Graph homomorphisms between trees

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Graph homomorphism

Homomorphism: adjacency-preserving map

$$f : V(G) \to V(H)$$

$$uv \in E(G) \Longrightarrow f(u)f(v) \in E(H)$$

$$\mathrm{Hom}(G, H) := \text{the set of homomorphisms from } G \text{ to } H$$

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Note: $\mathrm{End}(G)$ forms a monoid

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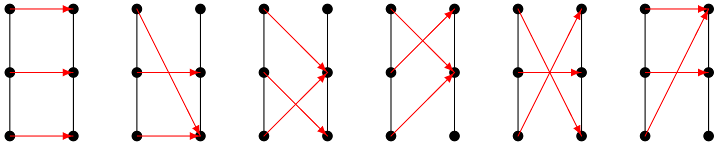


Figure : 6 endomorphisms of the path P_3.

The Path and the Star

P_n :=Path on n vertices
S_n :=Star on n vertices

Theorem (Lin & Zeng, 2011)

$$|\operatorname{End}(P_n)| = \begin{cases} (n+1)2^{n-1} - (2n-1)\binom{n-1}{(n-1)/2} & \text{if } n \text{ is odd} \\ (n+1)2^{n-1} - n\binom{n}{n/2} & \text{if } n \text{ is even} \end{cases}$$

$$|\operatorname{End}(S_n)| = (n-1)^{n-1} + (n-1)$$

How to compute $|\operatorname{End}(T_n)|$ for general trees T_n?

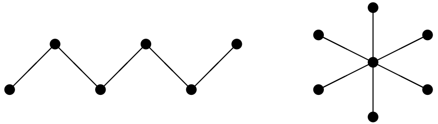


Figure : The path P_6 and the star S_7.

A conjecture

Conjecture (from 2008 to 2013)

For all trees T_n on n vertices we have

$$|\mathrm{End}(P_n)| \leq |\mathrm{End}(T_n)| \leq |\mathrm{End}(S_n)|.$$

The Tree-walk algorithm

How to count $\hom(T, G)$ for a tree T and a graph G?
Let $v \in V(T)$ and $V(G) = \{1, 2, \ldots, m\}$. Define

$$\mathbf{h}(T, v, G) := (h_1, h_2, \ldots, h_m),$$

where

$$h_i = |\{f \in \mathrm{Hom}(T, G) \mid f(v) = i\}|.$$

We call $\mathbf{h}(T, v, G)$ the hom-vector at v from T to G. It is clear that $\hom(T, G) = \|\mathbf{h}(T, v, G)\| = \sum h_i$.

An example:

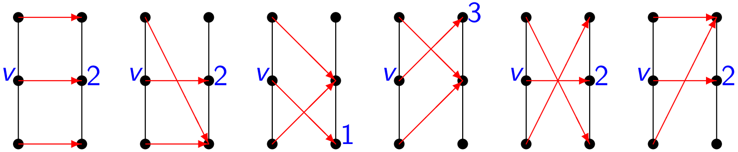


Figure : 6 endomorphisms of the path P_3.

$$\mathbf{h}(P_3, v, P_3) = (1, 4, 1)$$

The Tree-walk algorithm

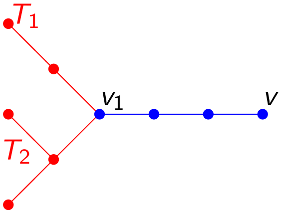
Lemma

Let G be a labeled graph and $A = A_G$ the adjacency matrix of G. Then the (i,j)-entry of the matrix A^n counts the number of walks in G from vertex i to vertex j with length n.

By this lemma, we have $\mathbf{h}(P_n, v, G) = \mathbf{1}A^{n-1}$, where v is the initial (terminal) vertex of the path P_n and $\mathbf{1}$ denotes the row vector with all entries 1.

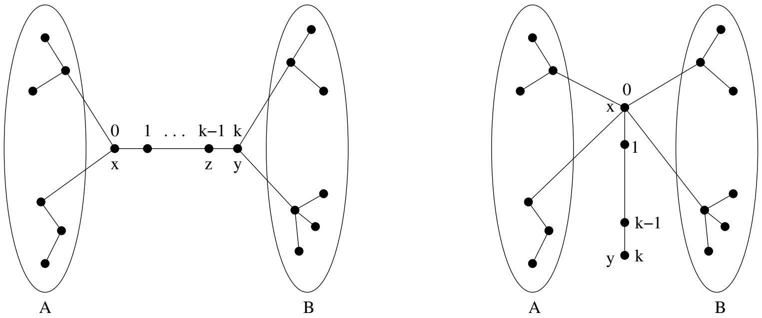
We will generalize this to compute $\mathbf{h}(T, v, G)$.

