suffers from the lack of a general BV (Bounded Variation) estimate, at least on general grids. It has motivated convergence study on arbitrary grids by mean of measure value solutions [16], [17] at the numerical level, by the so called kinetic formulation [6], [3], and via Kuznetsov error approximation [5], [1]. Nevertheless BV estimates exist for Cartesian meshes [9], [10], thus allowing to obtain optimal bounds for the numerical error on Cartesian meshes.

This work is an attempt to set a convenient framework for the development and analysis of genuinely multidimensional schemes on arbitrary grids. Our main result is an extension of TVD schemes and TVD estimate, called in the following LVD (Longitudinal Variation Diminishing) schemes and LVD estimate. Note that it does not enter in contradiction with Lax's remark: T is replaced by L, V is unchanged, the gain is D. An abstract of the paper is the following.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, and let (Ω_j) be an arbitrary mesh of Ω. For any time evolution equation, consider a finite volume approximation: α_j (resp. $\overline{\alpha}_j$) is the current (resp. updated) numerical solution in the cell Ω_j. Assume that a scheme allows to compute $\overline{\alpha}_j$'s from the α_j, such that

$$\forall j,\ \exists \gamma_j \in [0,1],\ \ \overline{\alpha}_j = (1-\gamma_j)\alpha_j + \gamma_j \sum_k p_{jk}\alpha_k, \tag{1}$$

where $P = (p_{jk})$ is a stochastic matrix (cf. [26]):

$$\sum_k p_{jk} = 1,\ \forall j \text{ and } p_{jk} \geq 0\ \forall j,k.$$

The dimension of the matrix P is the number of cells, assumed to be finite.

Definition 1 *We say that a scheme which may written as (1) satisfies a generalized Harten formalism.*

LVD estimate for generalized Harten formalism. *For a scheme (1) there exists non negative weights $\Lambda_j \geq 0$ (with at least one which is non zero) which depend on P and do not depend on the γ_j's, α_j's and $\overline{\alpha}_j$'s, such that the estimate (2) holds*

$$\sum_j \Lambda_j \left| \overline{\alpha}_j - \sum_k p_{jk}\overline{\alpha}_k \right| \leq \sum_j \Lambda_j \left| \alpha_j - \sum_k p_{jk}\alpha_k \right|. \tag{2}$$

The weights $\Lambda = (\Lambda_j)$ are solution of a global eigenvector problem for the eigenvalue 1

$$P^t\Lambda = \Lambda.$$

In the following P is such that (2) is an estimate of the Total Variation *Longitudinally to the streamlines.* This is the reason why we propose to retain this LVD (Longitudinal Variation Diminishing) terminology.

LVD estimate for linear advection. *Consider 2D linear advection with periodic boundary conditions*

$$\partial_t \alpha + \vec{a}.\vec{\nabla}\alpha = 0, \quad (t,x) \in [0,T] \times \Omega_{per}.$$

$\vec{a}$ is given and constant. Consider the standard finite volume upwind approximation on arbitrary grid of this problem: α_j is the current numerical solution in the cell Ω_j; $\overline{\alpha}_j$ is the updated numerical solution in the same cell. Denote $I^-(j)$ the set of neighboring incoming cells (i.e. $k \in I^-(j)$ if and only if $m_{jk} = -\int_{\overline{\Omega_j}\cap\overline{\Omega_k}} (\vec{a}, \vec{n}_j) > 0$), and define $p_{jk} = \frac{m_{jk}}{\sum_{l\in I^-(j)} m_{jl}}$ if $k \in I^-(j)$ and $p_{jk} = 0$ if $k \notin I^-(j)$.

Then a) the upwind scheme may be rewritten as (1) using the matrix P given above, b) for this particular matrix P the weights Λ_j (2) are given explicitly

$$\Lambda_j = \sum_{l\in I^-(j)} m_{jl}.$$

As a consequence the LVD estimate (2) for the upwind scheme applied to linear advection may be rewritten as

$$\sum_j \left| (\sum_{k\in I^-(j)} m_{jk})\overline{\alpha}_j - \sum_{k\in I^-(j)} m_{jk}\overline{\alpha}_k \right| \le \sum_j \left| (\sum_{k\in I^-(j)} m_{jk})\alpha_j - \sum_{k\in I^-(j)} m_{jk}\alpha_k \right|. \tag{3}$$

In 1D one easily checks on simple examples ($a > 0$) that $p_{jk} = \delta_{j,j-1}$ and $\Lambda_j = 1$. So (2) is in 1D

$$\sum_j |\overline{\alpha}_j - \overline{\alpha}_{j-1}| \le \sum_j |\alpha_j - \alpha_{j-1}|. \tag{4}$$

It explains why inequality (2) is a multidimensional generalization of the well known 1D TVD inequality (4). The upwind scheme discussed in the following is the simplest example of a scheme for which the generalized Harten formalism (1) is true.

In section 2, we introduce some notations: linear advection is, in this work, the model problem. In section 3 we propose a 2D generalization on arbitrary grids of TVD schemes, based on an extension of the formalism [22], [24]. These schemes are non linear in the general case and satisfy a generalized Harten formalism. In section 4 we derive a natural Variation Diminishing estimate (2) for this family of schemes on arbitrary grids. We propose to call it LVD estimate due to the presence of various weights. The proof of the LVD estimate relies on essentially three points: a) rewriting the generalized Harten formalism with a stochastic matrix P; b) the Perron-Frobenius theorem for the study in the general case of the maximal left eigenvector of this matrix; c) explicit calculation of the maximal left eigenvector for the matrix P defined by the numerical approximation of linear advection. Simple examples on square grids show that the LVD estimate is a natural extension on arbitrary grids of the TVD estimate. Finally in section 5 and for sake of completeness, a simple consequence of the LVD estimate on arbitrary grids is WBV (Weak Bounded Variation) estimates [1] with better constants (see also [36]). It gives a proof of weak convergence on 2D arbitrary uniformly regular triangular mesh for all linear non linear LVD schemes defined in section 3.

It is worthwhile to notice that the standard proof of convergence via WBV estimates assumes enough dissipation of the scheme: see [1] for a complete discussion. Our proof does not assume such a dissipation process: it is an important advantage of LVD estimates. It leaves place for the study of convergent non linear and non dissipative schemes for linear advection and transport equation on arbitrary grids (recent progress has been made on finite difference grids [24], [22]). We delay to a forthcoming work the question of finding optimal non dissipative LVD schemes for "real computations".

2 Notations and model problem

We consider the following linear advection model problem

$$\begin{cases} \partial_t \alpha + \vec{a}.\vec{\nabla}\alpha = 0, & (t,x) \in [0,T] \times \Omega, \\ \alpha(t=0,x) = \alpha_0(x), & x \in \Omega. \end{cases} \tag{5}$$

For sake of simplicity we consider the 2D case

$$\Omega = [0,1] \times [0,1] \subset \mathbb{R}^2, \tag{6}$$

assume that $\vec{a} \neq 0$ is constant in space and time, and supplement (5) with periodic boundary conditions

$$\begin{cases} \alpha(t,0,x_2) = \alpha(t,1,x_2), & (t,x_2) \in [0,T] \times [0,1], \\ \alpha(t,x_1,0) = \alpha(t,x_1,1), & (t,x_1) \in [0,T] \times [0,1]. \end{cases} \tag{7}$$

Let $(\Omega_j)_{j \in J}$ be a finite mesh of Ω

$$\begin{cases} \Omega_j \cap \Omega_k = \emptyset, & \forall j,k, \quad j \neq k, \\ \cup_{j \in J} \overline{\Omega_j} = \overline{\Omega} = \Omega. \end{cases} \tag{8}$$

The shape of any cell is arbitrary. Most usual cases are square cells (finite difference) or triangle cells (finite volume).

