1D Burgers Turbulence as a Model Case for the Kolmogorov Theory

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Estimates for the norms of the Solution: Preliminaries

In [Bor2], we obtain sharp upper and lower estimates on the Sobolev norms $W^{m,p}$ of the (1DB) solution.

Notation:

- $|\cdot|_p$: the Lebesgue norm in the space $L_p(S^1)$.
- $|\cdot|_{m,p}$: the Sobolev norm in the space $W^{m,p}(S^1)$.
- $\|\cdot\|_m$: the Sobolev norm in the space $H^m(S^1) = W^{m,2}(S^1)$.
- $\langle \dots \rangle$: taking the expected value (at a given time moment).
- $\{\dots\}$: averaging both over the time period $[t, t+T_0]$, where $t \geq T_0$ and T_0 is a constant, and in ensemble (taking the expected value).

The Gagliardo-Nirenberg inequality

We have the following inequality, denoted by (GN):

Lemma 1.1

For a smooth zero mean value function v on S^1,

$$|v|_{\beta,r} \leq C\,|v|_{m,p}^{\theta}\,|v|_{q}^{1-\theta},$$

where $m > \beta \geq 0$, and r is defined by

$$\beta - \frac{1}{r} = \theta\Big(m - \frac{1}{p}\Big) + (1-\theta)\Big(0 - \frac{1}{q}\Big),$$

under the assumption $\theta = \beta/m$ if $p = 1$ or $p = \infty$, and $\beta/m \leq \theta < 1$ otherwise. The constant C depends on m, p, q, β, θ.

Estimates for the norms of the solution

Theorem 1

$\{|u|_p^n\} \overset{n}{\sim} 1,\ \forall n \geq 0,\ \ p \in [1, +\infty].$

Theorem 2

$\{\max_{S^1} |u_x^+|^n\} \overset{n}{\sim} 1,\ \{\max_{S^1} |u_x^-|^n\} \overset{n}{\sim} \nu^{-n},\ \forall n \geq 0.$

Theorem 3

$\{|u|_{m,\infty}^n\} \overset{m,n}{\sim} \nu^{-mn},\ \forall m \geq 1,\ \ n \geq 0.$

Actually, we can get estimates of the same type for all $m \geq 1$ and p provided $p > 1$, as well as for $m = 0$ or $m, p = 1$. Moreover, lower estimates for $m = 0$ or $m, p = 1$ as well as upper estimates hold without averaging in time. The viscosity ν is raised to the power $-n\gamma$, where $\gamma = \max(0, m - 1/p)$.

Kruzhkov Maximum Principle (I)

Consider unforced (1DB) on $S = (t, x) \in [\Theta, \Theta + 1] \times S^1$:

$$u_t + uu_x = \nu u_{xx}.$$

Now take the derivative in x and consider a point where u_x reaches a maximum, such that $t > \Theta$ (we may assume that $u \not\equiv 0$: thus this maximum is str. positive):

$$\underbrace{(u_x)_t}_{\geq 0} + \underbrace{(u_x)^2}_{>0} + u \underbrace{u_{xx}}_{0} = \underbrace{\nu (u_x)_{xx}}_{\leq 0}.$$

Contradiction $\Rightarrow u_x$ verifies the maximum (but not the minimum) principle. Presence of the term $(u_x)_t + (u_x)^2 \Rightarrow u_x$ behaves as t^{-1}? Consider $v = (t - \Theta) u_x$. The function v can only reach a str. positive maximum for $t > \Theta$. Then we would have:

$$\underbrace{v_t}_{\geq 0} + u \underbrace{v_x}_{0} + (t - \Theta)^{-1}(-v + v^2) = \underbrace{\nu v_{xx}}_{\leq 0}.$$

Thus $v \leq 1$ on S. In other words, $u_x \leq (t - \Theta)^{-1} \Rightarrow$ "damping".

