Sharpening methods for finite volume schemes

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Abstract

We review sharpening methods for finite volume schemes, with an emphasis on the basic structure of sharpening methods. It covers high order methods and non linear techniques for linear advection, Glimm's method, anti-diffusion techniques, the interaction of these techniques with the PDE structures. Additional approaches like level sets, interface reconstruction and Vofire are also discussed. We also present the algorithmic structure of the downwind method for a simple two components problem.

keywords Sharpening methods, Finite Volume schemes, anti-diffusion, interface tracking.

1 Introduction

The present paper deals with sharpening methods for finite volume schemes (FV) understood as discretization strategies for the enhancement of sharp profiles in numerical simulations. We restrict the scope to finite volume schemes since they are the numerical method of choice for compressible computation fluid dynamics where the exact or approximate solutions may exhibit strong gradients that account for shocks or contact discontinuities.

We will more specifically focus on the calculations of interfaces associated with linearly degenerate fields (contact discontinuities), material contact discontinuities or free boundaries that are tracked across the computational domain. Although we shall consider numerical methods that are compliant with shock capturing, we shall not discuss the approximations of shocks in this paper.

Interface tracking has motivated a considerable amount of contributions since the early days of scientific computing and numerical analysis. Therefore reviewing exhaustively all the methods that have been published to date seems quite unrealistic and we apologize in advance to the community for all the works that will not be mentioned in the sequel. We propose to sketch a map of these methods by relying on mathematical and algorithmic arguments that can be used to analyze the efficiency. We hope that this effort may also

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help classifying the methods that will not be discussed in this document and help understanding the sharpening mechanisms at play within the numerical schemes that are available in the literature.

The paper is organized as follows. Most of the common material (that is high order and nonlinear techniques, the Glimm's scheme, the notion of anti-diffusion, level-sets, multidimensional issues) is presented for linear equations in section 2. The introduction of sharpening methods in nonlinear systems is evoked in section 3. References are provided inside the text.

2 Sharpening methods for linear equations

Sharpening methods for linear equations use two important generic ideas: the first one is to use high order schemes, and it may seem paradoxical at first sight; the second idea is based on compression with nonlinear techniques; other strategies rely on the Glimm's scheme, on PDEs to represent the interface, or reconstruct locally as in the volume of fluid (VOF) method. Most of the ideas can be presented on the advection equation with velocity $u \in \mathbb{R}$, which serves as a model problem. It writes

$$\partial_t c(t,x) + u\partial_x c(t,x) = 0, \quad x \in \mathbb{R}, \quad t > 0,$$

together with a Cauchy datum $c(0,x) = c^{\rm ini}(x)$.

2.1 High order methods

References to high order discretization of nonlinear equations are [Tor97, RTT08, TT07, TT05]. The fact that high order methods have the ability to sharpen discontinuities is kind of a paradox. Indeed, local Taylor expansions show bad convergence behavior for profiles involving discontinuities or strong gradients.

We give hereafter a simple explanation of the corresponding sharpening based on the theory of linear Strang's stencils. Let $\Delta t > 0$ and Δx be respectively the time and space steps. We consider a series of instants $t^n = n\Delta t$ and the classical discretization of the real line into intervals $[x_{j-1/2}, x_{j+1/2}]$, whereby $x_j = j\Delta x$ and $x_{j+1/2} = (j+1/2)\Delta x$. We note c_j^n an approximation of c at instant t^n within the cell $[x_{j-1/2}, x_{j+1/2}]$ and set $c^n = (c_j^n)$. The initial numerical datum can be taken as $c_j^0 = c^{ini}(x_j)$ (this is especially done when dealing with smooth solutions and high order methods) or $c_j^0 = \int_{x_{j-1/2}}^{x_{j+1/2}} c^{ini}(x)\,dx/\Delta x$ (usually when dealing with non-smooth data). The analysis is here limited to explicit and compact schemes with a stencil of $p+1$ contiguous cells. In a simplified finite difference form on a Cartesian grid, the family of linear schemes may read

$$c_j^{n+1} = \sum_{r=k-p}^{k} \alpha_r c_{j+r}^n, \qquad \alpha_r = \alpha_r(\nu). \tag{1}$$

The coefficients α_r are functions of the Courant-Friedrichs-Lewy (CFL) number $\nu = u\Delta t/\Delta x$. It is possible to write a scheme with order p in time and space using (1). Once p has been chosen, k determines the shift of the scheme. Basic examples are the well-known upwind scheme $c_j^{n+1} = (1-\nu)c_j^n + \nu c_{j-1}^n$, when $(p,k) = (1,0)$, $\alpha_{-1} = \nu$ and $\alpha_0 = 1-\nu$, the Lax-Wendroff scheme [LW60]

$c_j^{n+1} = H^{\mathrm{LW}}(c^n)_j = (1-\nu^2)c_j^n + \frac{\nu+\nu^2}{2}c_{j-1}^n + \frac{\nu^2-\nu}{2}c_{j+1}^n$, with $(p,k) = (2,1)$, $\alpha_{-1} = (\nu^2+\nu)/2$, $\alpha_0 = 1-\nu^2$ and $\alpha_1 = (\nu^2+\nu)/2$, and the Beam-Warming scheme [WB76]: $c_j^{n+1} = H^{\mathrm{BW}}(c^n)_j = \left(1 - \frac{3}{2}\nu + \frac{1}{2}\nu^2\right)c_j^n + (2\nu-\nu^2)c_{j-1}^n + \frac{\nu^2-\nu}{2}c_{j-2}^n$, with $(p,k) = (2,0)$, $\alpha_{-2} = (\nu^2-\nu)/2$, $\alpha_{-1} = 2\nu-\nu^2$ and $\alpha_0 = 1-3\nu/2+\nu^2/2$. Under the hypothesis that $\sum_r \alpha_r = 1$, which is a natural assumption that ensures the conservativity of the algorithm, these schemes may be rewritten also as finite volume methods in their classical form

$$\frac{c_j^{n+1} - c_j^n}{\Delta t} + u\frac{c_{j+\frac{1}{2}}^n - c_{j-\frac{1}{2}}^n}{\Delta x} = 0. \tag{2}$$

The conversion between the two forms is let to the reader because it does not have impact on the following discussion. A third order in time and space $O3$ scheme $(p,k) = (3,1)$ is defined by a convex combination [Des08] of the Lax-Wendroff scheme and the Beam-Warming scheme: $c_j^{n+1} = (1-\alpha)H^{\mathrm{LW}}(c^n)_j + \alpha H^{\mathrm{BW}}(c^n)_j$ with $\alpha = \frac{1+\nu}{3}$. The seminal works of Iserles and Strang [Str68, IS83] show that the order in time and space, p, can be arbitrary large. Nevertheless, *the only pairs (p,k) for which there exists schemes such that the l^2 norm is non-increasing at any iterate for all $\nu \leq 1$ are $p = 2k+1$, $p = 2k$ and $p = 2k+2$.* In the following, we call IS-schemes such schemes.

The stability in L^1 of IS-schemes has been given in [Des09]: *Assume moreover the order is odd, that is $p = 2k+1$. Then the scheme is stable in all L^q: there exists a constant $D_p > 0$ such that $||c^n||_{L^q} \leq D_p||c^0||_{L^q}$ $\forall n$, $\forall \nu \in]0,1]$, $\forall c^0$ and $\forall q \in [1,\infty]$.*

Equipped with these fundamental results, a convergence result that provides a sharp convergence estimate for an initial datum with bounded variation (BV datum) can be stated [Des08]. The proof is done by regularization of the BV profile and use of the L^1 stability. In this result, c^n is to be understood as the constant by cell function that takes the value c_j^n in the cell number j, namely $[(j-1/2)\Delta x, (j+1/2)\Delta x)$.