Two cells are neighboring cells if and only if they have an edge in common (taking in account periodic boundary conditions). Each cell has a finite number of neighbors: $I(j)$ is the set of indices of the neighbors of cell j. The outgoing normal from Ω_j on the edge $\overline{\Omega_j} \cap \overline{\Omega_k}$ is denoted as $\vec{n}_{jk}$. Of course the outgoing normal from Ω_j is the opposite of the outgoing normal from Ω_k for $k \in I(j)$

$$\vec{n}_{jk} + \vec{n}_{kj} = 0. \tag{9}$$

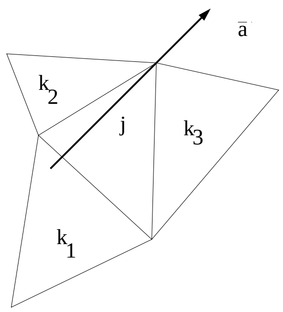

Figure 1: $I^+(j) = \{k_2, k_3\}$, $I^-(j) = \{k_1\}$

We introduce some very natural notations

$$\begin{cases} l_{jk} = l_{kj} = \mathbb{R}\text{-Lebesgue measure of } \overline{\Omega_j} \cap \overline{\Omega_k}, & \text{a length,} \\ s_j = \mathbb{R}^2\text{-Lebesgue measure of } \Omega_j, & \text{a surface.} \end{cases} \tag{10}$$

We also define

$$\begin{cases} I^+(j) = \{k \in I(j); (\vec{a}, \vec{n}_{jk}) > 0\}, \\ I^0(j) = \{k \in I(j); (\vec{a}, \vec{n}_{jk}) = 0\}, \\ I^-(j) = \{k \in I(j); (\vec{a}, \vec{n}_{jk}) < 0\}. \end{cases} \tag{11}$$

and

$$m_{jk} = m_{kj} = l_{jk} |(\vec{a}, \vec{n}_{jk})|. \tag{12}$$

$I^+(j)$ (resp. $I^-(j)$) is the set of outgoing (resp. incoming) cells from Ω_j. An example on triangle is given in figure 1. With all these notations a standard finite volume like method may be defined as

$$s_j \frac{\overline{\alpha}_j - \alpha_j}{\Delta t} + \sum_{k \in I^+(j)} m_{jk} \alpha_{jk} - \sum_{k \in I^-(j)} m_{jk} \alpha_{jk} = 0, \quad \forall j \in J, \tag{13}$$

where $m_{jk}\alpha_{jk}$ is the flux value integrated along the edge $\overline{\Omega_j} \cap \overline{\Omega_k}$, to be determined. In order to save notations the index of iteration n has been omitted: α_j stands for the current value in the cell Ω_j, $\alpha_j = \alpha_j^n$; $\overline{\alpha}_j$ stands for the updated value in the same cell, $\overline{\alpha}_j = \alpha_j^{n+1}$; since we study explicit schemes, α_{jk} stands for α_{jk}^n. In the following we consider symmetric values of the fluxes

$$\alpha_{jk} = \alpha_{kj} \text{ for } k \in I(j), \tag{14}$$

thus the scheme (13) is conservative.

Of course the standard upwind value of the flux

$$\alpha_{jk} = \alpha_k \text{ for } k \in I^-(j) \tag{15}$$

gives the well known upwind scheme

$$s_j \frac{\overline{\alpha}_j - \alpha_j}{\Delta t} + \sum_{k \in I^+(j)} m_{jk} \alpha_j - \sum_{k \in I^-(j)} m_{jk} \alpha_k = 0, \quad \forall j \in J. \tag{16}$$

The following formula will play an important role in the analysis.

Lemma 1 *One has the equality*

$$\sum_{k\in I^+(j)} m_{jk} = \sum_{k\in I^-(j)} m_{jk}, \ \ \forall j. \tag{17}$$

It is a well-known consequence of the divergence theorem

$$0 = \int_{\Omega_j} \mathbf{div}\, \vec{a} = \int_{\partial\Omega_j} (\vec{a}, \vec{n}_{jk}) = \sum_{k\in I^+(j)} m_{jk} - \sum_{k\in I^-(j)} m_{jk}.$$

Here $(.,.)$ denotes the standard scalar product.

It is straightforward to check that the upwind scheme (16) may be rewritten as

$$\overline{\alpha}_j = (1-\gamma_j)\alpha_j + \gamma_j \sum_k p_{jk}\alpha_k, \tag{18}$$

with

$$\gamma_j = \frac{\Delta t \sum_{k\in I^-(j)} m_{jk}}{s_j}$$

and

$$\begin{cases} p_{jk} = \frac{m_{jk}}{\sum_{k\in I^-(j)} m_{jk}}, & \forall k \in I^-(j), \\ p_{jk} = 0, & \forall k \notin I^-(j). \end{cases}$$

Assuming a CFL condition, then $0 \le \gamma_j \le 1$ for all j. It is clear from intuitive geometrical reasons that $\sum_{k\in I^-(j)} m_{jk} > 0$, so the definition of p_{jk} makes sense. See (65) in the appendix for a rigorous proof.

3 Some non linear schemes

In this section we propose other values of the fluxes than the upwind ones (15). Our purpose is to show that the LVD estimate, consequence (next section) of the generalized Harten formalism (18), is not restricted to the upwind scheme. These new fluxes define non linear schemes, even if linear advection is a linear equation: it is already the case in the 1D TVD theory [20].

The construction is an extension on arbitrary grids of the recent work [24], [22] about non dissipative TVD schemes on regular grids. We assume that the fluxes α_{jk} have to satisfy a compatibility principle for all $k \in I^+(j)$, a compatibility principle for all $k \in I^-(j)$, plus a kind of L^∞ estimate.

3.1 Compatibility principle for all $k \in I^+(j)$

We impose to the flux α_{jk} to be a convex combination of α_j and α_k. The compatibility principle means that the combination coefficient is the same for all $k \in I^+(j)$

$$\alpha_{jk} = (1-\beta_j)\alpha_j + \beta_j\alpha_k, \ \ \beta_j \in [0,1], \ \ \forall k \in I^+(j), \ \ \forall j \in J. \tag{19}$$

3.2 Compatibility principle for all $k \in I^-(j)$

We assume that the fluxes satisfy

$$\left(\sum_{k\in I^-(j)} m_{jk}\right) \min\left(\alpha_j, \sum_k p_{jk}\alpha_k\right) \le \sum_{k\in I^-(j)} m_{jk}\alpha_{jk} \tag{20}$$

$$\le \left(\sum_{k\in I^-(j)} m_{jk}\right) \max\left(\alpha_j, \sum_k p_{jk}\alpha_k\right),$$

where

$$\begin{cases} p_{jk} = \frac{m_{jk}}{\sum_{k\in I^-(j)} m_{jk}}, & \forall k \in I^-(j), \\ p_{jk} = 0, & \forall k \notin I^-(j). \end{cases} \tag{21}$$

We will see in the following what this assumption means.

Since $p_{jk} = 0 \ \forall k \notin I^-(j)$, then the sum in (20) is restricted to $k \in I^-(j)$. Note the interesting property

$$\sum_k p_{jk} = 1 \ \ \forall j, \ \ p_{jk} \ge 0 \ \ \forall j, k.$$

A rigorous proof of $\sum_{k\in I^-(j)} m_{jk} > 0$ is given in section A, formula (65).

If ever only one index is in $I^-(j)$ (in other words $I^-(j) = \{k_0\}$), then $p_{jk_0} = \frac{m_{jk_0}}{m_{jk_0}} = 1$. In this case (20) may be rewritten as

$$\min(\alpha_j, \alpha_{k_0}) \le \alpha_{jk} \le \max(\alpha_j, \alpha_{k_0}),$$

and is a direct consequence of (19). Thus the constraint (20) is active only if $card(I^-(j)) > 1$.

A simple manner to enforce (20) is to adopt the convention:

$$\text{if } \left(\sum_{k\in I^-(j)} m_{jk}(\alpha_k - \alpha_j)\right)(\alpha_{k'} - \alpha_j) < 0 \text{ for some } k' \in I^-(j), \tag{22}$$

then we impose

$$\beta_k = 0 \ \ \forall k \in I^-(j). \tag{23}$$

A simple interpretation of (20) is the following: if the upwind scheme (16) predicts an increasing (resp. decreasing) value of $\overline{\alpha}_j$ then all the incoming fluxes have to follow this prediction.

3.3 L^∞ estimate

We would like to impose the following L^∞ estimate

$$\min\left(\alpha_j, \sum_k p_{jk}\alpha_k\right) \le \overline{\alpha}_j \le \max\left(\alpha_j, \sum_k p_{jk}\alpha_k\right). \tag{24}$$

Inequality (24) is indeed a L^∞ stability estimate, since it implies

$$\min\left(\alpha_j, \min_{k\in I^-(j)} \alpha_k\right) \leq \overline{\alpha}_j \leq \max\left(\alpha_j, \max_{k\in I^-(j)} \alpha_k\right). \tag{25}$$

A simple way to enforce (24) for a general scheme (13) is to impose some constraints on the fluxes. We do this with the help of the formalism developed in [22]. First of all we use (13) and rewrite (24) as

$$\sum_{k\in I^-(j)} m_{jk}\alpha_{jk} + \frac{s_j}{\Delta t}\left(\alpha_j - \max\left(\alpha_j, \sum_k p_{jk}\alpha_k\right)\right) \leq \sum_{k\in I^+(j)} m_{jk}\alpha_{jk}, \tag{26}$$

and

$$\sum_{k\in I^+(j)} m_{jk}\alpha_{jk} \leq \sum_{k\in I^-(j)} m_{jk}\alpha_{jk} + \frac{s_j}{\Delta t}\left(\alpha_j - \min\left(\alpha_j, \sum_k p_{jk}\alpha_k\right)\right). \tag{27}$$

