UNIVERSITÉ DE NICE-SOPHIA ANTIPOLIS - UFR SCIENCES
École Doctorale de Sciences Fondamentales et Appliquées

THÈSE

Présentée pour obtenir le titre de

Docteur en SCIENCES
de l'Université de Nice Sophia Antipolis

Spécialité :

MATHÉMATIQUES

par

Grégoire LOEPER

Applications de l'équation de Monge-Ampère à la modélisation des fluides et des plasmas

Soutenue publiquement le 1er Décembre 2003
devant le jury composé de :

M. Jacques BLUM	Université de Nice	invité
M. Yann BRENIER	Université de Nice	Directeur
M. Pierre DEGOND	CNRS, Université Paul Sabatier	Rapporteur
M. Philippe DELANOE	CNRS, Université de Nice	Examinateur
M. Uriel FRISCH	CNRS, Observatoire de la Côte d'Azur	Examinateur
M. Philippe GHENDRIH	CEA Cadarache	invité
M. Frédéric POUPAUD	Université de Nice	Examinateur
M. Cédric VILLANI	ENS de Lyon	Rapporteur

à 14 heures au laboratoire J.A. Dieudonné, Parc Valrose, 06108 NICE Cedex 2

Résumé

L'objet de cette thèse est d'utiliser les outils du transport optimal et l'équation de Monge-Ampère dans l'étude de certaines équations aux dérivées partielles de la mécanique des fluides, modélisant fluides, plasmas et matière stellaire. On y étudiera notamment :

- des modèles dynamiques provenant du transport optimal, représentant une relaxation géométrique de l'équation d'Euler incompressible,

- les équations semi-géostrophiques, utilisées en météorologie,

-le problème de reconstruction en cosmologie, qui généralise le transport optimal à des coûts de transport dépendant d'une énergie interne du système.

Mots-clés

Mécanique des fluides, équations cinétiques, transport optimal, calcul variationnel, analyse asymptotique, équations aux dérivées partielles non-linéaires, équation de Monge-Ampère.

Abstract

In this thesis we use optimal transportation techniques and Monge-Ampère equation to study some partial differential equations arising in fluid mechanics, plasma physics and cosmological modelling. Our work studies :

- a geometrical relaxation of the Euler incompressible equation, derived using optimal transportation,

- the semi-geostrophic model, used in meteorology,

- the reconstruction problem in cosmology, that generalizes optimal transportation to costs depending on an internal energy of the system.

Applications de l'équation de Monge-Ampère à la modélisation des fluides et des plasmas.

Grégoire Loeper
Université de Nice-Sophia-Antipolis

REMERCIEMENTS

Je tiens à remercier tout d'abord Yann Brenier qui a accepté de diriger ma thèse. Je lui suis reconnaissant de m'avoir fait découvrir des sujets qui m'ont passionné, et de m'avoir fait partager sa vision éclairante des mathématiques.

Je remercie Pierre Degond et Cédric Villani qui ont accepté de rapporter sur cette thèse, pour leur lecture attentive et leurs commentaires et je remercie Philippe Delanöe, Uriel Frisch, Frédéric Poupaud, Jacques Blum et Philippe Gendrih pour leur participation au jury.

Je remercie toute l'équipe du laboratoire J.A. Dieudonné qui m'a accueilli pendant ces trois années. J'ai pu profiter du contact stimulant de nombreuses personnes je remercie ici pour des discussions enrichissantes Alexis Vasseur, Francesca Rapetti, Jacques Blum, Philippe Delanöe, Jean-Francois Collet, Thierry Goudon, Michel Rascle, Frédéric Poupaud, Jean-Francois Burnol, Mathieu Brassart et Jean-Francois Aujol. Merci aussi à Annie Borel et Janine Lachkar qui m'ont aidé dans les tâches administratives. Je ne mentionne pas ici les membres du laboratoire avec qui les liens sortent du cadre mathématique, notamment les membres de la section escalade. Je remercie également Jo Del Rey et l'équipe de Rubgy de l'Université de Nice avec qui j'ai partagé des moments inoubliables au cours de ces trois années.

Je remercie l'équipe de l'Observatoire de la Côte d'Azur, notamment Uriel Frisch et Andreï Sobolevskii avec lesquels les échanges furent nombreux et fructueux.

Je remercie également le Département de mathématiques d'Austin et Luis Caffarelli pour leur hospitalité.

Je remercie l'Institut Fields de Toronto qui m'accueille au moment où j'écris ces lignes et plus particulièrement Robert McCann.

Cette soutenance marquant la fin de ma vie étudiante, il est également temps de remercier ceux qui m'ont permis de mener ces études, je pense à mes parents et grand-parents.

Enfin merci à Marine, de m'avoir soutenu pendant ce travail, et de continuer.

Grégoire Loeper
Toronto, Novembre 2003

Table des matières

CHAPITRE 1

Introduction

L'objet de cette thèse est d'utiliser les outils du transport optimal, et donc l'équation de Monge-Ampère qui y est associée, dans l'étude de certaines équations aux dérivées partielles (EDP) de la mécanique des fluides modélisant fluides, plasmas et cosmologie. Le transport optimal lui-même est un sujet à part entière qui répond à un problème pratique compréhensible : transporter à un moindre coût. Si la motivation originelle du problème était de transporter de la terre ou du sable, la problèmatique du transport optimal est présente sous des formes moins évidentes dans la nature, ou l'évolution est souvent guidée par un principe de moindre action. Pour un certain choix du coût de transport, la solution du problème de transport optimal est liée à la résolution de l'équation de Monge-Ampère

$$\det \partial_{x_i}\partial_{x_j}\Psi = f.$$

Cette dernière était déjà étudiée pour des applications géométriques comme le problème de Minkowski (trouver une hypersurface à courbure de Gauss prescrite). Son étude représente une part non négligeable du domaine des EDP elliptiques et on s'appuiera sur les nombreux résultats d'existence et de régularité déjà obtenus.

Récemment de nombreuses applications du transport optimal ont été trouvées comme en météorologie, avec les équations semi-géostrophiques de Hoskins. Il apparait souvent comme un outil puissant pour fournir des démonstrations élégantes et directes d'inégalités d'analyse fonctionnelle (inégalités de type Log-Sobolev), où d'inégalités géométriques (ex. Brun Minkowski) ou pour fournir une interprétation élégante et nouvelle de phénomènes pourtant bien connus (les équations des milieux poreux sont interprétées comme des flots gradients pour une certaine métrique).

Dans cette thèse nous tenterons d'apporter une contribution à la compréhension de problèmes dont la relation avec le transport optimal est déjà connue comme les équations semi-géostrophiques (chapitres 4 et 5), mais aussi de l'appliquer à d'autres problèmes de modélisation en tentant de faire apparaître le bien fondé de ces applications. La première application sera l'approximation des équations d'Euler incompressible (chapitres 2 et 3). L'équation de Monge-Ampère apparaitra naturellement dans un modèle dynamique dont on montrera qu'il représente une approximation naturelle des équations d'Euler incompressible.

Une autre application nouvelle sera l'étude du mouvement de la matière stellaire (équation d'Euler-Poisson) (chapitre 6 et article joint en annexe). On traitera le problème inverse pour le système Euler-Poisson gravitationnel : trouver une solution de ce système en prescrivant non pas densité et vitesse initiales, mais densités intiale et finale. En effet il apparaît que ce problème peut se formuler comme un problème de transport optimal où le coût ne dépend pas seulement de la distance mais aussi d'une énergie interne du système.

Enfin le dernier chapitre (chapitre 7) de la thèse, qui n'est pas lié au transport optimal, concerne l'étude du comportement asymptotique d'un plasma (gaz ionisé) soumis à un champ électrique turbulent et à un fort champ magnétique. Cette étude vise des applications à la modélisation des réacteurs à fusion nucléaire.

Mis à part le chapitre 1 d'introduction, chaque chapitre de la thèse correspond à un article qui peut donc être lu indépendamment.

1. Concepts mathématiques de mécanique des milieux continus

1.1. Notations. Se placant dans $\mathbb{R}^d$ on désignera la position par $x = (x_1, ..., x_d)$ et le temps par $t \in \mathbb{R}$. L'opérateur de dérivée partielle par rapport à une coordonnée x_i sera noté ∂_i et par rapport au temps ∂_t. L'opérateur gradient qui à $f : \mathbb{R}^d \to \mathbb{R}$ associe $(\frac{\partial f}{\partial x_i}), i = 1..d$ sera noté ∇_x et pour un champ de vecteurs $v = v^i, i = 1..d$ on notera $\nabla_x \cdot v$ la divergence de v égale à $\sum_{i=1}^d \frac{\partial v^i}{\partial x_i}$. Si cela ne porte pas à confusion on notera seulement ∇ au lieu de ∇_x.

1.2. Transport de densités. Soient X, Y deux espaces topologiques, L une application borélienne de X dans Y, et μ une mesure bornée sur X. Soit $C_b(Y)$ l'ensembrle des fonctions continues et bornées sur Y ; on notera $\nu = L\#\mu$ la mesure image de μ par L définie par

$$d\nu = L\#d\mu \Longleftrightarrow \forall f \in C_b(Y), \int_Y f d\nu = \int_X f(L(x))d\mu.$$

1.3. Formulations lagrangiennes, hydrodynamiques et cinétiques. On veut décire mathématiquement le mouvement d'un continuum de matière. Pour cela on se donne A un espace métrique compact muni d'une mesure de probablité da. A sera l'ensemble des "noms" des particules. Soit L_0 de A dans $\mathbb{R}^d$, qui donne la position initiale de chaque particule. Si $(t, x) \to v(t, x)$ est un champ de vecteurs suffisamment régulier (disons C^1) sur $\mathbb{R}^d$ alors la solution (unique localement en temps par le théorème de Cauchy-Lipschitz) de l'équation différentielle ordinaire

$$\partial_t L(t, a) = v(t, L(t, a))$$
$$L(0, a) = L_0(a)$$

donnera pour tout $a \in A$ la position à l'instant t de la particule a située en $L_0(a)$ à l'instant initial et se déplacant avec le champ de vitesse v. De plus à tout instant t on aura

$$\rho(t, \cdot) = L(t, \cdot)\#da$$

et la paire ρ, v satisfaira l'equation de conservation de la masse où équation de continuité

$$\partial_t \rho + \nabla \cdot (\rho v) = 0. \tag{1}$$

$\partial_{tt} L(t, a)$, l'accélération de la particule a, sera donnée dans sa formulation "eulérienne" par

$$\partial_{tt} L(t, a) = (\partial_t v + v \cdot \nabla v)(t, L(t, a))$$

où $v \cdot \nabla v$ est le vecteur de $\mathbb{R}^d$ de j-ième coordonnée $\sum_{i=1}^d v^i \partial_i v^j$. Cette formulation appelée eulérienne nécessite que la vitesse soit déterminée par la position ce qui est la norme pour un fluide mais pas pour un plasma en général (où plusieurs particules peuvent se trouver au même endroit avec des vitesses différentes). Dans cas d'un plasma sans collision on peut s'affranchir de cette contrainte en considérant pout $t \in I$ une application $P(t)$ de A dans $\mathbb{R}^d \times \mathbb{R}^d$, $a \to P(t, a) = (X, \Xi)(t, a)$ où la première coordonnée donne la position et la deuxième la vitesse. En considérant $(t, x, \xi) \to \gamma(t, x, \xi) \in \mathbb{R}^d$ un champ C^1 sur $\mathbb{R}^d \times \mathbb{R}^d$, la solution de l'équation différentielle ordinaire

$$\partial_t X(t, a) = \Xi(t, a)$$
$$\partial_t \Xi(t, a) = \gamma(t, X(t, a), \Xi(t, a))$$

donnera pour tout $a \in A$ et à tout instant t la position et la vitesse de la particule a accélérée dans le champ γ. De plus à tout instant t on aura

$$f(t, \cdot, \cdot) = P(t, \cdot, \cdot)\#da$$

et la paire (f, γ) satisfera l'équation dite cinétique (ou de Vlasov, ou de Liouville selon les terminologies en vigueur)

$$\partial_t f(t, x, \xi) + \xi \cdot \nabla_x f(t, x, \xi) + \gamma \cdot \nabla_\xi f(t, x, \xi) = 0. \tag{2}$$

Si $f(t,x,\xi) = \rho(t,x) \otimes \delta(\xi = v(t,x))$ pour un certain champ de vecteur v alors alors l'équation (2) suivie par f implique v suit l'équation

$$\partial_t v(t,x) + v(t,x) \cdot \nabla v(t,x) = \gamma(t,x,v).$$

La formulation eulérienne (ou hydrodynamique) est donc un cas particulier de la formulation cinétique où la mesure $f(t,\cdot,\cdot)$ est supportée par le graphe d'un champ de vecteur.