Theorem 1. *Assume $c^{\mathrm{ini}} \in L^\infty \cap BV$ (in space dimension 1, this is just the BV space). Consider an IS-scheme, with $p = 2k+1$ odd. Assume $\nu \leq 1$. Then*

$$||c^n - c(n\Delta t)||_{L^1} \leq C_p|c^{\mathrm{ini}}|_{BV}\left(\Delta x^a T^b + \Delta x\right) \tag{3}$$

with $a = \frac{p}{p+1}$ and $b = \frac{1}{p+1}$.

Here, as the estimate is for non-smooth data and thus is of order less than 1, the initial numerical datum can be chosen both as point values or mean values.

Using very high order schemes means choosing p very large. In this case $\frac{p}{p+1}$ is very close to 1. This is optimal because an error of order 1 is what we get by a 1 cell translation of the Heavyside function. In a nutshell: very high odd order advection schemes have nearly optimal order of convergence in L^1 even for discontinuous initial data. It means that the very high order feature of such schemes is able to sharpen discrete profiles with strong gradients. Perhaps even more important for applications is the very small dependence with respect to the time T since $\frac{1}{p+1}$ is close to zero for large p. This means that the difference between the true solution and the numerical solution does not evolve significantly in time. That is the sharpening effect is time independent. This theoretical behavior is the solution of the apparent paradox explained at the beginning of the section.

Nevertheless the drawbacks of these high orders (linear) FV methods is that they do not satisfy the maximum principle, according to a well-known theorem by Godunov.

2.2 Compression within a BV setting

It is known since [Har84, YWH84] and [lR77] that the bounded variation (BV) setting is a convenient framework of the construction of numerical nonlinear FV schemes with good sharpening properties. For one-dimensional problems the BV setting is strongly related to the preservation of the maximum principle [Roe85, Swe84, Swe85]. In some cases the sharpening effect is so pronounced that it is called squaring.

The general situation can be explained a follows. Consider the FV formulation

$$\frac{c_j^{n+1}-c_j^n}{\Delta t}+u\frac{c_{j+\frac{1}{2}}^n-c_{j-\frac{1}{2}}^n}{\Delta x}=0, \qquad u>0. \tag{4}$$

Scheme (4) can be recast into

$$c_j^{n+1}=c_j^n-\nu\left(c_{j+\frac{1}{2}}^n-c_{j-\frac{1}{2}}^n\right), \quad \nu=u\frac{\Delta t}{\Delta x}. \tag{5}$$

The numerical fluxes $c_{j+\frac{1}{2}}^n$ are yet to be defined at this point of the construction. The design principle is to impose the maximum principle under the form

$$\min(c_j^n,c_{j-1}^n)\equiv m_{j-1/2}^n\le c_j^{n+1}\le M_{j-1/2}^n\equiv\max(c_j^n,c_{j-1}^n), \tag{6}$$

which is legitimate for advection to the right. If the advection is to the left ($u<0$) one takes $\min(c_j^n,c_{j+1}^n)\equiv m_{j+1/2}^n\le c_j^{n+1}\le M_{j+1/2}^n\equiv\max(c_j^n,c_{j+1}^n)$. We consider the classical formula (see [Swe84])

$$c_{j+\frac{1}{2}}^n=c_j^n+\frac{1}{2}(1-\nu)(c_{j+1}^n-c_j^n)\varphi_{j+\frac{1}{2}}^n, \quad \forall j. \tag{7}$$

where the correction factor $\varphi_{j+\frac{1}{2}}^n$ is a limiter or slope limiter. It is usually defined as a function of the local slope ratio

$$\varphi_{j+\frac{1}{2}}^n=\varphi(r_{j+\frac{1}{2}}^n), \quad r_{j+\frac{1}{2}}^n=\frac{c_j^n-c_{j-1}^n}{c_{j+1}^n-c_j^n}.$$

There are natural additional constraints for the definition of the slope limiter. A first one writes $\varphi(1)=1$: it gives back the Lax-Wendroff flux in case $r=1$, and, generally, the second order when the datum is smooth. A second constraint can be $\varphi(r)=0$ for any $r\le 0$: this is a way to enforce a local preservation of the maximum principle, this is explained in the classical textbook [Tor97]. Another idea could be to add $\varphi(r)=r\varphi\left(\frac{1}{r}\right)$ [Tor97]. There are variants where these conditions are relaxed, see for example [DM96, SST15].

Most of the formulas published in the literature make use of the minmod function. Its value is given as follows: if $ab\le 0$ then $\mathrm{minmod(a,b)}=0$; if $a>0$ and $b>0$, then $\mathrm{minmod(a,b)}=\min(\mathrm{a,b})$; if $a<0$ and $b<0$, then $\mathrm{minmod(a,b)}=\max(\mathrm{a,b})$. Then the multidimensional function minmod : $\mathbb{R}^{\mathrm{p}}\to\mathbb{R}$ is defined recursively for $p\ge 2$ independently of the ordering by

$$\mathrm{minmod}(\mathbf{a})=\mathrm{minmod}(\mathrm{minmod}(\mathbf{b}),\mathrm{c})) \quad \text{for } \mathbf{a}=(\mathbf{b},\mathrm{c})\in\mathbb{R}^{\mathrm{p}},\ \ \mathbf{b}\in\mathbb{R}^{\mathrm{p}-1},\ \mathrm{c}\in\mathbb{R}.$$

A first classical result is that if the slope limiter satisfies

$$0 \leq \varphi(r) \leq 2\text{minmod}\,(1, \text{r}) \tag{8}$$

then the scheme (4) with the flux (7) satisfies the maximum principle. Even if this is a very classical result, we provide a proof since it will explain how to modify (8) for deriving schemes with even stronger sharpening effect. One has

$$\begin{aligned} c_j^{n+1} &= c_j^n - \nu \left(c_j^n + \frac{1}{2}(1-\nu)(c_{j+1}^n - c_j^n)\varphi_{j+\frac{1}{2}}^n \right. \\ &\qquad \left. - c_{j-1}^n - \frac{1}{2}(1-\nu)(c_j^n - c_{j-1}^n)\varphi_{j-\frac{1}{2}}^n \right) \\ &= c_j^n - \nu \left(1 + \frac{1}{2}(1-\nu) \left(\frac{\varphi_{j+\frac{1}{2}}^n}{r_{j+\frac{1}{2}}^n} - \varphi_{j-\frac{1}{2}}^n \right) \right) (c_j^n - c_{j-1}^n), \end{aligned}$$

that is $c_j^{n+1} = (1 - L_j^n)c_j^n + L_j^n c_{j-1}^n, \quad L_j^n = \nu + \frac{\nu(1-\nu)}{2} \left(\frac{\varphi_{j+\frac{1}{2}}^n}{r_{j+\frac{1}{2}}^n} - \varphi_{j-\frac{1}{2}}^n \right)$. The maximum principle is satisfied provided $0 \leq L_j^n \leq 1$, that is

$$0 \leq \nu + \frac{\nu(1-\nu)}{2} \left(\frac{\varphi_{j+\frac{1}{2}}^n}{r_{j+\frac{1}{2}}^n} - \varphi_{j-\frac{1}{2}}^n \right) \leq 1.$$

Assume (8) holds. Then $0 \leq \varphi_{j-\frac{1}{2}}^n \leq 2$ and $1 - \frac{1-\nu}{2}\varphi_{j-\frac{1}{2}}^n \geq 1 - (1-\nu) \geq 0$, thus $0 \leq C_j^n$. One notices that (8) also yields $0 \leq \varphi_{j+\frac{1}{2}}^n \leq 2r_{j+\frac{1}{2}}^n$. Therefore $1 + \frac{1-\nu}{2} \frac{\varphi_{j+\frac{1}{2}}^n}{r_{j+\frac{1}{2}}^n} \leq 1 + (1-\nu) = 2 - \nu$. Finally

$$\nu + \frac{\nu(1-\nu)}{2} \frac{\varphi_{j+\frac{1}{2}}^n}{r_{j+\frac{1}{2}}^n} \leq 2\nu - \nu^2 \leq 1, \quad \forall \nu \in [0,1],$$

which ends the proof.