Kruzhkov Maximum Principle (II)

This method was used (in the unforced case) in a paper of Kruzhkov (1964). It can be adapted to the case when there is an additive white force. For $\Theta \geq 1$, we get:

$$\max_{s \in [\Theta, \Theta+1],\ x \in S^1} u_x(s,x) \leq C\left(\max_{s \in [\Theta-1, \Theta+1]} |w(s)|_{C^3} + 1\right), \quad (1)$$

where w is the Wiener process such that $\eta = w_t$ (we fix $w(\Theta - 1) = 0$.) Thus, $\max u_x$ has finite moments.
This method also works for nonlinearities of the type $f'(u)u_x$, where $f'(u) \leq C(|u|^{2-\delta} + 1),\ \delta > 0$.
This estimate on u_x^+ yields an upper estimate on $|u|_p$ and on $|u|_{1,1}$, since we are working with 1-periodic zero mean functions. Indeed:

$$|u|_\infty \leq \int |u_x| = 2 \int u_x^+.$$

Upper estimates for Sobolev norms

Upper estimates of higher Sobolev norms follow from the estimate for $|u|_{1,1}$ and dissipation relations of higher order. In other words, we estimate $d\left\langle \|u(t)\|_m^2 \right\rangle/dt$ using the Gagliardo-Nirenberg inequality.

Denote

$$x(t) = \left\langle \|u(t)\|_m^2 \right\rangle; \; y(t) = \left\langle \|u(t)\|_{m+1}^2 \right\rangle.$$

We claim that the following implication holds:

$$x(t) \geq C'\nu^{-(2m-1)} \Longrightarrow \frac{d}{dt}x(t) \leq -(2m-1)x(t)^{2m/(2m-1)}, \quad (2)$$

where C' is a fixed strictly positive number, chosen later.

This inequality yields:

$$x(t) \leq \max(C'\nu^{-(2m-1)}, t^{-(2m-1)}).$$

Lower estimates for Sobolev norms (I)

The Itô formula yields:

$$\frac{d}{dt}\int_{S^1} u^2 = \underbrace{-2\int_{S^1} uf'(u)u_x}_{0} + 2\nu \int_{S^1} uu_{xx} + stochastic\ integral + C$$

$$= -2\nu \int_{S^1} u_x^2 + stochastic\ integral + C.$$

Taking the expected value and then integrating in time, we get:

$$\langle |u(t+T_0)|_2^2 \rangle - \langle |u(t)|_2^2 \rangle = -2\nu T_0 \{\|u\|_1^2\} + CT_0.$$

By the Kruzhkov maximum principle, for $t \geq 1$ we have:

$$\langle |u(t+T_0)|_2^2 \rangle \leq \langle (\max_x u_x(t+T_0,x))^2 \rangle \leq C.$$

Consequently, for T_0 large enough:

$$\{\|u\|_1^2\} \geq \frac{CT_0 - C}{2T_0}\nu^{-1} \geq C\nu^{-1}.$$

Lower estimates for Sobolev norms (II)

To obtain other estimates for Sobolev norms, we use Hölder and (GN) and we combine the lower estimate above with the upper estimate for $|u|_{1,1}$. For instance, since

$$\|u\|_1 \lesssim |u|_{1,1}^{2/3} \|u\|_2^{1/3},$$

we get

$$\{\|u\|_2^2\} \gtrsim \{\|u\|_1^2\}^3 \{|u|_{1,1}^2\}^{-2} \gtrsim \nu^{-3}.$$

Notation

$S_p(\ell)$: $\int_{S^1} \{|u(x+\ell)-u(x)|^p\} dx,\ p \geq 0.$

$E(k)$: average of $\{\frac{1}{2}|\hat{\mathbf{u}}(n)|^2\}$ over n such that $|n| \in [C^{-1}k,\ Ck]$, where $C > 0$ is a constant.

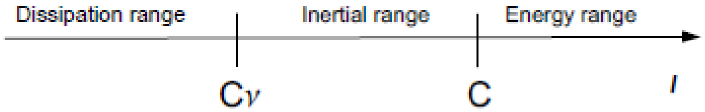
Note that in the definitions of ranges, we also have to replace $\langle \cdot \rangle$ by $\{\cdot\}$!

