STABILITY OF A THERMODYNAMICALLY COHERENT MULTI-PHASE MODEL

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Abstract

We analyze a hyperbolic system of conservation laws in dimension one, which is a drastic simplification of a multi-phase or multi-velocity fluid model. The domain of hyperbolicity is compact, which is a characteristic of multi-phase models. Our main result is the stability of the domain of hyperbolicity. Due to the degeneracy of the model on the boundary of the hyperbolicity domain, rarefaction waves are not unique. We also propose a numerical scheme for approximate resolution of the model and prove the stability of this scheme.

Keywords: Hyperbolic system of conservation laws, multi-phase flows, hyperbolicity domain stability, numerical schemes, Lax-Friedrichs (relaxation) schemes.

1. Introduction

This work deals with a stability analysis for a model hyperbolic system of conservation laws in dimension one, which is a drastic simplification of a multi-phase or multi-velocity model. It is well known that systems of conservation laws for multi-phase flows are subjected to a certain number of pathologies, at least from the mathematical point of view: some of these systems are non-hyperbolic even for reasonable physical values (it means that these systems are mathematically ill-posed), some of them are non-conservative: as a consequence the solution of the Riemann problem is not unique. It is nowadays an open challenge to design well-posed multi-phase models and to analyze their well-posedness, in conjunction with the development of numerical methods, see Després[1], Ghidaglia, Le Coq and Toumi[2], Godunov and Romensky[5], Gouin and Gavrilyuk[7], Ishii[8], Romensky, Resnyanski and Milton[18], Sainsaulieu[20] and Abgrall and Saurel[21].

Our model problem written in non-dimensional variables is

$$\left\{\begin{array}{l} \partial_t a + \partial_x \left[(a^2-1)b\right] = 0, \\ \partial_t b + \partial_x \left[(b^2-1)a\right] = 0. \end{array}\right. \tag{1.1}$$

This system may be understood as an extension of a scalar equation with discontinuous coefficients (in our notations)

$$\partial_t a + \partial_x \left[b(x)(a^2-1)\right] = 0 \tag{1.2}$$

for which we refer to Seguin and Vovelle[24], Towers[23] and references therein. Our system is member of the general class

$$\left\{\begin{array}{l} \partial_t a + \partial_x P(a,b) = 0, \\ \partial_t b + \partial_x Q(a,b) = 0, \end{array}\right. \tag{1.3}$$

where $P(a,b)$ and $Q(a,b)$ are polynomial with respect to (a,b). The case $P(a,b) = \frac{\partial C(a,b)}{\partial a}$ and $Q(a,b) = \frac{\partial C(a,b)}{\partial b}$ where $C(a,b)$ is a polynomial, cubic with respect to (a,b) is studied in Chen and Kan[25] and corresponds to hyperbolic system with umbilic degeneracy. The case $P(a,b) = -b^3 - \alpha b$, $Q(a,b) = -a$ is representative of non linear elasticity and is presented in LeFloch[13]. See also the new book of LeFloch[12] and references therein. An interest of the model system (1.1) compared with all given references is that the domain of real physical eigenvalues is compact in $\mathbb{R}^2$

$$-1 \leqslant a \leqslant 1, \quad -1 \leqslant b \leqslant 1, \tag{1.4}$$

which is characteristic of systems of equations for multi-velocity and multi-phase flows. One question under interest is the stability of this hyperbolicity domain. A general reference for elliptic-hyperbolic problem is Keyfitz and Lopes Filho[10]. The stability of the invariant region of a particular hyperbolic-elliptic problem is in Pego and Serre [17]. A fundamental reference for the stability of shock waves for conservation laws is Liu [15] and Liu and Wang [16].

The plan of this work is as follows. In section we explain how to derive the model problem from an isobar-isothermal conservative multi-phase model. A particularity of the isobar-isothermal conservative multi-phase model is that the flux depends on a potential which is physically equivalent to the entropy of the mixing: the model is thermodynamically coherent in the sense of Godunov[6]. In section we discuss the hyperbolicity domain of the model problem (1.1), together with the domain of strict convexity of the entropy. The hyperbolicity domain depends crucially on the entropy of the mixing. Shock curves and rarefaction waves are given. Due to the degeneracy of the model on the boundary of the hyperbolicity domain, rarefaction waves are not unique. In section we apply a theorem of Smoller[22] for the study of the stability of the hyperbolicity domain of the model problem with parabolic regularization. In section we prove that the numerical solution of the model problem is stable provided the numerical dissipation is large enough: relaxation schemes are attractive in this context, but many other schemes are possible. In section numerical results are presented.

2. The isobar-isothermal conservative multi-phase model

The basic multi-phase one-dimensional model (2.5), given in Després[1], that we consider is

$$\begin{cases} \partial_t(\rho) + \partial_x(\rho u) = 0, \\ \partial_t(\rho c_2) + \partial_x(\rho c_2 u_2) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + P) = 0, \\ \partial_t(w) + \partial_x(uw + \mu_1 - \mu_2 - \frac{(1-2c_2)}{2} w^2) = 0, \\ \partial_t(\rho e) + \partial_x \left(\rho u e + P u + \rho w (1-c_2) c_2 (\mu_1 - \mu_2 - \frac{(1-2c_2)}{2} w^2)\right) = 0, \end{cases} \tag{2.5}$$

where ρ is the total density, $c_1 = 1 - c_2$ and c_2 are the mass fractions of each fluid, u_1 and u_2 their velocities, P is the total pressure tensor, μ_1 and μ_2 are the generalized chemical potentials, $w = u_1 - u_2$ is the differential velocity and e is the density of total energy. This system intends to be representative of multi-velocity flows in hot and dense plasma. The motivation for this kind of systems of conservation laws is to obtain a **thermodynamically coherent** and **conservative** modeling for multi-velocity flows: we refer to Godunov and Romensky[5] for a general presentation of thermodynamically coherent multi-velocity models (very much in the spirit of the seminal work of Godunov[6]): see Romensky, Resnyanski and Milton[18] for some numerical simulations in a water-air context.

Naturally system (2.5) needs some closure relations which are given in (2.6) and (2.9): (2.6) is a set of natural closure relations

$$\begin{cases} \frac{1}{\rho} = c_1 \tau_1 + c_2 \tau_2, \\ u = c_1 u_1 + c_2 u_2, \\ \varepsilon = c_1 \varepsilon_1 + c_2 \varepsilon_2, \\ e = \varepsilon + \frac{1}{2} u^2 + \frac{c_2(1-c_2)}{2} w^2. \end{cases} \tag{2.6}$$

The two last equations in (2.6) are simple generalization of the definition of total energy; the first equation in (2.6) expresses the additivity of volumes for a two-fluid mixture; the second one expresses the additivity of impulse. Other closure relations (equations (2.9)) are given by a thermo-dynamical analysis. We assume that each fluid has its own thermodynamical functions and incorporate some general considerations about the entropy of a mixture: we consider that the entropy of a mixture should be greater than the sum of partial entropies $s \geqslant c_1 S_1 + c_2 S_2$. It is possible to model such phenomena with an additional mixing contribution $s = c_1 S_1 + c_2 S_2 + S_{\text{mix}}(c_1, c_2)$, where S_{mix} is a non-negative concave function. Classically one considers that S_{mix} is given by a formula similar to the celebrated Boltzmann entropy

$$S_{\text{mix}} \approx -c_1 \log c_1 - c_2 \log c_2. \tag{2.7}$$

In order to avoid technical difficulties which are probably not the essential point in this work, we focus on

$$S_{\text{mix}} = kc_1c_2 = kc_1(1-c_1) = kc_2(1-c_2), \quad k \geqslant 0. \tag{2.8}$$

The interest of (2.8) against (2.7) is that S_{mix} given in (2.8) is non-singular everywhere, even for $c_1 = 0$ or $c_2 = 0$. In Després[1] it is proved that closure relations compatible with (2.5) and (2.6) are

$$\left\{\begin{array}{l} p_1(\tau_1, \varepsilon_1) = p_2(\tau_2, \varepsilon_2), \\ T_1(\tau_1, \varepsilon_1) = T_2(\tau_2, \varepsilon_2) = T, \\ \mu_1 = -T\frac{\partial}{\partial c_1}(c_1 S_1 + c_2 S_2 + S_{\text{mix}}) = -TS_1 + \varepsilon_1 + p\tau_1 - kTc_2, \\ \mu_2 = -T\frac{\partial}{\partial c_2}(c_1 S_1 + c_2 S_2 + S_{\text{mix}}) = -TS_2 + \varepsilon_2 + p\tau_2 - kTc_1, \\ P = p_1 + c_2(1-c_2)\rho(u_1 - u_2)^2. \end{array}\right. \tag{2.9}$$