A huge number of formulas has been proposed in the literature. We just review the most usual ones. The Minmod flux writes

$$\varphi(r) = \text{minmod}(1, \text{r}). \tag{9}$$

The Superbee flux writes

$$\varphi(r) = \max(0, \min(1, 2r), \min(2, r)). \tag{10}$$

Remark 2 (Squaring/sharpening behavior of Superbee)**.** *The notion of sharpening is not present at this stage of the discussion. It is introduced by noticing that the SuperBee limiter is squaring. This has been reported in the literature in [Tor97] and many other texts. Squaring means that if an initial smooth profile is chosen, for example in the form of a Gaussian, then the numerical solution has the tendency to converge to a mass preserving square profile for $t \to \infty$. This behavior necessarily increases the L^2 norm of the profile.*

Squaring is usually considered as a consequence of the strong nonlinearity of SuperBee. Even if it is a well documented behavior, we know of no definitive proof. But on the contrary, it is easy to understand that the minmod limiter cannot sharpen. To this end we consider the semi-discrete (that is continuous in time) version of the scheme

$$\frac{\mathrm{d}}{\mathrm{d}t}c_j(t) + u\frac{c_{j+\frac{1}{2}}(t) - c_{j-\frac{1}{2}}(t)}{\Delta x} = 0, \quad j \in \mathbb{Z}. \tag{11}$$

Since Δt vanishes, the flux (7) is simplified taking $\nu = 0$.

Lemma 3. *The semi-discrete scheme (11) with the flux $c_{j+\frac{1}{2}} = c_j + \frac{1}{2}(c_{j+1} - c_j)\varphi_{j+\frac{1}{2}}$ and the Minmod limiter (9) satisfies the a priori estimate*

$$\frac{d}{dt}\left(\sum_{j\in\mathbb{Z}} |c_j(t)|^2\right) \leq 0. \tag{12}$$

So, as a corollary of remark 2, this scheme cannot sharpen.

The same property holds for similar schemes with a limiter $0 \leq \varphi(r) \leq 1$ for all r. The proof proceeds as follows. One has

$$\frac{\Delta x}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{j\in\mathbb{Z}} |c_j|^2\right) = \Delta x \sum_j c_j \frac{\mathrm{d}}{\mathrm{d}t}c_j = -u\sum_j c_j\left(c_{j+\frac{1}{2}} - c_{j-\frac{1}{2}}\right)$$

$$= -u\sum_j c_j\left(c_j - c_{j-1}\right) - u\sum_j c_j\left(c_{j+\frac{1}{2}} - c_j\right) + u\sum_j c_j\left(c_{j-\frac{1}{2}} - c_{j-1}\right).$$

It is easy to check the identities

$$\begin{cases} \sum_j c_j\left(c_j - c_{j-1}\right) & = \frac{1}{2}\sum_j |c_j - c_{j-1}|^2, \\ \sum_j c_j\left(c_{j+\frac{1}{2}} - c_j\right) & = \frac{1}{2}\sum_j c_j(c_{j+1} - c_j)\varphi_{j+\frac{1}{2}}, \\ & = \frac{1}{2}\sum_j c_{j-1}(c_j - c_{j-1})\varphi_{j-\frac{1}{2}}, \\ \sum_j c_j\left(c_{j-\frac{1}{2}} - c_{j-1}\right) & = \frac{1}{2}\sum_j c_j(c_j - c_{j-1})\varphi_{j-\frac{1}{2}}. \end{cases}$$

Therefore by summation and rearrangements

$$\frac{\Delta x}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{j\in\mathbb{Z}} |c_j|^2\right) = -\frac{u}{2}\sum_j |c_j - c_{j-1}|^2\left(1 - \varphi_{j-\frac{1}{2}}\right) \leq 0,$$

which shows that the L^2 norm decreases. It makes squaring impossible. The proof is ended. It can be generalized to the fully discrete scheme with the same conclusion. A corollary is as follows.

Lemma 4 (Necessary condition for sharpening)**.** *A slope limiter that sharpens is necessarily such that $\varphi(r) > 1$ for some $r \in \mathbb{R}$. This condition is satisfied by the SuperBee formula (10), for which $\lim_{r\to\infty}\varphi(r) = 2$.*

2.3 Inequality and anti-diffusion

This sharpening strategy is more radical. It is naturally introduced in the context of BV schemes [DL01a], see also [Tor97], and has been adapted to ENO techniques [Shu09] in [XS06, XS05]. We refer to [SX14, CPT12, CM11, KL10, Shu09, JL07, XS06, XS05, BFK14, GLT13] for the use of such methods for different problems.

We shall note $m_{j+1/2} = \min(c_j^n, c_{j+1}^n)$, $M_{j+1/2} = \max(c_j^n, c_{j+1}^n)$

$$\lambda_{j+1/2} = \frac{\Delta x}{u\Delta t}(c_j^n - M_{j-1/2}) + M_{j-1/2}^n \text{ and } \Lambda_{j+1/2} = \frac{\Delta x}{u\Delta t}(c_j^n - m_{j-1/2}) + m_{j-1/2}^n. \tag{13}$$

We observe that $\lambda_{j+1/2} \leq \Lambda_{j+1/2}$ if the CFL condition $u\Delta t \leq \Delta x$ is satisfied. A basic property writes as follows.

Lemma 5. *Under CFL, the upwind flux choice $c_{j+1/2}^n = c_j^n$ belongs to the interval $[\lambda_{j+1/2}, \Lambda_{j+1/2}] \cap [m_{j+1/2}, M_{j+1/2}]$, which ultimately ensures the maximum principle [DL01a, XS06, XS05].*

In this context, one introduces compression, or sharpening, or anti-diffusion, by using the most extreme formulated choice. Let $\omega_{j+1/2}$ and $\Omega_{j+1/2}$ such that $[\omega_{j+1/2}, \Omega_{j+1/2}] = [\lambda_{j+1/2}, \Lambda_{j+1/2}] \cap [m_{j+1/2}, M_{j+1/2}]$. One obtains

$$c_{j+1/2}^n = \begin{cases} \Omega_{j+1/2}, & \text{if } \Omega_{j+1/2} \leq c_{i+1}^n, \\ c_{j+1}^n, & \text{if } \omega_{j+1/2} \leq c_{i+1}^n \leq \Omega_{j+1/2}, \\ \omega_{j+1/2}, & \text{if } c_{i+1}^n \leq \omega_{j+1/2}. \end{cases}$$

An equivalent definition (still for the case $u > 0$) is given in the following lemma.

Lemma 6. *The limited downwind flux defined above is equivalent to the so-called UltraBee flux limiter flux (see [Tor97]) defined as*

$$c_j^{n+1} = c_j^n - \nu(c_j^n - c_{j-1}^n) - \frac{\nu(1-\nu)}{2}(\varphi_{j+1/2}^n(c_{j+1}^n - c_j^n) - \varphi_{j-1/2}^n(c_j^n - c_{j-1}^n))$$

with $\varphi_{j+1/2}^n = \varphi(r_{j+1/2}^n, \nu)$ and $\varphi(r, \nu) = minmod(\frac{2r}{\nu}, \frac{2}{1-\nu})$.