2. Brève introduction au transport optimal

La théorie du transport optimal remonte au problème de Monge dit des déblais et des remblais et consiste à trouver la manière la moins coûteuse de déplacer un tas de sable dans un trou de même volume. En termes plus mathématiques, on cherchera à transporter une mesure de probabilité sur une autre mesure de probabilité. Le coût du transport d'une particule du point x au point y sera $c(x,y)$ et en général la fonction $c(x,y)$ sera une fonction de la distance $d(x,y)$. Une approche naturelle consiste à chercher la solution du problème suivant : trouver une application $\bar{T}$ telle que $\bar{T}\#\mu = \nu$ et telle que

$$\int_X c(x,\bar{T}(x))d\mu(x) = \inf_{T\#\mu=\nu} \left\{ \int_X c(x,T(x))d\mu(x) \right\}.$$

Un progrès important fut fait par Kantorovitch qui formula le problème de la manière suivante : soient μ et ν deux mesures de probabilité sur un Banach X, on définit $\Gamma(\mu,\nu)$ comme l'ensemble des mesures de probabilité sur $X \times X$ ayant μ et ν pour marginales, i.e.

$$\begin{aligned} & \gamma \in \Gamma(\mu,\nu) \iff \\ \forall f \in C_b(X), \quad & \int_{X\times X} f(x)d\gamma(x,y) = \int_X f(x)d\mu(x) \\ \text{et} \quad & \int_{X\times X} f(y)d\gamma(x,y) = \int_X f(y)d\nu(y). \end{aligned}$$

On voit que ce problème est une relaxation du précédent car à toute application T vérifiant $T\#\mu = \nu$ on peut associer la mesure $\gamma_T = \mu \otimes \delta(y = T(x))$ qui appartient à $\Gamma(\mu,\nu)$. Le problème de transport optimal est alors le suivant : étant donné une fonction coût $c : X \times X \to \mathbb{R}^+$, trouver une probabilité γ_0 minimisant

$$C(\gamma) = \int_{X\times X} c(x,y)d\gamma(x,y).$$

parmi tous les $\gamma \in \Gamma(\mu,\nu)$. Dans cette thèse on s'interessera tout particulèrement au cas dit quadratique où la fonction coût est la distance euclidienne au carré entre x et y : $c(x,y) = \frac{1}{2}|x-y|^2$ et $X = \mathbb{R}^d$. Il a été montré dans [**10**] que si $d\mu$ est absolument continue par rapport à la mesure de Lebesgue, la probabilité optimale est alors supportée par le graphe du gradient d'une fonction convexe :

$$\gamma_{opt} = \mu \otimes \delta(y = \nabla\Phi(x))$$

et $\nabla\Phi$ est l'unique ($d\mu$ presque partout) gradient de fonction convexe satsifaisant

$$\forall f \in C^0(\mathbb{R}^d), \int_{\mathbb{R}^d} f(x)d\nu(x) = \int_{\mathbb{R}^d} f(\nabla\Phi(x))d\mu(x). \tag{3}$$

Cette formulation est une forme faible de l'équation de Monge-Ampère

$$\frac{d\nu}{dx}(\nabla\Phi)\det D^2\Phi = \frac{d\mu}{dx}.$$

où $\frac{d\nu}{dx}, \frac{d\mu}{dx}$ sont les densités (si elles existent) des mesures μ, ν. Notons que Ψ la transformée de Legendre de Φ, définie par

$$\Psi(y) = \sup_{x\in\mathbb{R}^d}\{x\cdot y - \Phi(x)\}$$

sera solution du problème en intervertissant μ et ν, et satisfera donc

$$\forall f \in C^0(\mathbb{R}^d),\ \int_{\mathbb{R}^d} f(x)d\mu(x) = \int_{\mathbb{R}^d} f(\nabla\Psi(x))d\nu(x). \tag{4}$$

Une application frappante de ce résultat est la factorisation polaire des applications :
Soit Ω un ouvert borné de $\mathbb{R}^d$. On introduit $G(\Omega)$ l'ensemble des applications $g : \Omega \to \Omega$ préservant la mesure de Lebesgue de Ω notée $\mathbf{1}_\Omega dx$:

$$\forall g \in G(\Omega),\ \forall f \in C^0(\Omega),\ \int_\Omega f(g(x))dx = \int_\Omega f(x)dx.$$

On suppose que $\partial\Omega$ est de mesure de Lebesgue nulle. Soit $\mathbf{m} : \Omega \to \mathbb{R}^d$ telle que la mesure image de $\mathbf{m}$ donnée par $\mathbf{m}\#\mathbf{1}_\Omega dx$ soit absolument continue par rapport à la mesure de Lebesgue de $\mathbb{R}^d$, alors il existe une unique décomposition de $\mathbf{m}$ de la forme

$$\mathbf{m} = \nabla\Phi \circ \pi$$

avec Φ convexe de $\mathbb{R}^d$ dans $\mathbb{R}$. $\nabla\Phi$ est unique $\mathbf{1}_\Omega dx$ presque partout. De plus Φ est la solution du problème précédent avec μ la mesure de Lebesgue de Ω et ν la mesure image de μ par $\mathbf{m}$.
Notons que $G(\Omega)$ est inclus dans l'espace de Hilbert $H = L^2(\Omega, \mathbb{R}^d)$. Alors π a la propriété supplémentaire d'être la projection orthogonale de $\mathbf{m}$ sur $G(\Omega)$:

$$\|\mathbf{m} - \pi\|_{L^2(\Omega)} = \inf\{\|\mathbf{m} - g\|_{L^2(\Omega)},\ g \in G(\Omega)\}.$$

2.0.1. *Le cas périodique.* Dans le cas où $\mathbb{R}^d$ est remplacé par une variété compacte lisse et sans bord, il a été montré dans [**49**] que les résultats énoncés ci-dessus se transposent intégralement. C'est notamment le cas pour le tore plat $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. La solution du problème de transport sera alors le gradient d'une fonction convexe, Φ, avec $\Phi(x) - |x|^2/2$ périodique. Cette dernière condition permet de définir sans ambigüité la classe de $\nabla\Phi(x)$ dans $\mathbb{T}^d$ pour $x \in \mathbb{R}^d$. $\nabla\Phi$ sera alors l'unique gradient de fonction convexe, vérifiant

$$\forall \vec{p} \in \mathbb{Z}^d,\ \Phi(x+\vec{p}) - |x+\vec{p}|^2/2 = \Phi(x) - |x|^2/2 \tag{5}$$

$$\forall f \in C^0(\mathbb{T}^d),\ \int_{\mathbb{T}^d} f(x)d\nu(x) = \int_{\mathbb{T}^d} f(\nabla\Phi(x))d\nu(x). \tag{6}$$

3. Les équations d'Euler/Vlasov-Monge-Ampère

La factorisation polaire permet d'introduire le concept de géodésique approchée sur $G(\Omega)$ qui fait l'objet de la première partie de la thèse.

3.0.2. *Motivations.* Signalons tout d'abord la motivation de cette étude qui est son lien avec l'équation d'Euler incompressible rappelée ici :

$$\begin{aligned}&\partial_t v + v \cdot \nabla v = \nabla p\\&v \cdot n = 0 \text{ sur } \partial\Omega\\&\nabla \cdot v = 0.\end{aligned}$$

Ce système donne le mouvement d'un fluide parfait incompressible dans un domaine Ω. Il a été interprété par Arnold ([**2**]) comme décrivant les géodésiques pour la métrique $L^2(\Omega)$ sur le groupe des difféomorphismes de Ω de jacobien 1. Cette célèbre équation fait l'objet de nombreuses études et l'existence globale de solution faibles ou fortes en dimension 3 reste à ce jour un problème ouvert. Ce système décrit également le comportement asymptotique de nombreux systèmes issus de la mécanique des fluides et des plasmas : citons par exemple le cas d'un fluide parfait compressible dont le taux de compressibilité tend vers 0, le cas d'un fuide incompressible mais visqueux dont la viscosité tend vers 0, ainsi qu'un plasma où la permitivité électrique du milieu tend vers 0. La convergence repose donc sur l'interprétation physique de l'équation d'Euler : elle décrit le mouvement d'un fluide incompressible.

Ici nous allons construire un système qui constitue une approximation géométrique de l'équation d'Euler dans le sens de l'interprétation qui en a été donnée par Arnold.

On peut commencer par étudier un problème modèle. Soit M une sous-variété lisse de $\mathbb{R}^d$, considérons le système dynamique

$$\partial_{tt} X(t) = \frac{1}{2\epsilon^2} \nabla_x [d^2(X(t), M)].$$

Ce système s'écrit également de la manière suivante :

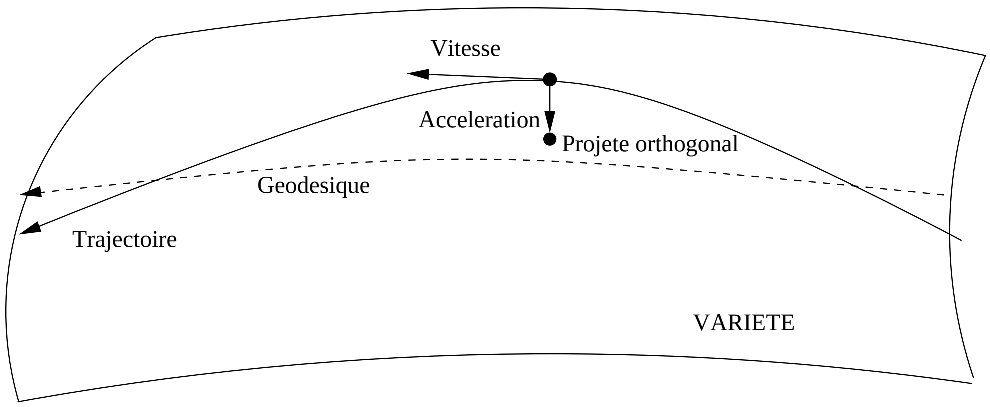
$$\partial_{tt} X(t) = \frac{1}{\epsilon^2}(\pi_X - X) \tag{7}$$

avec π_X la projection orthogonale (au sens de la distance euclidienne de $\mathbb{R}^d$) de X sur M. De plus il est Hamiltonien, son énergie étant donnée par

$$H(t) = \frac{1}{2}|\dot{X}|^2 + \frac{d^2(X, M)}{2\epsilon^2}$$

et indépendante du temps. On peut alors montrer que $t \to X(t)$ se comporte quand ϵ tend vers 0 et pour une donnée initiale bien préparée (X_0 proche de M et $\dot{X}_0$ presque parallèle à M en X_0) comme une géodésique de M.