The compatibility principle (20) allows to eliminate incoming fluxes in (26-27). We derive a sufficient double inequality

$$\begin{aligned}&\left(\sum_{k\in I^-(j)} m_{jk}\right)\max\left(\alpha_j, \sum_k p_{jk}\alpha_k\right)\\ &+\frac{s_j}{\Delta t}\left(\alpha_j - \max\left(\alpha_j, \sum_k p_{jk}\alpha_k\right)\right) \leq \sum_{k\in I^+(j)} m_{jk}\alpha_{jk},\end{aligned} \tag{28}$$

and

$$\begin{aligned}\sum_{k\in I^+(j)} m_{jk}\alpha_{jk} \leq &\left(\sum_{k\in I^-(j)} m_{jk}\right)\min\left(\alpha_j, \sum_k p_{jk}\alpha_k\right)\\ &+\frac{s_j}{\Delta t}\left(\alpha_j - \min\left(\alpha_j, \sum_k p_{jk}\alpha_k\right)\right).\end{aligned} \tag{29}$$

We thus have

Lemma 2 *If (20) and (28-29) are true then (26-27) is true so the L^∞ estimate (25) is true.*

Note that (28-29) is an inequality only for the β_j variable, due the compatibility principle (19) for $k \in I^+(j)$

$$\sum_{k\in I^+(j)} m_{jk}\alpha_{jk} = \sum_{k\in I^+(j)} m_{jk}\alpha_j + \beta_j \sum_{k\in I^+(j)} m_{jk}(\alpha_k - \alpha_j).$$

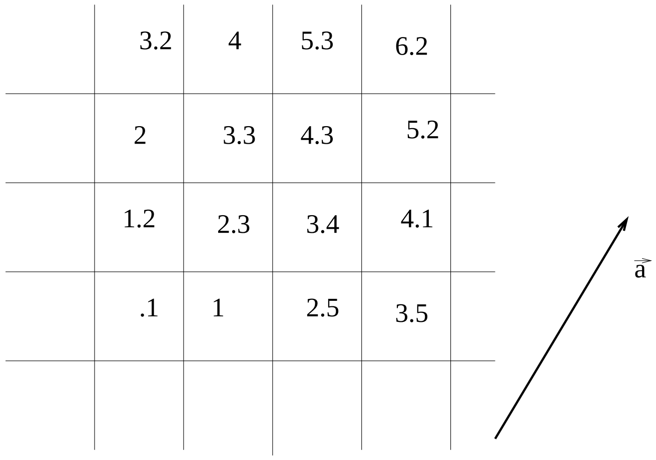

Figure 2: Example of monotone profile on a cartesian mesh

We rewrite (28-29) as

$$\left(\frac{s_j}{\Delta t}-\sum_{k\in I^-(j)}m_{jk}\right)\left(\alpha_j-\max\left(\alpha_j,\sum_k p_{jk}\alpha_k\right)\right)\leq\beta_j\sum_{k\in I^+(j)}m_{jk}\left(\alpha_k-\alpha_j\right) \tag{30}$$

and

$$\beta_j\sum_{k\in I^+(j)}m_{jk}\left(\alpha_k-\alpha_j\right)\leq\left(\frac{s_j}{\Delta t}-\sum_{k\in I^-(j)}m_{jk}\right)\left(\alpha_j-\min\left(\alpha_j,\sum_k p_{jk}\alpha_k\right)\right) \tag{31}$$

(recall lemma 1: $\sum_{k\in I^+(j)}m_{jk}=\sum_{k\in I^-(j)}m_{jk},\ \ \forall j$).

The following result states that at least $\beta_j=0$ (i.e. the upwind scheme) is a solution of (19), (20) and (30-31). In other terms all inequalities are compatible.

Theorem 1 *Assume that the CFL condition*

$$\frac{\sum_{k\in I^+(j)}m_{jk}}{s_j}\Delta t\leq 1,\quad \forall j\in J \tag{32}$$

is satisfied. Then $\beta_j=0$ solves the inequalities (19), (20) and (30-31) for all $j\in J$.

Due to the CFL condition the left hand side of (30) is non positive and the right hand side of (31) is non negative. So $\beta_j=0$ is a solution.

However the upwind scheme ($\beta_j=0\ \forall j$) is not the only solution of inequalities (19), (20) and (30-31).

For sake of simplicity consider a cartesian mesh (figure 2) and assume that a given numerical profile $(\alpha_j)_j$ is monotone, in the sense that

$$\alpha_k < \alpha_j, \quad \forall j, \ \forall k \in I^-(j). \tag{33}$$

One is frequently faced with this monotone situation in applications (or with the revers situation: $\alpha_k > \alpha_j$). In this case (22) is never true: in other words, (20) is automaticly true. So it is sufficient to consider only (30) and (31). Now the term $\beta_j \sum_{k \in I^+(j)} m_{jk} (\alpha_k - \alpha_j)$ is the product of β_j with a positive coefficient $\sum_{k \in I^+(j)} m_{jk} (\alpha_k - \alpha_j)$. Thanks to a strict CFL condition

$$\frac{\sum_{k \in I^+(j)} m_{jk}}{s_j} \Delta t < 1$$

and to (33), the left hand side of (30) is negative and the right hand side of (31) is positive. We deduce that all β_j in the interval

$$\left\{ \begin{array}{l} \beta_j \in [0, \beta_j^{max}], \\ \beta_j^{max} = \left(\frac{s_j}{\Delta t} - \sum_{k \in I^-(j)} m_{jk} \right) \frac{\alpha_j - \min\left(\alpha_j, \sum_k p_{jk} \alpha_k\right)}{\sum_{k \in I^+(j)} m_{jk} (\alpha_k - \alpha_j)} > 0, \end{array} \right. \tag{34}$$

are other values such that (19), (20) and (30-31) are true. As a consequence the L^∞ estimate (24) is true for this choice of β_j. We have proved

Corollary 1 *There exists other fluxes and other schemes than the upwind scheme such that (24) is true. In the monotone situation (33), formula (34) is an example.*

We stop here this discussion and prefer to concentrate on the consequences of (24).

4 Longitudinal Variation Diminishing estimate

Now we derive the LVD estimate for all schemes (13) such that the L^∞ estimate (24) is true. Necessarily $\overline{\alpha}_j$ is a convex combination of the upper and lower bounds of (24).

$$\forall j, \ \ \exists \gamma_j \in [0,1], \ \ \overline{\alpha}_j = (1-\gamma_j)\alpha_j + \gamma_j \sum_k p_{jk} \alpha_k. \tag{35}$$

This is the generalized Harten formalism. In dimension 1, (35) implies the TVD estimate (4).

Let us gather all these quantities using some vector and matrix notations

$$\begin{cases} P=(p_{jk}), & \text{a square matrix,}\\ I=\mathrm{diag}(1), & \text{the identity matrix,}\\ D=\mathrm{diag}(\gamma_j), & \text{a diagonal matrix,}\\ X=(\alpha_j), & \text{a vector,}\\ \overline{X}=(\overline{\alpha}_j), & \text{a vector,}\\ Y=\left(\left|((I-P)X)_j\right|\right), & \text{a vector,}\\ \overline{Y}=\left(\left|\left((I-P)\overline{X}\right)_j\right|\right), & \text{a vector.} \end{cases}$$

All these objects are of dimension $cardJ$, number of cells. As already mentioned a matrix P such that $p_{jk} \geq 0$, $\forall j,k$, and $\sum_k p_{jk} = 1$, $\forall j$ is called a stochastic matrix [26].

We rewrite (35) as

$$\overline{X} = X - D(I-P)X.$$

Thus

$$(I-P)\overline{X} = (I-P)X-(I-P)D(I-P)X = (I-D)(I-P)X+PD(I-P)X. \quad (36)$$

Now we introduce the natural vector ordering

$$\forall \text{ vectors } (X,Y), \quad X \leq Y \text{ if and only if } X_j \leq Y_j \ \forall j,$$

and the natural matricial ordering

$$\forall \text{ matrices } (P,Q), \quad P \leq Q \text{ if and only if } P_{jk} \leq Q_{jk} \ \forall j,k.$$

Taking the absolute value of each coefficient of the vector equation (36) we obtain

$$\overline{Y} \leq (I-D)Y + PDY.$$

We have used the positivity of P and the very important property $0 \leq D \leq I$, which is a consequence of its definition. In summary we have

$$\overline{Y} \leq Y + (P-I)DY. \quad (37)$$

4.1 Basic properties of the matrix P

Since P is a matrix with non negative coefficients such that $\sum_k p_{jk} = 1$, $\forall k$, then

$$\|P\|_\infty = 1 \quad (38)$$

for the induced matrix l^∞ norm

$$||PX||_\infty = \max_{X\neq 0} \frac{||PX||_\infty}{||X||_\infty}, \text{ where } ||X||_\infty = \max_j |X_j|.$$

Since

$$E = \begin{pmatrix} 1 \\ 1 \\ ... \\ 1 \end{pmatrix} \tag{39}$$

is clearly a right eigenvector of the matrix P (indeed, $PE = E$), we know that there exists at least one left eigenvector of the matrix P, for the same eigenvalue. We denote this left eigenvector as Λ,

$$\Lambda^t P = \Lambda^t. \tag{40}$$

If we assume that all the components of Λ are non negative,

$$\Lambda_j \geq 0,$$

then we deduce from (37)

$$(\Lambda, \overline{Y}) \leq (\Lambda, Y) + ((P^t - I)\Lambda, DY) = (\Lambda, Y), \tag{41}$$

where $(.,.)$ denotes the standard l^2 scalar product. This is exactly what we call the Longitudinal Variation Diminishing estimate. Note that Λ depends only on P and not on (α_j). It depends only on $\vec{a}$ and on the mesh. In particular it is constant in time.