The limiter is now function of the slope r and of the Courant number ν. The scheme is called *limited downwind* in the following.

Lemma 7. *This limited downwind scheme is exact for step initial conditions [DL01a].*

Confirmation is by starting from an initial data which is not a step function, but a (discretized) smooth function. One observes (under a surprising technical condition CFL$\neq 1/2$) that the smooth profile is replaced a step function close by a step function with an approximation error is $O(\Delta x)$. After that first stage the step function is perfectly transported. So in some sense the UltraBee limiter is a perfect sharpener. The sharpening effect is so pronounced that it may resemble an instability, but it is not.

This technique was incorporated in FV algorithms for the simulation of two-component fluid flows, for the mass fraction, volume fraction, or color function of components, in, e.g., [DL07], [KL10], and extended to *multi*-component in [JL07] and [BFK14].

This was also modified to apply to non-linear discontinuities such as classical shocks, in [AC16], and non-classical shocks in the scalar context, [BCLL08], and in the context of systems in [Agu16].

2.4 Glimm's method

At this stage of the discussion the problem is the following: either one accepts to violate the maximum principle although this can be very critical, for example when the transported unknown is the mass or volume fraction of a fluid in a multi-component flow, or one has to use linear first order or a non-linear scheme (see Section 2.2). Yet there exists an alternative, that was first proposed by Glimm, in [Gli65] for theoretical analysis purposes. This method avoids the numerical diffusion of first order stable schemes because it does not involve any "projection" on the mesh, and it does not create new values of the solution a each time step in the case of linear transport.

To describe it briefly, let us consider once again the upwind scheme written as $c_j^{n+1} = (1-\nu)c_j^n + \nu c_{j-1}^n$ with ν the CFL number. The smearing of the profiles comes from the (strictly) convex combination that appears in the formula. This scheme can be interpreted as a two-step scheme: exact transport of the profile for a time Δt, and then projection on the mesh (the upwind scheme is the Godunov scheme). Glimm proposes to avoid the projection by taking one of the two values that are present in cell j after one time step: c_{j-1}^n or c_j^n. The choice is performed randomly: c_{j-1}^n is chosen with probability ν, and c_j^n is chosen with probability $1-\nu$. This interpretation is correct since $0 < 1 \leq \nu$ (resp. $0 \leq 1-\nu < 1$) under CFL. In the more general context of nonlinear problems, the algorithm is based on the resolution of the Riemann problems at each interface and on the choice of a random variable δ^n(different from one time step to the other), chosen according to the uniform law between 0 and Δx. Then the updated value of the unknown in the cell j is defined by taking the value of the solution of the Riemann problem at time Δt at position $(j-1/2)\Delta x + \delta^n$. This was shown by Glimm to converge, with probability 1, and it is clear that it does not *smear* profiles, at least when the profile is a step and in the linear context. Let us note that this random procedure has the drawback that the scheme is non-conservative, however this does not prevent the scheme to converge to the entropy solution for nonlinear problems. Note also that the randomness is not mandatory: the only property that is required for $(\delta^n)_n$ is that it is an equidistributed (with low discrepancy) sequence. The Van der Corput sequence, which is such a deterministic sequence, is shown to give qualitatively very good (better than a random sequence) results in [Col82].

One can notice that, for the linear transport equation (1), the upwind scheme is the *expectation* of Glimm's scheme. This observation was used to prove error estimates for the upwind scheme on general meshes, using central-limit type estimates, in [DL11].

In space dimension 1 and in the context of linearly degenerate fields (which correspond to material discontinuities) an FV algorithm based on a Lagrange-remap (formulated as Lagrange-transport) strategy with a random sampling technique for the transport part, for the simulation of two-component compressible fluid flows is derived in [CC12], with very good efficiency. See also [Cha07, CG08, BHJ$^+$13].

In the more particular context of *non-linear* material discontinuities that are

present in some viscous-dispersive limits of systems with fields that are neither genuinely non-linear nor linearly degenerate, with so-called *non-classical shocks* (see [LeF02]), the random choice method was shown in [CL03] to give very good (and convergent) results, which is very difficult in this context.

The tentation to use such a scheme in higher dimension is great, but it is known since Chorin ([Cho76]) that it is not satisfactory for genuinely nonlinear conservation laws. Colella proposed in [Col82] a modification of the random choice algorithm, that involves the Godunov method and that seem to be convergent. Unfortunately this modification of Glimm's algorithm does not allow to preserve sharp fronts. Nevertheless, for linear or linearly degenerate fields, this random choice procedure shows great efficiency, at least on Cartesian grids. This has been investigated and analyzed, for example in [HJ14] and [HJ13].

2.5 PDE models and sharpening methods

Level sets methods are discussed in [OF03, OS88, Set96]. This is a very popular set of numerical methods for interface modeling that has been applied to many problems. In the present context, the idea is to rely on a partial differential equation to transport a color function (our definition of a color function f is that it takes value in $[0,1]$, so that: if a point x is such that $f(x)=0$ has color equal to 0; if $f(x)=1$ then x has a color equal to 1; and finally $0<f(x)<1$ corresponds to intermediate colors). No colors below 0 and above 1 are considered in this presentation, but it is not mandatory. A typical elementary question with the level set approach is about the influence of the numerical parameters on the level set. In certain cases the answer is that the method can be insensitive to this parameters.

To understand this property we consider the simplest color function at initial time $c^{\rm ini}(x)=H(x)$, that is $c^{\rm ini}(x)=0$ for $x<0$ and $c_0(x)=1$ for $x>1$. Instead of manipulating the upwind first order scheme, we use its modified equation (that is to say, the PDE it is consistent with at the *second* order in time and space). We thus consider the function c_μ solution of the advection equation with viscosity

$$\partial_t c_\mu + u\partial_x c_\mu = \mu \partial_{xx} c_\mu, \quad \mu = \frac{\Delta x}{2}(1-\nu),$$

where $0\le\nu\le 1$ is the CFL number. The modified equation is a second order approximation of the upwind scheme. The interface is recovered at any time t by as the $1/2$ level set $\Gamma_\mu(t)=x$ such that $c_\mu(x,t)=1/2$. It is easy to prove that x exists and is unique for $t>0$ and $0\le\nu<1$: this is a consequence of well known integral representation formula detailed below. One has more.

Lemma 8. *For all $t>0$ and $0\le\nu<1$, the $1/2$ level set is exact: that is $\Gamma_\mu(t)=ut$.*

One has with the fundamental solution of the heat equation

$$c_\mu(x,t)=\frac{1}{\sqrt{4\pi\mu t}}\int_{\mathbb{R}}\exp\left(-(x-y-ut)^2/(4\mu t)\right)H(y)dy.$$

So

$$c_\mu(ut,t)=\frac{1}{\sqrt{4\pi\mu t}}\int_0^\infty \exp\left(-y^2/(4\mu t)\right)dy=\frac{1}{\sqrt{2\pi}}\int_0^\infty\exp\left(-y^2/2\right)dy=\frac{1}{2}.$$

Even if this argument is very elementary, it explains that level set methods have the ability to predict the interface with great accuracy, even if the underlying scheme for the transport of the color function is low order. In the context of this review paper, it is perfect sharpening.

2.6 Nature of the grid/mesh

The discussion so far was restricted to one dimensional grids. The extension of the previous FV algorithms to general multidimensional grids poses two fundamental difficulties.