Since $\mu_1 - \mu_2$ depends on the particular form we chose for the entropy of the mixing, it is clear that the choice of the particular formula (2.8) is here preferable in the sense that μ_1 and μ_2 are finite even for $c_1 = 0$ or $c_2 = 0$. It is not the case if one chooses $S_{\text{mix}} \approx -c_1 \log c_1 - c_2 \log c_2$. From now on, we consider only (2.8). Note that (2.5) admits a Lagrangian reformulation, given in

$$\left\{\begin{array}{l} \mathrm{D}_t(\tau) - \mathrm{D}_m(u) = 0 \\ \mathrm{D}_t(c_2) - \mathrm{D}_m(\rho w(1-c_2)c_2) = 0 \\ \mathrm{D}_t(u) + \mathrm{D}_m(P) = 0 \\ \mathrm{D}_t(\tau w) + \mathrm{D}_m(\mu_1 - \mu_2 - \frac{(1-2c_2)}{2}w^2) = 0 \\ \mathrm{D}_t(e) + \mathrm{D}_m\left(Pu + \rho w(1-c_2)c_2(\mu_1 - \mu_2 - \frac{(1-2c_2)}{2}w^2)\right) = 0. \end{array}\right. \tag{2.10}$$

Here D_t is the material derivative and D_m is the derivation with respect to the mass variable

$$\mathrm{D}_t = \partial_t + u\partial_x \quad \mathrm{D}_m = \frac{1}{\rho}\partial_x. \tag{2.11}$$

A possibility in order to take into account phase transition and/or drag force is to consider

$$\left\{\begin{array}{l} \mathrm{D}_t(\tau) - \mathrm{D}_m(u) = 0 \\ \mathrm{D}_t(c_2) - \mathrm{D}_m(\rho w(1-c_2)c_2) = -A(c_2 - \frac{1}{2}) \\ \mathrm{D}_t(u) + \mathrm{D}_m(P) = 0 \\ \mathrm{D}_t(\tau w) + \mathrm{D}_m(\mu_1 - \mu_2 - \frac{(1-2c_2)}{2}w^2) = -Bw \\ \mathrm{D}_t(e) + \mathrm{D}_m\left(Pu + \rho w(1-c_2)c_2(\mu_1 - \mu_2 - \frac{(1-2c_2)}{2}w^2)\right) = 0. \end{array}\right. \tag{2.12}$$

Here $A > 0$ characterizes phase transition: assumed for example that only phase number two is present $c_2 \approx 1$; thus the right hand side $-A(c_2 - \frac{1}{2}) < 0$ tends to produce phase number one. On the other hand $-Bw$ ($B > 0$) intends to model some drag force since it lowers $|w|$. The mathematical analysis of (2.5) or (2.10) is far from being trivial. In order to pave the way for future developments, we consider in this work that (2.5) or (2.10) is composed more or less of two subsystems: the first one is the classical Euler system for inviscid compressible gas, the second one is a system of conservation laws with two unknowns $(\rho c_2, w = u_1 - u_2)$. From (2.10) we get in Lagrange coordinates

$$\left\{ \begin{array}{l} \mathrm{D}_t(c_2) - \mathrm{D}_m(\rho w(1-c_2)c_2) = 0 \\ \mathrm{D}_t(\tau w) + \mathrm{D}_m(\mu_1' - \mu_2' - \frac{(1-2c_2)}{2} w^2) = 0, \end{array} \right. \tag{2.13}$$

with $\mu_{1,2}' = -kTc_{2,1}$. For the sake of simplicity we assume that $\rho = \frac{1}{\tau}$ and $T \approx \overline{T}$ are almost constant, and that u is negligible. Then we get

$$\left\{ \begin{array}{l} \partial_t(c_2) - \partial_x(w(1-c_2)c_2) = 0 \\ \partial_t(w) + \partial_x(k\overline{T}(1-2c_2) - \frac{(1-2c_2)}{2} w^2) = 0. \end{array} \right. \tag{2.14}$$

Using now non-dimensional variables

$$a = 2c_2 - 1 \text{ and } b = \frac{w}{\sqrt{2k\overline{T}}}, \tag{2.15}$$

and the non-dimensional coordinate

$$\tilde{x} = \sqrt{\frac{2}{k\overline{T}}} x, \tag{2.16}$$

we obtain the system of conservation laws

$$\left\{ \begin{array}{l} \partial_t a + \partial_{\tilde{x}} \left[(a^2-1)b \right] = 0, \\ \partial_t b + \partial_{\tilde{x}} \left[(b^2-1)a \right] = 0. \end{array} \right. \tag{2.17}$$

In the following we omit the tilde and x will stand for the non-dimensional coordinate (instead of $\tilde{x}$).

An extension is to add non-dimensional drag force and phase transition in the model. From (2.12) and using non-dimensional drag force and phase transition, we arrive at the model problem with a source term which depends on $\alpha > 0$

$$\left\{ \begin{array}{l} \partial_t a + \partial_x \left[(a^2-1)b \right] = -\alpha a, \\ \partial_t b + \partial_x \left[(b^2-1)a \right] = -\alpha b. \end{array} \right. \tag{2.18}$$

What is remarkable in (2.17) or (2.18) is the perfect mathematical symmetry with respect to a and b, even if the physical meaning of these variables is clearly very different: just recall that $c_2 = \frac{a+1}{2}$ is a concentration and $u_1 - u_2 = \sqrt{2k\overline{T}} b$ is a differential velocity.

3. Analysis

In this section we analyze the hyperbolic structure of (1.1).

3.1. *Hyperbolicity*

Consider the Jacobian matrix for system (1.1),

$$\frac{\partial \left((a^2-1)b, (b^2-1)a \right)}{\partial (a,b)} = \begin{pmatrix} 2ab & a^2-1 \\ b^2-1 & 2ab \end{pmatrix}.$$

The characteristic polynomial is:

$$P(X) = (2ab - X)^2 - (a^2-1)(b^2-1). \tag{3.19}$$

Thus the Jacobian matrix has two eigenvalues $\lambda_1 \leqslant \lambda_2$ not necessarily distinct

$$\lambda_1 = 2ab - \sqrt{(1-a^2)(1-b^2)} \text{ and } \lambda_2 = 2ab + \sqrt{(1-a^2)(1-b^2)}. \tag{3.20}$$

The related eigenvectors R_1 and R_2 are

$$R_1 = \begin{pmatrix} \sqrt{1-a^2} \\ \sqrt{1-b^2} \end{pmatrix} \text{ and } R_2 = \begin{pmatrix} \sqrt{1-a^2} \\ -\sqrt{1-b^2} \end{pmatrix}. \tag{3.21}$$

Lemma 1 *The eigenvalues are real and distinct if and only if* $(a,b) \in \{(1-a^2)(1-b^2) > 0\}$*. The physical domain of strict hyperbolicity is* $(a,b) \in \mathcal{D}$ *where by definition*

$$\mathcal{D} = \{-1 < a < 1 \ \textit{and} \ -1 < b < 1\}. \tag{3.22}$$

By definition $a = 2c_2 - 1$ lies in $[-1,1]$ because c_2 is a mass fraction and lies "naturally" in $[0,1]$. Since the strict hyperbolicity domain is defined by $(1-a^2)(1-b^2) > 0$, the result is obvious. The domain of physical interest is then the closure of $\mathcal{D}$, that is

$$\overline{\mathcal{D}} = \{-1 \leqslant a \leqslant 1 \text{ and } -1 \leqslant b \leqslant 1\}.$$

In the case where $c_2 = 1$ or 0 (i.e. only one phase is present, which is clearly an important regime from the physical point of view and has to be taken in consideration), $a = \pm 1$.

3.2. *Non-linearity of the system*

Here we point out that the system is in some sense pathological: the fields are neither genuinely non-linear nor linearly degenerated. Indeed, let us compute the scalar products $\nabla\lambda_1.R_1$ and $\nabla\lambda_2.R_2$

$$\nabla\lambda_1.R_1 = \begin{pmatrix} 2b + a\sqrt{\frac{1-b^2}{1-a^2}} \\ 2a + b\sqrt{\frac{1-a^2}{1-b^2}} \end{pmatrix} \begin{pmatrix} \sqrt{1-a^2} \\ \sqrt{1-b^2} \end{pmatrix} = 3b\sqrt{1-a^2} + 3a\sqrt{1-b^2}, \ \ (a,b) \in \mathcal{D}, \tag{3.23}$$

and

$$\nabla\lambda_2.R_2 = \begin{pmatrix} 2b - a\sqrt{\frac{1-b^2}{1-a^2}} \\ 2a - b\sqrt{\frac{1-a^2}{1-b^2}} \end{pmatrix} \begin{pmatrix} \sqrt{1-a^2} \\ -\sqrt{1-b^2} \end{pmatrix} = 3b\sqrt{1-a^2} - 3a\sqrt{1-b^2}, \ \ (a,b) \in \mathcal{D}. \tag{3.24}$$

Be careful that both eigenvalues are differentiable in $]-1,1[^2$ but are not on the boundary of the square. It is the reason why (3.24-3.23) makes sense only for $(a,b) \in \mathcal{D}$. Thus both scalar products $\nabla\lambda_1.R_1$ and $\nabla\lambda_2.R_2$ can be zero: the one associated to λ_1 vanishes if and only if $b = -a$ which is the second diagonal of the hyperbolicity domain. The other one vanishes if and only if $b = a$ which is the first diagonal. See figure 1. We refer to LeFloch[12] for a general study of this kind of degeneracy.

3.3. *Entropy*

Since $\frac{c_2(1-c_2)}{2}w^2 = k^2\frac{(1-a^2)b^2}{4}$ is the differential kinetic energy for the isobar-isothermal model with five unknowns, it is natural (doing for example the parallel with Euler equations and p-system) to think that the same quantity (or a very close one) may be an entropy for the reduced model problem. It is indeed the case.

