1D Burgers Turbulence as a Model Case for the Kolmogorov Theory

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Outline

The Kolmogorov Theory and Intermittency

1D Burgers Turbulence: Physical Predictions

1D Burgers Turbulence: Results

3D Incompressible Navier-Stokes Equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nu \Delta \mathbf{u} + \eta; \quad \text{div } \mathbf{u} = 0. \qquad \text{(NSE)}$$

Supplemented by boundary conditions.

$\mathbf{u}(t, \mathbf{x})$ velocity	$\nu > 0$ constant viscosity coefficient
$p(t, \mathbf{x})$ pressure	$(\nu \ll 1)$
	$\eta(t, \mathbf{x})$ random forcing, smooth as a function of $\mathbf{x}$

The idea is to study the statistical behaviour of $\mathbf{u}$ as ν varies, all other parameters being fixed.

The K41 Theory

In Fourier space, a scale is, roughly speaking, the inverse of the Fourier frequency under consideration. In this talk, we only consider space scales, not time scales.
Thus, in Fourier space, small-scale quantities are quantities such as the Fourier coefficients $\hat{\mathbf{u}}(\mathbf{k})$ for large $\mathbf{k}$.
In turbulence research, physical-space quantities of the type $\mathbf{u}(\mathbf{x}+\mathbf{r})-\mathbf{u}(\mathbf{x})$ for small $\mathbf{r}$ are also called small-scale quantities.

Small-scale behaviour for a velocity field of turbulent fluid is a very old problem (1930s-: Taylor, Onsager, Batchelor...)
3 papers by Kolmogorov in 1941. Various physical assumptions (cf [Frisch 1995], [Tsinober 2001]), including a time-stationary regime.
We limit ourselves to the case of space-periodic flows.

Notation

$X \overset{a}{\sim} Y$: There exists $C > 0$ such that $C^{-1}X \leq Y \leq CX$. C only depends on the parameter a, which is never the viscosity coefficient ν. The abbreviations $\overset{a}{\gtrsim}$ et $\overset{a}{\lesssim}$ are defined similarly.

$\langle \ldots \rangle$: Expected value (when considering a random force).

The K41 Theory: Ranges.

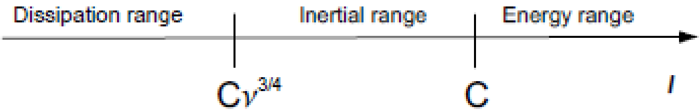
The dissipation length scale l_d is the scale such that, for $\|\mathbf{k}\| \succeq l_d^{-1}$, the Fourier coefficients of a function $\mathbf{u}$ decrease at a super-algebraic rate, uniformly in ν. In K41, $l_d = C\nu^{3/4}$. The range $\mathbb{I}_{diss} = [0, l_d]$ is called the dissipation range.

The energy range $\mathbb{I}_{energy} = [l_e, 1]$ consists of scales such that the corresponding Fourier modes contain most of the L^2 norm of $\mathbf{u}$:

$$\sum_{\|\mathbf{k}\| \leq l_e^{-1}} \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle \gg \sum_{\|\mathbf{k}\| > l_e^{-1}} \langle |\hat{\mathbf{u}}(\mathbf{k})|^2 \rangle.$$

In K41, $l_e = C$.

$\mathbb{I}_{inertial} = [l_d, l_e]$ is called the inertial range.
In K41, $\mathbb{I}_{inertial} = [C\nu^{3/4}, C]$.



The K41 Theory: Predictions in Physical Space

We begin by considering moments of the longitudinal increment:

$$S^{\|}(\mathbf{x},\mathbf{r}) = \frac{(\mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x}))\cdot\mathbf{r}}{\|\mathbf{r}\|}.$$

For $p \geq 0$, the p-th moment of $S^{\|}(\mathbf{x},\mathbf{r})$ is called the structure function of p-th order:

$$S_p^{\|}(\mathbf{x},\mathbf{r}) = \left\langle \left| \frac{(\mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x}))\cdot\mathbf{r}}{\|\mathbf{r}\|} \right|^p \right\rangle .$$

For $\mathbf{r}$, $\|\mathbf{r}\| = \ell \in \mathbb{I}_{inertial}$, we have

$$S_p^{\|}(\mathbf{x},\mathbf{r}) \overset{p}{\sim} \ell^{p/3}.$$

The K41 Theory: Predictions in Fourier Space

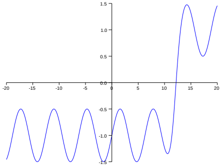
We define the energy spectrum $E(k)$ as the average of $\langle \frac{1}{2}|\hat{\mathbf{u}}(\mathbf{n})|^2\rangle$ over $\mathbf{n}$ such that $\|\mathbf{n}\| \in [C^{-1}k,\ Ck]$. For k such that $k^{-1} \in \mathbb{I}_{inertial}$, we have:

$$E(k) \sim k^{-5/3}.$$

(Obukhov 1941).