A first one is that sharpening techniques are highly nonlinear methods. A good sharpening technique is in practice equipped with a method which controls the oscillations due to strong nonlinear interactions. In dimension $d = 1$, this principle is mostly based on the BV setting. The issue is that this bound on the total variation is lost in dimension $D = 2$ and greater. This has been proved in a famous article [GL85] on a Cartesian grid. This unfortunate situation has the consequence that the preservation of the maximum principle does not yield a control of some special oscillations which develop mostly tangentially to the isolines of the exact profiles: an important reference in this direction is the series [TK05, KM05b, KM05a]. See also [DLLM10].

A confirmation of this behavior is the 2D algorithm in [DL01a]. It is shown that the extension of the Ultra-Bee scheme with directional splitting is exact for squares. But unfortunately this algorithm is not equipped with a control of 2D variations. It can be interpreted as a distant consequence of the [GL85] theorem. In consequence this algorithm is useless for calculations of profiles with values which are not exactly 0 and or 1. Even if the initial data is an indicatrix function, its boundary is not necessarily a 2D step function: in this situation one observes oscillations at the boundary between 0 and 1: these oscillations are perfectly bounded in L^∞ norm because directional splitting preserves the maximum principle; but they are not bounded in the BV semi-norm because the BV semi-norm is a global quantity destroyed by directional splitting. An attempt is been made in [DL01b] to overcome this failure, but the numerical results are deceptive (not published), probably due to the curse explained by the [GL85] theorem.

2.7 Interface reconstruction and VOF

The simple line interface calculation (SLIC) [NW76] is an extremely popular method that presents a nearly all purpose methodology for FV interface sharpening. The design principle of SLIC is to reconstruct parallel and/or anti parallel perfect interfaces in Cartesian cells from the knowledge of volume fractions. In dimension 1, for a step initial conidtion, it is equivalent to the limited downwind scheme (that can be seen as a reconstruction algorithm, where the reconstructed solution is a step function in every cell). Even if it is an extremely simple method, the results are quite good when comparing with the implementation cost and run time. This is probably the reason why it is still a reference. With respect to SLIC, the volume of fluid (VOF) [HN81] has the huge advantage to reconstruct interface with any direction. Even without discussing the simplicity of the method, it is clear that this information is a kind of first order interface

reconstruction while SLIC can be considered as a zeroth order interface reconstruction. Another feature of VOF is that the normal direction of the interface is computed from the discrete gradient of some volume fractions. It is possible to optimize the performance of VOF by changing the parameters of the discrete gradient operator and of the method used to evolve the volume fractions.

It must be noted that SLIC is not PDE based and VOF is only partially PDE based. In consequence it is not really possible to perform a convergence analysis of the algorithms, but only on parts of them. The Youngs algorithm [You84] has a similar nature.

2.8 Vofire

We give some details of the Vofire method, which is a multidimensional nonlinear FV scheme. The geometrical idea relies on the following observation: in dimension greater than 2, the numerical diffusion can be decomposed into two different diffusions: the longitudinal diffusion, along the velocity field, which is typically one-dimensional, and the transverse diffusion, which is really due to the fact that the mesh is multi-dimensional. This distinction between the two phenomena could appear arbitrary, but is in accordance with basic numerical tests. Consider for example an initial condition which is the characteristic function of the square $]0.25, 0.75[\times]0.25, 0.75[$. This profile is advected with the upwind scheme. The velocity direction $\mathbf{u}$ has a great influence on the result. It is illustrated on figure 1.

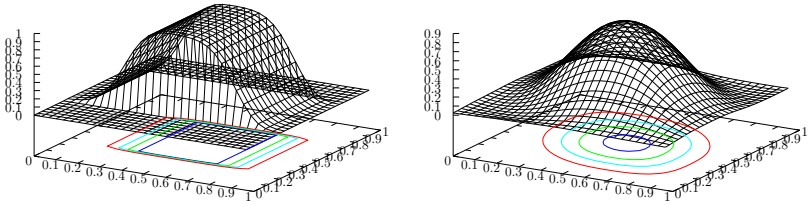

Figure 1: Upwind scheme. The initial condition is the characteristic function of a square. Final time $t = 1$. Periodic boundary conditions. On the left: the velocity $\mathbf{u} = (1, 0)^T$ is aligned with the mesh; the result displays only longitudinal diffusion. On the right: the velocity $\mathbf{u} = (1, 1)^T$ is not aligned with the mesh. The consequence is that there is both longitudinal and transverse diffusion.

We here propose to restrict to triangular meshes, on which it is simpler to expose the Vofire technique. Thus we consider the following type of mesh structure:

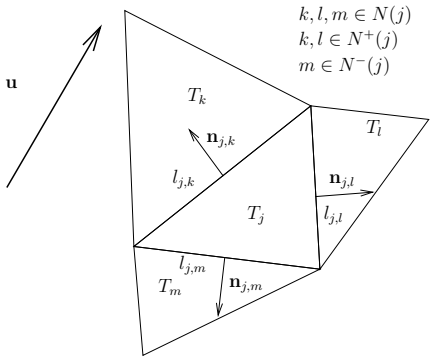

Figure 2: Mesh and notations (for the sake of brievity, these notations will not be explained further in the text, as they are very classical).

The idea to "fight" against these two diffusion phenomena is to use, as for the limited downwind seen as a (discontinuous) reconstruction scheme, a reconstruction procedure in every cell. This reconstruction will be here two-fold: it will consist in a first reconstruction that will be transverse, and in a second one that will be done along the velocity field. The velocity field $\mathbf{u}$ is constant for simplicity, but this assumption can be removed. Note that the transverse diffusion actually depends more on the shape of the cells than on the velocity, so that this assumption of constant velocity is not so much restrictive here. For a given celle T_j, $N^-(j)$ denotes the set of cells that are adjacent to T_j and *upwind*: $N^-(j) = \{T_m \text{ such that } \overline{T_j} \cap \overline{T_m} \text{ is of non-zero one dimensional Lebesgue measure and } u \cdot n_{j,m} < 0\}$, and $N^+(j)$ denotes the set of downwind cells to T_j. As in dimension 1, the fundamental requirement of the scheme is that it satisfies an upwind maximum principle:

$$\min\left(c_j^n, \min_{k\in N^-(j)} c_k^n\right) \leq c_j^{n+1} \leq \max\left(c_j^n, \max_{k\in N^-(j)} c_k^n\right) \quad \text{for any } j.$$

The most important part of the procedure, regarding the multidimensional properties of the scheme, is the first one, that concerns the transverse reconstruction. As we will see, after this reconstruction, the algorithm will be one-dimensional, and one-dimensional techniques (such as the limited downwind scheme for instance) will be applied.

Recall that, for expository purposes, the mesh is assumed to be made with triangles, in dimension 2. The transverse reconstruction consists in breaking a cell in two parts by a segment parallel to the velocity, and modifying the value of the unknown in each of these two sub-cells. Each triangle T_j has at least one downwind neighbor and at most two. If it has only one downwind neighbor, we do not perform the transverse reconstruction (we do not cut the cell). This can be explained by the fact that when there is only one downwind neighbor, the "information" contained in the cell is not spread transversally by any scheme (with small stencil). Let us thus assume that T_j has two downwind neighbors, T_k and T_l. It has then one upwind neighbor, T_m. We consider the intersection point of the two edges relative to the downwind neighbors and cut T_j along the line passing on this intersection point and parallel to $\mathbf{u}$. The two sub-cells are

denoted $T_{j,k}$ and $T_{j,l}$: $T_{j,k}$ has T_k as (unique) downwind neighbor, and $T_{j,l}$ has T_l as (unique) downwind neighbor. This partitioning is illustrated on figure 3.