Lemma 2 *An entropy-entropy flux pair for (1.1) is* $S(a,b) = -(1-a^2)(1-b^2)$ *and* $F(a,b) = 2abS$*. Furthermore* S *is strictly convex in* $\mathcal{D}$ *if and only if* $(a,b) \in \mathcal{D}_e \subsetneq \mathcal{D}$*, where by definition*

$$\mathcal{D}_e = \{(a,b) \in \mathcal{D}, \ \ (1-a^2)(1-b^2) > 4a^2b^2\}.$$

For smooth solutions of (1.1), one has

$$\partial_t S = 2a(1-b^2)\partial_t a + 2b(1-a^2)\partial_t b = 2a(1-b^2)\partial_x((1-a^2)b) + 2b(1-a^2)\partial_x((1-b^2)a) = -\partial_x F,$$

which shows that (S,F) is an entropy-entropy flux pair. Thus S is a possible entropy with flux F for (1.1). The Hessian matrix of S is

$$\frac{\partial^2 S}{\partial(a,b)^2} = \begin{pmatrix} 1-b^2 & -2ab \\ -2ab & 1-a^2 \end{pmatrix}. \tag{3.25}$$

It is obvious that the trace of $\partial^2 S/\partial(a,b)^2$ is positive in $\mathcal{D}$. Then S is strictly convex if and only if

$$det\left(\frac{\partial^2 S}{\partial(a,b)^2}\right) = (1-a^2)(1-b^2) - 4a^2b^2 > 0. \tag{3.26}$$

The domain of strictly concavity of S is then included in the hyperbolicity domain. By means of straightforward calculations, one has that $\lambda_1\lambda_2 = 4a^2b^2 - (1-a^2)(1-b^2)$ which means that the domain of strict convexity is also equal to the domain where the eigenvalues have different sign. See figure 1. However (S,F) is not the only entropy-entropy flux pair. For example $(s,f) = (ab, s^2 + \frac{S}{2})$ is another entropy-entropy flux pair such that $\partial_t s + \partial_x f = 0$ for smooth solutions of (1.1). The domain of strict convexity of s is the empty set, so the physical meaning of s is very poor with respect to the one of S.

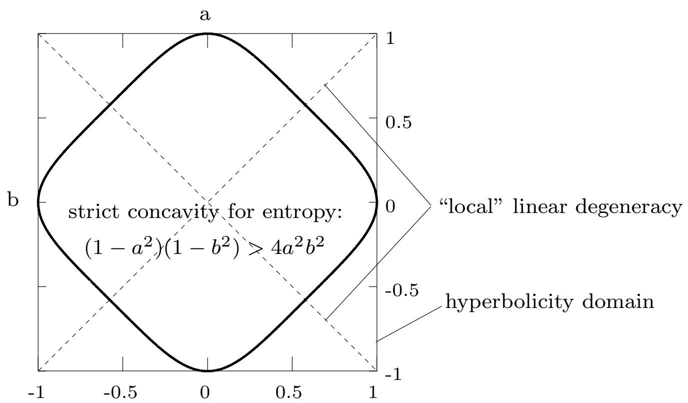

Figure 1: Hyperbolicity, strict concavity of entropy and linear degeneracy.

3.4. *Shocks and rarefaction waves*

3.4.1. *Rarefaction waves*

Rarefaction waves are smooth autosimilary solutions: see LeVeque[14]. A rarefaction wave for (1.1) is a smooth solution $(\tilde{a}(t,x), \tilde{b}(t,x))$ of (2.17) such that $(\tilde{a}(t,x), \tilde{b}(t,x)) = (a(x/t), b(x/t))$. Differentiating $(\tilde{a}, \tilde{b})$ relatively to t and x we get $\partial_t(\tilde{a}(t,x), \tilde{b}(t,x)) = -x/t^2(a'(x/t), b'(x/t))$ and $\partial_x(\tilde{a}(t,x), \tilde{b}(t,x)) = 1/t^2(a'(x/t), b'(x/t))$. Together with (1.1) it gives

$$\partial_t \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} + \begin{pmatrix} 2\tilde{a}\tilde{b} & \tilde{a}^2 - 1 \\ \tilde{b}^2 - 1 & 2\tilde{a}\tilde{b} \end{pmatrix} \partial_x \begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = 0.$$

So $(a(x/t), b(x/t))$ must verify

$$-\frac{x}{t^2}\begin{pmatrix} a'(x/t) \\ b'(x/t) \end{pmatrix} + \frac{1}{t}\begin{pmatrix} 2a(x/t)b(x/t) & a(x/t)^2 - 1 \\ b(x/t)^2 - 1 & 2a(x/t)b(x/t) \end{pmatrix}\begin{pmatrix} a'(x/t) \\ b'(x/t) \end{pmatrix} = 0.$$

Defining $\xi = x/t$, multiplying by t and rearranging gives

$$\begin{pmatrix} 2a(\xi)b(\xi) & a(\xi)^2 - 1 \\ b(\xi)^2 - 1 & 2a(\xi)b(\xi) \end{pmatrix}\begin{pmatrix} a'(\xi) \\ b'(\xi) \end{pmatrix} = \xi \begin{pmatrix} a'(\xi) \\ b'(\xi) \end{pmatrix}.$$

So $(a'(\xi), b'(\xi))$ is equal to zero or proportional to an eigenvector of the Jacobian matrix: in this second case one has $(a(\xi), b(\xi)) = \alpha(\xi) R_{1,2}(\xi)$.

First case: (a, b) is inside the square From $\frac{d\lambda_{1,2}(\xi)}{d\xi} = 1$ and assuming that $\lambda_{1,2}$ is differentiable, one gets $\alpha(\xi) = \frac{1}{\nabla\lambda_{1,2}(a(\xi),b(\xi))\cdot R_{1,2}}$. We finally obtain the equations for the rarefaction waves, valid only for $(a, b) \in \mathcal{D}$,

$$(a(\xi), b(\xi)) = \frac{1}{\nabla\lambda_{1,2}(a(\xi), b(\xi)) \cdot R_{1,2}} R_{1,2}(\xi). \tag{3.27}$$

Second case: (a, b) is the boundary of the square we assume that a piece of a rarefaction wave lies on the boundary, for example where $b = 1$. We look for an autosimilary smooth solution of (2.17), $(\tilde{a}(t, x), \tilde{b}(t, x)) = (a(x/t), b(x/t)) = (a(x/t), 1)$. The partial differential equation on $\tilde{a}$ reads $\partial_t \tilde{a} + 2\tilde{a}\tilde{b}\partial_x \tilde{a} + (\tilde{a} - 1)\partial_x \tilde{b} = 0$, or more simply $\partial_t \tilde{a} + 2\tilde{a}\partial_x \tilde{a} = 0$, which leads to $-\frac{x}{t^2} a'(x/t) + \frac{2a(x/t)}{t} a'(x/t) = 0$. Finally the equation is $a'(\xi)(2a(\xi) - \xi) = 0$. For $a' \neq 0$ it gives

$$a(\xi) = \frac{\xi}{2}. \tag{3.28}$$

It remains to solve (3.27-3.28). Due to symmetries in the equations, we restrict the discussion to the top quarter of the domain $\overline{\mathcal{D}}$, namely

$$\mathcal{Q} = \overline{\mathcal{D}} \cap \{-b < a < b\}. \tag{3.29}$$

Lemma 3 2-rarefaction waves in $\mathcal{D}$. *Let $(a_0, b_0) \in \mathcal{Q} \cap \mathcal{D}$ inside the square. The 2-rarefaction wave $(a, b) \in \mathcal{D}$ connected to (a_0, b_0) is the piece of an ellipse with equation (3.32), located between a point $(\overline{a}, 1)$ and a point $(\tilde{a}, \tilde{a})$ on the degeneracy line $a = b$.*

Lemma 4 2-rarefaction waves in $\overline{\mathcal{D}}$. *Let $(a_0, b_0) \in \mathcal{Q} \cap \mathcal{D}$ inside the square. The 2-rarefaction wave $(a, b) \in \overline{\mathcal{D}}$ connected to (a_0, b_0) is composed of*

- *the piece of an ellipse with equation (3.32), located between a point $(\overline{a}, 1)$ and a point $(\tilde{a}, \tilde{a})$ on the degeneracy line $a = b$;*
- *the segment $[(-1, 1), (\overline{a}, 1)]$ which is on the boundary of the square.*

Let (a_0, b_0) on the boundary of the square, such that $b_0 = 1$. There exist an infinity of 2-rarefaction waves in the plane $(a, b) \in \overline{\mathcal{D}}$. All of them are composed of

- *a line segment $[(-1, 1), (\overline{a}, 1)]$ for some $\overline{a} \in [a_0, 1]$;*
- *the portion of the ellipse with equation (3.32) where d_1 and d_2 are such that this ellipse passes through $\overline{a} \in [a_0, 1]$, located between $\overline{a} \in [a_0, 1]$ and the degeneracy line where $a = b$.*

See figures 3 and 4 for an illustration.