4.2 More properties of the matrix P

So the key point is to prove that the left eigenvector Λ is indeed a non negative vector. Reminiscence of the Krein-Rutman [25] theorem or of the Perron-Frobenius theorem [25], [26] gives some hints that this property is true. Let us recall the following result which is a corollary of the Perron-Frobenius theorem [26].

Theorem 2 *Let $A \neq 0$ be a non negative square matrix with non negative coefficients*

$$A_{jk} \geq 0, \forall j, k.$$

Then there exists one maximal real eigenvalue

$$\lambda = \rho(A) > 0,$$

associated with a non negative eigenvector $\Lambda \neq 0$

$$\Lambda_j \geq 0 \ \ \forall j,$$

such that

$$A\Lambda = \lambda\Lambda.$$

Applying this theorem to $A = P^t$ and since $\rho(P^t) = \rho(P) = 1$, it explains why $\Lambda_j \geq 0$, $\Lambda \neq 0$, is true (Λ being defined by (40) and $\lambda = 1$). If we assume moreover that the matrix A is irreducible, the Perron-Frobenius theorem states that $\Lambda > 0$, i.e. $\Lambda_j > 0$ for all j.

With straightforward notations we denote $X^n = (\alpha_j^n)$ the numerical solution of the scheme (13) at the n^{th} time step. We assume it enters the generalized Harten formalism:

$$\forall n,\ \forall j,\ \ \exists \gamma_j^n \in [0,1],\ \ \alpha_j^{n+1} = (1-\gamma_j^n)\alpha_j^n + \gamma_j^n \sum_k p_{jk}\alpha_k^n. \tag{42}$$

$Y^n = |(I-P)X^n|$ is defined by

$$Y_j^n = |\sum_k p_{jk}(\alpha_j^n - \alpha_k^n)|$$

Iterating in time (41) and since Λ depends only on the constant matrix P, we get

Theorem 3 *A numerical solution of any scheme verifying (42) satisfies the LVD (Longitudinal Variation Diminishing) estimate*

$$(\Lambda, Y^n) \leq (\Lambda, Y^{n-1}) \leq ... \leq (\Lambda, Y^0) \tag{43}$$

that is

$$\sum_j \Lambda_j |\sum_k p_{jk}(\alpha_j^n - \alpha_k^n)| \leq \sum_j \Lambda_j |\sum_k p_{jk}(\alpha_j^0 - \alpha_k^0)|. \tag{44}$$

4.3 More about the left eigenvector

Thus the previously cited corollary of the Perron-Frobenius theorem gives an abstract framework such that a generic diminishing estimate holds for every scheme which may be rewritten as (42). Hopefully it is possible, in the case of linear advection, to give the exact value of the weights.

Theorem 4 *Consider the matrix P given by (21). Then a solution of (40) is*

$$\Lambda_j = \sum_{k \in I^-(j)} m_{jk} > 0. \tag{45}$$

All Λ_j's defined by (45) are positive due to (64). The equation $P^t\Lambda = \Lambda$ means that

$$\sum_k p_{kj}\Lambda_k = \Lambda_j,\ \ \forall j,$$

that is

$$\sum_{k \in I^+(j)} \left(\frac{m_{jk}}{\sum_{r \in I^-(k)} m_{kr}} \right) \Lambda_k = \Lambda_j,\ \ \forall j \tag{46}$$

Define

$$\mu_j = \frac{\Lambda_j}{\sum_{r\in I^-(j)} m_{jr}}, \quad \forall j.$$

Then we rewrite (46) as

$$\sum_{k\in I^+(j)} (m_{kj})\,\mu_k = \left(\sum_{r\in I^-(j)} m_{jr}\right)\mu_j, \quad \forall j. \tag{47}$$

Due to the divergence identity (17) one has

$$\sum_{r\in I^-(j)} m_{jr} = \sum_{r\in I^+(j)} m_{jr} = \sum_{k\in I^+(j)} m_{kj}.$$

Thus (47) is

$$\sum_{k\in I^+(j)} (m_{kj})\,\mu_k = \left(\sum_{k\in I^+(j)} m_{kj}\right)\mu_j, \quad \forall j. \tag{48}$$

Finally $\mu_j = 1$ for all j is a solution. It proves (45).

Corollary 2 *Consider a numerical solution of the scheme (13) such that (35) is true (assuming a CFL condition, the upwind scheme is an example of such a scheme). Then the following diminishing estimate holds*

$$\sum_j \left| \left(\sum_{k\in I^-(j)} m_{jk}\right)\overline{\alpha}_j - \sum_{k\in I^-(j)} m_{jk}\overline{\alpha}_k \right| \tag{49}$$

$$\leq \sum_j \left| \left(\sum_{k\in I^-(j)} m_{jk}\right)\alpha_j - \sum_{k\in I^-(j)} m_{jk}\alpha_k. \right|$$

The proof is by direct calculation: consider (44), use (45) and combine with the definition (21) of P.

At this point we would like to make a comment: at first sight (49) may be considered as a numerical approximation of the L^1 norm of $\mathbf{div}(\alpha\vec{a}) = \vec{a}.\vec{\nabla}\alpha$. It would be elegant to find a continuous definition of the same quantity, such that both definitions (the discrete one and the continuous one) give the same result (as it is the case in 1D). Actually it is not so clear for us how to find this kind of continuous definition in the multi-D case.

4.4 The matrix P for square cells

Here we discuss the simple example where the cells are squares. In some sense this example corresponds to finite differences. It is clear that in this case $\Lambda_j = \Lambda_k$ for all j,k (see (45)). So the diminishing inequality simplifies into

$$\sum_j |\sum_k p_{jk}(\alpha_j^n - \alpha_k^n)| \leq \sum_j |\sum_k p_{jk}(\alpha_j^0 - \alpha_k^0)|. \tag{50}$$

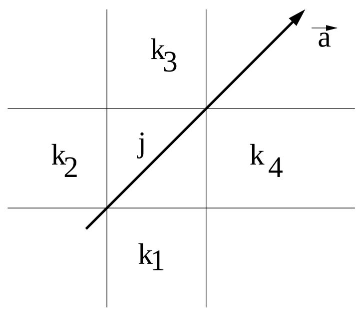

Figure 3: Example: $\vec{a} = (\sqrt{2}/2, \sqrt{2}/2)$

We consider two cases.

a) Assume that $\vec{a} = (\sqrt{2}/2, \sqrt{2}/2)$. So

$$p_{jk} = \frac{1}{2} \; \forall k \in I^-(j), \;\; p_{jk} = 0 \text{ otherwise.}$$

p_{jk} is non zero if and only if the cell k is immediately under or on the left of the cell j. In the example of figure 3

$$p_{jk_1} = p_{jk_2} = \frac{1}{2} \text{ and } p_{jk_3} = p_{jk_4} = 0.$$

By summation we thus obtain

$$\sum_j \left| \alpha_j^n - \frac{1}{2} \sum_{k \in I^-(j)} \alpha_k^n \right| \leq \sum_j \left| \alpha_j^0 - \frac{1}{2} \sum_{k \in I^-(j)} \alpha_k^0 \right| . \tag{51}$$

b) Assume that $\vec{a} = (1, 0)$. So

$$p_{jk} = 1 \; k \in I^-(j), \;\; p_{jk} = 0 \text{ otherwise.}$$

p_{jk} is non zero if and only if the cell k is immediately on the left of the cell j. In the example of figure 4

$$p_{jk_2} = 1 \text{ and } p_{jk_1} = p_{jk_3} = p_{jk_4} = 0.$$

We thus obtain

$$\sum_{\text{lines}} \left(\sum_{j,k \text{ neighboring on the line}} |\alpha_j^n - \alpha_k^n| \right)$$
$$\leq \sum_{\text{lines}} \left(\sum_{j,k \text{ neighboring on the line}} |\alpha_j^0 - \alpha_k^0| \right) . \tag{52}$$

In this case LVD is TVD only line by line.

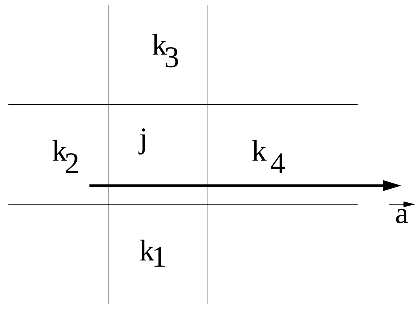

Figure 4: Example: $\vec{a} = (1, 0)$

Here we see that LVD is TVD *longitudinally to the streamlines* due to the weights p_{jk} and $|(\vec{a}, \vec{n}_{jk})|$. This is the reason why we propose to retain this LVD (Longitudinal Variation Diminishing) terminology.