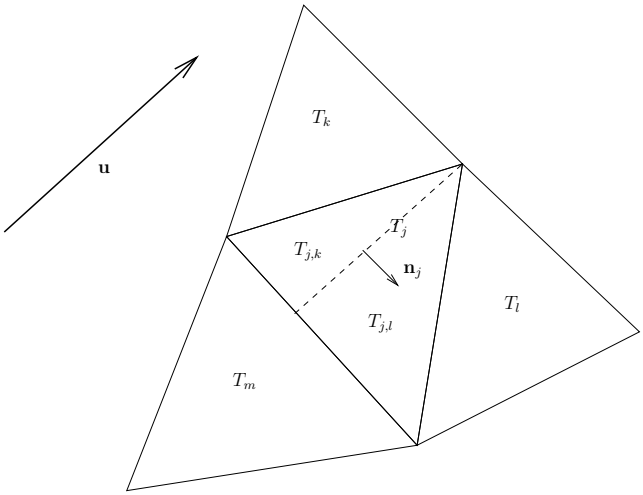

Figure 3: Transverse reconstruction.

The essential property of this cutting is that now every subcell has exactly one downwind and one upwind neighbor, as will be used below: this is due to the fact that the new normal vector $\mathbf{n}_j$ (see Figure 3) is orthogonal to the velocity, so that there will be no flux through the new interface. We use symbols $s_{j,k}$ and $s_{j,l}$ to denote the areas of sub-cells $T_{j,k}$ and $T_{j,l}$ respectively. Clearly $s_{j,k} + s_{j,l} = s_j$ and $s_{j,k} > 0$ and $s_{j,l} > 0$. The aim is to define a reconstructed value $c^R_{j,k}$ in $T_{j,k}$ and a reconstructed value $c^R_{j,l}$ in $T_{j,l}$. We impose

$$s_{j,k}c^R_{j,k} + s_{j,l}c^R_{j,l} = s_j c^n_j \tag{14}$$

to guarantee the local conservativity. Let us write

$$\left\{ \begin{array}{l} c^R_{j,k} = c^n_j + \lambda_{j,k}\left(c^n_k - c^n_j\right), \quad 0 \leq \lambda_{j,k} \leq 1, \\ c^R_{j,l} = c^n_j + \lambda_{j,l}\left(c^n_l - c^n_j\right), \quad 0 \leq \lambda_{j,l} \leq 1, \end{array} \right. \tag{15}$$

which means that $c^R_{j,k}$ and $c^R_{j,l}$ must satisfy a condition of local consistency. We introduce the idea of anti-dissipative schemes which will serve to find a unique value of $\lambda_{j,k}$ and $\lambda_{j,l}$.

Remark 9. *Our objective is to choose $\lambda_{j,k}$ and $\lambda_{j,l}$ in order to obtain an anti-dissipative scheme, with a very low level of numerical diffusion. This is the reason why we seek the largest possible $\lambda_{j,k}$ and $\lambda_{j,l}$ in the interval $[0, 1]$. This is the same principle as in section 2.3. But of course we cannot take $\lambda_{j,k} = \lambda_{j,l} = 1$ directly because we ask the reconstruction to be conservative: see Equation (14).*

So our goal is to have the largest $\lambda_{j,k}$ and $\lambda_{j,l}$ in the interval $[0, 1]$, but still satisfying the maximum principle. Equation (14) can be rewritten as $s_{j,k}c^R_{j,k} + s_{j,l}c^R_{j,l} - s_j c^n_j = s_{j,k}\left(c^R_{j,k} - c^n_j\right) + s_{j,l}\left(c^R_{j,l} - c^n_j\right) = 0$, that is

$$\left[s_{j,k}\left(c^n_k - c^n_j\right)\right]\lambda_{j,k} + \left[s_{j,l}\left(c^n_l - c^n_j\right)\right]\lambda_{j,l} = 0.$$

As we will see, either the data c_j^n is a local transverse maximum or minimum and then (14) implies $\lambda_{j,l} = \lambda_{j,k} = 0$ (it means there is actually no reconstruction), or the data is transversally monotone, and then at least $\lambda_{j,l} = 1$ or $\lambda_{j,k} = 1$. The solution is computed as follows.

1) If $s_{j,k}\left(c_k^n - c_j^n\right) s_{j,l}\left(c_l^n - c_j^n\right) \geq 0$**,** c_j^n is a local extremum in the transverse direction. Then we do not reconstruct, which means $\lambda_{j,k} = \lambda_{j,l} = 0$ and

$$c_{j,l}^R = c_{j,k}^R = c_j^n. \tag{16}$$

2) If $-\dfrac{s_{j,k}\left(c_k^n - c_j^n\right)}{s_{j,l}\left(c_l^n - c_j^n\right)} > 1$**,** the solution is obtained by taking $\lambda_{j,l} = 1$,

$$c_{j,l}^R = c_l^n, \qquad c_{j,k}^R = c_j^n - \frac{s_{j,l}}{s_{j,k}}\left(c_l^n - c_j^n\right) = (s_j c_j^n - s_{j,l} c_l^n)/s_{j,k}. \tag{17}$$

3) If $-\dfrac{s_{j,k}\left(c_k^n - c_j^n\right)}{s_{j,l}\left(c_l^n - c_j^n\right)} < 1$**,** the solution is obtained by taking $\lambda_{j,k} = 1$,

$$c_{j,k}^R = c_k^n, \qquad c_{j,l}^R = c_j^n - \frac{s_{j,k}}{s_{j,l}}\left(c_k^n - c_j^n\right) = (s_j c_j^n - s_{j,k} c_k^n)/s_{j,l}. \tag{18}$$

As the situation is now one-dimensional for each subcell T_{jk} and T_{jl}, one can analyze the scheme where the first stage is this reconstruction followed by a second stage which is the upwind scheme. It is obvious that this scheme will provide the maximum principle, as the reconstruction does. What is not so obvious is that the CFL stability condition for the upwind scheme on this new (and finer) mesh *is the same as for the initial mesh.* A simple proof is as follows.

Proof. The reconstructed quantities (15) respect the maximum principle. By construction the scheme is equal to a two-steps algorithm: first step, use the upwind scheme for a mesh which is locally cut in smaller cells, as it is described in figure 3, and with cell quantities equal to the reconstructed quantities; second step, project onto the original coarse mesh. Therefore it is sufficient to check that the CFL condition is the same for the original mesh (4 cells in figure 3) and for the new mesh (5 cells in figure 3).

Since $\mathbf{u}$ is constant, then $\sum_{k\in N^+(j)} l_{j,k}(\mathbf{u}^T\mathbf{n}_{j,k}) = -\sum_{k\in N^-(j)} l_{j,k}(\mathbf{u}^T\mathbf{n}_{j,k})$. The standard CFL condition for the upwind scheme for the cell T_j thus has the form $\frac{\Delta t}{s_j}\sum_{k\in N^+(j)} l_{j,k}(\mathbf{u}^T\mathbf{n}_{j,k}) \leq 1$, that is

$$\frac{\Delta t}{s_j}\left(l_{j,k}(\mathbf{u}^T\mathbf{n}_{j,k}) + l_{j,l}(\mathbf{u}^T\mathbf{n}_{j,l})\right) \leq 1. \tag{19}$$

The CFL condition for the sub-cells $T_{j,k}$ and $T_{j,l}$ are respectively

$$\frac{\Delta t}{s_{j,k}} l_{j,k}(\mathbf{u}^T\mathbf{n}_{j,k}) \leq 1 \quad \text{and} \quad \frac{\Delta t}{s_{j,l}} l_{j,l}(\mathbf{u}^T\mathbf{n}_{j,l}) \leq 1. \tag{20}$$

Let $l_j = \text{length}\left(\overline{T_{j,k}} \cap \overline{T_{j,l}}\right)$ be the length of the segment separating $T_{j,k}$ and $T_{j,l}$. One has $s_{j,k} = \frac{l_j}{2|\mathbf{u}|} l_{j,k}(\mathbf{u}^T\mathbf{n}_{j,k})$ and $s_{j,l} = \frac{l_j}{2|\mathbf{u}|} l_{j,l}(\mathbf{u}^T\mathbf{n}_{j,l})$ and $s_j = s_{j,k} + s_{j,l} = \frac{l_j}{2|\mathbf{u}|}\left(l_{j,k}(\mathbf{u}^T\mathbf{n}_{j,k}) + l_{j,l}(\mathbf{u}^T\mathbf{n}_{j,l})\right)$. The two inequalities of (20) and inequality (19) thus rewrite $|\mathbf{u}|\frac{2\Delta t}{l_j} \leq 1$. So they are equivalent and the proof is ended.