Main Results: Length Scales

From Theorems 1-3, we derive the following estimates, confirming the physical predictions.

Theorem 4

For a solution of (1DB), $\mathbb{I}_{energy} = [C, 1]$ *and the dissipation length scale is* $C\nu$*.*
Therefore $\mathbb{I}_{inertial} = [C\nu,\ C]$*.*



Main Results in Physical Space

Theorem 5

For $\ell \in \mathbb{I}_{inertial}$, the structure functions satisfy:

$$S_p(\ell) \overset{p}{\sim} \begin{cases} \ell^p, \ 0 \leq p \leq 1. \\ \ell, \ p \geq 1. \end{cases}$$

Corollary 6

For $\ell \in \mathbb{I}_{inertial}$, the flatness satisfies:

$$F(\ell) \sim \ell^{-1}.$$

Main Results in Physical Space: Proof (I)

Upper bounds follow from Theorems 1-3 by Hölder's inequality. Lower bounds in Theorem 5 follow from the following fact:

Lemma 3.1

For a given solution $u(s) = u^\omega(s)$ and $K > 1$, we denote by O_K the set of all $(s, \omega) \in [t, t + T_0] \times \Omega$ such that

$$K^{-1} \leq |u(s)|_\infty \leq \max u_x(s) \leq K \tag{3}$$

$$K^{-1}\nu^{-1} \leq -\min u_x(s) \leq K\nu^{-1} \tag{4}$$

$$|u(s)|_{2,\infty} \leq K\nu^{-2}. \tag{5}$$

Then, there exist constants $\tilde{C}, K_1 > 0$ such that if $K \geq K_1$ and $\nu < K_1^{-2}$, then $\rho(O_K) \geq \tilde{C}$. Here, ρ denotes the product measure of the Lebesgue measure and $\mathbb{P}$ on $[t, t + T_0] \times \Omega$.

Main Results in Physical Space: Proof (II)

We consider a "typical" solution and we use an argument in [AFLV].
Since $\ell \in \mathbb{I}_{inert}$, ℓ is at least of the same order as the width of a region where $-u_x$ is large ("cliff"), but also $\lesssim 1$.
Thus, the probability that $[x, x+\ell]$ covers a large part of a "cliff" is at least $C\ell$.

In this case we have $|u(x+\ell) - u(x)|^p \overset{p}{\sim} 1$. Thus, $S_p(\ell) \overset{p}{\gtrsim} \ell$, which proves the lower estimate for $p \geq 1$.

For $p < 1$, this argument does not work. In fact, geometric intuition cannot help, since the main contribution to $S_p(\ell)$ comes from "ramps", and we do not have enough information on their structure. Thus, we have to use a different method (Hölder's inequality).

Main Results: Fourier Space

Theorem 7

For $k^{-1} \in \mathbb{I}_{inertial}$, the energy spectrum satisfies:

$$E(k) \sim k^{-2}.$$

Idea of the proof: We use the Wiener-Khinchin Formula:

$$\int_{S^1} |u(x+\ell) - u(x)|^2 dx = 4 \sum_{n \in \mathbb{Z}} \sin^2(\pi n \ell) |\hat{u}(n)|^2.$$

Generalisations

We can generalise our results to the "kicked force" case, since we have the same Sobolev norm estimates (cf. [Bor1]).

We obtain similar results (with a dependence on $|u(0,\cdot)|_1, |u(0,\cdot)|_{1,\infty}$) for the unforced case (cf. [Biryuk 2001, Bor3]).

Upper estimates are similar to their stochastic analogues (and are easier to prove). Lower estimates only hold on a given time interval $[T_1, T_2]$.

We expect similar results to hold for the multidimensional potential Burgers equation:

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \Delta \mathbf{u} + \eta_t; \quad \mathbf{u} = -\nabla \phi$$

as well as for the equation with a fractional Laplacian $-(-\Delta)^{\alpha}$ (the critical case $\alpha = 1/2$ is especially interesting because the scaling seems to tell that there is no length scale).

Bibliography

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