Intermittency and Corrections to K41

K41 predictions are in agreement with experiments for $S_p^{\|},\ p = 2, 3$ and the energy spectrum, but not for $S_p^{\|},\ p \geq 4$. Corrections to K41 ([Kolmogorov 1962], [Frisch, Parisi 1985]) explain these discrepancies by spatial intermittency, i.e. bursty behaviour.



This type of intermittency can be quantified by flatness $F(\ell) = S_4^{\|}(\ell)/S_2^{\|}(\ell)^2$: the larger the flatness, the more bursty is the function.

1D Periodic Stochastic Burgers Equation

$$u_t + f'(u)u_x = \nu u_{xx} + \eta,\ \ t \geq 0,\ \ x \in S^1 = \mathbb{R}/\mathbb{Z}. \qquad \text{(1DB)}$$

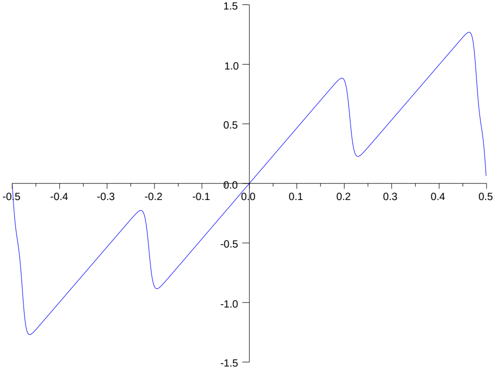
f smooth, strongly convex, of moderate growth. $\nu > 0,\ \nu \ll 1$.
$\eta(t,x) = \eta^\omega(t,x)$: smooth in space random force, white in time.
$\eta = w_t$, where w is a spatially smooth L_2-valued Wiener process.
Initial condition $u_0 = u(0,\cdot) \in L_1(S^1)$.

For simplicity, we assume that $\int_{S^1} \eta(t,\cdot) = 0, \forall t$; $\int_{S^1} u(0,\cdot) = 0$.
Thus $\int_{S^1} u(t,\cdot) = 0,\ \forall t$.

Here we consider the case $f(u) = u^2/2$. Same type of nonlinearity and dissipation as (NSE); no pressure. Therefore, natural model pour (NSE). Studied by physicists such as Burgers, Kida, Kraichnan, Zeldovich, Frisch, Parisi, Gotoh, Polyakov...

Shocks after a finite time for $\nu = 0$. For $\nu > 0$, instant smoothening but steep cliffs.
Again, only ν varies: we fix η and we do not care about u_0.

Typical Profile of a Burgers Solution



Amplitude of solution ~ 1. Cliffs (quasi-shocks): number of cliffs ~ 1, jump ~ -1, width $\sim \nu$.

Burgers turbulence or "Burgulence": see [Bec-Khanin 2007].

Ramp-cliff structure $\Rightarrow$ intermittency.

Predictions for (1DB)

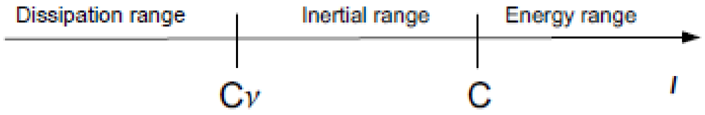
Physical predictions for dissipation length scale, increments, flatness, energy spectrum.
Unforced case with random initial data: [Kraichnan 1968], [Kida 1979], [Aurell-Frisch-Lutsko-Vergassola 1992].
Arguments are easily adapted to the case when there is an additive forcing term, smooth in space and white in time.
If we add such a term: [E-Khanin-Mazel-Sinai 1997] (stationary solution for $\nu = 0$), [Kraichnan-Gotoh 1998].
No rigorous proof ever given in the case $\nu > 0$ with white in time smooth in space forcing, for finite-time evolution.

Predictions for (1DB): Length Scales

Energy range (corresponding to Fourier modes where most of the L^2 norm is contained): $\mathbb{I}_{energy} = [C,\ 1]$.
K41: $[C,\ 1]$.

Dissipation length scale (the scale l_d such that the energy spectrum decreases super-algebraically for Fourier modes corresponding to $|k| \succeq l_d^{-1}$): $C\nu$.
K41: $C\nu^{3/4}$.