Figure 1 : Principe de l'approximation géométrique



Nous étudierons un système analogue où $\mathbb{R}^d$ sera remplacé par $H = L^2(\Omega, \mathbb{R}^d)$ avec la métrique correspondante et M sera remplacé par $G(\Omega)$. X étant un élément de $L^2(\Omega, \mathbb{R}^d)$, sa projection π_X sera alors donnée par la factorisation polaire, i.e. $\pi_X = \nabla\Psi \circ X$, où pour tout t, $\Psi(t, \cdot)$ satisfait (4) avec $\mu = \mathbf{1}_\Omega dx$, $\nu = X(t, \cdot)\#\mu$. Le système (7) prend alors la forme suivante :

$$\partial_{tt} X(t, a) = \frac{1}{\epsilon^2}[\nabla\Psi(t, X(t, a)) - X(t, a)] \tag{8}$$

$$\nabla\Psi(t, X(t, \cdot)) \in G(\Omega), \Psi \text{ convexe.} \tag{9}$$

Si le résultat précédent s'étend à ce cas où la variété limite est de dimension infinie, et d'une régularité inconnue, alors comme les géodésiques de $G(\Omega)$ sont les solutions de l'équation d'Euler incompressibles, on aura montré que le système consiste en une approximation géométrique d'Euler incompressible et que ses solutions convergent vers celles d'Euler incompressible quand ϵ tend vers 0.

Ce résultat sera montré de plusieurs manières différentes : on étudiera deux versions du système (8, 9) : une version cinétique et une version hydrodynamique. On se placera à présent dans le tore plat $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ avec d quelconque.

3.1. L'équation de Vlasov-Monge-Ampère. Cette partie correspond au chapitre de la thèse. Elle fait l'objet d'un article co-écrit avec Y. Brenier ([**16**]).

L'intérèt d'un modèle cinétique est qu'il présente des propriétés analytiques qui le rendent sous certains aspects plus faciles à manier qu'un modèle fluide. La densité est notamment plus facilement contrôlable. C'est pourquoi nous commencerons ainsi notre étude. On considèrera $f : (t, x, \xi) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to f(t, x, \xi) \in \mathbb{R}^+$ avec t le temps, x la position et ξ la vitesse. La densité sera donnée par

$$\rho(t, x) = \int_{\mathbb{R}^d} f(t, x, \xi)\, d\xi.$$

L'équation de Vlasov-Monge-Ampère est alors la suivante :

$$\partial_t f + \xi \cdot \nabla_x f + \frac{1}{\epsilon^2}[\nabla\Psi(x) - x] \cdot \nabla_\xi f = 0 \tag{10}$$

$$\nabla\Psi(t,.)\#d\rho(t,.) = \mathbf{1}_\Omega dx, \;\; \Psi \text{ convexe.} \tag{11}$$

Ce système peut s'interpréter comme une version non-linéaire du système Vlasov-Poisson (12, 13) introduit plus bas. Notons que dans ce cas le champ éléctrique est borné dans L^∞ car $\nabla\Psi(t,x) \in \Omega$ et Ω est supposé borné. On montrera l'existence de solutions faibles globales renormalisées, et l'existence locale de solutions fortes. On procèdera ensuite à l'analyse asymptotique $\epsilon \to 0$. On montrera que le système converge effectivement vers la solution de l'équation d'Euler sur l'intervalle de temps où celle-ci est régulière et si les données initiales sont bien préparées. La convergence aura lieu au sens suivant :

$$\begin{aligned} G(t) &= \frac{1}{2}\int f(t,x,\xi)|\xi - \bar{v}(t,x)|^2 \; dx d\xi + \frac{1}{2\epsilon^2}\int \rho(t,x)|\nabla\Psi(t,x) - x|^2 \\ &\leq C(G(0)\mathrm{e}^{Ct} + \epsilon^2) \end{aligned}$$

où la constante C dépend des normes des dérivées de $\bar{v}$, la solution de l'équation d'Euler. La technique utilisée pour la preuve impose de se placer au voisinage d'une solution régulière de l'équation d'Euler incompressible. Cette condition remplace en fait la condition que variété limite soit lisse : il suffit de se placer au voisinage d'une géodésique régulière. Si la variété était lisse, toutes les géodésiques seraient régulières.

3.2. Le système Euler/Vlasov-Poisson. Le système de Vlasov-Poisson

$$\partial_t f + \xi \cdot \nabla_x f + \frac{1}{\epsilon^2}\nabla\phi \cdot \nabla_\xi f = 0 \tag{12}$$

$$\Delta\phi = \rho - 1 \tag{13}$$

décrit le comportement des électrons dans un plasma, les ions étant supposés au repos. La limite quasi-neutre $\epsilon \to 0$ a fait l'objet de nombreuses études et il a été montré dans [**13**] que pour des données initiales bien préparées, la solution converge vers la solution d'Euler incompressible. La convergence de Vlasov-Monge-Ampère vers Euler s'inspire fortement de cette preuve.

Nous étudierons le système Euler-Poisson (sans pression)

$$\partial_t \rho + \nabla \cdot (\rho v) = 0 \tag{14}$$

$$\partial_t v + v \cdot \nabla v = \frac{1}{\epsilon^2}\nabla\phi \tag{15}$$

$$\Delta\phi = \rho - 1 \tag{16}$$

qui est la version hydrodynamique de (12, 13), i.e. qui correspond au cas de données monocinétiques. On montrera alors que pour des données bien préparées, la solution de ce système converge vers la solution d'Euler incompressible en normes $L^\infty([0,T], H^s(\mathbb{T}^d))$ pour s grand, toujours en s'appuyant sur la régularité de la limite. Ce résultat, bien que présentant un intérêt indépendant, permettra d'obtenir une meilleure description de la limite quasi-neutre (i.e. $\epsilon \to 0$) de (10, 11) :

3.3. Lien entre Vlasov-Monge-Ampère et Euler-Poisson. Dans la limite $\epsilon \to 0$, les solutions de Vlasov-Monge-Ampère et de Euler-Poisson convergent toutes deux vers la solution d'Euler incompressible. On montrera de plus que la solution de Vlasov-Monge-Ampère se rapproche asymptotiquement à un ordre plus élevé de la solution d'Euler-Poisson lors de cette limite. La raison formelle de ce dernier résultat est que l'expression $\det(I + \epsilon^2 D^2\phi)$ se développe à l'ordre 1 en ϵ^2 en $1 + \epsilon^2\Delta\phi$. Comme $\phi = \frac{1}{\epsilon^2}(\Psi - x^2/2)$ satisfait formellement $\det(\mathbf{I} + \epsilon^2 D^2\phi) = \rho$ alors pour ρ proche de 1 cette équation devient formellement $1+\epsilon^2\Delta\phi = \rho$ et on retrouve l'équation de Poisson (13). Finalement les résultats obtenus pour des solutions faibles de (10, 11) seront également obtenus pour les solutions fortes du système suivant :

3.4. L'équation d'Euler-Monge-Ampère. Cette partie correspond au chapitre 3.
On étudie la forme hydrodynamique de (10, 11).

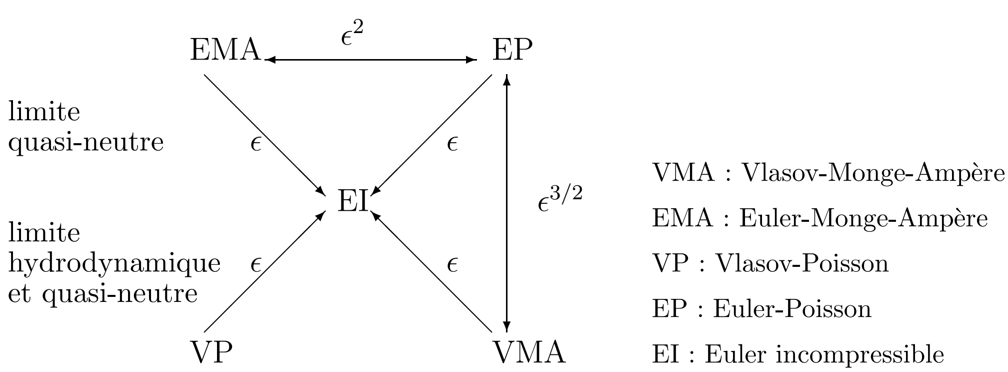
$$\partial_t\rho + \nabla\cdot(\rho v) = 0 \tag{17}$$

$$\partial_t v + v\cdot\nabla v = \frac{1}{\epsilon^2}[\nabla\Psi(x) - x] \tag{18}$$

$$\nabla\Psi(t,.)\#d\rho(t,.) = \mathbf{1}_\Omega dx. \tag{19}$$

En utilisant une technique similaire à celle utilisée pour étudier (15, 16) on montrera l'existence locale de solutions fortes, la convergence en normes $L^\infty([0,T], H^s(\mathbb{T}^d))$ pour s grand vers la solution d'Euler incompressible si cette dernière est régulière sur $[0,T]$. (On obtiendra ainsi l'existence de solutions quasi globales au voisinage des solutions globales régulières d'Euler). L'asymptotique à un ordre plus élevé vers (15, 16) sera également obtenues en norme d'ordre élevé. Ci-dessous un schéma récapitule les différents résultats de convergence. Les ordres de grandeur indiquent l'écart en vitesse. Noter que la convergence de VP vers EI a été prouvée par Brenier dans [**13**].

Figure 2 : Schéma récapitulatif des convergences



4. Le problème de reconstruction en cosmologie

4.1. La reconstruction MAK. Cette partie est le fruit d'une collaboration avec Y. Brenier, U. Frisch, M. Hénon, S. Matarrese, R. Mohayaee, A. Sobolevskii et fait l'objet d'un article ([**15**]), fourni en annexe.

4.1.1. *Le problème de reconstruction.* Il est admis que juste après le Big Bang et avant le découplage baryons/photons, l'univers présente une répartition quasi-homogène de matière noire froide sans collisions entre particules. On connaît également la répartition actuelle de matière dans l'univers, grâce aux observations et sous certaines hypothèses admises. La question est alors de déterminer l'évolution de la répartition de matière entre l'origine de l'univers ($t = 0$) et aujourd'hui ($t = T$). La connaisance de cette évolution permettrait notamment de connaître les fluctuations initiales de densité, ainsi que les fluctuations de la vitesse actuelle autour du mouvement d'expansion global.