The Tree-walk algorithm



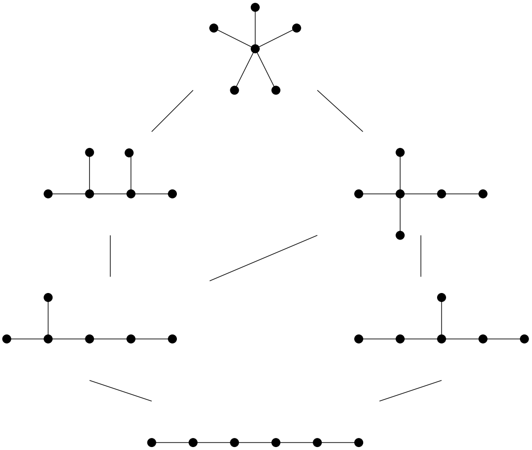
- $\mathbf{h}(T_1 \cup T_2, v_1, G) = \mathbf{h}(T_1, v_1, G) * \mathbf{h}(T_2, v_1, G)$
  $\mathbf{a} * \mathbf{b} := (a_1 b_1, \ldots, a_n b_n)$ is Hadamard product
- $\mathbf{h}(T, v, G) = \mathbf{h}(T_1 \cup T_2, v_1, G) A^3$

KC-transformation



KC-transformation (Csikvári): A transformation on trees with respect to the path $0, 1, \ldots, k$

KC-transformation



The KC-transformation give rise to a graded poset of trees on n vertices with the star as the largest and the path as the smallest element.

KC-transformation: Closed Walks

C_n:=Cycle on n vertices

Theorem (Csikvári, 2010)

Let T be a tree and T' be a KC-transformation of T. Then

$$\hom(C_m, T') \geq \hom(C_m, T)$$

for any $m \geq 1$.

The extremal problem about the number of closed walks in trees:

Corollary (Csikvári, 2010)

Let T_n be a tree on n vertices. We have

$$\hom(C_m, P_n) \leq \hom(C_m, T_n) \leq \hom(C_m, S_n).$$

KC-transformation: Walks

Theorem (Bollobás & Tyomkyn, 2011)

Let T be a tree and T' be a KC-transformation of T. Then

$$\hom(P_m, T') \geq \hom(P_m, T)$$

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Corollary (Bollobás & Tyomkyn, 2011)

Let T_n be a tree on n vertices. We have

$$\hom(P_m, P_n) \leq \hom(P_m, T_n) \leq \hom(P_m, S_n).$$

A natural question arises: Does the above inequalities still true when replacing P_m by any tree?

Generalize to Tree-Walks

Starlike tree: at most one vertex of degree greater than 2

Theorem

Let T be a tree and T' the KC-transformation of T with respect to a path of length k. Then the inequality

$$\hom(H, T') \geq \hom(H, T)$$

holds when

- *k is even and H is any tree*
- *or k is odd and H is a starlike tree.*

Corollary

Let H be a starlike tree and T_n be a tree on n vertices. Then

$$\hom(H, P_n) \leq \hom(H, T_n) \leq \hom(H, S_n).$$

Counterexamples to the odd case

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Counterexamples to the second inequality:

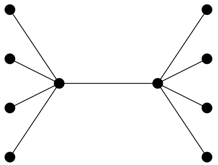

Figure : The doublestar S^*_{10}

For $k \geq 5$ we have $\hom(S^*_{2k}, S^*_{2k}) > \hom(S^*_{2k}, S_{2k})$. Note that S_{2k} can be obtained from S^*_{2k} by a KC-transfromation.

$$\hom(T, P_n) \leq \hom(T, T_n) \leq \hom(T, S_n)?$$

Counterexamples to the second inequality:

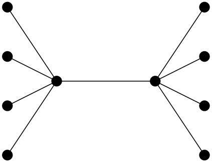

Figure : The doublestar S_{10}^*

For $k \geq 5$ we have $\hom(S_{2k}^*, S_{2k}^*) > \hom(S_{2k}^*, S_{2k})$. Note that S_{2k} can be obtained from S_{2k}^* by a KC-transfromation.
Question: Is the first inequality holds for any tree T?

Sidorenko's theorem on extremality of stars

Theorem (Sidorenko, 1994)

Let G be an arbitrary graph and let T_m be a tree on m vertices. Then

$$\hom(T_m, G) \leq \hom(S_m, G).$$

- Fiol & Garriga (2009) reproved the special case $T_m = P_m$.
- Use Wiener index and some easy observations we (rediscover and) give a new proof of Sidorenko's theorem.
- We constructe some special trees G to disprove the inequality

$$\hom(P_m, G) \leq \hom(T_m, G).$$

Markov chains and homomorphisms

Definition (Markov chains)

Let G be a graph with $V(G) = \{1, 2, \ldots, n\}$. Then $P = (p_{ij})$ is a Markov chain on G if:

$$\sum_{j \in N(i)} p_{ij} = 1 \quad \text{for all} \quad i \in V(G),$$

where $p_{ij} \geq 0$ and $p_{ij} = 0$ if $(i, j) \notin E(G)$.