Proof of 3. Starting from a given point $(a_0, b_0) \in]0, 1[^2$, the 2-rarefaction curves parametrized with ξ are solutions of the ordinary differential following Cauchy problem

$$\left\{ \begin{array}{l} \begin{pmatrix} a \\ b \end{pmatrix}'(\xi) = \dfrac{1}{\nabla\lambda_2(a(\xi), b(\xi)).R_2(a(\xi), b(\xi))} R_2(a(\xi), b(\xi)) \\ \begin{pmatrix} a \\ b \end{pmatrix}(0) = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \end{array} \right.$$

The solutions of this system in $]0, 1[^2$ are the 2-rarefaction waves. In the next we integrate this system, find the analytic expression of the 2-rarefaction curves and show that they lie on ellipses whose axes are the diagonals of the square. Note that these system can be integrated only when

$\nabla\lambda_2(a(\xi), b(\xi)).R_2(a(\xi), b(\xi)) \neq 0$, i.e. outside the sets of local linear degeneracy (the first diagonal of the square). In the following computation we thus assume that $(a(\xi), b(\xi))$ does not lie on this diagonal. Let us now compute the rarefaction wave for the field associated to λ_2. The problem is to solve

$$\left\{ \begin{array}{l} a' = \dfrac{\sqrt{1-a^2}}{3(b\sqrt{1-a^2} - a\sqrt{1-b^2})}, \\ b' = \dfrac{-\sqrt{1-b^2}}{3(b\sqrt{1-a^2} - a\sqrt{1-b^2})}. \end{array} \right. \tag{3.30}$$

We of course assume (a_0, b_0) does not lie on the first diagonal. First notice that this system is locally Lipschitz continuous in $]0,1[^2$, i.e. if $|a| \neq 1$ and $|b| \neq 1$. Thus there exists a unique maximal solution to it in $]0,1[^2$ (from the Cauchy-Lipschitz theorem). The system (3.30) rewrites

$$\frac{b'}{\sqrt{1-b^2}} = -\frac{a'}{\sqrt{1-a^2}}$$

and can be integrated as

$$\arcsin(b) = \arccos(a) + C,$$

C being a constant given by the initial condition of the Cauchy problem: $C = \arcsin b_0 - \arccos a_0$. The expression of b as a function of a is then $b = \sin(\arccos(a) + C)$. The integral curve is (a, b) such that $\arcsin(b) - \arcsin(b_0) = \arccos(a) - \arccos(a_0)$. We now show that this integral curve lies on an ellipse. As consequences of $b = \sin(\arccos(a) + C)$, we can state that $b = \sin(\arccos(a))\cos(C) + a\sin(C)$ and $\sqrt{1-b^2} = \cos(\arccos(a) + C) = a\cos(C) - \sin(\arccos(a))\sin(C)$, and mixing these two equalities we get

$$\sqrt{1-b^2} = \frac{a}{\cos(C)} - b\tan(C),$$

Assuming that $\cos(C) \neq 0$, so that $\tan(C)$ is well defined. We first do this assumption and will investigate the case $\cos(C) = 0$ in a second step. Putting this expression to the square we see that $1 - b^2 = a^2/\cos^2(C) + b^2\tan^2(C) - 2ab\sin(C)/\cos^2(C)$, leading (after a short algebraic computation) to

$$\frac{a^2}{\cos^2(C)} + \frac{b^2}{\cos^2(C)} - 2ab\frac{\sin(C)}{\cos^2(C)} = 1. \tag{3.31}$$

Recall that $|\sin(C)| \neq 1$ because we assumed $\cos(C) \neq 0$ and note

$$\left\{ \begin{array}{l} d_1 = \frac{2\cos^2(C)}{1-\sin(C)}, \\ d_2 = \frac{2\cos^2(C)}{1+\sin(C)}. \end{array} \right.$$

It is then easy to observe that equation (3.31) is equivalent to

$$\frac{(a+b)^2}{d_1} + \frac{(a-b)^2}{d_2} = 1, \tag{3.32}$$

which shows that $(a(\xi), b(\xi))$ lies on an ellipse oriented along the two diagonals of the square with length axes $\sqrt{d_1}$ and $\sqrt{d_2}$. In the case where $\cos(C) = 0$, a and b verify $b = a\sin(C)$, leading to $b = \pm a$ (note that $\cos(C) = 0 \Longleftrightarrow b_0 = \pm a_0$) but this will not be taken into account because we took (a_0, b_0) not on a diagonal of the square. We now can see that the considered ellipse (for $\cos(C) \neq 0$) stays in the unit square $\overline{\mathcal{D}}$ and has exactly 4 intersecting points with its boundary (in other words, it is intersecting it tangentially). For this we prove that there exists one and only one point (a, b) on the ellipse such that $a = 1$ and the cases $a = -1$, $b = 1$ and $b = -1$ are following by the same way. So let us assume that $a = 1$. We are looking for b such that

$$\frac{(1+b)^2}{d_1} + \frac{(1-b)^2}{d_2} = 1,$$

that is $d_2(1+b)^2 + d_1(1-b)^2 = d_1d_2$, equivalent to $(d_1+d_2)b^2 + 2(d_2-d_1)b + d_1 + d_2 + d_1d_2 = 0$. The discriminant of this equation is $\Delta = 4d_1d_2(d_1+d_2-4)$ and with the definitions of d_1 and d_2 we find that $d_1 + d_2 = 4$, so that $\Delta = 0$. Thus the above polynomial equation has one and only one solution $b = (d_1-d_2)/(d_1+d_2) = 4\sin(C)$. The ellipse supporting the rarefaction waves is plotted on figure 2, together with the Hugoniot curves (see next section).

Proof of lemma 4. We begin by the following simple remark: assume for example that a portion of a rarefaction curve lies on the line $b = 1$: thus we get a non-trivial rarefaction wave on the boundary ; $a(\xi) = \xi/2$. We now distinguish two cases in order to connect this rarefaction wave on the boundary to the rarefaction wave inside the domain given in lemma 3.

- Starting from a point (a_0, b_0) such that $1 > b_0 > a_0 > -b_0 < 0$, there exists a unique rarefaction wave, solution of (3.30) as long as it has not reached the boundary of the unit square neither the line of local linear degeneracy, i.e. $a = b$. The solution is continuous, so that there exists $\xi^* > 0$ such that $1 > b(\xi) > a(\xi) > -b(\xi) < 0 \ \forall \xi \in [-\xi^*, \xi^*]$. In this interval, the solution verifies $1 > b(\xi)^2 > a^2(\xi) > 0$, and consequently $b(\xi)\sqrt{1-a^2(\xi)} - a(\xi)\sqrt{1-b^2(\xi)} > 0$, so that $a'(\xi) > 0$ and $b'(\xi) < 0$. This means that the solution is going closer to the line of local degeneracy as ξ is increasing, and closer to the line $b = 1$ as ξ is decreasing. We know that this solution lies on an ellipse having one intersecting point with the line $b = 1$ and one with the line $a = 1$, so the ellipse has one intersecting point with the degeneracy line $a = b$. More precisely we can see that the line $a = b$ is reached for a finite $\tilde{\xi}$ by the 2-rarefaction wave: it follows from $a'(\xi) > 0$ and $b'(\xi) < 0$ that $\forall \xi > 0$ we have $a(\xi) > a_0$ and $b(\xi) < b_0$, and then $b(\xi)\sqrt{1-a^2(\xi)} - a(\xi)\sqrt{1-b^2(\xi)} < b_0\sqrt{1-a_0^2} - a_0\sqrt{1-b_0^2}$, and this fact put together with $-\sqrt{1-b_0^2} > -\sqrt{1-b^2(\xi)}$ for $\xi > 0$ gives
$$b'(\xi) < \frac{-\sqrt{1-b_0^2}}{3(b_0\sqrt{1-a_0^2} - a_0\sqrt{1-b_0^2})} < 0 \quad \forall \xi.$$
As a consequence, we have that there exists necessarily $\tilde{\xi}$ such that $a(\tilde{\xi}) = b(\tilde{\xi})$. When this solution reaches the line $a = b$, rarefaction waves are not well-defined anymore... Along the reverse side ($\xi < 0$), one can show too that the line $b = 1$ is reached for finite ξ: trivially $3(b\sqrt{1-a^2} - a\sqrt{1-b^2}) < 6$ and $\sqrt{1-a^2(\xi)} > \sqrt{1-a_0^2} \ \forall \xi < 0$ until $b(\xi) = 1$, and so $a'(\xi) > \sqrt{1-a_0^2}/6$. Consequently, $a(\xi)$ can be as small as wanted for ξ sufficiently small, except if $b = 1$ for a certain $\overline{\xi}$. Then, as the rarefaction wave reaches the boundary of the unit square, the "continuing" rarefaction wave is solution to (3.28). It is then a straight line on $b = 1$ reaching $(-1, 1)$.
- Starting from a point $(a_0, 1)$, the curve reaches $(-1, 1)$ on the left for $\xi < 0$. For $\xi > 0$, there is an infinity of solutions: each one can follow the line $b = 1$ and take of for every $\xi > 0$ and then follow a 2-ellipse.

Thus proposition 4 is proved.