Some modifications and improvements of the Vofire technique have been proposed in [BTVG10, MTF10] for example.

3 Coupling with hyperbolic nonlinear equations

An issue is to use the previous techniques in complex computational fluid dynamics FV codes with a respect of the key properties necessary for a correct simulation. Of course the notion of a correct simulation and the identification of the key ingredients of a correct coupling are far to be evident. We restrict the discussion to hyperbolic models for compressible fluids for which conservative issues are critical. Indeed the nonlinearity of the equations induce discontinuous solutions such as shocks and contact discontinuities: it is well accepted that the violation of the conservation is only at the cost of a strong deviation with respect to the solution of the Riemann problem (see [HL94] for a justification). These questions are fiercely debated when dealing with multi-fluid models.

There are common guidelines for incorporating sharpening techniques into discretization strategies of complex models. Usually one singles out transport effects in the system and update a set of key fluid parameters thanks to a numerical scheme that transports discontinuities as sharply as possible. A delicate matter is generally to preserve good stability and consistency properties of the overall numerical scheme.

3.1 An example of discretization for compressible flows with two components separated by a sharp interface

For the sake of illustrating these ideas, we consider, in space dimension 1, a simple model of compressible flows that involves two perfect gases that was studied in [Abg88, LF89]. The specific heat at constant volume and ratio of specific heat of the fluid $k = 1, 2$ are respectively $c_{vk} > 0$ and $\gamma_k > 1$. The density of the two-phase medium is noted ρ and $Y_1 = Y$ (resp. $Y_2 = 1 - Y$) is the mass fraction of the fluid $k = 1$ (resp. $k = 2$). We suppose that there is a thermal equilibrium between the gases and that the pressure P verifies Dalton's law, then we have

$$P = \frac{\sum_{k=1,2} Y_k(\gamma_k - 1)c_{vk}}{\sum_{k=1,2} Y_k c_{vk}} \rho e, \tag{21}$$

where e is the specific internal energy of the medium. One supposes that the components have the same velocity u and that no mass transfer occurs between the species. If one notes $\rho\mathbf{W} = [\rho Y, \rho u, \rho(e + u^2/2)]^T$, $\mathbf{T}(\mathbf{W}) = [0, 0, P, Pu]^T$ then the flow is governed by

$$\partial_t \rho + \partial_x(\rho u) = 0, \qquad \partial_t(\rho\mathbf{W}) + \partial_x(\rho\mathbf{W}u + \mathbf{T}) = 0. \tag{22}$$

System (22) is hyperbolic provided that $\gamma_k > 1$ and it is equipped with jump relations that enable the definition of weak solutions that verifies the transport equation

$$\partial_t Y + u\partial_x Y = 0. \tag{23}$$

Although this model is equipped with a mixture model (based on oversimplified assumptions), if one chooses an initial condition such that $Y(x, t = 0) \in \{0, 1\}$, then no physical mixing should occur in the domain as (23) guarantees that $Y(x, t) \in \{0, 1\}$ for $t > 0$. In this sense, (22) can be used as a model for a flows involving two compressible fluids separated by a sharp interface.

The decoupling between transport and other phenomenon can be achieved thanks to a Lagrange-Remap method. Let us note ξ the Lagrangian space

coordinate defined by $\mathrm{d}\xi(t;x_0)/\mathrm{d}t = u(\xi(t;x_0),t)$ with $\xi(t=0;x_0) = x_0$. If $(x,t) \mapsto b$ is any fluid parameter, we note $(\xi,t) \mapsto b^{\mathrm{Lag}}$ the Lagrangian field associated with b by $b^{\mathrm{Lag}}(\xi(t;x),t) = b(x,t)$. System (22) can be expressed in the so-called Lagrangian reference frame as follows

$$\rho^{\mathrm{Lag}}(\xi,0)\partial_t(1/\rho^{\mathrm{Lag}}) - \partial_\xi u^{\mathrm{Lag}} = 0, \qquad \rho^{\mathrm{Lag}}(\xi,0)\partial_t \mathbf{W}^{\mathrm{Lag}} + \partial_\xi \mathbf{T}^{\mathrm{Lag}} = 0. \quad (24)$$

Given a set of discrete values $(\rho,\rho\mathbf{W})_j^n$ that represent an approximation of the fluid state at instant $t = t^n$ within the cell i, the Lagrange-Remap method is a two-step algorithm [GR96, Des10]: first, we update the discrete unknowns to a value $(\rho,\rho\mathbf{W})_j^{\mathrm{Lag}}$ by approximating the solution of (24) over $[t^n, t^n + \Delta t]$. Let us remark that the evolution equation for Y expressed in (24) boils down to $\partial_t Y = 0$, therefore it is reasonable to expect that $Y_i^{\mathrm{Lag}} = Y_i^n$. The second step updates the fluid parameter to their values $(\rho,\rho\mathbf{W})_j^{n+1}$ by remapping the Lagrange values $(\rho,\rho\mathbf{W})_j^{\mathrm{Lag}}$ onto the Eulerian mesh. It can read as follows

$$\rho_j^{n+1} - \rho_j^n + \frac{\Delta t}{\Delta x}(\rho_{j+1/2}^{\mathrm{Lag}} u_{j+1/2}^n - \rho_{j-1/2}^{\mathrm{Lag}} u_{j-1/2}^n) = 0, \quad (25a)$$

$$(\rho\mathbf{W})_j^{n+1} - \rho_j^n \mathbf{W}_j^{\mathrm{Lag}} + \frac{\Delta t}{\Delta x}((\rho\mathbf{W})_{j+1/2}^{\mathrm{Lag}} u_{j+1/2}^n - (\rho\mathbf{W})_{j-1/2}^{\mathrm{Lag}} u_{j-1/2}^n) = 0. \quad (25b)$$

The values $u_{j-1/2}^n$ are approximations of the material velocity of the fluid at the cell interface $x = x_{j+1/2}$ that can be estimated with the discretization of (24). One can therefore consider that $u_{j-1/2}^n$ is known when performing (25). The only missing ingredient for achieving the remap procedure is thus the definition of the variable flux $(\rho,\rho\mathbf{W})_{j+1/2}^{\mathrm{Lag}} = [\rho,\rho Y,\rho u,\rho(e+u^2/2)]_{j+1/2}^{\mathrm{Lag}}$. For this problem it is clear that the anti-diffusive mechanism should concern the variable Y whose discontinuity carries the location of material interface between the fluids. Suppose given a definition for the fluxes $b_{j+1/2}^{\mathrm{Lag}}$ that is consistent for $b \in \{\rho,\rho u,\rho(e+u^2/2)\}$ and that $\rho_j^n > 0$ and $\rho_{j+1/2}^{\mathrm{Lag}} > 0$. Let us note $\mathfrak{m}_{j+1/2}^n = \min(Y_j^n, Y_{j+1}^n)$, $\mathfrak{M}_{j+1/2}^n = \max(Y_j^n, Y_{j+1}^n)$. Following the ideas introduced in section 2.3 in the case of pure transport problem, we aim at defining a flux $Y_{j+1/2}^{\mathrm{Lag}}$ that fulfills two requirements.