4.2. Un modèle d'interaction gravitationnelle. Le mouvement d'un continuum de matière soumis à sa propre gravitation peut être décrit par l'équation d'Euler-Poisson, mais les forces entre particules étant attractives, l'équation de Poisson aura un signe opposé au cas éléctrostatique étudié plus haut. Le système Euler-Poisson gravitationnel prendra alors la forme suivante :

$$\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho v) = 0 \\
&\partial_t (\rho v) + \nabla \cdot (\rho v \otimes v) = -\rho \nabla p \\
&\Delta p = 4\pi \mathcal{G} \rho
\end{aligned}$$

où $\mathcal{G}$ est la constante de gravitation. Il suppose que la formulation hydrodynamique est valide : la solution doit rester monocinétique, cela est d'ailleurs vérifié par les mesures, pour des échelles suffisament grandes.

4.2.1. *L'approximation de Zel'dovich.* En se placant dans un système de coordonnées qui prend en compte l'expansion de l'univers (appelées coordonnées co-mobiles) en étudiant les fluctuations de vitesse et de densité autour du mouvement d'expansion uniforme, une première approximation consiste à considérer que les particules (dans le nouveau repère) se déplacent en ligne droite et à vitesse constante. D' autre part, pour que le problême ne soit pas singulier à l' origine, la vitesse initiale (toujours dans ce repère co-mouvant) doit être potentielle, ce phénomène également admis porte le nom de slaving. La position à l'instant t d'une particule initialement située en $X(0)$ seradonc donnée par

$$X(t) = X(0) + t\nabla\phi(X(0)).$$

Signalons qu'une justification rigoureuse (au moins en dimension 1) de cette approximation a été donnée dans [**52**]. La condition nécessaire et suffisante pour que la solution reste monocinétique pour des temps $t \leq T$ est que la fonction $x \to |x|^2/2 + T\phi(x)$ soit convexe. Sous l'hypothèse monocinétique, $\nabla\Phi = x + T\nabla\phi$ sera donc l'unique gradient de fonction convexe tel que $\nabla\Phi \# \rho_0 = \rho_T$.

4.2.2. *Résolution numérique.* Ce problème est alors résolu numériquement : On approche tout d'abord les mesures initiale et finale par des nuages de N points de masse $\frac{1}{N}$: $\rho_0 = \sum_{i=1}^N \frac{1}{N}\delta_{x=x_i}$, $\rho_T = \sum_{j=1}^N \frac{1}{N}\delta_{y=y_j}$. le problème devient alors un problème d'assignement. On cherche une permutation optimale $\bar{\sigma}$ telle que

$$\sum_{i=1}^N |x_i - y_{\bar{\sigma}(i)}|^2 = \inf_{\sigma \in S(n)} \{\sum_{i=1}^N |x_i - y_{\sigma(i)}|^2\}.$$

Le problème est ensuite relaxé : on ne cherche plus nécessairement ne permutation, un point x_i peut être réparti sur plusieurs points $y_{\sigma(j)}$. On obtient ainsi le problème de Kantorovitch : minimiser

$$C = \sum_{i,j=1}^N c_{ij} f_{ij}$$

où les variables f_{ij} satisfont

$$f_{ij} \geq 0, \quad \sum_{i=1}^{N} f_{ij} = \sum_{j=1}^{N} f_{ij} = 1.$$

et avec $c_{ij} = \frac{1}{2}|x_i - y_j|^2$. On va résoudre ici le problème dual : maximiser

$$D = \sum_{i=1}^{N} \beta_i - \sum_{j=1}^{N} \alpha_j$$

sous la contrainte

$$\beta_i - \alpha_j \leq c_{ij}.$$

Ce problème est résolu suivant un algorithme du à Bertseakas et Hénon, qui converge en $O(N^3)$ itérations. Les test comparant avec une simulation du problème à N corps montrent un taux de reconstruction exacte de 60%. Ce taux est calculé en comparant la permutation obtenue avec la permutation exacte qui est connue, puis en prenant le rapport en nombre de points bien "reconstruits" sur nombre de points total.

4.3. Reconstruction avec les équations d'Euler-Poisson. C'est le chapitre 6 de la thèse. Mentionnée dans le précédent travail, et examinée plus en détail dans le chapitre suivant, la reconstruction sans approximation de Zel'dovich peut-être traitée avec succès et présente en outre un intérêt théorique indépendant. On se placera dans le tore unité $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, et le système Euler-Poisson gravitationnel prendra alors la forme suivante :

$$\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho v) = 0 \\
&\partial_t (\rho v) + \nabla \cdot (\rho v \otimes v) = -\rho \nabla p \\
&\Delta p = \rho - 1
\end{aligned}$$

avec la contrainte $\int_{\mathbb{T}^d} \rho(\cdot, x)\ dx \equiv 1$. Ce système est Hamiltonien avec un Hamiltonien (i.e. une énergie) donné par

$$\mathcal{H}(t) = \frac{1}{2} \int_{\mathbb{T}^d} \rho(t,x)|v(t,x)|^2 - |\nabla p(t,x)|^2\ dx.$$

Les solutions des systèmes Hamiltoniens peuvent être obtenus en cherchant les points critiques pour l'action du Lagrangien. Si celui-ci s'avère convexe, on cherchera alors les minimiseurs de l'action du Lagrangien. Un exemple classique est l'équation d'Euler incompressible : le Lagrangien se réduit à l'énergie cinétique, et on trouve comme solutions formelles les géodésiques de $G(\Omega)$. Si l'on enlève la contrainte d'incompressibilité on trouve le transport optimal pour le coût quadratique. En général pour des systèmes hyperboliques tels que ceux de la dynamique des gaz barotropes le Lagrangien est une fonctionnelle concave en la densité, et la recherche de points critiques devient scabreuse. En revanche dans le cas présent, le Lagrangien est donné par :

$$\mathcal{L}(\rho(t,\cdot), v(t,\cdot)) = \frac{1}{2} \int_{\mathbb{T}^d} \rho(t,x)|v(t,x)|^2 + |\nabla p(t,x)|^2\ dx.$$

Après le changement de variables $\rho, v \to \rho, J = \rho v$ on obtient un Lagrangien convexe, et donc on cherchera les minimiseurs de l'action

$$\mathcal{I}(\rho_0, \rho_T) = \int_0^T \mathcal{L}(\rho(t,\cdot), v(t,\cdot))\, dt$$

sous les contraintes

$$\partial_t \rho + \nabla \cdot (\rho v) = 0$$
$$\rho_{t=0} = \rho_0$$
$$\rho_{t=T} = \rho_T.$$

4.4. Résultats. En utilisant des techniques de dualité on montrera tout d'abord l'existence et l'unicité d'un minimiseur pour le problème énoncé ci-dessus. Le champ de vitesse sera potentiel, avec potentiel ϕ solution en un sens qui sera examiné attentivement de l'équation de Hamilton-Jacobi définie ρ presque partout :

$$\rho(\partial_t \phi + \frac{1}{2}|\nabla \phi|^2 + p) = 0 \tag{20}$$

$$\partial_t \phi + \frac{1}{2}|\nabla \phi|^2 + p \leq 0. \tag{21}$$

Des propriétés de régularité interessantes seront alors montrées. La solution ne pourra pas développer de chocs à l'intérieur de l'intervalle $]0, T[$, et la densité ρ sera de plus bornée dans $L^\infty([\tau, T-\tau] \times \mathbb{T}^d)$ pour tout $\tau \in]0, T/2]$. Un point remarquable est que cette borne sera indépendante de la donnée ρ_0, ρ_T. Ce genre de résultat est à rapprocher de ceux d'Evans et Gomes [**33**] dans leur étude des ensembles d'Aubry-Mather et de la théorie KAM faible.

On voit que dans l'équation (20) le comportement de ϕ n'est imposé que ρ presque partout, et qu'une certaine latitude est laissée lorsque ρ peut s'annuler sur des ensembles de mesure non nulle. On montrera cependant que l'on peut choisir ϕ comme étant la solution de viscosité de

$$\partial_t \phi + \frac{1}{2}|\nabla \phi|^2 + p = 0.$$

Les résultats obtenus sur ρ permettront d'obtenir de la régularité pour p puis pour ϕ.

Enfin il sera noté au passage que ce problème induit naturellement une interpolation entre densités, et qu'au cours de ce déplacement, des quantités telles $\int_{\mathbb{T}^d} [\rho(t,x)]^k\, dx,\ k \geq 1$ seront convexes par rapport à t. On retrouve ainsi des résultats similaires à ceux de [**48**].

Rappelons que si l'on enlève le terme de Poisson on retrouve le problème de transport optimal de densité, sous la formulation de [**7**] : on cherche à minimiser

$$\mathcal{I}(\rho_0, \rho_T) = \int_0^T \int_{\mathbb{T}^d} \frac{1}{2}\rho(t,x)|v(t,x)|^2\, dtdx$$

sous les contraintes

$$\partial_t \rho + \nabla \cdot (\rho v) = 0$$
$$\rho_{t=0} = \rho_0$$
$$\rho_{t=T} = \rho_T.$$

Le potentiel ϕ de la vitesse vérifie alors

$$\rho(\partial_t\phi + \frac{1}{2}|\nabla\phi|^2) = 0,$$
$$\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 \leq 0$$

Les techniques employées fonctionnent ici encore pour obtenir les résultats de convexité de [**48**].

5. Régularité de la factorisation polaire pour des applications dépendant d'un paramètre.

C'est le chapitre 4. Si X est une application de $\Omega \subset \mathbb{R}^d$ dans $\mathbb{R}^d$, $\mathbf{1}_\Omega dx$ est la mesure de Lebesgue de Ω, et si la mesure $X_\#\mathbf{1}_\Omega dx$ est absolument continue par rapport à la mesure de Lebesgue, alors X se factorise de manière unique en

$$X = \nabla\Phi \circ g$$

avec Φ convexe et $g \in G(\Omega)$ préservant la mesure de Lebesgue de Ω. La question posée ici est la suivante : si l'application X dépend d'un paramètre, : $X : (t, a) \in I \subset \mathbb{R} \times \Omega \to \mathbb{R}^d$, quelle est la dépendance par rapport à t de sa factorisation polaire.

5.1. Motivations.

5.1.1. *Les équations Semi-Géostrophiques.* Une des motivations de cette question est l'étude des équations Semi-Géostrophiques, utilisées en météorologie [**27**]. Dans le cas le plus simple, ces équations se formulent de la manière suivante : étant donné un ouvert $\Omega \subset \mathbb{R}^2$ de mesure 1, on cherche une mesure de probabilité ρ satisfaisant

$$\partial_t\rho + \nabla\cdot(\rho v) = 0 \tag{22}$$
$$\bar{v}(t, x) = (\nabla\Psi(t, x) - x)^\perp \tag{23}$$
$$\nabla\Psi(t, \cdot)\#\rho(t, \cdot) = \mathbf{1}_\Omega dx, \;\; \Psi \text{ convexe.} \tag{24}$$

La suffixe $^\perp$ signifie tourné de $\pi/2$. Si $t, a \to X(t, a)$ pour $t \in I, a \in \Omega$ (remarquons que les noms de particules appartiennent ici à Ω) satisfait

$$X(t, \cdot)\#\mathbf{1}_\Omega da = \rho(t, \cdot)$$
$$\partial_t X(t, a) = \bar{v}(t, X(t, a))$$

alors pour tout t, $\nabla\Psi(t, \cdot)$ est l'unique gradient de fonction convexe tel que $\nabla\Psi(t, X(t, \cdot)) \in G(\Omega)$. On a également $\Psi(t, \cdot) = \Phi^*(t, \cdot)$ où pour tout $t \in I$, $X(t, \cdot) = \nabla\Phi(t, g(t, \cdot))$, $\Phi(t, \cdot)$ convexe et $g(t, \cdot) \in G(\Omega)$. Le champ de vitesse $\bar{v}$ est donc donné à chaque instant par la factorisation polaire de X.