Remark 1 . *Concerning the 1-rarefaction wave, the following result is proved by symmetry for 1-rarefaction waves in $\overline{\mathcal{D}}$. Let $(a_0, b_0) \in \mathcal{Q} \cap \mathcal{D}$ inside the square. The 1-rarefaction wave $(a, b) \in \overline{\mathcal{D}}$ connected to (a_0, b_0) is composed of*

- *the piece of an ellipse with equation (3.32)*
$$\frac{(a+b)^2}{d_1'} + \frac{(a-b)^2}{d_2'} = 1 \tag{3.33}$$
where
$$d_1' = \frac{2\sin^2(C')}{1+cos(C')}, d_2' = \frac{2\sin^2(C')}{1-cos(C')}, C = \arcsin(b_0) - \arcsin(a_0),$$
piece located between a point $(\tilde{a}, \tilde{a})$ on the degeneracy line $a = -b$ and a point $(\overline{a}, 1)$;

- *the segment $[(\overline{a}, 1), (1, 1)]$ which is on the boundary of the square.*

Let (a_0, b_0) on the boundary of the square, such that $b_0 = 1$. There exist an infinity of 1-rarefaction waves in the plane $(a, b) \in \overline{\mathcal{D}}$. All of them are composed of

- *a line segment $[(\overline{a}, 1), (1, 1)]$ for some $\overline{a} \in [-1, a_0]$;*
- *the portion of the ellipse with equation (3.33) where d_1' and d_2' are such that this ellipse passes through $\overline{a} \in [-1, a_0]$, located between $\overline{a} \in [-1, a_0]$ and the degeneracy line where $a = -b$.*

3.4.2. *Shock curves*

Let $(a_0, b_0) \in D$. The Rankine-Hugoniot relation for shocks between this state and another one $(a, b) \in D$ is

$$\begin{cases} \sigma(a_0 - a) &= (a_0^2 - 1)b_0 - (a^2 - 1)b, \\ \sigma(b_0 - b) &= (b_0^2 - 1)a_0 - (b^2 - 1)a. \end{cases} \tag{3.34}$$

where $\sigma \in \mathbb{R}$ is the speed of the shock. Then, eliminating σ (taking $a \neq a_0$ and $b \neq b_0$), we obtain

$$(1 - aa_0)(b - b_0)^2 = (1 - bb_0)(a - a_0)^2. \tag{3.35}$$

Call $b_1(a_0, b_0, a - a_0)$ and $b_2(a_0, b_0, a - a_0)$ the two roots of the previous second order polynomial in b, we get

$$\begin{aligned} b_1(a_0, b_0, a - a_0) &= b_0 + \frac{-b_0(a - a_0)^2 + \sqrt{\Delta}}{2(1 - aa_0)}, \\ \text{and } b_2(a_0, b_0, a - a_0) &= b_0 + \frac{-b_0(a - a_0)^2 - \sqrt{\Delta}}{2(1 - aa_0)}, \end{aligned} \tag{3.36}$$

where $\Delta = b_0^2(a - a_0)^4 + 4(1 - aa_0)(1 - b_0^2)(a - a_0)^2$. In $\mathcal{D}$, $\Delta \geqslant 0$ thus $b_1 \geqslant b_2$. Then the solutions of (3.35) are $b = b_1(a_0, b_0, a - a_0)$ or $b = b_2(a_0, b_0, a - a_0)$.

Remark *Another possibility to solve (3.35) is based on the following remark: equation (3.35) is equivalent to $\alpha = \pm\beta$ where $\alpha = \frac{a - a_0}{\sqrt{1 - aa_0}}$ and $\beta = \frac{b - b_0}{\sqrt{1 - bb_0}}$. It is a classroom exercise to prove that the change of coordinate $(a, b) \to (\alpha, \beta)$ is invertible. The reason we prefer (3.36) is that it is more adapted for the analysis of the Lax admissibility condition for shocks.*

Consider the two parametrizations (3.36) for b_1 and b_2, and define for simplicity $a - a_0 = \varepsilon$. The shock speeds are $\sigma = \sigma_1(a_0, b_0, \varepsilon)$ or $\sigma = \sigma_2(a_0, b_0, \varepsilon)$, where

- if $\varepsilon \geqslant 0$,

$$\begin{aligned} \sigma_1(a_0, b_0, \varepsilon) &= \lambda_1(a_0, b_0) + \varepsilon\frac{3}{2}\left[b_0 - a_0\sqrt{\frac{1 - b_0^2}{1 - a_0^2}}\right] + o(\varepsilon^2), \\ \text{and } \sigma_2(a_0, b_0, \varepsilon) &= \lambda_2(a_0, b_0) + \varepsilon\frac{3}{2}\left[b_0 - a_0\sqrt{\frac{1 - b_0^2}{1 - a_0^2}}\right] + o(\varepsilon^2), \end{aligned} \tag{3.37}$$

- if $\varepsilon < 0$,

$$\begin{aligned} \sigma_1(a_0, b_0, \varepsilon) &= \lambda_2(a_0, b_0) + \varepsilon\frac{3}{2}\left[b_0 + a_0\sqrt{\frac{1 - b_0^2}{1 - a_0^2}}\right] + o(\varepsilon^2), \\ \text{and } \sigma_2(a_0, b_0, \varepsilon) &= \lambda_1(a_0, b_0) + \varepsilon\frac{3}{2}\left[b_0 + a_0\sqrt{\frac{1 - b_0^2}{1 - a_0^2}}\right] + o(\varepsilon^2), \end{aligned} \tag{3.38}$$

The curve associated to the first eigenvalue λ_1 is obtained by the juxtaposition of b_2 for $\varepsilon < 0$ and b_1 for $\varepsilon > 0$ (the second eigenvalue λ_2 corresponds to b_1 for $\varepsilon < 0$ and b_2 for $\varepsilon > 0$).

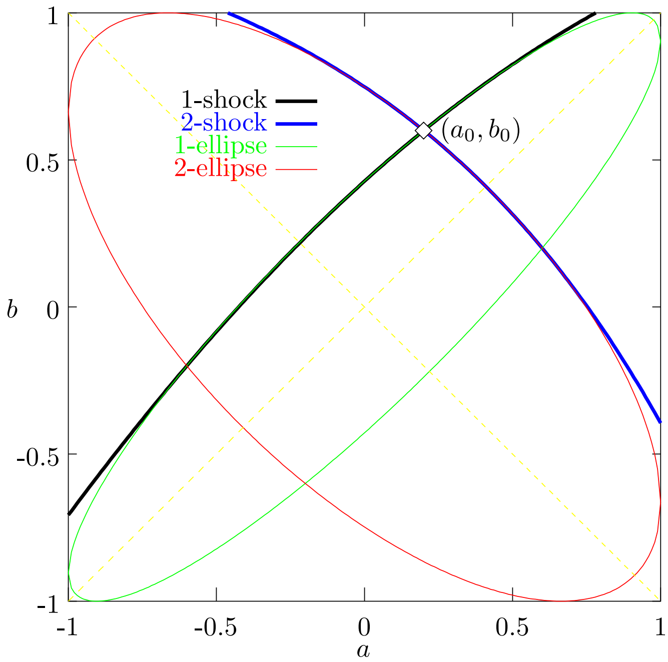

Figure 2: Hugoniot curves and rarefaction curves

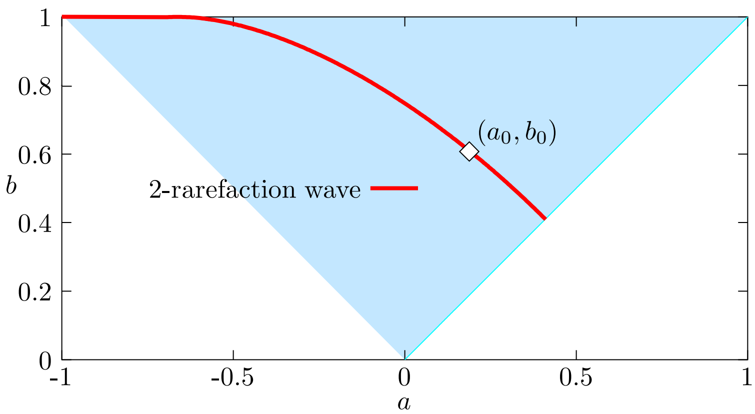

Figure 3: 2-rarefaction wave through (a_0, b_0) (the shaded region represents the set of Cauchy data considered in proposition 4).

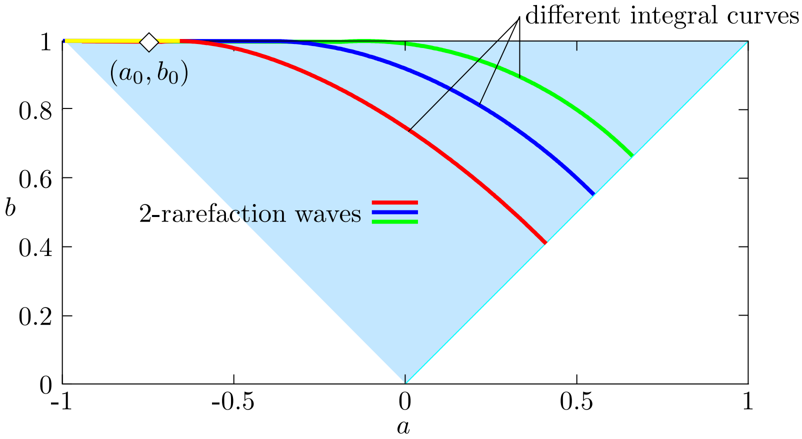

Figure 4: 2-rarefaction waves through (a_0, b_0) (the shaded region represents the set of Cauchy data considered in proposition 4).