5 A proof of weak convergence for LVD schemes

A LVD scheme refers to a scheme (13) which satisfies the LVD estimates of theorem 3, consequence of (42). Proving strong convergence with optimal rate of convergence for these kind of non linear schemes on arbitrary grids is still an open problem nowadays. Many researchers have stressed that non optimal bounds for the error are probably due to the lack of a BV estimate. Since the core of our work is precisely the derivation of such a LVD estimate for arbitrary grids, there is some hope that optimal bounds will take advantage of the approach developed in this work. Moreover optimal error estimates for monotone schemes need Kruzkov entropy inequalities ([9], [10], [1], [5]), which are far from the scope of this paper.

From both examples on square grids (51-52) it is clear that LVD does not imply TVD: at most TVD line by line in the case (52). As a consequence it is not possible to rely on Helly's theorem (compact embedding of $BV \cap L^1 \subset L^1$, see [20]) in order to prove strong convergence for general grids.

So we delay the question of strong convergence, and rely on an analysis of [1] in order to simply prove weak convergence via Weak Bounded Variations estimates.

An interest of the following proof is that we do not assume that the scheme is monotone as in [1]. The LVD estimate and $\alpha_0 \in L^\infty(\Omega) \cap BV(\Omega)$ is sufficient.

More precise definition of uniformly regular meshes could be found in [27]. For the sake of simplicity, we consider in the following only meshes with triangle cells or quadrilateral cells.

Theorem 5 *Be* $\Omega = [0, 1] \times [0, 1] \subset \mathbb{R}^2$. *Let us consider a sequence of triangular or quadrangular meshes with* $\Delta x \to 0$*:* Δx *is a characteristic length of the mesh.*

We assume that the sequence of meshes is uniformly regular in the sense that

$$\exists (c_1, c_2, c_3) \in \left(\mathbb{R}^{+,*}\right)^3,$$

such that for every mesh in the sequence

$$l_{jk} \le c_1 \Delta x, \ c_2 \Delta x^2 \le s_j \le c_3 \Delta x^2, \ \ \forall j, k. \tag{53}$$

Let $\alpha_0 \in L^\infty(\Omega) \cap BV(\Omega)$*. Let* $\alpha_{\Delta x}$ *be a sequence of numerical solutions, such that*

a) *the initial value is given by the total mass approximation*

$$\alpha_{\Delta x}{}_j^0 = \frac{1}{s_j} \int_{\Omega_j} \alpha_0 \tag{54}$$

(note that s_j *and* Ω_j *depend on* Δx*),*

b) *for each* Δx*,* $\alpha_{\Delta x}^n$ *is given by the scheme (13-14),*

c) *the flux is a convex combination of the unknowns on both sides*

$$\min(\alpha_j, \alpha_k) \le \alpha_{jk} \le \max(\alpha_j, \alpha_k),$$

as in (19),

d) *equality (42) is true for all* j *and* n*.*

The upwind scheme (15-54) together with the CFL condition (32) is an example of such a sequence of numerical solutions. Another non linear example is given in corollary 1.

Let us define

$$\alpha^{\Delta x} = \sum_{j,n} \alpha_{\Delta x}{}_j^n \times 1_j^n \in L^\infty([0,T] \times \Omega),$$

that is

$$\alpha^{\Delta x}(x,t) = \alpha_{\Delta x}{}_j^n \quad \forall (x,t) \in \Omega_j \times]n\Delta t, (n+1)\Delta t[.$$

Let $\alpha \in L^\infty([0,T] \times \Omega)$ *be the solution of*

$$\left\{ \begin{array}{l} \partial_t \alpha + \vec{a}.\nabla \alpha = 0, \\ \alpha(0,x) = \alpha_0(x), \end{array} \right. \tag{55}$$

with periodic boundary conditions: $\alpha(x,t) = \alpha_0(t - (\vec{a}, x))$*. Then* $\alpha^{\Delta x}$ *converges in* $L^{\infty,*}([0,T] \times \Omega)$ *to* α*, that is,*

$$\forall \varphi \in L^1([0,T] \times \Omega), \ \ \lim_{\Delta x \to 0} \left(\int_{[0,T] \times \Omega} (\alpha^{\Delta x} - \alpha) \varphi dx dt \right) = 0. \tag{56}$$

In [1] the previous result of convergence is based on the first of these two following WBV estimates

$$\sum_{p=0}^{p=Q} \Delta t \sum_j \sum_{k\in I^-(j)} m_{jk}|\alpha_{\Delta x}{}_j^p - \alpha_{\Delta x}{}_k^p| \leq \frac{C}{\Delta x^{\frac{1}{2}}},\ Q\Delta t = T, \tag{57}$$

and

$$\sum_{p=0}^{p=Q} \sum_j s_j |\alpha_{\Delta x}{}_j^{p+1} - \alpha_{\Delta x}{}_j^p| \leq \frac{C}{\Delta x^{\frac{1}{2}}}. \tag{58}$$

The constant $C > 0$ depends only on α_0, $\vec{a}$, $T = Q\Delta t$ and on (c_1, c_2, c_3) characterizing the regularity of the mesh. In fact in [1] C depends also on an additional parameter which states that the CFL number must be strictly less than one. Moreover it is possible for the upwind scheme (15) to replace the bound $\frac{C}{\Delta x^{\frac{1}{2}}}$ in (58) by C. Thus the constant in our WBV estimates (57-58) is slightly better than in the original work [1]. Note that we impose $\alpha_0 \in L^\infty(\Omega) \cap BV(\Omega)$ in our hypothesis while [1] needs only $\alpha_0 \in L^\infty(\Omega)$: it is a consequence of the use of the LVD estimate.

Lemma 3 *Assume all the hypothesis of the previous theorem. Then the WBV estimates (57-58) are true with a bound $C > 0$ which depends only on α_0, $\vec{a}$, $T = Q\Delta t$ and (c_1, c_2, c_3).*

To prove the WBV estimates it is sufficient to consider the LVD estimate (44) which is a consequence of (19) (see theorem 3).

$$\sum_j |\sum_{k\in I^-(j)} m_{jk}(\alpha_{\Delta x}{}_j^n - \alpha_{\Delta x}{}_k^n)| \leq \sum_j |\sum_{k\in I^-(j)} m_{jk}(\alpha_{\Delta x}{}_j^0 - \alpha_{\Delta x}{}_k^0)|$$

$$\leq |\vec{a}| \left(\sum_j \sum_{k\in I^-(j)} l_{jk}|\alpha_{\Delta x}{}_j^0 - \alpha_{\Delta x}{}_k^0| \right) = |\vec{a}| \|\alpha_{\Delta x}{}^0\|_{BV(\Omega)}.$$

We have used the definition of the BV norm, true for a piecewise constant function (see [8])

$$\|\alpha_{\Delta x}\|_{BV(\Omega)} = \sum_j \sum_{k\in I^-(j)\bigcup I^0(j)} l_{jk}|\alpha_{\Delta x j} - \alpha_{\Delta x k}|.$$

Here we use the continuity of the L^2 projection on arbitrary uniformly regular grids in BV space (see appendix C):

$$\exists C > 0,\ \ \|\alpha_{\Delta x}{}^0\|_{BV(\Omega)} \leq C\|\alpha_0\|_{BV(\Omega)},\ \ \forall \Delta x > 0. \tag{59}$$

Thus

$$\sum_j |\sum_{k\in I^-(j)} m_{jk}(\alpha_{\Delta x}{}_j^n - \alpha_{\Delta x}{}_k^n)| \leq C \tag{60}$$

(here and in the following C denotes an undefined constant).

Now let us define

$$\alpha_{\Delta x}^n = \sum_n \alpha_{\Delta x}{}_j^n \times 1_j \in L^\infty(\Omega),$$

that is

$$\alpha_{\Delta x}^n(x,t) = \alpha_{\Delta x}{}_j^n \quad \forall x \in \Omega_j.$$

Then we use $\|\alpha_{\Delta x}^n\|_{L^\infty} \le \|\alpha_0\|_{L^\infty}$ (which is a consequence of (25) and (54)) and a discrete integration by part (see formula (67) in appendix B) to obtain

$$\sum_j \sum_{k\in I^-(j)} m_{jk}(\alpha_{\Delta x}{}_j^n - \alpha_{\Delta x}{}_k^n)^2 = 2\sum_j \sum_{k\in I^-(j)} m_{jk}(\alpha_{\Delta x}{}_j^n - \alpha_{\Delta x}{}_k^n)\alpha_{\Delta x}{}_j^n$$

$$\le 2\|\alpha_{\Delta x}{}^n\|_{L^\infty} \sum_j \Big| \sum_{k\in I^-(j)} m_{jk}(\alpha_{\Delta x}{}_j^n - \alpha_{\Delta x}{}_k^n)\Big| \le C.$$

From the Cauchy-Schwarz inequality one has

$$\sum_j \sum_{k\in I^-(j)} m_{jk}|\alpha_{\Delta x}{}_j^n - \alpha_{\Delta x}{}_k^n|$$

$$\le \left(\sum_j \sum_{k\in I^-(j)} m_{jk}\right)^{\frac{1}{2}} \left(\sum_j \sum_{k\in I^-(j)} m_{jk}(\alpha_{\Delta x}{}_j^n - \alpha_{\Delta x}{}_k^n)^2\right)^{\frac{1}{2}}.$$

Inequality (66) of the appendix, true for a triangular or quadrangular uniformly regular mesh, gives

$$\sum_j \sum_{k\in I^-(j)} m_{jk}|\alpha_{\Delta x}{}_j^n - \alpha_{\Delta x}{}_k^n| \le \frac{C}{\Delta x^{\frac{1}{2}}}. \tag{61}$$

After summation in time we obtain (57).