5.1.2. *Une décomposition de Hodge liée à un problème elliptique dégénéré.* Formellement en dérivant la factorisation polaire de X on trouverait :

$$\partial_t X(t, a) = \partial_t\nabla\Phi(t, g(t, a)) + D^2\Phi(t, g(t, a))\partial_t g(t, a)$$

et en supposant g inversible alors on aurait

$$\partial_t X(t, g^{-1}(t, b)) = \partial_t\nabla\Phi(t, b) + D^2\Phi(t, b)\partial_t g(t, (g^{-1}(t, b)).$$

Formellement $\partial_t g(t, g^{-1}(t,\cdot))$ est un champ de vecteurs sur Ω à divergence nulle. Si X est le flot d'un champ de vecteur, i.e.

$$\partial_t X(t,a) = v(t, X(t,a))$$

alors on obtient une décomposition de type Helmoltz pour v de la forme :

$$v(t, \nabla\Phi) = \nabla p + D^2\Phi\, w$$

avec $\nabla \cdot w = 0$ et $w \cdot \partial\Omega = 0$. Il existe un problème elliptique naturellement associé à cette question : $\Psi(t,x)$ étant l'unique fonction convexe satisfaisant pour tout t

$$\det D^2\Psi(t,x) = \rho(t,x)$$

au sens de la factorisation polaire, $\partial_t\Psi$ satisfait

$$M_{ij}\partial_{ij}\partial_t\Psi = \partial_t\rho \tag{25}$$

où M_{ij} est la comatrice de $D^2\Psi$. Ce problème elliptique est en général dégénéré, mais il fournira néanmoins des estimations à priori qui mèneront au résultat de régularité.

5.1.3. *Résultat.* Dans le cas le plus général on suppose seulement que $\partial_t X \in L^\infty(I \times \Omega)$ et $\rho = X\#dx \in L^\infty(I \times \mathbb{R}^d)$ supportée dans $B(0,R)$ pour tout t. On a alors

$$\|\partial_t \nabla\Phi\|_{L^\infty(I,\mathcal{M}(\Omega))} \leq C(R,d)\|\rho\|^{\frac{1}{2}}_{L^\infty(I\times B_R)}\|\partial_t X\|_{L^\infty(I\times B_r)}$$

et $\phi \in C^\alpha(I, C^0(\bar\Omega))$ pour un certain $\alpha \in]0,1[$.
Sous l'hypothèse supplémentaire que la densité ρ est comprise entre deux valeurs strictement positives sur un ouvert convexe, alors on obtiendra que $\nabla\Phi, \nabla\Psi \in C^\alpha_{loc}(\mathbb{R}\times\Omega)$. En dimension 2 si $0 < \lambda_1 \leq \rho \leq \lambda_2$ on aura de plus $\partial_t\nabla\Phi \in L^\infty(I, L^p_{loc}(\Omega))$ pour un certain $p \in]1,2[$ dépendant de $\frac{\lambda_2}{\lambda_1}$.
Signalons que ces résultats s'appliquent de manière immédiate aux équations semi-géostrophiques qui vérifient toutes les hypothèses requises : la vitesse est bornée et les bornes inférieures et supérieures pour la densité sont indépendantes du temps.

5.1.4. *Remarque : Analogie avec Euler incompressible en dimension 2.* En dimension 2, l'équation d'Euler incompressible admet une formulation vorticité : On cherche une fonction Θ s'annulant sur $\partial\Omega$ telle que

$$\begin{aligned} &v = \nabla^\perp\Psi \\ &\partial_t\rho + \nabla\cdot(\rho v) = 0 \\ &\Delta\Psi = \rho. \end{aligned}$$

Le système semi-géostrophique s'obtient donc à partir d'Euler en remplacant l'équation de Poisson $\Delta\Psi = \rho$ par l'équation de Monge-Ampère $\det D^2\Psi = \rho$ cette dernière devant être entendue au sens $\nabla\Psi\#\rho = \mathbf{1}_\Omega$. Dans le cas Euler, $\partial_t\Psi$ satisfait l'équation de Poisson $\Delta\partial_t\Psi = \partial_t\rho$ qui remplace (25).

6. Contribution à l'étude des équations semi-géostrophiques

Cette partie correspond au chapitre 5. Dans cette courte partie, on montrera que les equations semi-géostrophiques introduites ci-dessus (22, 23, 24) admettent des solutions à valeurs dans les mesures bornées et des solutions régulières (densité continue ou Lipschitz). L'existence de solutions faibles pour ce système avait déjà été traitée ([**6**], [**26**],[**47**]). La technique utilisée pour montrer l'existence de solutions classiques est analogue à celle utilisée pour le sytème Vlasov-Monge-Ampère au chapitre 2.

7. Turbulence dans un plasma

C'est le chapitre 7. Ce travail a été effectué avec Alexis Vasseur. Il concerne la limite turbulente d'un plasma bidimensionnel soumis simultanément à un fort champ magnétique transverse (limite gyrocinétique) et à un champ électrique fortement oscillant. On montre que dans cette limite le plasma satisfait une équation de diffusion en énergie.

7.1. Motivations. Ce travail est motivé par l'étude de la turbulence éléctrique dans les tokamaks. Les tokamaks sont des réacteurs à fusion nucléaire qui sont encore à l'état de prototype. Pour que les réactions nucléaires puissent avoir lieu, les composants doivent se trouver à une température telle qu'ils sont uniquement présents sous forme de gaz ionisé (plasma). L'idée est alors de confiner le plasma au centre d'un réacteur torique (le tokamak) grâce à un fort champ magnétique dans la direction toroïdale. On peut considérer une section du tore et le problème devient alors bi-dimensionnel, avec un champ magnétique transverse. Les ions étant supposés au repos, l'équation cinétique satisfaite par la fonction de distribution des électrons
$f : (t, x, v) \in (\mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^2) \to f(t, x, v) \in \mathbb{R}^+$ est alors donnée par

$$m\left(\frac{\partial f}{\partial t} + v \cdot \nabla_x f\right) + q\left(Bv^\perp + \nabla V^{\rm turb}(t,x)\right) \cdot \nabla_v f = 0, \qquad (26)$$

m désigne la masse des électrons, q leur charge, B la norme du champ magnétique transverse, $v^\perp$ est la vitesse tournée de $\pi/2$ et $\nabla V^{\rm turb}(t,x)$ le champ électrique turbulent. Dans cette étude on négligera les effets non-linéaires du couplage entre densité et champ électrique : le champ éléctrique stochastique sera donné et indépendant de f. Le paramètre $\epsilon = \frac{m}{qB}$ désigne la fréquence cyclotronique, c'est à dire la fréquence de gyration d'un électron lorsque le champ électrique est nul. Quand ϵ tend vers 0, on montrera que le système satisfait à la limite l'équation de diffusion

$$\partial_t \rho - \partial_e(a(e)\partial_e \rho) = 0 \qquad (27)$$

où ρ est ici donné par $\rho(t,x,e) = \frac{1}{2\pi}\int_{|v|^2/2=e} f(t,x,v)\,dv$. Le coefficient $a(e)$ sera obtenu comme une fonction explicite de la corrélation V^{turb}, et sa dépendance en la variable e permet alors d'expliquer des phénomènes de diffusion anormale, là où la théorie quasi-linéaire obtenait des coefficients constants. Cette équation de diffusion est similaire au modèle SHE (spherical harmonics expansion) obtenue dans [**28**] pour modéliser le comportement asymptotique de certains semi-conducteurs.

CHAPITRE 2

The Vlasov-Monge-Ampère equation

A geometric approximation to the Euler equations : the Vlasov-Monge-Ampère system

Yann Brenier, Grégoire Loeper,
UMR 6621, Parc Valrose, 06108 Nice, France

Résumé. This paper studies the Vlasov-Monge-Ampère system (VMA), a fully non-linear version of the Vlasov-Poisson system (VP) where the (real) Monge-Ampère equation $\det \frac{\partial^2 \Psi}{\partial x_i \partial x_j} = \rho$ substitutes for the usual Poisson equation. This system can be derived as a geometric approximation of the Euler equations of incompressible fluid mechanics in the spirit of Arnold and Ebin. Global existence of weak solutions and local existence of smooth solutions are obtained. Links between the VMA system, the VP system and the Euler equations are established through rigorous asymptotic analysis.

1. Introduction

The classical Vlasov-Poisson (VP) system describes the evolution of an electronic cloud in a neutralizing uniform background through the following equations

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla_x f + \nabla_x \varphi \cdot \nabla_\xi f = 0 \tag{28}$$

$$\epsilon^2 \Delta \varphi = \rho - 1, \tag{29}$$

where $f(t, x, \xi) \geq 0$ denotes the electronic density at time $t \geq 0$, point $x \in \mathbb{R}^d$, velocity $\xi \in \mathbb{R}^d$ (usually $d = 3$), $\rho(t, x) \geq 0$ denotes the 'macroscopic' density

$$\rho(t, x) = \int_{\mathbb{R}^d} f(t, x, \xi) d\xi, \tag{30}$$

and $\varphi(t, x)$ denotes the electric potential at time t and point x generated, through the Poisson equation (29), where ϵ is a coupling constant, by the difference between the electronic density $\rho(t, x)$ and the neutralizing background density, which is supposed to be uniform and normalized to unity. Standard notations $\nabla = (\partial_1, ..., \partial_d)$ and $\Delta = \partial_1^2 + ... + \partial_d^2$ have been used and $\cdot$ stands for the inner product in $\mathbb{R}^d$. The mathematical theory of the VP system is now well understood. In particular, existence of global smooth solutions in three space dimensions has been proved in [**56**] (see also [**46**], [**60**]). In the present paper, a fully nonlinear version of the VP system is addressed :

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla_x f + \nabla_x \varphi \cdot \nabla_\xi f = 0 \tag{31}$$

$$\det(\mathbb{I} + \epsilon^2 D^2 \varphi) = \rho, \tag{32}$$

where the (real) Monge-Ampère equation (32) substitutes for the Poisson equation (29). Here, $D^2\varphi(t, x)$ stands for the $d \times d$ symmetric matrix made of all second order $x-$partial derivatives of φ, $\mathbb{I}$ stands for the $d \times d$ identity matrix and det for the determinant of a square matrix. The occurrence of the Monge-Ampère equation in mathematical modeling is rather unusual. Notice, however, that a very similar structure can be found in meteorology with Hoskins' semi-geostrophic equations (cf. [**6**], [**26**] and the included references).