3.4.3. *Shock curves and the Lax condition*

We now have to determine admissible shocks among all possible Rankine-Hugoniot satisfying discontinuities. Since it is not possible to use entropy condition (recall the entropy S is not strictly convex in the whole square $\mathcal{D}$), we here use the Lax condition.

The Lax criteria of admissibility for shocks is

$$\lambda_1(a_0, b_0) > \sigma > \lambda_1(a, b) \quad (\text{resp. } \lambda_2(a_0, b_0) > \sigma > \lambda_2(a, b)) \tag{3.39}$$

where (a, b) is a right state connected to the left state (a_0, b_0) by a 1-shock (resp. 2-shock). Using symmetry in a and b, we reduce for simplicity the study to $(a_0, b_0) \in \mathcal{Q} \cap \mathcal{D}$: $-b_0 < a_0 < b_0 < 1$.

Lemma 5 *Let $(a_0, b_0) \in \mathcal{Q} \cap \mathcal{D}$ be a left state. Let $\mathcal{N}$ be a neighborhood of (a_0, b_0). Then $(a, b) \in \mathcal{Q} \cap \mathcal{D} \cap \mathcal{N}$ is a solution of the Rankine-Hugoniot equations (3.34) with (3.39) if and only if $a < a_0$.*

Since $(a_0, b_0) \in \mathcal{Q} \cap \mathcal{D}$, we have $b_0 > |a_0|\sqrt{\dfrac{1 - b_0^2}{1 - a_0^2}}$. Thus relations (3.37) and (3.38) imply, for small shocks,

1. if $\varepsilon > 0$: $\sigma_1(\varepsilon) > \lambda_1(a_0, b_0)$ and $\sigma_2(\varepsilon) > \lambda_2(a_0, b_0)$.
2. if $\varepsilon < 0$: $\sigma_1(\varepsilon) < \lambda_2(a_0, b_0)$ and $\sigma_2(\varepsilon) < \lambda_1(a_0, b_0)$.

Assume that $a > a_0$. If $b = b_1(a_0, b_0, \varepsilon)$ where $\varepsilon = a - a_0$, $\sigma = \sigma_1(a_0, b_0, \varepsilon) > \lambda_1(a_0, b_0)$: by symmetry (a_0, b_0) is such that $b_0 = b_2(a, b, -\varepsilon)$ so $\sigma < \lambda_1(a, b)$. Thus $\lambda_1(a_0, b_0) < \sigma < \lambda_1(a, b)$ and (3.39) is not true. Similarly if $a > a_0$ and $b = b_2(a_0, b_0, \varepsilon)$ we get: $\lambda_2(a_0, b_0) < \sigma < \lambda_2(a, b)$ and (3.39) is not true.

Consider now that (a, b) is such that $a < a_0$. If $b = b_1(a_0, b_0, \varepsilon)$, then $\sigma = \sigma_1(a_0, b_0, \varepsilon) < \lambda_2(a_0, b_0)$. By symmetry $b_0 = b_2(a, b, -\varepsilon)$ so $\sigma > \lambda_2(a, b)$. Thus the Lax condition (3.39) is true. Similarly if $a < a_0$ and $b = b_2(a_0, b_0, \varepsilon)$, then $\sigma < \lambda_1(a_0, b_0)$: by symmetry $b_0 = b_1(a, b, -\varepsilon)$ so $\sigma > \lambda_1(a, b)$: this is the Lax condition (3.39). It ends the proof.

4. Stability for the parabolic system

In this section we study the Cauchy problem for the regularized system

$$\left\{\begin{array}{l}\partial_t a + \partial_x \left[(a^2-1)b\right] = \varepsilon \partial_{xx} a,\\[2ex] \partial_t b + \partial_x \left[(b^2-1)a\right] = \varepsilon \partial_{xx} b.\end{array}\right. \tag{4.40}$$

We assume that the Cauchy data is periodic

$$a(0,x) = a_0(x) = a_0(x+1) \quad b(0,x) = b_0(x) = b_0(x+1) \quad \forall x \in \mathbb{R}. \tag{4.41}$$

Following Smoller[22], we add a forcing term on the right hand side in order to circumvent technical difficulties. Thus we modify (4.40) by adding the source term, $\alpha > 0$,

$$\left\{\begin{array}{l}\partial_t a + \partial_x \left[(a^2-1)b\right] = \varepsilon \partial_{xx} a - \alpha a,\\[2ex] \partial_t b + \partial_x \left[(b^2-1)a\right] = \varepsilon \partial_{xx} b - \alpha b.\end{array}\right. \tag{4.42}$$

Theorem 1 *Assume the initial condition (4.41) is physically admissible: $-1 \leqslant a_0(x) \leqslant 1$ and $-1 \leqslant b_0(x) \leqslant 1$ a.e. Assume $\alpha > 0$ and $\varepsilon > 0$. Then the unique solution of* (4.42) *stays in this square: $-1 \leqslant a(t,x) \leqslant 1$ and $-1 \leqslant b(t,x) \leqslant 1$ a.e.*

For the sake of simplicity we assume that the Cauchy data is smooth, so that the unique solution of (4.42) is smooth. Let us consider the function $G(a,b) = a^2 - 1$. If we prove that $G(a(t,x), b(t,x))$ is non-positive for almost every $x \in \Omega$ and $t \geqslant 0$, then it proves that $-1 \leqslant a(t,x) \leqslant 1$ a.e. Using a symmetric argument for the b variable, it then proves that $-1 \leqslant a_0(x) \leqslant 1$ and $-1 \leqslant b_0(x) \leqslant 1$ a.e., which means that the square $[-1;1]^2$ is an invariant region for equation (4.42).

So let us assume that there exists a point (t,x) such that $(a,b)(t,x) \notin \overline{\mathcal{D}}$, and let us define

$$t_0 = \mathrm{Inf}\left\{t; \quad \exists x \in \mathbb{R} \text{ such } (a,b)(t,x) \notin \overline{\mathcal{D}}\right\}. \tag{4.43}$$

The time t_0 is in some sense the first time where G becomes non-negative. By definition

$$\forall t \leqslant t_0, \quad \forall x \qquad G(a(t,x), b(t,x)) \leqslant 0, \tag{4.44}$$

and there exists a sequence (t^n, x^n) and x_0

$$G(a(t^n,x^n), b(t^n,x^n)) > 0, \quad t^n \to t_0 \text{ and } x^n \to x_0. \tag{4.45}$$

The periodicity of the Cauchy data (4.41) is important here to have the periodicity of the solution, so that $x^n \in \mathbb{T} = [0,1[_{per}$: by compacity, one extracts the subsequence $x^n \to x_0 \in \mathbb{T}$. By continuity $G(a(t_0,x_0), b(t_0,x_0)) = 0$, i.e. $a(t_0,x_0) = \pm 1$. Let us assume that $a(t_0,x_0) = 1$. One has the formula

$$\frac{\partial G(a(t,x), b(t,x))}{\partial t} = 2a\frac{\partial a}{\partial t} = 2a\left[\varepsilon \partial_{xx} a - 2ab\partial_x a - (a^2-1)\partial_x b - \alpha a\right] \tag{4.46}$$

so

$$\frac{\partial G(a(t_0,x_0), b(t_0,x_0))}{\partial t} = 2\varepsilon \partial_{xx} a(t_0,x_0) - 4b(t_0,x_0)\partial_x a(t_0,x_0) - 2\alpha. \tag{4.47}$$

Let us now study the sign of $\partial_{xx} a(t_0,x_0)$ and $\partial_x a(t_0,x_0)$. One defines $h(x) = G(a(t_0,x), b(t_0,x))$, so that $h(x_0) = 0$ and $h'(x) = 2a(t_0,x)\partial_x a(t_0,x)$.

First one has $h'(x_0) = 0$. Indeed assume $h'(x_0) > 0$ then, h being continuous, $h(x) > 0$ for $x > x_0$ and $|x - x_0|$ small enough. Thus $G(a(t_0,x), b(t_0,x)) > 0$ for such an x and, by regularity (in time), $G(a(t,x), b(t,x)) > 0$ for $|t - t_0|$ small enough, in particular for some $t < t_0$ and this violates (4.44). Similarly, $h'(x_0) < 0$ is impossible. $h'(x_0) = \partial_x a(t_0,x_0) = 0$.

Second one has $h''(x_0) \leqslant 0$. If $h''(x_0) > 0$ then $h(x) > 0$ for $|x - x_0|$ small enough and we arrive at the same conclusion as above (i.e. $G(a(t_0, x), b(t_0, x)) > 0$ which is in contradiction with (4.44)). Thus $h''(x_0) \leqslant 0$, i.e. $\partial_{xx} a(t_0, x_0) \leqslant 0$.