Inertial range: $\mathbb{I}_{inertial} = [C\nu,\ C]$.
K41: $[C\nu^{3/4},\ C]$.



Predictions for (1DB): Physical Space (I)

Structure functions, i.e. moments of increments:
We define $S_p(\ell)$ as

$$\int_{S^1} \langle |u(x+\ell) - u(x)|^p \rangle dx.$$

Then, for $\ell \in \mathbb{I}_{inertial}$, we have

$$S_p(\ell) \stackrel{p}{\sim} \begin{cases} \ell^p, \; 0 \leq p \leq 1. \\ \ell, \; p \geq 1. \end{cases}$$

K41: $S_p(\ell) \stackrel{p}{\sim} \ell^{p/3}, \; \forall p.$

Flatness: We define $F(\ell) = S_4(\ell)/S_2(\ell)^2$. Then for $\ell \in \mathbb{I}_{inertial}$, we have

$$F(\ell) \sim \ell^{-1}.$$

K41: $F(\ell) \sim 1.$

Predictions pour (1DB): Physical Space (II) Arguments of [AFLV]

Aurell-Frisch-Lutsko-Vergassola begin by observing that $\ell \in \mathbb{I}_{inert} = [C\nu, C]$ is larger than the length of a "cliff" and smaller than the length of a "ramp". Thus, there are 3 possibilities for $[x, x+\ell]$:

1) $[x, x+\ell]$ covers a large part of a "cliff".
Probability $\simeq C\ell$. $|u(x+\ell)-u(x)|^p \overset{p}{\sim} 1$.

2) $[x, x+\ell]$ covers a small part of a "cliff".
Contribution of this term is negligible.

3) $[x, x+\ell]$ lies entirely on a "ramp".
Probability $\simeq 1 - C\ell$. $|u(x+\ell)-u(x)|^p \overset{p}{\sim} \ell^p$.

Thus, $S_p(\ell) \overset{p}{\sim} \ell + \ell^p$. $S_p(\ell) \overset{p}{\sim} \begin{cases} \ell^p, & 0 \leq p \leq 1. \\ \ell, & p \geq 1. \end{cases}$

Predictions for (1DB): Fourier Space

Energy spectrum: Let $E(k)$ denote the average of $\langle \frac{1}{2} |\hat{\mathbf{u}}(n)|^2 \rangle$ over n such that $|n| \in [C^{-1}k,\ Ck]$. Then, for $k^{-1} \in \mathbb{I}_{inertial}$:

$$E(k) \sim k^{-2}.$$

K41: $E(k) \sim k^{-5/3}$.

Estimates for the norms of the Solution: Preliminaries

In [Bor2], we obtain sharp upper and lower estimates on the Sobolev norms $W^{m,p}$ of the (1DB) solution. These estimates are uniform with respect to the initial condition $u(0,\cdot)$. The upper and lower estimates coincide up to a multiplicative constant factor, which does not depend on ν.

Notation:

- $|\cdot|_p$: the Lebesgue norm in the space $L_p(S^1)$.
- $|\cdot|_{m,p}$: the Sobolev norm in the space $W^{m,p}(S^1)$.
- $\{\dots\}$: averaging both over the time period $[t, t+T_0]$, where $t \geq T_0$ and T_0 is a constant, and in ensemble (taking the expected value).

Estimates for the norms of the solution

Theorem 1
$\{|u|_p^n\} \overset{n}{\sim} 1, \ \forall n \geq 0, \ \ p \in [1, +\infty].$

Theorem 2
$\{\max_{S^1} |u_x^+|^n\} \overset{n}{\sim} 1, \ \{\max_{S^1} |u_x^-|^n\} \overset{n}{\sim} \nu^{-n}, \ \forall n \geq 0.$

Theorem 3
$\{|u|_{m,\infty}^n\} \overset{m,n}{\sim} \nu^{-mn}, \ \forall m \geq 1, \ \ n \geq 0.$

Exact upper estimates are obtained by using Kruzhkov's maximum principle(good bound for u_x^+), and some stochastic methods. They still hold if time averaging is replaced by maximising in time over $[t, t + T_0]$.

Exact lower estimates are obtained by stochastic methods.

What do these bounds tell us?

$\{|u|_p^n\} \overset{n}{\sim} 1,\ \forall n \geq 0,\ \ p \in [1, +\infty].$
$\Rightarrow$ amplitude of $u \sim 1$.

$\{(\max_{S^1} u_x^+)^n\} \overset{n}{\sim} 1,\ \{(\max_{S^1} u_x^-)^n\} \overset{n}{\sim} \nu^{-n},\ \forall n \geq 0.$
$\Rightarrow$ positive part of $u_x \sim 1$; negative part of $u_x \sim \nu^{-1}$.

