Exponential convergence to the stationary measure and hyperbolicity of the minimisers for random Lagrangian systems.

Alexandre Boritchev, University Lyon 1

Part of this work in common with Konstantin Khanin (Toronto).

Supported by the French National Research Agency contract ANR WKBHJ (P.I. A.Fathi, P.Thieullen).

What happens in the deterministic case?

Random Hamilton-Jacobi equation

Minimisers and hyperbolicity

Variational description

Consider ϕ satisfying the Hamilton-Jacobi equation for a mechanical Hamiltonian on a compact manifold :

$$\phi_t + \frac{1}{2}\phi_x^2 = -V, \ x \in M.$$

The Legendre transform of $H(t, x, \phi_x) = \phi_x^2/2 + V$ gives us :

$$L(t, x, \dot{x}) = \frac{1}{2}\dot{x}^2 - V$$

For an initial condition $g(x)$ at time t_1, we get the following variational description of the solutions :

$$\phi(t_2, x) = \min_{\gamma(t_2)=x} \left(g(\gamma(t_1)) + \int_{t_1}^{t_2} L(t, \gamma, \dot{\gamma}) \right).$$

Generalising to the case of random forcing is not a problem.

Minimisers (I)

For an initial condition $g(x)$ at time t_1, we have the following variational description :

$$\phi(t_2, x) = \min_{\gamma \in \Gamma} \left(g(\gamma(t_1)) + \int_{t_1}^{t_2} L(t, \gamma, \dot{\gamma}) \right).$$

Here Γ is the set of curves such that $\gamma(t_2) = x$. Such curves are called g-minimisers.

When we minimise the action $\int L$ on a time interval with fixed endpoints and without fixing an initial condition, the corresponding curves are called minimisers.

Minimisers (II)

In the same way, one can also define one-sided minimisers :

$$\gamma^x : [t_0, +\infty) \ (or \ (-\infty, t_0]) \to S^1$$

(one minimises the action for compactly supported in time perturbations of γ^x such that $\tilde{\gamma}^x(t) = x$) and the global minimisers :

$$\gamma : (-\infty, +\infty) \to S^1$$

(without x or t : again, one minimises the action for compactly supported in time perturbations of γ).

A g-minimiser is a minimiser.
The expected restriction properties hold.

Deterministic generic setting

We consider the equation with a deterministic generic forcing. In other words :

$$\phi_t + \frac{1}{2}\phi_x^2 = -V(x), \tag{1}$$

where V is smooth and has a unique non-degenerate maximum. We assume that this maximum is reached for $x = 0$ and equals 0 and denote $-V''(0) =: \lambda^2 > 0$.
Now we linearise in a neighbourhood of $(x, v) = (0, 0)$ and we consider the Euler-Lagrange equation :

$$\frac{dx}{dt} = v, \ \frac{dv}{dt} = -V_x.$$

Therefore in the (x, v) coordinates, the point $(0, 0)$ is "exponentially attractive", i.e. a minimiser on $[0, T]$ is in a $exp(-\lambda T/2)$-neighbourhood of $(0, 0)$ at time $T/2$.

Exponential convergence of the minimisers

There exists a unique global minimiser : the line $\tilde{\gamma} \equiv 0$. Moreover :

Theorem 1

There exist a constant $C > 0$ s.th. for a minimiser $\gamma : [0, +\infty) \mapsto S^1$ one has :

$$|\gamma(t) - \tilde{\gamma}(t)| = |\gamma(t)| \leq C \exp(-\lambda t/2), \; t \geq 0.$$

Exponential convergence of the solutions

Corollary 1

There exist a constant $\tilde{C} > 0$ *s.th. if we consider a pair of initial conditions* ϕ_0, ψ_0*, then the corresponding solutions* $\phi,\ \psi : [0, +\infty) \mapsto S^1$ *of (1) converge to each other exponentially up to an additive constant, i.e. :*

$$\max_{x \in S^1} |\phi(t,x) - \psi(t,x) - A(t)| \leq \tilde{C} \exp(-\lambda t),\ \ t \geq 0,$$

where $A = \phi(t/2, 0) - \psi(t/2, 0)$.

The results above were obtained (in a more general setting) by Iturriaga-Sanchez Morgado in 2009.

The stochastic periodic 1D Hamilton-Jacobi equation

$$\phi_t + \phi_x^2/2 = (\nu \phi_{xx}) + \eta, \ \ t \geq 0, \ \ x \in S^1 = \mathbb{R}/\mathbb{Z}. \qquad \text{(1DB)}$$

$\eta(t,x) = \eta^\omega(t,x)$: smooth in space random force, irregular in time (white noise or "kicked").

We can consider a more general nonlinear term $f(\phi_x)$ (under Tonelli-type convexity-growth assumptions on f).