Concerning (58): it is a direct consequence of (60), (61), and the definition of the scheme

$$s_j(\alpha_{\Delta x}{}_j^{n+1} - \alpha_{\Delta x}{}_j^n) = -\Delta t \sum_{k\in I^-(j)} m_{jk}(\alpha_{\Delta x}{}_j^n - \alpha_{\Delta x}{}_k^n)$$

$$-\Delta t \sum_{k\in I^+(j)} m_{jk}(\alpha_{\Delta x}{}_{jk}^n - \alpha_{\Delta x}{}_j^n) + \Delta t \sum_{k\in I^-(j)} m_{jk}(\alpha_{\Delta x}{}_{jk}^n - \alpha_{\Delta x}{}_k^n).$$

The first right hand side contribution is bounded by (60). Since the flux is a convex combination of the upwind and downwind values (compatibility principle (19)), the second and third right hand side contributions are bounded thanks to (61). It gives

$$\sum_j s_j|\alpha_{\Delta x}{}_j^{n+1} - \alpha_{\Delta x}{}_j^n| \le 3\Delta t \sum_j \sum_{k\in I^-(j)} m_{jk}|\alpha_{\Delta x}{}_j^n - \alpha_{\Delta x}{}_k^n| \le \frac{C}{\Delta x^{\frac{1}{2}}}\Delta t.$$

After summation in time it gives (58). Considering the upwind scheme the second and third right hand side contributions vanish: it gives for the upwind scheme only

$$\sum_j s_j |\alpha_{\Delta x}{}_j^{n+1} - \alpha_{\Delta x}{}_j^n| \leq \Delta t \sum_j \left| \sum_{k \in I^-(j)} m_{jk} (\alpha_{\Delta x}{}_j^n - \alpha_{\Delta x}{}_k^n) \right| \leq C \Delta t,$$

which gives a sharper estimate (C instead of $\frac{C}{\Delta x^{\frac{1}{2}}}$) in (58). This ends the proof of lemma 3.

Final proof of theorem 5

Since $\alpha_0 \in L^\infty(\Omega)$ then the approximation $\alpha^{\Delta x}$ is uniformly bounded in $L^\infty([0,T] \times \Omega)$

$$\exists C > 0, \|\alpha^{\Delta x}\|_{L^\infty([0,T]\times\Omega)} \leq C.$$

It implies the existence of $\alpha \in L^\infty([0,T] \times \Omega)$ such that $\alpha^{\Delta x}$ converges to α in the $L^{\infty,*}([0,T] \times \Omega)$ sense (up to an extracted subsequence). Now essentially copying the proof in [1] it gives the result (55). Let $\varphi \in C^1([0,T] \times \Omega)$ be a smooth test function. We assume that $\varphi(x,T) = 0 \ \forall x \in \Omega$. We multiply (13) by $\frac{\Delta t}{s_j} \varphi(x,(n+1)\Delta t)$, integrate over $x \in \Omega_j$ and sum for all j and for all $0 \leq n \leq Q = \frac{T}{\Delta t}$. It gives

$$A_{\Delta x} + B_{\Delta x} + D_{\Delta x} = 0,$$

where

$$A_{\Delta x} = \sum_n \sum_j (\alpha_{\Delta x}{}_j^{n+1} - \alpha_{\Delta x}{}_j^n) \int_{\Omega_j} \varphi(x,(n+1)\Delta t) dx,$$

$$B_{\Delta x} = \sum_n \sum_j \frac{\Delta t}{s_j} \left(\sum_{k \in I^-(j)} m_{jk} (\alpha_{\Delta x}{}_j^n - \alpha_{\Delta x}{}_k^n) \right) \int_{\Omega_j} \varphi(x,(n+1)\Delta t) dx,$$

and

$$D_{\Delta x} = \sum_n \sum_j \left(\sum_{k \in I^+(j)} m_{jk} (\alpha_{\Delta x}{}_{jk}^n - \alpha_{\Delta x}{}_j^n) - \sum_{k \in I^-(j)} m_{jk} (\alpha_{\Delta x}{}_{jk}^n - \alpha_{\Delta x}{}_k^n) \right) \tag{62}$$
$$\times \frac{\Delta t}{s_j} \int_{\Omega_j} \varphi(x,(n+1)\Delta t) dx.$$

Following [1] we note that

$$A_{\Delta x} = -\int_{[0,T]} \int_\Omega \alpha^{\Delta x}(x,t) \partial_t \varphi(x,t) dx dt - \int_\Omega \alpha_0(x) \varphi(x,0) dx$$
$$+ \sum_j \int_{\Omega_j} \alpha_0(x) \left(\varphi(x,0) - \frac{1}{s_j} \int_{\Omega_j} \varphi(y,0) dy \right) dx.$$

Since $\alpha^{\Delta x}$ converges to α in the $L^{\infty,*}([0,T]\times\Omega)$ sense, we get

$$A_{\Delta x} \to -\int_{[0,T]}\int_\Omega \alpha(x,t)\partial_t\varphi(x,t)dxdt - \int_\Omega \alpha_0(x)\varphi(x,0)dx \quad \text{as } \Delta x \to 0$$

(recall that $\partial_t\varphi \in L^1$). Similarly we have

$$B_{\Delta x} = -\sum_n \int_{n\Delta t}^{(n+1)\Delta t}\int_\Omega \alpha^{\Delta x}(x,t)\vec{a}.\vec{\nabla}\varphi(x,t)dxdt$$

$$+\sum_n\sum_j \Delta t\left(\sum_{k\in I^-(j)} m_{jk}({\alpha_{\Delta x}}_j^n - {\alpha_{\Delta x}}_k^n)\right.$$

$$\left.\times\left(\frac{1}{s_j}\int_{\Omega_j}\varphi(x,(n+1)\Delta t)dx - \frac{1}{l_{jk}\Delta t}\int_{n\Delta t}^{(n+1)\Delta t}\int_{\overline{\Omega_j}\cap\overline{\Omega_k}}\varphi(x,t)dldt\right)\right).$$

Since φ is smooth there exists $C>0$ such that

$$\left|\frac{1}{s_j}\int_{\Omega_j}\varphi(x,(n+1)\Delta t)dx - \frac{1}{l_{jk}\Delta t}\int_{n\Delta t}^{(n+1)\Delta t}\int_{\overline{\Omega_j}\cap\overline{\Omega_k}}\varphi(x,t)dldt\right| \le C\Delta x, \quad \forall j,k.$$

Thus using (60)

$$\left|\sum_n\sum_j \Delta t\left(\sum_{k\in I^-(j)} m_{jk}({\alpha_{\Delta x}}_j^n - {\alpha_{\Delta x}}_k^n)\right.\right.$$

$$\left.\left.\times\left(\frac{1}{s_j}\int_{\Omega_j}\varphi(x,(n+1)\Delta t)dx - \frac{1}{l_{jk}\Delta t}\int_{n\Delta t}^{(n+1)\Delta t}\int_{\overline{\Omega_j}\cap\overline{\Omega_k}}\varphi(x,t)dldt\right)\right)\right|$$

$$\le \frac{C}{\Delta x^{\frac{1}{2}}}C\Delta x \le C\Delta x^{\frac{1}{2}}.$$

Since $\alpha^{\Delta x}$ converges to α in the $L^{\infty,*}([0,T]\times\Omega)$ sense, we get

$$B_{\Delta x} \to -\int_{[0,T]}\int_\Omega \alpha(x,t)\vec{a}.\vec{\nabla}\varphi(x,t)dxdt \quad \text{as } \Delta x \to 0.$$

If we assume that $\lim_{\Delta x\to 0} D_{\Delta x} = 0$, then

$$-\int_{[0,T]}\int_\Omega \alpha(x,t)\partial_t\varphi(x,t)dxdt - \int_\Omega \alpha_0(x)\varphi(x,0)dx$$

$$-\int_{[0,T]}\int_\Omega \alpha(x,t)\vec{a}.\vec{\nabla}\varphi(x,t)dxdt = 0,$$

$$\forall \varphi \in C^1([0,T]\times\Omega),\ \varphi(T)=0.$$

Thus it proves that α is the weak solution of (55).