Formally, as the coupling constant ϵ is small, the VP and VMA equations asymptotically approach each other up to order $O(\epsilon^4)$. Indeed, linearizing the determinant about the identity matrix leads to

$$\det(\mathbb{I} + \epsilon^2 D^2 \varphi) = 1 + \epsilon^2 \Delta \varphi + O(\epsilon^4). \tag{33}$$

The formal limit, as $\epsilon = 0$, reads

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla_x f + \nabla_x \varphi \cdot \nabla_\xi f = 0 \tag{34}$$

$$\rho = 1, \tag{35}$$

where constraint (35) substitutes for both the Poisson and the Monge-Ampère equations. The limit system, that we call constrained Vlasov system, can be seen as a 'kinetic' extension of the Euler equations of classical incompressible fluid mechanics,

$$\partial_t v + (v \cdot \nabla) v = -\nabla p, \tag{36}$$

$$\nabla \cdot v = 0, \tag{37}$$

where $v(t,x) \in \mathbb{R}^d$ and $p(t,x) \in \mathbb{R}$ respectively are the velocity and the pressure of the fluid at time t and position x. Indeed, any smooth solution (v,p) provides a 'monokinetic' solution to the constrained Vlasov system defined by

$$f(t,x,\xi) = \delta(\xi - v(t,x)), \quad \varphi = -p.$$

Here a monokinetic solution means a delta-valued solution in the ξ variable. In a similar way, there is a monokinetic version of the VP system, the so-called (pressureless) Euler-Poisson (EP) system, which reads

$$\partial_t v + (v \cdot \nabla) v = \nabla \varphi, \tag{38}$$

$$\partial_t \rho + \nabla \cdot (\rho v) = 0 \tag{39}$$

$$\epsilon^2 \Delta \varphi = \rho - 1. \tag{40}$$

A rigorous asymptotic analysis of the VMA system as $\epsilon \to 0$ will be provided (sections 5.1 and 5.2), in the case when the initial electronic density

$$f(t=0,x,\xi) = f^0(x,\xi), \tag{41}$$

is asymptotically monokinetic, namely approaching $\delta(\xi - v_0(x))$, for some smooth divergence free velocity field v_0, as ϵ tends to zero. Before this asymptotic analysis, we want to explain the geometric origin of the VMA system. It has been known, since Arnold's celebrated work (cf. [**2**]), that the Euler equations correspond to geodesics curves along a suitable group of volume preserving maps, lengths being measured in the L^2 sense. We will show (section 2) that the VMA system just describes approximate geodesics obtained through a very natural penalty method, where ϵ stands for the penalty parameter. For this geometric interpretation to be valid, the Monge-Ampère equation (32) must be understood in the following weak sense : for each fixed t, $\varphi(t,\cdot)$ is the unique (up to an additive constant) function such that $\Phi(x) = x^2/2 + \epsilon^2 \varphi(t,x)$ is convex in x and

$$\forall g \in C^0(\mathbb{R}^d), \int_{\mathbb{R}^d} g(\nabla \Phi(x)) \rho(t,x) dx \;=\; \int_\Omega g(y) dy, \tag{42}$$

where Ω is a fixed bounded open convex set where the neutralizing background of the electrons is assumed to be located. (This definition is made precise in section 2.3.) Notice that, by construction, $\nabla\Phi$ must be valued in the closure of Ω and, therefore, the potential φ enjoys the following property

$$|x + \epsilon^2 \nabla_x \varphi(t,x)| \leq \sup_{y \in \Omega} |y| < +\infty.$$

There is no similar bound for the electrostatic potential of the classical VP system. Thus, in some sense, the VMA system can be seen as a nonlinearly cutoffed version of the VP system.

Beyond the geometric derivation of the VMA system, our main analytic results are as follows :

- The VMA system admits global energy preserving weak solutions.
- The VMA system admits local strong solutions in periodic domains.
- For well prepared, nearly monokinetic, initial data, the solutions of the VMA system converge when ϵ goes to 0 to those of the Euler equations.
- In this asymptotic, the EP system is a higher order approximation of the VMA system.

The paper is organized as follows : in the first section, we introduce the concept of approximate geodesics for volume preserving maps, and derive the VMA system. The second section is devoted to the proof of existence of global energy preserving weak solutions. In the third section, we prove existence of local strong solutions, in the case of a periodic domain. In the final section, we study the asymptotic behavior of the VMA system as ϵ goes to 0.

2. The geometric origin of the Vlasov-Monge-Ampère system

2.1. The Euler equations. The motion of an incompressible fluid in a domain $\Omega \subset \mathbb{R}^d$ is described by the Euler equations (E) :

$$\partial_t v + (v \cdot \nabla)v = -\nabla p, \tag{43}$$

$$\nabla \cdot v = 0, \tag{44}$$

with $t \in \mathbb{R}$, $x \in \Omega$, where $v = v(t,x)$ stands for the velocity field and $p = p(t,x)$ for the scalar pressure field. These equations have a nice geometrical interpretation going back to Arnold (see [**2**]). Introducing $G(\Omega)$ the group of all volume preserving diffeomorphisms of Ω with jacobian determinant equal to 1, and measuring lengths in the L^2 sense, we may define (at least formally) geodesic curves along $G(\Omega)$. It turns out that the Euler equations just describe these curves.

2.2. Approximate geodesics. A general strategy to define approximate geodesics along a manifold M (in our case $M = G(\Omega)$) embedded in a Hilbert space H (here $H = L^2(\Omega, \mathbb{R}^d)$) is to introduce a penalty parameter $\epsilon > 0$ and the following *unconstrained* dynamical system in H

$$\partial_{tt} X + \frac{1}{2\epsilon^2} \nabla_X \left(d^2(X, M) \right) = 0. \tag{45}$$

In this equation, the unknown $t \to X(t)$ is a curve in H, $d(X, M)$ is the distance (in H) of X to the manifold M, i.e. in our case as $M = G(\Omega)$,

$$d(X, G(\Omega)) = \inf_{g \in G(\Omega)} \|X - g\|_H, \tag{46}$$

and, finally, ∇_X denotes the gradient operator in H. This penalty approach has been used for the Euler equations by Brenier in [**13**]. It is similar-but not identical- to Ebin's slightly compressible flow theory [**31**], and is a natural extension of the theory of constrained finite dimensional mechanical systems [**58**]. The penalized system is formally hamiltonian in variables $(X, \partial_t X)$ with Hamiltonian (or energy) given by :

$$E = \frac{1}{2}\|\partial_t X\|_H^2 + \frac{1}{2\epsilon^2} d^2(X, G(\Omega)).$$

(Multiplying equation (45) by $\partial_t X$, we formally get that the energy is conserved.) Therefore it is plausible that the map $X(t)$ will remain close to $G(\Omega)$ if properly initialized at $t = 0$. A formal computation shows that, given a point X for which there is a unique closest point π_X to X in the H closure of $G(\Omega)$, we have :

$$\nabla_X \left(d(X,G)\right) = \frac{1}{d(X,G)}(X - \pi_X). \tag{47}$$

Thus the equation (45) formally becomes :

$$\partial_{tt} X + \frac{1}{\epsilon^2}(X - \pi_X) = 0. \tag{48}$$

To understand why solutions to such a system may approach geodesics along $G(\Omega)$ as ϵ goes to 0, just recall that, in the simple framework of a surface S embedded in the 3 dimensional Euclidean space, a geodesic $t \to s(t)$ along S is characterized by the fact that for every t, the plane defined by $\{\dot{s}(t), \ddot{s}(t)\}$ is orthogonal to S. In our case, $\partial_{tt} X(t)$ is nearly orthogonal to $G(\Omega)$ thanks to (48) meanwhile $X(t)$ remains close to $G(\Omega)$.
The approximate geodesic equation was introduced in [**13**] in order to allow a spatial approximation of $G(\Omega)$ by the group of permutations of N points A_j chosen to form a discrete grid on Ω. On such a discrete group, the concept of geodesics becomes unclear meanwhile approximate geodesics still make sense. They can be interpreted as trajectories of a cloud of N particles X_i moving in the Euclidean space $\mathbb{R}^{dN}$, which substitutes for H. These particles solve the following coupled system of harmonic oscillators

$$\epsilon^2 \frac{d^2 X_i}{dt^2} + X_i - A_{\sigma_i} = 0,$$

where σ is a time dependent permutation minimizing, at each fixed time t, $\Sigma \left|X_i - A_{\sigma(i)}\right|^2$ among all other permutations of the first N integers. The convergence of this discrete model to the incompressible Euler equations for well prepared initial data was proved in [**13**]. In order to study the continuous version (48), a specific study of the projection problem (46) is needed.

2.3. The polar decomposition theorem. Let us first recall a general measure theoretic definition :

DEFINITION 2.1. *Let A and B be two topological spaces, let ρ be a Borel finite measure of A and X a Borel map $A \to B$, we call the push-forward of ρ by X and note $X\#d\rho$ the Borel measure η on B defined by*

$$\forall f \in C^0(B), \int_B f(y) d\eta(y) = \int_A f(X(x)) d\rho(x)$$

Let us now consider the case of a bounded open subset Ω of the Euclidean space $\mathbb{R}^d$ equipped with the Lebesgue measure that we denote dx. We say that a Borel map $s : \overline{\Omega} \to \overline{\Omega}$ is volume (or Lebesgue measure) preserving if $s\#dx = dx$ i.e. if for all $g \in C^0(\overline{\Omega})$ one has $\int_\Omega g(x)dx = \int_\Omega g(s(x))dx$, or equivalently for any Borel subset B of $\overline{\Omega}$ one has $|s^{-1}(B)| = |B|$. The set of all measure preserving maps of Ω is a closed subset of the Hilbert space $H = L^2(\Omega, \mathbb{R}^d)$ and will be denoted by $S(\Omega)$. Notice that $S(\Omega)$ is only a semi-group for the composition rule and contains the group of volume preserving diffeomorphisms $G(\Omega)$. It is known [**51**] that, at least in the case when Ω is convex and $d \geq 2$, $S(\Omega)$ is exactly the closure of $G(\Omega)$ in $L^2(\Omega, \mathbb{R}^d)$, which implies $d(., G(\Omega)) = d(., S(\Omega))$.
The polar decomposition theorem for maps [**10**] (extended to Riemannian manifolds in [**49**]) will be crucial for our analysis of the VMA system :

THEOREM 2.2. *Let Ω be a bounded convex open subset of $\mathbb{R}^d$, let $X \in L^2(\Omega;\mathbb{R}^d)$ and $\rho_X = X\#dx$, where dx is the Lebesgue measure on Ω. Assume ρ_X to be a Lebesgue integrable function, or, equivalently, X to satisfy the non-degeneracy condition :*

$$\forall E \subset \mathbb{R}^d \text{ Borel },\ |E| = 0 \Rightarrow |X^{-1}(E)| = 0. \tag{49}$$

Then there exists a unique pair $(\nabla\Phi_X, \pi_X)$ where Φ_X is a convex function, and $\pi_X \in S(\Omega)$ such that

$$X = \nabla\Phi_X \circ \pi_X. \tag{50}$$

In this 'polar decomposition', π_X is also characterized as the unique closest point to X on $S(\Omega)$ in the L^2 sense and Φ_X is characterized to be (up to an additive constant) the unique convex function on Ω satisfying

$$\int_{\mathbb{R}^d} g(x)d\rho_X = \int_\Omega g(X(y))dy = \int_\Omega g(\nabla\Phi_X(y))dy, \tag{51}$$

for any $g \in C^0(\mathbb{R}^d)$ such that $|g(x)| \le C(1+|x|^2)$.
In addition, the Legendre-Fenchel transform Ψ_X of Φ_X defined by

$$\Psi_X(x) = \sup_{y\in\Omega}\{x\cdot y - \Phi_X(y)\} \tag{52}$$

is Lipschitz continuous on $\mathbb{R}^d$, with Lipschitz constant bounded by $\sup_{x\in\Omega}|x|$ and has the following properties :
$\nabla\Phi_X(x) \in \Omega$ holds true for ρ_X a.e. x,

$$\int_{\mathbb{R}^d} g(\nabla\Psi_X)\rho_X(x)dx = \int_\Omega g(\nabla\Psi_X(X(x)))dx = \int_\Omega g(x)dx \tag{53}$$

for any $g \in C^0(\overline{\Omega})$, and

$$\nabla\Phi_X(\nabla\Psi_X(x)) = x, \quad \rho_X(x)dx \text{ a.e}, \tag{54}$$

$$\nabla\Psi_X(\nabla\Phi_X(y)) = y, \quad dy \text{ a.e} \tag{55}$$

$$\pi_X(y) = \nabla\Psi_X(X(y)), \quad dy \text{ a.e.} \tag{56}$$

We make here several remarks on theorem 2.2 :

Link with the Monge-Ampère equation : We can interpret (51) as a weak version of the Monge-Ampère equation :

$$\rho_X(\nabla\Phi)\det D^2\Phi = 1$$

and (53) can be seen as a weak version of another Monge-Ampère equation :

$$\det D^2\Psi = \rho_X$$
$$\nabla\Psi \text{ maps } \mathrm{supp}(\rho_X) \text{ in } \Omega.$$

The pair Φ_X, Ψ_X depends in fact only of Ω and the measure $\rho_X = X\#dx$, and if condition (49) fails, then existence and uniqueness of the projection π_X may fail, but existence and uniqueness of $\nabla\Phi_X$ remain true.