Different types of forcing

-"Kicked" force :

$$\eta^{\omega}(x) = \sum_{i=1}^{+\infty} \delta_{t=i}\zeta_i^{\omega}(x),$$

where ζ_i^{ω} are non-trivial smooth i.i.d. random variables in $L_2(S^1)$ with support in $C^{\infty}(S^1)$ and finite moments in all Sobolev spaces $H^m(S^1)$.

Example : At each integer time, a kick equals 0, $\cos(2\pi x)$, $\sin(2\pi x)$ or $\sin(2\pi x) + \cos(2\pi x)$, with probability $1/4$ each.

-White noise-type force : $\eta^{\omega}(x) = w_t^{\omega}(x)$, where w^{ω} is an $L_2(S^1)$-valued Wiener process in time with all moments of $w(t)$ bounded in $H^m(S^1)$, $m \geq 1$.

Stationary measure

The arguments below in this part of the talk hold uniformly with respect to the viscosity coefficient $\nu \geq 0$ (i.e. for the stochastic HJ equation with or without the viscous term $\nu\phi_{xx}$) and on the torus $\mathbb{T}^d,\ d \geq 1$.

Solutions ϕ of the stochastic HJ equation define a Markov process. We have ν-uniform upper estimates [Bor12, Bor13] : the Bogolyubov-Krylov argument implies the existence of a stationary measure.

Speed of convergence

The semigroup G_t^ω corresponding to the Markov process is L_∞-nonexpanding :

$$|G_t^\omega \phi_0 - G_t^\omega \tilde{\phi}_0|_\infty \leq |\phi_0 - \tilde{\phi}_0|_\infty.$$

Then a coupling argument (cf. [Kuksin-Shirikyan '12]) gives us algebraic convergence to the unique stationary measure if 0 is in the support of the forcing.

Idea : The distance between two solutions corresponding to different initial values and the same forcing becomes small since the solutions themselves become small after a long "small-force" period, and then this distance is nonincreasing.

Speed of convergence : comments

The assumption $0 \in Supp$ is (more or less the only) necessary assumption.

Analogous arguments are used in [Gomes-Iturriaga-Khanin-Padilla '05] (and also by Dirr-Souganidis, Debussche-Vovelle...) but there is no explicit estimate of the speed of convergence.

Random dynamics

Two types of questions :
-Existence-uniqueness-properties of the stationary measure
-Properties of the minimisers (in particular : existence, uniqueness and hyperbolicity of the global minimiser).

Sinai '91 (using the Cole-Hopf transform) ;
E-Khanin-Mazel-Sinai '00 (hyperbolicity).
Multi-d case : [Iturriaga-Khanin '03],
[Gomes-Iturriaga-Khanin-Padilla '05].

In these papers there are additional assumptions on the forcing.

Minimisers and the sets $\Omega_{s,t}$

Definition 1

For an initial condition $\psi_0(0,\cdot) : S^1 \to \mathbb{R}$ and $0 < s < t$, let $\Omega_{s,t}$ be the set of points reached, at time s, by the ψ_0-minimisers on $[0,t]$.

One notices that $\Omega_{s,t}$ is closed.

For a closed subset Z of S^1, the diameter of Z is defined as :

$$d(Z) = 1 - m(Z),$$

where $m(Z)$ is the maximal length among connected components of $S^1 - Z$.

Other possible definition : the minimal length of a closed interval in S^1 containing Z.

Assumptions on the potentials

The goal is to prove exponential convergence of the minimisers to the global minimiser.

Idea : In the kicked case, if we denote by μ the probability measure on $L_2(S^1)$ for the kicks, we want the following :
-The forcing can be arbitrarily small : $0 \in Supp\ \mu$.
-The forcing is not too structured : there exist potentials $f_i,\ i = 1, 2, 3$ reaching their nondegenerate maxima x_i at 3 different points, s.th. $f_i \in Supp\ \mu$.

For instance, a forcing generated by $(\cos\ 4\pi x,\ \sin\ 4\pi x)$ does not work (periodicity) ; $(\cos\ 2\pi x,\ \sin\ 2\pi x)$ works.

Assumptions on the potentials (II)

Assumptions 1

"Kicked case" :

(i) The kicks at integer time moments j have the form

$$F^\omega(j) = \sum_{k=1}^{K} c_k^\omega(j) F^k,$$

where the F^k are smooth functions defined on $S^1 = \mathbb{R}/\mathbb{Z}$.
The vectors $(c_k^\omega(j))_{1\leq k\leq K}$ are i.i.d. $\mathbb{R}^K$-valued random variables with finite moments. Their distribution on $\mathbb{R}^K$, denoted μ, is absolutely continuous with respect to $\mu_{Lebesgue}$.
(ii) We have $0 \in Supp\ \mu$.
(iii) The mapping $x \mapsto (F^1(x), ..., F^K(x))$ is an embedding (injective and homeomorphism onto its image).