So it remains to prove that the extra term $D_{\Delta x}$ tends to 0. We sum (62) by part

$$D_{\Delta x} = \sum_n \Delta t \sum_{k \in I^+(j)} m_{jk} ({\alpha_{\Delta x}}_{jk}^n - {\alpha_{\Delta x}}_j^n)$$

$$\times \left(\frac{1}{s_j} \int_{\Omega_j} \varphi(x, (n+1)\Delta t) dx - \frac{1}{s_k} \int_{\Omega_k} \varphi(x, (n+1)\Delta t) dx \right).$$

Since φ is smooth

$$\exists C > 0, \quad \left| \frac{1}{s_j} \int_{\Omega_j} \varphi(x, (n+1)\Delta t) dx - \frac{1}{s_k} \int_{\Omega_k} \varphi(x, (n+1)\Delta t) dx \right| \leq C \Delta x.$$

Combining with (60) we get

$$|D_{\Delta x}| \leq \sum_{p=0}^{Q} \Delta t \frac{C}{\Delta x^{\frac{1}{2}}} C \Delta x \leq C \Delta x^{\frac{1}{2}}.$$

It implies $\lim_{\Delta x \to 0} D_{\Delta x} = 0$ and ends the proof of theorem 5.

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A Regular grids

For sake of completeness, we prove here some elementary properties about uniformly regular triangular or quadrangular grids. More precise definition of uniformly regular meshes may be found in [27].

Let us consider a sequence of meshes with $\Delta x \to 0$: Δx is a characteristic length of the mesh. We assume that the sequence of meshes is uniformly regular in the sense that

$$\exists (c_1, c_2, c_3) \in \left(\mathbb{R}^{+,*}\right)^3, \text{ such that } l_{jk} \le c_1 \Delta x, \; c_2 \Delta x^2 \le s_j \le c_3 \Delta x^2, \;\; \forall j, k. \tag{63}$$

First we prove that such meshes satisfy

$$0 < C_1 \sum_{l \in I^-(k)} m_{kl} \le \sum_{l \in I^-(j)} m_{jl} \le C_2 \sum_{l \in I^-(k)} m_{kl}, \;\; \forall j, k, \tag{64}$$

with C_1 and C_2 independent of the characteristic length of the mesh Δx.

Let $\vec{f_j}(\mathbf{x}) = (\mathbf{x} - \mathbf{x}_j, \vec{a})\vec{a} \in \mathbb{R}^2$ be a linear vectorial function of $\mathbf{x} \in \mathbb{R}^2$. Here $\mathbf{x}_j \in \mathbb{R}^2$ is the coordinate of any point inside the triangle Ω_j. From the divergence theorem we get

$$\int_{\Omega_j} \mathbf{div} \vec{f_j} = \int_{\partial \Omega_j} \vec{f_j} . \vec{n}_j,$$

where $\vec{n}_j$ is the outgoing normal. It is equivalent to

$$s_j(\vec{a}, \vec{a}) = \sum_{k \in I(j)} (\vec{a}, \vec{n}_{jk}) \int_{\partial \Omega_j} (\mathbf{x} - \mathbf{x}_j, \vec{a}).$$

Due to the various constants in (53) or (63) we obtain

$$\begin{aligned} c_2 \Delta x^2 (\vec{a}, \vec{a}) &\le \left(\sum_{k \in I(j)} |(\vec{a}, \vec{n}_{jk})| l_{jk} \right) \max_{\mathbf{x} \in \partial \Omega_j} |(\mathbf{x} - \mathbf{x}_j, \vec{a})| \\ &\le C \Delta x \sum_{k \in I(j)} |(\vec{a}, \vec{n}_{jk})| l_{jk} \end{aligned}$$

(recall that the number of faces for each cell is less than 4).

Since

$$\begin{aligned} 2 \sum_{k \in I^-(j)} |(\vec{a}, \vec{n}_{jk})| l_{jk} &= \sum_{k \in I^+(j)} |(\vec{a}, \vec{n}_{jk})| l_{jk} + \sum_{k \in I^-(j)} |(\vec{a}, \vec{n}_{jk})| l_{jk} \\ &= \sum_{k \in I(j)} |(\vec{a}, \vec{n}_{jk})| l_{jk}, \end{aligned}$$

we deduce

$$\exists c_4 > 0, \;\; 0 < c_4 \Delta x \le \sum_{k \in I^-(j)} |(\vec{a}, \vec{n}_{jk})| l_{jk}, \; \forall j. \tag{65}$$

On the other hand $\sum_{k\in I^-(j)} |(\vec{a},\vec{n}_{jk})| l_{jk}$ is bounded from above

$$\exists c_5>0,\ \sum_{k\in I^-(j)} |(\vec{a},\vec{n}_{jk})| l_{jk} \le c_5 \Delta x,\ \ \forall j.$$

It proves (64) with uniform constants $C_1=\frac{c_4}{c_5}$ and $C_2=\frac{c_5}{c_4}$.

For a uniformly regular triangular or quadrangular mesh, we bound the number of cells

$$\mathrm{card}(J) \le \frac{\mathrm{mes}(\Omega)}{c_2 \Delta x^2} \le \frac{C}{\Delta x^2}.$$

It implies that

$$\left(\sum_j \sum_{k\in I^-(j)} m_{jk}\right)^{\frac12} \le \left(\frac{C}{\Delta x^2} \max_{j\in J} \sum_{k\in I^-(j)} m_{jk}\right)^{\frac12}$$

$$\le \left(\frac{C}{\Delta x^2} |\vec{a}| U c_1 \Delta x\right)^{\frac12},$$

where U is 3 (resp. 4) for triangular (resp. quadrangular) meshes. Thus

$$\left(\sum_j \sum_{k\in I^-(j)} m_{jk}\right)^{\frac12} \le \frac{C}{\Delta x^{\frac12}}. \tag{66}$$

Here the constant C depends on $mes(\Omega)$, $\vec{a}$ and C_1.

B A discrete integration by part formula

Here we need to prove

$$\sum_j \sum_{k\in I^-(j)} m_{jk}(x_j-x_k)^2 = 2\sum_j \sum_{k\in I^-(j)} m_{jk}(x_j-x_k)x_j \tag{67}$$

which is used in the proof of lemma 3. One has

$$\sum_j \sum_{k\in I^-(j)} m_{jk}(x_j-x_k)x_j = \sum_j \sum_{k\in I^-(j)} m_{jk}x_j^2 - \sum_j \sum_{k\in I^-(j)} m_{jk}x_kx_j$$

$$= \sum_j \sum_{k\in I^-(j)} m_{jk}x_j^2 - \sum_j \sum_{k\in I^-(j)} m_{jk}\left(\frac12 x_j^2 - \frac12 (x_j-x_k)^2 + \frac12 x_k^2\right)$$

$$= \frac12 \sum_j \sum_{k\in I^-(j)} m_{jk}x_j^2 + \frac12 \sum_j \sum_{k\in I^-(j)} m_{jk}(x_j-x_k)^2 - \frac12 \sum_j \sum_{k\in I^-(j)} m_{jk}x_k^2.$$

Once more due to the divergence lemma

$$\frac{1}{2}\sum_j \sum_{k\in I^-(j)} m_{jk}x_k^2 = \frac{1}{2}\sum_k \sum_{j\in I^-(k)} m_{jk}x_j^2 = \frac{1}{2}\sum_j \sum_{k\in I^+(j)} m_{jk}x_j^2 = \frac{1}{2}\sum_j \sum_{k\in I^-(j)} m_{jk}x_j^2.$$

Finally we obtain after simplification

$$\sum_j \sum_{k\in I^-(j)} m_{jk}(x_j - x_k)x_j = \frac{1}{2}\sum_j \sum_{k\in I^-(j)} m_{jk}(x_j - x_k)^2$$

which is (67).

C Continuity of the L^2 projection into BV space

Here we have to prove (59).

This is a standard property, well known in finite element space, see [4] and references therein. However and since most of previous proof of convergence for finite volume approximations avoided the BV framework [1], [3], [5], we give here a simple proof.