Definition (Stationary distribution)

Distribution $Q = (q_i)$ is the stationary distribution of P if:

$$\sum_{j \in N(i)} q_j p_{ji} = q_i \quad \text{for all} \quad i \in V(G).$$

Definition (Entropy)

We define the following three entropies:

$$H(Q) = \sum_{i \in V(G)} q_i \log \frac{1}{q_i},$$

and

$$H(D|Q) = \sum_{i \in V(G)} q_i \log d_i,$$

where d_i is the degree of i and let

$$H(P|Q) = \sum_{i \in V(G)} q_i \left(\sum_{j \in N(i)} p_{ij} \log \frac{1}{p_{ij}} \right).$$

Markov chains and homomorphisms

Theorem

If T_m is a tree with ℓ leaves and m vertices, where $m \geq 3$, then

$$\hom(T_m, G) \geq \exp\left(H(Q) + \ell H(D|Q) + (m-1-\ell) H(P|Q) \right).$$

Corollary

Let G be a graph with e edges and degree sequence $(d_1, \ldots, d_n)$. Then for any tree T_m with m vertices we have

$$\hom(T_m, G) \geq 2e \cdot C^{m-2},$$

where $C = \left(\prod_{i=1}^{n} d_i^{d_i}\right)^{1/2e}$.

Sketch of proof: Consider the classical Markov chain: $p_{i,j} = \frac{1}{d_i}$ if $j \in N(i)$.

Trees with 4 leaves

Theorem

If T_n is a tree on n vertices with at least 4 leaves, then

$$\hom(T_m, T_n) \geq \hom(T_m, P_n).$$

Proof: Indeed,

$$\hom(T_m, T_n) \geq (n-2)2^{m-1} + 2 = \hom(S_m, P_n) \geq \hom(T_m, P_n),$$

where the first inequality by the following lemma and the second inequality by Sidorenko's theorem on extremality of stars.

Lemma

Let T_m and T_n be trees on m and n vertices, respectively. If the tree T_n has at least four leaves, then

$$\hom(T_m, T_n) \geq (n-2)2^{m-1} + 2.$$

Lemma

If the tree T_n has at least four leaves, then

$$\hom(T_m, T_n) \geq (n-2)2^{m-1} + 2.$$

Ideals of the proof:

- Fact: If G is a graph and G_1, G_2 are induced subgraphs of G with possible intersection, then for any graph H we have

  $$\hom(H, G) \geq \hom(H, G_1) + \hom(H, G_2) - \hom(H, G_1 \cap G_2).$$

  We can reduce T_n to trees with exactly 4 leaves.
- Use a generalization of even KC-transformation, that we call LS-switch, we can further reduce T_n to 6 classes of special trees with 4 leaves.
- Construct some special Markov chains on the 6 classes of trees and use the lower bound related to Markov chains.

A dual inequality

Bollobás & Tyomkyn's theorem:

$$\hom(P_n, P_m) \le \hom(P_n, T_m) \le \hom(P_n, S_m).$$

Theorem

Let T_m be a tree on m vertices and let T'_m be obtained from T_m by a KC-transformation.

(i) *If n is even, or n is odd and $\mathrm{diam}(T_m) \le n-1$, then*

$$\hom(T_m, P_n) \le \hom(T'_m, P_n).$$

(ii) *For any m, n,*

$$\hom(P_m, P_n) \le \hom(T_m, P_n) \le \hom(S_m, P_n).$$

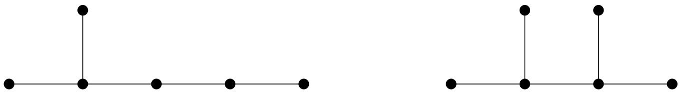

Figure : The trees T_6 (left) and T_6' (right).

The KC-transformation does not always increase the number of homomorphisms to the path P_n when n is odd. In the figure, we have $\hom(T_6, P_3) = 20 > 16 = \hom(T_6', P_3)$.

A dual inequality

$$\hom(P_m, P_n) \le \hom(T_m, P_n) \le \hom(S_m, P_n)$$

Our proof is very complicated...