Assumptions on the potentials (III)

Assumptions 2

White noise case :

(i) The potential is of the form

$$F^{\omega}(x,t) = \sum_{k=1}^{K} \dot{W}_k^{\omega}(t) F^k(x),$$

where the F^k *are smooth on* S^1 *and the* $\dot{W}_k^{\omega}$ *are independent white noises.*

(ii) The mapping

$$x \mapsto (F^1(x), ..., F^K(x))$$

is an embedding.

The separation property

Lemma 1

There exist $\alpha_0 > 0$, three pairwise disjoint open intervals J_i, $i = 1, 2, 3$, and three potentials $\tilde{F}_i$, $i = 1, 2, 3$ s.t. :
1) The potentials $\tilde{F}_i$ are in the support of forcing.
2) Each of the functions $\tilde{F}_i$ reaches its minimum, denoted by m_i, at a single point x_i ; moreover these points are different.

Lemma 2

Assumptions 1 or 2 imply the separation property.

Asymptotic behaviour of the minimisers

Here we fix $-\infty < s \leq t < +\infty$ and we assume that the minimisers verify the assumptions mentioned above.

Lemma 3

Fix $s \in \mathbb{R}$. Then ω-a.s., there exists a random variable with finite moments $\tilde{C} > 0$ s.th. :

$$d(\Omega_{s,t}) \leq \tilde{C} \exp(-\lambda(t-s)/2), \quad t \geq s.$$

Corollary 2

Ω-a.s., there exists a random variable with finite moments $\tilde{C} > 0$ s.th. for any one-sided minimiser γ on $[0, +\infty)$:

$$|\gamma(t) - \tilde{\gamma}(t)| \leq \tilde{C}(s, \omega) \exp(-\lambda t/2), \quad t \geq 0,$$

where $\tilde{\gamma}$ is the unique global minimiser.

Remarks

Hyperbolicity : Using the Pesin theory, Corollary 3 implies hyperbolicity of the global minimiser for the Euler-Lagrange dynamics in the phase space (x, u, t).

Multi-d : A global minimiser exists and is unique ; it is hyperbolic [Khanin-Zhang]. However, we do not *a priori* have exponential contraction for the minimisers and the shock structure is not well-understood.

Noncompact setting :
Hoang-Khanin '03 : Forcing with a well-localised minimum.
Bakhtin-Cator-Khanin '13 : Poisson process in space-time. The minimisers coalesce exponentially.
Bakhtin '15-'16 : further extension. Positive viscosity !

Exponential convergence

For $\nu = 0$ et $d = 1$ we have exponential convergence to the stationary measure [Bor '16].

In other words : two solutions corresponding to different initial conditions and the same forcing converge to each other exponentially in $L_\infty/\mathbb{R}$ like in Iturriaga-Sanchez Morgado.

An analoguous result was proved by Bec-Frisch-Khanin (2001) in L_∞ far away from the shocks.

Exponential convergence : proof

Proof : For two different initial conditions, consider the position of two minimisers on $[0, T_1 + T_2]$ at time T_1. They are localised in the $exp(-CT_2)$-small sets $\Omega_{T_1, T_1+T_2}(\psi_0^i), i = 1, 2$.

On the other hand, the limit points $\lim_{s\to+\infty} \Omega_{T_1,s}(\psi_0^i),\ i = 1, 2$ are $exp(-CT_1)$-close (they are both $\gamma(T_1)$ for some one-sided minimiser γ).

Finally, we fix $T_1 = T_2 = T/2$ and we get that at time $T/2$ minimisers on $[0, T]$ are $\exp(-CT/2)$-close, and then we conclude as in [ISM09].

Perspectives

We want to prove that under the assumptions 1 or 2, the convergence is exponential (uniformly in ν). Get rid of these assumptions ?

Main obstacle : the viscous Lagrangian representation of Feynman-Kac type is less straightforward. We have no minimisers since now we minimise an expected value.

Bibliography

A. Boritchev, K. Khanin, *On the hyperbolicity of minimizers for 1D random Lagrangian systems*, Nonlinearity 26 (2013), 65-80.

A. Boritchev, *Sharp estimates for turbulence in white-forced generalised Burgers equation*, GAFA 23 (2013), 6, 1730-1771

A. Boritchev, *Multidimensional potential Burgers turbulence*, CMP 342(2016) : 441-489.

A. Boritchev, *Exponential convergence to the stationary measure for a class of 1D Lagrangian systems with random forcing*, arXiv :1601.01937

Weinan E, K. Khanin, A. Mazel, Ya. Sinai, *Invariant measures for Burgers equation with stochastic forcing*, Annals of Mathematics 151 (2000), 877-960.

R. Iturriaga, H. Sánchez-Morgado, *Hyperbolicity and exponential convergence for the Lax-Oleinik semigroup* JDE 246 (5) 2009, 1744-1753.