$\{|u|_{m,\infty}^n\} \overset{m,n}{\sim} \nu^{-mn},\ \forall m \geq 1,\ \ n \geq 0.$
$\Rightarrow$ seemingly, $u(x)$ behaves "as $g(x\nu^{-1})$".
Problem: positive part of u_x. To understand the small-scale structure of u, we need to work more.

Stationary measure

Solutions u define a Markov process in $L_1(S^1)$. The corresponding semigroup S_t is contracting:

$$|S_t u_0 - S_t \tilde{u}_0|_1 \leq |u_0 - \tilde{u}_0|_1.$$

Theorems 1-3 yield, by the Bogolyubov-Krylov argument, the existence of a stationary measure. Its uniqueness, and the rate of convergence to it, follow from estimates of the same type as (**??**) (idea: distance between solutions is made small since the solutions themselves become small, and then this distance is nonincreasing.)

Obtaining lower bounds

The Itô formula yields:

$$\langle |u(t+T_0)|_2^2 \rangle - \langle |u(t)|_2^2 \rangle = -2\nu T_0 \{ |u|_{1,2}^2 \} + CT_0.$$

By the Kruzhkov maximum principle, for $t \geq 1$ we have:

$$\langle |u(t+T_0)|_2^2 \rangle \leq \langle (\max_x u_x(t+T_0,x))^2 \rangle \leq C.$$

Consequently, for T_0 large enough, the terms in the right-hand side of the Itô formula are asymptotically equal, and therefore:

$$\{ |u|_{1,2}^2 \} \geq C\nu^{-1}.$$

Notation

$S_p(\ell)$: $\int_{S^1} \{|u(x+\ell) - u(x)|^p\} dx, \ p \geq 0.$

$E(k)$: average of $\{\frac{1}{2}|\hat{\mathbf{u}}(n)|^2\}$ over n such that $|n| \in [C^{-1}k, \ Ck]$, where $C > 0$ is a constant.

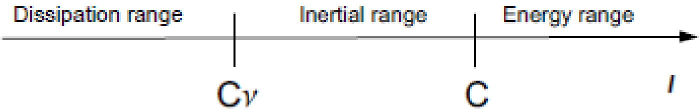
Note that in the definitions of ranges, we also have to replace $\langle \cdot \rangle$ by $\{\cdot\}$!

Main Results: Length Scales

From Theorems 1-3, we derive the following estimates, confirming the physical predictions.

Theorem 4

For a solution of (1DB), $\mathbb{I}_{energy} = [C, 1]$ *and the dissipation length scale is* $C\nu$*.*
Therefore $\mathbb{I}_{inertial} = [C\nu,\ C]$*.*



Main Results in Physical Space

Theorem 5

For $\ell \in \mathbb{I}_{inertial}$, the structure functions satisfy:

$$S_p(\ell) \overset{p}{\sim} \begin{cases} \ell^p, \ 0 \leq p \leq 1. \\ \ell, \ p \geq 1. \end{cases}$$

Corollary 6

For $\ell \in \mathbb{I}_{inertial}$, the flatness satisfies:

$$F(\ell) \sim \ell^{-1}.$$

Upper bounds follow from Theorems 1-3 by Hölder's inequality. Some lower bounds follow from geometrical arguments in [AFLV] for "typical" solutions.

Main Results: Fourier Space

Theorem 7

For $k^{-1} \in \mathbb{I}_{inertial}$, the energy spectrum satisfies:

$$E(k) \sim k^{-2}.$$

Generalisations

We can generalise our results to the "kicked force" case, since we have the same Sobolev norm estimates (cf. [Bor1]).
We obtain similar results for the unforced case (cf. [Biryuk 2001, Bor3]).
We expect similar results to hold for the multidimensional potential Burgers equation:

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \Delta \mathbf{u} + \eta_t; \quad \mathbf{u} = -\nabla \phi.$$

as well as for the equation with fractional Laplacian.

Concluding Remarks

Our results give exact and rigorously proved small-scale estimates for 1D Burgers turbulence. They confirm previous physical predictions under very general conditions on the initial data, for a physically reasonable class of forces. Our small-scale estimates also hold for solutions of the inviscid equation, and for the stationary solution for $\nu > 0$.

PDE and SPDE methods are used to confirm geometric intuition and quantify the dissipation length scale and the intermittency factor (flatness) for (1DB).
We hope that our estimates can be generalised to other equations admitting a "good" maximum principle.

Bibliography

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