A dual inequality

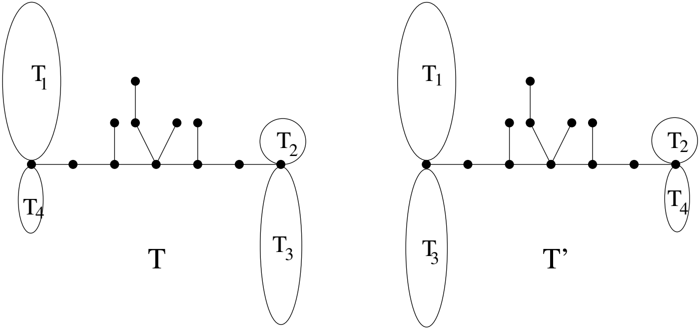

Figure : LS-switch.

A dual inequality

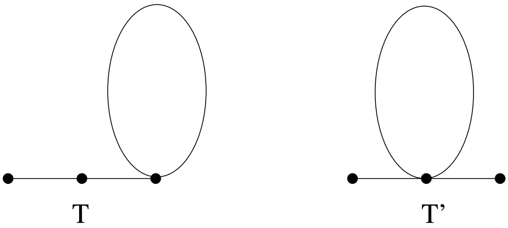

Figure : Short-path shift.

A dual inequality

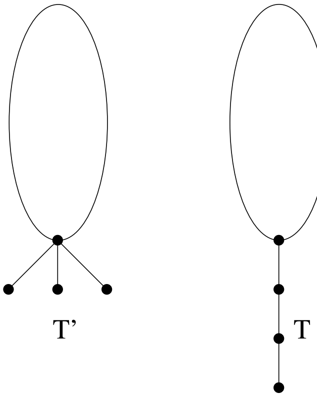

Figure : Claw-deletion.

A dual inequality

Theorem

Let T_m be a tree on m vertices and let T'_m be obtained from T_m by a KC-transformation.

(i) *If n is even, or n is odd and $\mathrm{diam}(T_m) \leq n-1$, then*

$$\hom(T_m, P_n) \leq \hom(T'_m, P_n).$$

(ii) *For any m, n,*

$$\hom(P_m, P_n) \leq \hom(T_m, P_n) \leq \hom(S_m, P_n).$$

Ideals of the proof:

- (i) by the symmetry and unimodality of $\mathbf{h}(T_m, P_n)$.
- (ii) $\hom(P_m, P_n) \leq \hom(T_m, P_n)$
  - n even: by (i).
  - n odd: by the symmetry, bi-unimodal, log-concavity of $\mathbf{h}(T_m, P_n)$ using the LS-switch, Short-path shift and Claw-deletion.

$$\begin{array}{ccccc}
\hom(P_n, P_n) & \le & \hom(P_n, T_n) & \le & \hom(P_n, S_n) \\
\wedge\!| & & ? & & \wedge\!| \\
\hom(T_n, P_n) & \le & \hom(T_n, T_n) & X & \hom(T_n, S_n) \\
\wedge\!| & & \wedge\!| & & \wedge\!| \\
\hom(S_n, P_n) & \le & \hom(S_n, T_n) & \le & \hom(S_n, S_n)
\end{array}$$

Figure : Trees with the same size

The number of endomorphisms:

$$|\,\mathrm{End}(P_n)| \le |\,\mathrm{End}(T_n)| \le |\,\mathrm{End}(S_n)|.$$

Summary of our results: trees with different sizes

$$
\begin{array}{ccccc}
\hom(P_m, P_n) & \leq & \hom(P_m, T_n) & \leq & \hom(P_m, S_n) \\
\wedge| & & X & & \wedge| \\
\hom(T_m, P_n) & \overset{(*)}{\leq} & \hom(T_m, T_n) & X & \hom(T_m, S_n) \\
\wedge| & & \wedge| & & \wedge| \\
\hom(S_m, P_n) & \leq & \hom(S_m, T_n) & \leq & \hom(S_m, S_n)
\end{array}
$$

Figure : Trees with different sizes. The $(*)$ means that there are some well-determined (possible) counterexamples which should be excluded.

Further work

Conjecture

Let T_n be a tree on n vertices, where $n \geq 5$. Then for any tree T_m we have

$$\hom(T_m, P_n) \leq \hom(T_m, T_n).$$

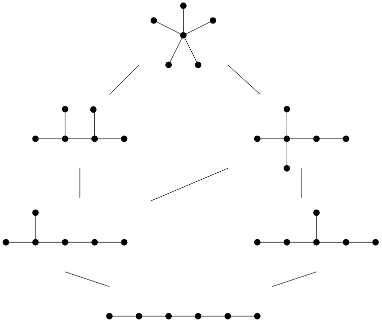

Figure : EL-Shellable? Cohen-Maculay? Möbius functions on the intervals alternate in sign?

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Merci!