Then (4.46) gives $\dfrac{\partial G}{\partial t}(a, b)(t_0, x_0) \leqslant -2\alpha a(t_0, x_0)^2 = -2\alpha < 0$. By continuity $\dfrac{\partial G(a, b)}{\partial t}(t, x) < 0$ for all (t, x) in a neighborhood $\mathcal{N}(t_0, x_0)$ of (t_0, x_0). Then for all $t > t_0$ and $(t, x) \in \mathcal{N}(t_0, x_0)$, one has

$$G(a(t, x), b(t, x)) = G(a(t_0, x), b(t_0, x)) + \frac{\partial G(a, b)}{\partial t}(\theta_x t + (1 - \theta_x)t_0, x) < 0, \quad \theta_x \in]0, 1[,$$

due to (4.44), and this is finally in contradiction with (4.45).

We then have proved that the solution of (4.40) stays behind line $a = 1$. Similarly $-1 \leqslant a$ and $-1 \leqslant b \leqslant 1$. It ends the proof.

5. Numerical scheme

In this section we study the stability of a numerical scheme for computing approximate solutions of (2.17). The scheme is constructed using the relaxation method and is equal to the Lax-Friedrichs scheme.

5.1. *Construction of the scheme*

Let us rewrite

$$\left\{ \begin{array}{l} \partial_t a + \partial_x \left[(a^2 - 1)b\right] = 0, \\ \partial_t b + \partial_x \left[(b^2 - 1)a\right] = 0 \end{array} \right.$$

as the relaxation limit of the larger system

$$\left\{ \begin{array}{l} \partial_t a + \partial_x \tilde{a} = 0, \\ \partial_t \tilde{a} + u^2 \partial_x a = \frac{1}{\varepsilon}((a^2 - 1)b - \tilde{a}), \\ \partial_t b + \partial_x \tilde{b} = 0, \\ \partial_t \tilde{b} + u^2 \partial_x b = \frac{1}{\varepsilon}((b^2 - 1)a - \tilde{b}) \end{array} \right. \tag{5.48}$$

where $\varepsilon \to 0$ by positive values and $u > 0$. This system rewrites

$$\left\{ \begin{array}{l} \partial_t A + \partial_x \left[\begin{pmatrix} 0 & 1 \\ u^2 & 0 \end{pmatrix} A \right] = \frac{1}{\varepsilon} \begin{pmatrix} 0 \\ (a^2 - 1)b - \tilde{a} \end{pmatrix} \\ \partial_t B + \partial_x \left[\begin{pmatrix} 0 & 1 \\ u^2 & 0 \end{pmatrix} B \right] = \frac{1}{\varepsilon} \begin{pmatrix} 0 \\ (b^2 - 1)a - \tilde{b} \end{pmatrix} \end{array} \right.$$

where $A = \begin{pmatrix} a \\ \tilde{a} \end{pmatrix}$ and $B = \begin{pmatrix} b \\ \tilde{b} \end{pmatrix}$. Let us now focus on the equation for A, the one for B being independent. To solve numerically this system, we split the linear part and the relaxation part. In a first step we solve the system

$$\partial_t A + \partial_x \left[\begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix} A \right] = 0. \tag{5.49}$$

A second step is devoted to take into account the right-hand side of the system

$$\partial_t A = \frac{1}{\varepsilon} \begin{pmatrix} 0 \\ (a^2 - 1)b - \tilde{a} \end{pmatrix}. \tag{5.50}$$

First step: solving (5.49). To solve the differential part, we diagonalize it: a short and obvious calculation shows that

$$R^{-1}\begin{pmatrix} 0 & 1 \\ u^2 & 0 \end{pmatrix} R = \begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix}$$

where

$$R = \begin{pmatrix} 1 & 1 \\ u & -u \end{pmatrix}, \qquad R^{-1} = \frac{1}{2}\begin{pmatrix} 1 & \frac{1}{u} \\ 1 & -\frac{1}{u} \end{pmatrix}.$$

The vector $R^{-1}A = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ is a solution of the linear diagonal system

$$\begin{cases} \partial_t a_1 + u\partial_x a_1 = 0 \\ \partial_t a_2 - u\partial_x a_2 = 0 \end{cases}$$

These two equations are solved using the simplest method, the upwind scheme. Given a regular mesh with space increment Δx, given a time increment Δt with $u\Delta t/\Delta x \leqslant 1$, and given the values of the approximate solution at time $n\Delta t$, the approximate solution in cell j at time $n + 1/2\Delta t$ is given by

$$\begin{cases} {a_1}_j^{n+1/2} = {a_1}_j^n - u\Delta t/\Delta x({a_1}_j^n - {a_1}_{j-1}^n), \\ {a_2}_j^{n+1/2} = {a_2}_j^n + u\Delta t/\Delta x({a_2}_{j+1}^n - {a_2}_j^n). \end{cases}$$

Of course the same algorithm is used for the vector $R^{-1}B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. Next we multiply the previous results by R, which allows to have $a_j^{n+1/2}$, $\tilde{a}_j^{n+1/2}$, $b_j^{n+1/2}$ and $\tilde{b}_j^{n+1/2}$.

Second step: solving (5.50). For the sake of simplicity we consider only the limit case $\varepsilon \to 0$. One gets

$$\begin{cases} a_j^{n+1} = a_j^{n+1/2}, \\ \tilde{a}_j^{n+1} = ((a_j^{n+1/2})^2 - 1)b_j^{n+1/2}, \\ b_j^{n+1} = b_j^{n+1/2}, \\ \tilde{b}_j^{n+1} = ((b_j^{n+1/2})^2 - 1)a_j^{n+1/2}, \end{cases}$$

After elimination of $\tilde{a}_j^{n+1}$ and $\tilde{b}_j^{n+1}$ we get the scheme

$$\begin{cases} a_j^{n+1} = a_j^n - \frac{\Delta t}{\Delta x}(c_{j+\frac{1}{2}}^n - c_{j-\frac{1}{2}}^n) = 0, \\ b_j^{n+1} = b_j^n - \frac{\Delta t}{\Delta x}(d_{j+\frac{1}{2}}^n - d_{j-\frac{1}{2}}^n) = 0, \end{cases} \text{with fluxes} \begin{cases} c_{j+\frac{1}{2}}^n = \frac{c_j^n + c_{j+1}^n}{2} - u\frac{a_{j+1}^n - a_j^n}{2}, & c = (a^2-1)b, \\ d_{j+\frac{1}{2}}^n = \frac{d_j^n + d_{j+1}^n}{2} - u\frac{b_{j+1}^n - b_j^n}{2}, & d = (b^2-1)a. \end{cases} \tag{5.51}$$

5.2. *Numerical stability*

Stability in the sense of dissipative properties of this scheme is proved in Jin and Xin[9] under the condition

$$u > \max_{(a,b)\in\overline{\mathcal{D}}}(|\lambda_1(a,b)|, |\lambda_2(a,b)|) = 2. \tag{5.52}$$

The value 2 is given by inspection of formulae (3.20): first, 2 is clearly the maximum of the eigenvalue on the boundary $(1-a^2)(1-b^2) = 0$; second, if an eigenvalue is extremal at a point inside the square, its gradient at this point is zero. For example $\nabla\lambda_1 = 0$ if and only if $2b + a\sqrt{\frac{1-b^2}{1-a^2}} = 0$ and $2a + b\sqrt{\frac{1-a^2}{1-b^2}} = 0$, that is $a = b = 0$. So the extremal value of the first eigenvalue inside the square is $\lambda_1 = 0$. Since it is the same for λ_2, we infer that the maximum of the moduli of the eigenvalues is reached on the boundary. Thus $\max_{(a,b)\in\overline{\mathcal{D}}}(|\lambda_1(a,b)|, |\lambda_2(a,b)|) = 2$.

We now state that the scheme is stable in the sense that numerical solutions stay in $\overline{\mathcal{D}}$.

Theorem 2 *Assume that $u \geqslant 6$ and assume the CFL condition $u\frac{\Delta t}{\Delta x} \leqslant 1$. Then if all values $(a_j^n, b_j^n) \in \overline{\mathcal{D}}$ are inside the physical domain of interest, then $(a_j^{n+1}, b_j^{n+1}) \in \overline{\mathcal{D}}\ \forall j$.*

Remark *We do not know if the constant 6 is optimal. Numerical experiments show it is probably not the case.*

By symmetry, it is of course sufficient to prove that $a_j^n \leqslant 1\ \forall j$ *then* $a_j^{n+1} \leqslant 1\ \forall j$ One has

$$a_j^{n+1} = a_j^n - \frac{\Delta t}{2\Delta x}\left[((a_{j+1}^n)^2 - 1)b_{j+1}^n - ((a_{j-1}^n)^2 - 1)b_{j-1}^n\right] - u\frac{\Delta t}{\Delta x}\left(a_j^n - \frac{1}{2}a_{j-1}^n - \frac{1}{2}a_{j+1}^n\right).$$