Theorem 2.2 and the subsequent remarks allows us to introduce the following notation that will be used throughout the paper :

DEFINITION 2.3. *Let Ω be a fixed bounded convex open set of $\mathbb{R}^d$, let ρ be a positive measure on $\mathbb{R}^d$ of total mass $|\Omega|$. We call $\Phi[\Omega,\rho]$, or, in short, $\Phi[\rho]$, the unique up to a constant convex function on Ω satisfying*

$$\forall g \in C^0(\mathbb{R}^d) \cap L^1(d\rho), \int_{\mathbb{R}^d} g(x)d\rho(x) = \int_\Omega g(\nabla\Phi[\Omega,\rho](y))dy. \tag{57}$$

We call $\Psi[\Omega,\rho]$ its Legendre-Fenchel transform satisfying

$$\forall g \in C^0(\mathbb{R}^d) \cap L^1(\Omega, dy), \int_{\mathbb{R}^d} g(\nabla\Psi[\Omega,\rho](x))d\rho(x) = \int_\Omega g(y)dy \tag{58}$$

If no confusion is possible we may write Φ (resp. Ψ) instead of $\Phi[\Omega,\rho]$ (resp. $\Psi[\Omega,\rho]$).

We will use some additional results from [**10**]. The first one establishes the continuity of the polar decomposition :

THEOREM 2.4. *Let ρ_n be a sequence of bounded positive measures on $\mathbb{R}^d$, of total mass $|\Omega|$ such that $\forall n$, $\int(1+|x|^2)d\rho_n \leq \infty$, let $\Phi_n = \Phi(\Omega,\rho_n)$ and $\Psi_n = \Psi(\Omega,\rho_n)$ be as in definition 2.3. If for any $f \in C^0(\mathbb{R}^d)$ such that $|f(x)| \leq C(1+|x|^2)$, $\int f\rho_n \to \int \rho f$, then $\Phi_n \to \Phi[\Omega,\rho]$ uniformly on each compact set of Ω and strongly in $W^{1,1}(\Omega;\mathbb{R}^d)$, and $\Psi_n \to \Psi[\Omega,\rho]$ uniformly on each compact set of $\mathbb{R}^d$.*

The second one provides a 'dual' definition of the distance between a map X and the semi-group $S(\Omega)$

THEOREM 2.5. *let $X \in L^2(\Omega;\mathbb{R}^d)$ and $\rho = X\#dx$, where dx is the Lebesgue measure on Ω. Assume ρ to be a Lebesgue integrable function. Then*

$$\begin{aligned}\frac{1}{2}d^2(X,S(\Omega)) &= \int \big(|x|^2/2 - \Psi[\Omega,\rho](x)\big)\,\rho(x)dx + \int_\Omega \big(|y|^2/2 - \Phi[\Omega,\rho](y)\big)\,dy \\ &= \sup_{u,v} \int \big(|x|^2/2 - u(x)\big)\,\rho(x)dx + \int_\Omega \big(|y|^2/2 - v(y)\big)\,dy,\end{aligned}$$

where the supremum if performed over all pairs (u,v) of continuous functions on $\mathbb{R}^d$ such that $u(x)+v(y) \geq x\cdot y$ pointwise.

2.4. The Vlasov-Monge-Ampère system. Let us now derive the VMA system as the kinetic formulation of the approximate geodesic equation (48). First, from the polar decomposition theorem 2.2, equation (48) reads

$$\partial_{tt}X(t,x) = \nabla\varphi(t,X(t,x)), \tag{59}$$

where

$$\nabla\varphi(t,x) = \frac{\nabla\Psi[\Omega,\rho(t,\cdot)](x) - x}{\epsilon^2} \tag{60}$$

and $\Psi[\Omega,\rho]$ is as in definition (2.3). This means that $\nabla\varphi$ satisfies (32) in a weak form with the additional condition that the range of $x \to x + \epsilon^2\nabla\varphi(t,x)$ is contained in $\overline{\Omega}$.

Next, let $f^0 \geq 0$ be a given initial density function, that we assume to be in $L^\infty(\mathbb{R}^d\times\mathbb{R}^d)$, compactly supported and satisfying the compatibility condition :

$$\int f^0(x,\xi)dxd\xi = |\Omega|. \tag{61}$$

For each $t \geq 0$, let us define $(x, \xi) \to f(t, x, \xi)$ to be f^0 pushed forward by the following ODE

$$\partial_t X(t, x, \xi) = \Xi(t, x, \xi) \tag{62}$$

$$\partial_t \Xi(t, x, \xi) = (\nabla\varphi)(X(t, x, \xi)) \tag{63}$$

$$(X, \Xi)(t = 0, x, \xi) = (x, \xi). \tag{64}$$

Then f satisfies the following kinetic (or Liouville) equation :

$$\frac{\partial f}{\partial t} + \nabla_x \cdot (\xi f) + \nabla_\xi \cdot (\nabla\varphi f) = 0 \tag{65}$$

$$f(0, \cdot, \cdot) = f^0, \tag{66}$$

which must be understood in the following weak sense :

$$\begin{aligned} &\forall g \in C_c^\infty([0, +\infty) \times \mathbb{R}^d \times \mathbb{R}^d), \\ &\int_0^\infty dt \int_{\mathbb{R}^d \times \mathbb{R}^d} \left(\frac{\partial g}{\partial t} + \xi \cdot \nabla_x g + \nabla\varphi \cdot \nabla_\xi g \right) f dx d\xi \\ &= - \int_{\mathbb{R}^d \times \mathbb{R}^d} f_0(x, \xi) g(t = 0, x, \xi) dx d\xi. \end{aligned} \tag{67}$$

This linear Liouville equation is nonlinearly coupled to equation (60), where ρ is linked to f by equation (30). Finally, we have defined, through (60,65,66), the weak formulation of the VMA initial value problem.

The energy of the system is defined by

$$\begin{aligned} E(t) = \quad & \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} f(t, x, \xi) |\xi|^2 dx d\xi \\ + \quad & \frac{1}{2\epsilon^2} \int_{\mathbb{R}^d} \rho(t, x) \left| \nabla\Psi[\Omega, \rho](t, x) - x \right|^2 dx. \end{aligned} \tag{68}$$

3. Existence of global renormalized weak solutions

The main result of this section is as follows :

THEOREM 2.6. *Let $(x, \xi) \to f_0(x, \xi) \geq 0$ be in $L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, with compact support in both x and ξ, satisfying condition (61).*
Then the VMA system (60,65,66) admits a global weak solution $(f, \rho, \nabla\Psi) \in L^\infty$. In addition, each such weak solution enjoys the following properties

- *f is a continuous function of t, valued in $L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ weak star,*
- *the density ρ is a continuous function of t, valued in $L^\infty(\mathbb{R}^d)$ weak star,*
- *the support of $f(t, \cdot, \cdot)$ in (x, ξ) is compact, with a diameter growing no more than linearly in t.*
- *the total energy defined by (68) is conserved,*
- *the 'renormalization' property (in the sense of [**30**])*

$$\frac{\partial g(f)}{\partial t} + \nabla_x \cdot (\xi g(f)) + \nabla_\xi \cdot (\nabla\varphi g(f)) = 0$$

  holds true for all $g \in C^1(\mathbb{R})$,
- *the trajectories of (63,64) are uniquely defined for almost every initial condition (x, ξ).*
- *$t \to f_h(t, \cdot, \cdot)$ is just f^0 pushed forward along the trajectories of (63,64).*

Proof of theorem 2.6 :
We build a sequence of approximate solutions (f_h, Ψ_h) by time discretization and let the time step h go to zero. To handle the limiting process, the non-linear terms will be treated with the help of theorem 2.4. More precisely if one can extract a subsequence such that, for every t, $f_h(t,\cdot,\cdot)$ converges weakly, then we can deduce from theorem 2.4 that the corresponding sequence $\nabla\Psi_h(t,\cdot)$ will converge strongly, and so we can pass to the limit in the nonlinear term.

3.1. Construction of a sequence of approximate solutions. We consider $\eta \in C_c^\infty(\mathbb{R}^d)$ such that $\eta \geq 0$, $\int_{\mathbb{R}^d} \eta = 1$ and $\eta_h = \frac{1}{h^d}\eta(\frac{\cdot}{h})$. We then seek approximate solutions as solutions of the approximate problem

$$\frac{\partial f_h}{\partial t} + \xi \cdot \nabla_x f_h + \frac{\nabla\Psi_h(x) - x}{\epsilon^2} \cdot \nabla_\xi f_h = 0 \tag{69}$$

$$f_h(0,x,\xi) = f_h^0(x,\xi) = f_0 \star_{x,\xi} \eta_h \otimes \eta_h \tag{70}$$

$$\Psi_h(t) = \eta_h \star \Psi(\Omega, \rho(t = nh)) \text{ for } t \in [nh,(n+1)h[. \tag{71}$$

$\nabla\Psi_h$ being a smooth function of space this regularized equation admits a unique solution that one builds by the method of characteristics. Since the flow is divergence-free in the phase space , the solution f_h satisfy

$$\forall p \in [1,+\infty],\ \|f_h(t)\|_{L^p_{x,\xi}} = \|f_h(0)\|_{L^p_{x,\xi}}. \tag{72}$$

By construction (through theorem 2.2), $\nabla\Psi_h$ is valued in the convex bounded set $\overline{\Omega}$. Suppose that $f^0(x,\xi)$ vanishes outside of the set $\{x^2 + \epsilon^2\xi^2 \leq C^2\}$ for some constant $C > 0$ fixed and denote $R = \sup_{y\in\Omega} |y|$. Then we have

LEMMA 2.7. $\forall t \geq 0,\ f_h(t,\cdot,\cdot)$ *is supported in* $\{\sqrt{x^2+\epsilon^2\xi^2} \leq C + Rt/\epsilon\}$.