(i) $Y_{j+1/2}^{\mathrm{Lag}} \in [\mathfrak{m}_{j+1/2}^n, \mathfrak{M}_{j+1/2}^n]$;

(ii) the choice of $Y_{j+1/2}^{\mathrm{Lag}}$ and (25b) should ensure a discrete maximum principle for Y in the cell i (resp. $i+1$) if $u_{j+1/2}^n > 0$ and $u_{j-1/2}^n > 0$ (resp. $u_{j+1/2}^n < 0$ and $u_{j+3/2}^n < 0$).

For the sake of simplicity, we suppose that $\rho_{j+1/2}^{\mathrm{Lag}}$ is defined by the upwind flux, *i.e.* $\rho_{j+1/2}^{\mathrm{Lag}} u_{j+1/2}^n = \rho_j^{\mathrm{Lag}}(u_{j+1/2}^n)^+ + \rho_{j+1}^{\mathrm{Lag}}(u_{j+1/2}^n)^-$, then we can define the real interval $[d_{j+1/2}, D_{j+1/2}]$ as follows.

- If $u_{j+1/2}^n > 0$ and $u_{j+1/2}^n > 0$ (resp. $u_{j+1/2}^n < 0$), we set

$$d_{j+1/2} = Y_j^n + (\mathfrak{M}_{j-1/2} - Y_j^n)\left[1 - \frac{\Delta x}{u_{j+1/2}^n \Delta t}\right] \quad (\text{resp. } d_{j+1/2} = Y_j^n),$$

$$D_{j+1/2} = Y_j^n + (\mathfrak{m}_{j-1/2} - Y_j^n)\left[1 - \frac{\Delta x}{u_{j+1/2}^n \Delta t}\right] \quad (\text{resp. } D_{j+1/2} = Y_j^n).$$

- If $u^n_{j+1/2} < 0$ and $u^n_{j+3/2} < 0$ (resp. $u^n_{j+3/2} > 0$), we set

$$d_{j+1/2} = Y^n_{j+1} + (\mathfrak{M}_{j+3/2} - Y^n_{j+1}) \left[1 + \frac{\Delta x}{u^n_{j+1/2}\Delta t}\right] \quad (\text{resp. } d_{j+1/2} = Y^n_{j+1}\ ),$$

$$D_{j+1/2} = Y^n_{j+1} + (\mathfrak{m}_{j+3/2} - Y^n_{j+1}) \left[1 - \frac{\Delta x}{u^n_{j+1/2}\Delta t}\right] \quad (\text{resp. } D_{j+1/2} = Y^n_{j+1}\ ).$$

Let us note $[\omega_{j+1/2}, \Omega_{j+1/2}] = [\mathfrak{m}^n_{j+1/2}, \mathfrak{M}^n_{j+1/2}] \cap [d_{j+1/2}, D_{j+1/2}]$. Under the CFL condition

$$|u^n_{j+1/2}|\Delta t/\Delta x < 1, \tag{26}$$

one can check that $[\omega_{j+1/2}, \Omega_{j+1/2}] \neq \emptyset$ as Y^n_j (resp. Y^n_{j+1}) belongs to $[\omega_{j+1/2}, \Omega_{j+1/2}]$ if $u^n_{j+1/2} > 0$ (resp. $u^n_{j+1/2} < 0$). Choosing $Y^{\text{Lag}}_{j+1/2} \in [\omega_{j+1/2}, \Omega_{j+1/2}]$ ensures that (i) and (ii) are verified under the condition (26). In order to enable a sharp transport of Y, one just need to use the limited downwind choice within the interval $[\omega_{j+1/2}, \Omega_{j+1/2}]$, which boils down to set

$$Y^{\text{Lag}}_{j+1/2} = \min(\max(\omega_{j+1/2}, Y^{\text{Lag}}_{\text{down}}), \Omega_{j+1/2}), \tag{27}$$

where $Y^{\text{Lag}}_{\text{down}} = Y^{\text{Lag}}_{j+1}$ (resp. $Y^{\text{Lag}}_{\text{down}} = Y^{\text{Lag}}_{j}$) if $u^n_{j+1/2} > 0$ (resp. $u^n_{j+1/2} < 0$).

A numerical scheme based on a Finite Volume approximation of (24) and (25) with the limited downwind choice (27) was studied in [Lag00] for the model described in this section. It is worth mentioning that up to a careful discretization choice for (24) the overall algorithm is conservative with respect to $(\rho, \rho\mathbf{W})$. Let us also emphasize that the algorithm presented in this section is difficult to use in practice: spurious pressure and velocity oscillations at the material interface may occurs, which is a common issue for this type of problems [Abg96]. The same method was applied to similar two-phase models with an alternate mixture law in [Lag00, DL07] that guarantees that constant pressure and velocity profiles are preserved.

3.2 Example of other evolution equation involving sharp interfaces

It is not possible to give an exhaustive list of all possible sharpening techniques implementation, we will try to give hereafter an overview of the works that have been achieved the past years that is inevitably incomplete.

The approach of section 3.1 has been successfully extended to other systems like the five-equation model of [MSNA02, ACK02] in [KL10] and also for compressibles flows involving an arbitrary number of components separated by interfaces [JL07, BFK14]. Other techniques may be used to sharpen front in systems with interface. For example, considering again system (22), one can discretize directly the transport equation (23) with the limited downwind scheme of section 2.3 and use a classical Finite-Volume discretization for ρ, ρu and $\rho(e + u^2/2)$, at the cost of deriving a non-conservative numerical scheme. Other sharpening techniques can also be used for compressible two-phase flows with interface similar to (22): the THINC method that was first developed for incompressible flows [XHK05] has been adapted in [SX14] to the five-equation

model studied in [ACK02]. This method relies on controlling the spreading of the material interface thanks to an hyperbolic tangent profile. As mentioned in section 2.4, Glimm's method has also been used for discretizing sharply the evolution of an interface. Indeed, it is possible to sharply let evolve contact discontinuities in a system by providing a dedicated treatment based on a Glimm type random choice method [Cha07, CG08]. In [BHJ$^+$13] a random choice method is within a Lagrange-Remap strategy to perform the Remap step while preserving sharp profiles. The limited downwind strategy has been implemented to describe interface that are not solely passively advected like problems of reacting gas flows[TBC14]. A VOF-type reconstruction that relies on a level set description of the interface is proposed in [HKAH06] for the simulation of two-component compressible flows.

3.3 Cut-cells and CFL condition

Taking as a principle that sharpening techniques have the ability to reconstruct interfaces, it appears that an interface which moves dynamically in a Cartesian mesh may cut cells into smaller cells. Of course it is most of the time only a geometrical interpretation. However it has the unfortunate consequence that these small cut cells may have a dramatic influence on the CFL conditions through a complex nonlinear interaction of the parts of the global algorithm (note nevertheless that it is not the case with the Vofire algorithm). This feature is difficult to analyze rigorously in the context of sharpening methods. In practice one observes a posteriori the stability or the instability of the scheme/code. We refer to the chapter [Ber84] in this volume for a comprehensive presentation of the topic.

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