Let us define, for $\theta \in [0,1]$, $f(\theta) = (a(\theta)^2 - 1)b(\theta)$ where $a(\theta) = a_{j-1}^n + \theta(a_{j+1}^n - a_{j-1}^n)$ and $b(\theta) = b_{j-1}^n + \theta(b_{j+1}^n - b_{j-1}^n)$. A first order Taylor expansion of the flux with the formula $f(1) - f(0) = f'(\theta) = f_a(\theta)(a_{j+1}^n - a_{j-1}^n) + f_b(\theta)(b_{j+1}^n - b_{j-1}^n)$ gives

$$\begin{aligned}((a_{j+1}^n)^2 - 1)b_{j+1}^n - ((a_{j-1}^n)^2 - 1)b_{j-1}^n &= 2a(\theta_j^n)b(\theta_j^n)(a_{j+1}^n - a_{j-1}^n) + (a(\theta_j^n)^2 - 1)(b_{j+1}^n - b_{j-1}^n)\\ &= 2a(\theta_j^n)b(\theta_j^n)(a_{j+1}^n - a_{j-1}^n) + (a(\theta_j^n) + 1)(b_{j+1}^n - b_{j-1}^n)(a(\theta_j^n) - 1)\end{aligned}$$

where $a(\theta_j^n) = \theta_j^n a_{j-1}^n + (1-\theta_j^n)a_{j+1}^n$, $b(\theta_j^n) = \theta_j^n b_{j-1}^n + (1-\theta_j^n)b_{j+1}^n$ and most of all $0 < \theta_j^n < 1$. So

$$((a_{j+1}^n)^2 - 1)b_{j+1}^n - ((a_{j-1}^n)^2 - 1)b_{j-1}^n = s_j^n(a_{j+1}^n - 1) + t_j^n(a_{j-1}^n - 1)$$

where the first coefficient is $s_j^n = \left(2a(\theta_j^n)b(\theta_j^n) + (1-\theta_j^n)(a(\theta_j^n)+1)(b_{j+1}^n - b_{j-1}^n)\right)$ and the second coefficient is $t_j^n = \left(-2a(\theta_j^n)b(\theta_j^n) + \theta_j^n(a(\theta_j^n)+1)(b_{j+1}^n - b_{j-1}^n)\right)$. Then

$$a_j^{n+1} - 1 = \left(1 - u\frac{\Delta t}{\Delta x}\right)(a_j^n - 1) + \frac{\Delta t}{2\Delta x}\left(u - s_j^n\right)(a_{j+1}^n - 1) + \frac{\Delta t}{2\Delta x}\left(u + t_j^n\right)(a_{j-1}^n - 1). \quad (5.53)$$

If all coefficients in (5.53) are non-negative, then $a_j^{n+1} - 1 \leqslant 0$ provided $a_{j-1,j,j+1}^n - 1 \leqslant 0$: this is the stability property. Since $|s_j^n| \leqslant 6$ and $|t_j^n| \leqslant 6$ by definition, it is sufficient to impose $u \geqslant 6$ plus the CFL condition $u\frac{\Delta t}{\Delta x} \leqslant 1$ to get the stability property. The constant 6 is here only sufficient and may not be the optimal one. By symmetry it ends the proof.

6. Some numerical results

In this section we present some numerical results obtained with the scheme described in section to illustrate the behavior of solutions of the two-by-two system discussed in this paper. We just report 3 results:

- a simple Riemann problem in the strict hyperbolicity region without local linear degeneracy;
- a Riemann problem with a rarefaction wave touching its associated linear degeneracy line;
- a Riemann problem with a left state on the boundary of $\mathcal{D}$.

The results we present are computed with a large number of cells in order to show "almost converged" solutions.

6.1. *A simple Riemann problem*

The initial conditions we take here are

$$\begin{cases} a = 0, b = 0.9 \text{ if } x \in [0, 0.5],\\ a = 0.2, b = 0.3 \text{ if } x \in]0.5, 1],\end{cases}$$

The computation is done with 10000 cells and the final time is 0.3. The solution is composed of a shock and a rarefaction fan.

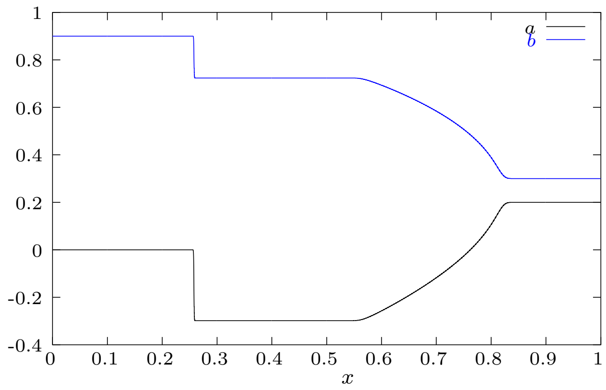

Figure 5: Simple Riemann problem: (a, b) at $t = 0.3$, 10000 cells.

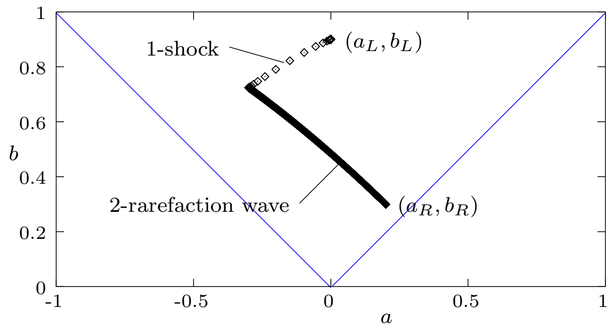

Figure 6: Simple Riemann problem: (a, b) in the phase plane.

6.2. *A Riemann problem with linear degeneracy*

The initial conditions are

$$\begin{cases} a = 0.3, b = 0.9 \text{ if } x \in [0, 0.5], \\ a = 0.9, b = 0.1 \text{ if } x \in]0.5, 1], \end{cases}$$

The computation is done with 10000 cells and the final time is 0.3. The solution is composed of a first classical shock (on the left of figure 7) plus a shock attached to a rarefaction fan. The attached shock is at $a = b$ which is the locus of local linear degeneracy of the non-linear wave.

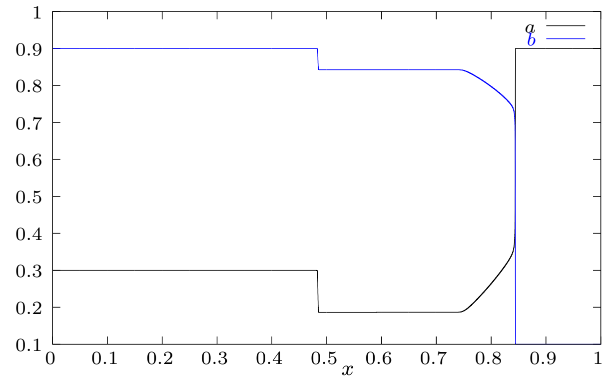

Figure 7: Riemann problem with linear degeneracy: (a, b) at $t = 0.3$, 10000 cells.

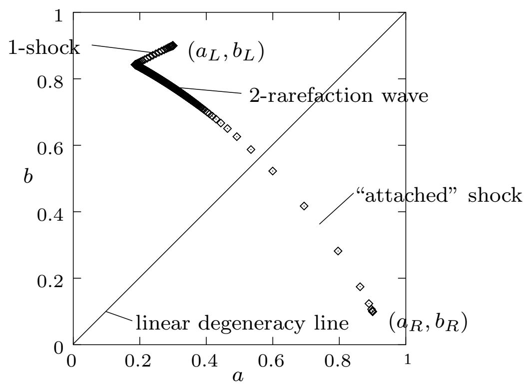

Figure 8: Riemann problem with linear degeneracy: (a, b) in the phase plane.

6.3. *A Riemann problem starting from the boundary*

We take for initial conditions

$$\begin{cases} a = -0.3, b = 1 \text{ if } x \in [0, 0.5], \\ a = 0.4, b = 0.8 \text{ if } x \in]0.5, 1], \end{cases}$$

The computation is done with 300000 cells to focus on the stability of the algorithm and the final time is 0.2. All points are inside the square, as predicted by theorem 2.

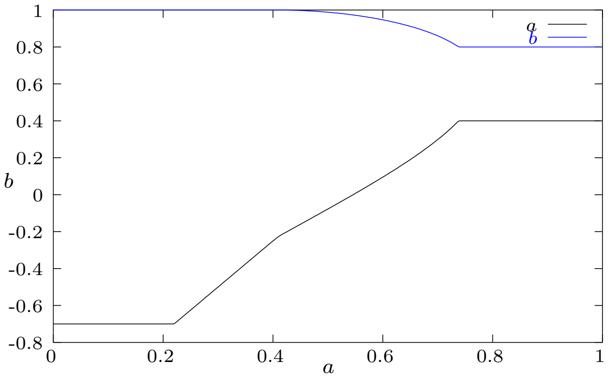

Figure 9: Riemann problem from the boundary: (a, b) at $t = 0.2$, 300000 cells.

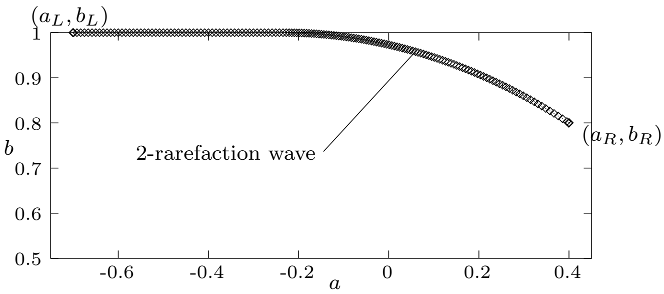

Figure 10: Riemann problem from the boundary: (a, b) in the phase plane.

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