We assume all the hypothesis of section A and split the proof in three steps.

a) Let us consider a given mesh in a sequence of uniformly regular meshes. For a given $M = (x, y) \in \Omega$ and a given $R \in \mathbb{R}^+$, we denote as $\mathcal{N}(M, R)$ the number of cells Ω_j in the mesh such that $\Omega_j \subset \mathcal{B}(M, R)$ where $\mathcal{B}(M, R)$ stands for the ball centered at M with radius R. We claim that

$$\forall C' > 0,\ \exists C(C') > 0 \text{ independent of the mesh size } \Delta x$$
$$\text{such that } \forall M \in \Omega, \text{ we have } \mathcal{N}(M, C'\Delta x) \le C(C'). \tag{68}$$

It is due to

$$\text{meas}(\mathcal{B}(M, R)) = \pi(C'\Delta x)^2 \ge \sum_{\Omega_j\subset\mathcal{B}(M,C'\Delta x)} \text{meas}(\Omega_j)$$

$$\ge \sum_{\Omega_j\subset\mathcal{B}(M,C'\Delta x)} c_2\Delta x^2 = \mathcal{N}(M, C'\Delta x)c_2\Delta x^2$$

Thus

$$\mathcal{N}(M, C'\Delta x) \le \frac{\pi(C')^2}{c_2}.$$

b) Now for two given adjacent cells Ω_j and Ω_k, we define $C(\Omega_j, \Omega_k)$ as the smallest rectangle such that $\Omega_j \subset C(\Omega_j, \Omega_k)$ and $\Omega_k \subset C(\Omega_j, \Omega_k)$. That is

$$C(\Omega_j, \Omega_k) = [a_{jk}, b_{jk}] \times [c_{jk}, d_{jk}]$$

with

$$\begin{aligned} a_{jk} &= \min_{(x,y)\in\Omega_j\bigcup\Omega_k} x, \ b_{jk} = \max_{(x,y)\in\Omega_j\bigcup\Omega_k} x, \\ c_{jk} &= \min_{(x,y)\in\Omega_j\bigcup\Omega_k} y, \ d_{jk} = \max_{(x,y)\in\Omega_j\bigcup\Omega_k} y. \end{aligned} \tag{69}$$

It is clear that there exists $C > 0$ such that for all adjacent cells (Ω_j, Ω_k)

$$\forall (M, M') \in C(\Omega_j, \Omega_k), \ \ |M - M'| \le C\Delta x. \tag{70}$$

Indeed, since $\exists c > 0$ such that the diameter of each cell is smaller than $c\Delta x$, then (70) holds with $C = 2c$ (the mesch is assumed to be triangular or quadrangular). As a consequence of this we have

$$meas(C(\Omega_j, \Omega_k)) = (b_{jk} - a_{jk}) \times (c_{jk} - d_{jk}) \le (C\Delta x)^2,$$

and, because of the hypothesis (63), $\exists \tilde{C}$ such that

$$meas(C(\Omega_j, \Omega_k)) = (b_{jk} - a_{jk}) \times (c_{jk} - d_{jk}) \le \tilde{C} \max(s_j, s_k). \tag{71}$$

c) Let us now consider $f \in L^\infty(\Omega) \cap BV(\Omega)$ and $f_{\Delta x}$ given by the standard L^2 projection

$$f_{\Delta x} = \sum_j \left(\frac{1}{s_j} \int_{\Omega_j} f \right) 1_{\Omega_j}. \tag{72}$$

Then

$$\begin{aligned} ||f_{\Delta x}||_{BV} &= \sum_j \sum_{k\in I^-(j)\bigcup I^0(j)} l_{jk} \left|\frac{1}{s_j}\int_{\Omega_j} f - \frac{1}{s_k}\int_{\Omega_k} f\right| \\ &= \sum_j \sum_{k\in I^-(j)\bigcup I^0(j)} \frac{l_{jk}}{s_j s_k} \left| \int_{M'\in\Omega_j} \int_{M\in\Omega_k} f(M) - f(M')\, dM\, dM' \right| \\ &\le \sum_j \sum_{k\in I^-(j)\bigcup I^0(j)} \frac{l_{jk}}{s_j s_k} \int_{M'\in\Omega_j} \int_{M\in\Omega_k} |f(M) - f(M')|\, dM\, dM' \\ &\le \sum_j \sum_{k\in I^-(j)\bigcup I^0(j)} \frac{l_{jk}}{s_j s_k} \int_{M'\in C(\Omega_j,\Omega_k)} \int_{M\in C(\Omega_j,\Omega_k)} |f(M) - f(M')|\, dM\, dM'. \end{aligned} \tag{73}$$

For $M = (x, y)$ and $M' = (x', y')$ we bound

$$\begin{aligned} |f(M) - f(M')| &= \left| \int_x^{x'} \partial_x f(s, y)\, ds + \int_y^{y'} \partial_y f(x', s)\, ds \right| \\ &\le \int_x^{x'} |\partial_x f(s, y)|\, ds + \int_y^{y'} |\partial_y f(x', s)|\, ds \\ &\le \int_x^{x'} ||\nabla f(s, y)||\, ds + \int_y^{y'} ||\nabla f(x', s)||\, ds \end{aligned}$$

that is (using notations (69)

$$|f(M) - f(M')| \leq \int_{a_{jk}}^{b_{jk}} ||\nabla f(s, y)||\, ds + \int_{c_{jk}}^{d_{jk}} ||\nabla f(x', s)||\, ds.$$

Next we incorporate this expression in (73) and get

$$||f_{\Delta x}||_{BV} \leq \sum_j \sum_{k \in I^-(j) \bigcup I^0(j)} \frac{l_{jk}}{s_j s_k}$$

$$\times \int_{M' \in C(\Omega_j, \Omega_k)} \int_{M \in C(\Omega_j, \Omega_k)} \left(\int_{a_{jk}}^{b_{jk}} ||\nabla f(s, y)||\, ds + \int_{c_{jk}}^{d_{jk}} ||\nabla f(x', s)||\, ds \right) dM\, dM'$$

$$= \sum_j \sum_{k \in I^-(j) \bigcup I^0(j)} \frac{l_{jk}}{s_j s_k}$$

$$\times \int_{x'=a_{jk}}^{b_{jk}} \int_{y'=c_{jk}}^{d_{jk}} \int_{x=a_{jk}}^{b_{jk}} \int_{y=c_{jk}}^{d_{jk}} \left(\int_{a_{jk}}^{b_{jk}} ||\nabla f(s, y)||\, ds + \int_{c_{jk}}^{d_{jk}} ||\nabla f(x', s)||\, ds \right) dy\, dx\, dy'\, dx'.$$

This can be written as

$$||f_{\Delta x}||_{BV} \leq \sum_j \sum_{k \in I^-(j) \bigcup I^0(j)} \frac{l_{jk}}{s_j s_k}$$

$$\times \left(\int_{x'=a_{jk}}^{b_{jk}} \int_{y'=c_{jk}}^{d_{jk}} \int_{x=a_{jk}}^{b_{jk}} \left(\int_{y=c_{jk}}^{d_{jk}} \int_{a_{jk}}^{b_{jk}} ||\nabla f(s, y)||\, ds\, dy \right) dx\, dy'\, dx' \right.$$

$$\left. + \int_{y'=c_{jk}}^{d_{jk}} \int_{x=a_{jk}}^{b_{jk}} \int_{y=c_{jk}}^{d_{jk}} \left(\int_{x'=a_{jk}}^{b_{jk}} \int_{c_{jk}}^{d_{jk}} ||\nabla f(x', s)||\, ds\, dx' \right) dy\, dx\, dy' \right).$$

Thus we have

$$||f_{\Delta x}||_{BV} \leq \sum_j \sum_{k \in I^-(j) \bigcup I^0(j)} \frac{l_{jk}}{s_j s_k} (b_{jk} - a_{jk} + d_{jk} - c_{jk})$$

$$\times (b_{jk} - a_{jk})(d_{jk} - c_{jk}) \int_{C(\Omega_j, \Omega_k)} |\nabla f(M)|\, dM,$$

and, using (71),

$$||f_{\Delta x}||_{BV} \leq \sum_j \sum_{k \in I^-(j) \bigcup I^0(j)} \frac{l_{jk}}{s_j s_k} (b_{jk} - a_{jk} + d_{jk} - c_{jk})$$

$$\times \max(s_j, s_k) \int_{C(\Omega_j, \Omega_k)} |\nabla f(M)|\, dM.$$

Due to the uniform regularity of the grid and (70) there exists $C > 0$ such that

$$\frac{l_{jk}}{s_j s_k}(|b_{jk} - a_{jk}| + |d_{jk} - c_{jk}|)\max(s_j, s_k) \le C, \quad \forall j, k \text{ and } \forall \Delta x.$$

Thus we obtain

$$||f_{\Delta x}||_{BV} \le C \sum_j \sum_{k \in I^-(j) \bigcup I^0(j)} \int_{C(\Omega_j, \Omega_k)} |\nabla f(M)|\, dM.$$

We rewrite this as

$$||f_{\Delta x}||_{BV} \le C \int_\Omega | \left(\sum_{j,k \text{ such that } M \in C(\Omega_j, \Omega_k)} 1 \right) \nabla f(M)|\, dM.$$

Now we use (70): if $(x, y) \in C(\Omega_j, \Omega_k)$ then

$$\Omega_j \subset \mathcal{B}(M, \text{diameter}(C(\Omega_j, \Omega_k))) \subset \mathcal{B}(M, C\Delta x).$$

Inequality (70) implies that the number of such cells is bounded

$$\exists C'' > 0, \qquad \sum_{j,k \text{ such that } M \in C(\Omega_j, \Omega_k)} 1 \le C''.$$

What is important is that C'' is independent of the size of the mesh Δx. So finally

$$||f_{\Delta x}||_{BV} \le (CC'') \int_\Omega |\nabla f(M)|\, dM = (CC'')||f||_{BV}.$$

Since all constants are independent of the mesh size it proves the continuity of the L^2 projection in BV. The constant given here is sufficient for our purposes, but is probably far from being optimal.