Proof : We just write

$$\epsilon^2\partial_{tt}X + X = \nabla\Psi_h(X)$$

in complex notation $-i\epsilon\partial_t Z + Z = F$, where $Z = X + i\epsilon\partial_t X$ and $F = \nabla\Psi_h(X)$, which is bounded by R. This leads to

$$Z(t) = Z(0)\exp(-it/\epsilon) + i\epsilon^{-1}\int_0^t \exp(-i(t-s)/\epsilon)F(s)ds$$

ant the desired bound easily follows. Notice here a sharp contrast with the classical VP system, for which the $\xi-$support of the solutions cannot be controlled so easily (except in the one dimensional case). □

Convergence of the sequence of approximate solutions.
Using (72) and lemma 2.7 there exists, for any $1 < p < \infty$, up to the extraction of a subsequence, $f \in L^p([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$ such that f_h converges weakly to f as $h \to 0$.

It remains to show that the product $f_h\nabla\Psi_h$ converges to the good limit. For this we need a stronger convergence for $\nabla\Psi_h$. We already know that $\nabla\Psi_h \in L^\infty([0,T]\times\mathbb{R}^d)$. We claim that for all $t > 0$, $\nabla\Psi_h(t,\cdot)$ strongly converges to $\nabla\Psi(t,\cdot)$ in L^q_{loc}, $\forall q \in [1,+\infty[$. Indeed, such a strong convergence of $\nabla\Psi_h$ follows from Theorem 2.4 provided that we have for all $t > 0$,

$$\int_{\mathbb{R}^d} g(x)\rho_h(t,x)dx \to \int_{\mathbb{R}^d} g(x)\rho(t,x)dx. \tag{73}$$

for any $g \in C_c(\mathbb{R}^d)$. For this we show that the sequence ρ^h is relatively compact in $C([0,T], L^p - w)$. This is done by the following lemma :

LEMMA 2.8. *For all $T > 0$, for all p with $1 \le p < \infty$ the sequence f_h (resp. ρ^h) satisfies*
– f_h (resp. ρ^h) is a bounded sequence in $L^\infty([0,T]; L^p(\mathbb{R}^d \times \mathbb{R}^d))$ (resp. in $L^\infty([0,T]; L^p(\mathbb{R}^d))$.
– $\partial_t f_h$ (resp. $\partial_t \rho^h$) is a bounded sequence in $L^\infty([0,T]; W^{-1,p}(\mathbb{R}^d \times \mathbb{R}^d)))$, (resp. in $L^\infty([0,T]; W^{-1,p}(\mathbb{R}^d))$.
and one can extract from f^h (resp. from ρ^h) a subsequence converging in $C([0,T], L^p(\mathbb{R}^d \times \mathbb{R}^d) - w)$ (resp. in $C([0,T], L^p(\mathbb{R}^d) - w)$).

Proof : the first point uses equation (72) and lemma 2.7. The second point uses equation (65) and the identity :

$$\partial_t \rho_h = -\nabla_x \cdot \int_{\mathbb{R}^d} \xi f_h d\xi,$$

then the last point is a classical result of functional analysis. □

This lemma and lemma 2.7 yield (73). Then using theorem 2.4 and if we denote ρ the limit of a subsequence of ρ_h we have convergence of the sequence $\nabla\Psi_h$ to $\nabla\Psi[\Omega, \rho]$ in $C([0,T], L^p(\mathbb{R}^d))$. We have extracted a subsequence f_h such that
– f_h converges in $C([0,T], L^p(\mathbb{R}^d \times \mathbb{R}^d) - w)$ for every $1 \le p < \infty$.
– ρ_h converges in $C([0,T], L^p(\mathbb{R}^d) - w)$ for every $1 \le p < \infty$.
– $\nabla\Psi_h(t, \cdot)$ converges in $L^p(\mathbb{R}^d)$ for every t and for every $1 \le p < \infty$.
Thus the limit $(f, \nabla\Psi)$ satisfies equations (65-66) and the first part of theorem 2.6 is proved.

3.2. Conservation of energy. We now give a rigorous proof of the conservation of energy following an argument going back to F. Otto (in an unpublished work on the semi-geostrophic equations). We recall the definition of the energy as

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} f(t,x,\xi)|\xi|^2 dx d\xi + \frac{1}{2\epsilon^2} \int_{\mathbb{R}^d} \rho(t,x)|\nabla\Psi(t,x) - x|^2 dx.$$

We call the first term the kinetic energy E_c and the second term, multiplied by ϵ^2, the (normalized) potential energy E_p. We have

PROPOSITION 2.9. *Let f be any solution of 65 such that on every interval $[0,T]$, $f(t,\cdot,\cdot)$ is uniformly compactly supported in $|x|, \|\xi| \le R(T)$ for some function $R(T)$. Then the energy of the solution f is conserved.*

Proof :
From theorem 2.5, we know that

$$\begin{aligned} E_p(t) &= \int \left(|x|^2/2 - \Psi(t,x)\right) \rho(t,x) dx + \int_\Omega \left(|y|^2/2 - \Phi(t,y)\right) dy \\ &= \sup_{u,v} \int \left(|x|^2/2 - u(x)\right) \rho(t,x) dx + \int_\Omega \left(|y|^2/2 - v(y)\right) dy, \end{aligned}$$

where the supremum if performed over all pairs (u, v) of continuous functions on $\mathbb{R}^d$ such that $u(x) + v(y) \ge x \cdot y$ pointwise. Thus for each $t, t_0 \in \mathbb{R}+$, we have

$$E_p(t) \quad \ge \int \left(|x|^2/2 - \Psi(t_0,x)\right) \rho(t,x) dx + \int_\Omega \left(|y|^2/2 - \Phi(t_0,y)\right) dy.$$

and this implies

$$\begin{aligned} E_p(t) - E_p(t_0) \quad &\geq \int_{\mathbb{R}^d} \left(|x|^2/2 - \Psi(t_0,x)\right)(\rho(t,x) - \rho(t_0,x))dx \\ &= \int_{t_0}^t \int_{\mathbb{R}^d} \partial_t \rho(s,x) \left(|x|^2/2 - \Psi(t_0,x)\right) dxds \\ &= \int_{t_0}^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \xi f(s,x,\xi)\left(x - \nabla\Psi(t_0,x)\right) dxd\xi ds. \end{aligned}$$

Notice that the product in the second line makes sense because $\partial_t \rho$ is in $W^{-1,p}$ for any $1 \leq p <\leq \infty$ and Ψ is in $W^{1,\infty}$, moreover $f(t,\cdot,\cdot)$ and therefore $\rho(t,\cdot)$ are compactly supported. Exchanging t_0 and t we would have found :

$$E_p(t_0) - E_p(t) \geq \int_t^{t_0} \int_{\mathbb{R}^d \times \mathbb{R}^d} \xi f(s,x,\xi)\left(x - \nabla\Psi(t,x)\right) dxd\xi ds,$$

moreover we have for the kinetic energy :

$$\epsilon^2(E_c(t) - E_c(t_0)) = \int_{t_0}^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \xi f(t,x,\xi) \cdot (\nabla\Psi(s,x) - x) dxd\xi ds$$

dividing by $t - t_0, t > t_0$ we find

$$\begin{aligned} &\epsilon^2 \frac{E(t) - E(t_0)}{t - t_0} \\ &\geq \frac{1}{t - t_0} \int_{t_0}^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \xi f(s,x,\xi) \cdot (\nabla\Psi(s,x) - \nabla\Psi(t_0,x)) dxd\xi ds \end{aligned}$$

and

$$\begin{aligned} &\epsilon^2 \frac{E(t) - E(t_0)}{t - t_0} \\ &\leq \frac{1}{t - t_0} \int_{t_0}^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \xi f(s,x,\xi) \cdot (\nabla\Psi(t,x) - \nabla\Psi(s,x)) dxd\xi ds \end{aligned}$$

We know from 3.1 that $\nabla\Psi(t,.)$ converges strongly in $L^p_{loc}(\mathbb{R}^d), 1 \leq p < \infty$ to $\nabla\Psi(t_0,.)$ as t goes to t_0, and so the right hand sides of the above inequalities converges to 0 and we conclude that

$$\lim_{t > t_0} \frac{E(t) - E(t_0)}{t - t_0} = 0$$

We could take $t < t_0$ and find the same result. Finally we conclude that

$$\frac{dE}{dt} \equiv 0 \tag{74}$$

□

3.3. Renormalized solutions and existence of trajectories. The study of renormalized solutions for transport equations has been introduced in [**30**] for vector fields in $W^{1,1}$ and with bounded divergence. These results have been extended by Bouchut [**9**] to the case of Vlasov type equations with acceleration field in BV. The fact that solutions of (65, 66) are renormalized solutions is an immediate consequence of the following theorem :

THEOREM 2.10. (F. Bouchut).
Let $f \in L^\infty(]0,T[, L^\infty_{loc}(\mathbb{R}^d \times \mathbb{R}^d))$ satisfy

$$\frac{\partial f}{\partial t} + \nabla_x \cdot (\xi f) + \nabla_\xi \cdot (E(t,x) f) = 0,$$

with $E(t,x) \in L^1(]0,T[; L^1_{loc}(\mathbb{R}^d)) \cap L^1(]0,T[; BV_{loc}(\mathbb{R}^d))$,
then for any $g \in C^1(\mathbb{R})$

$$\frac{\partial g(f)}{\partial t} + \nabla_x \cdot (\xi g(f)) + \nabla_\xi \cdot (E(t,x) g(f)) = 0.$$

In our case the BV bound on the acceleration $\nabla\Psi$ is a direct consequence of the fact that Ψ is a globally Lipschitz convex function. Finally, as in [**30**], it can be deduced from the renormalization property that
1) for almost every initial condition (x, ξ), there is a unique trajectory solving (63,64),
2) $t \to f(t)$ is just f^0 pushed forward along these trajectories.

A complete proof is given in appendix.

Remark : From the renormalization property it follows that, once the potential $\Psi(t,x)$ is known, there exists a unique solution to (65) in $L^\infty_{t,x,\xi}$. Of course, this does not imply at all the uniqueness of weak solutions to the Vlasov-Monge-Ampère system ! This paragraph ends the proof of theorem 2.6.

4. Strong solutions

In this section we show existence of strong solutions for small times. To do this we need regularity estimates on solutions of the Monge-Ampère equation and this will be simpler to handle in the periodic case.

4.1. Polar factorization of maps in a periodic domain. The polar decomposition theorem has been generalized by McCann [**49**] to general Riemannian manifolds, while the particular case of the flat torus had been addressed in [**24**]. Here we use both results. We thus consider the flat torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$.

DEFINITION 2.11. *We say that a mapping $Y : \mathbb{R}^d \to \mathbb{R}^d$ is $\mathbb{Z}^d$ additive if the mapping $x \to Y(x) - x$ is $\mathbb{Z}^d$ periodic. The set of all measurable $\mathbb{Z}^d$ additive mappings is denoted $\mathcal{P}$. For each $x \in \mathbb{R}^d$ we call $\hat{x}$ the class of x in $\mathbb{R}^d/\mathbb{Z}^d$, and for any $X \in \mathcal{P}$, $\hat{X}$ the mapping of $\mathbb{T}^d$ into itself defined by*

$$\forall x \in \mathbb{R}^d, \hat{X}(\hat{x}) = \hat{X(x)}.$$

We may say if no confusion is possible additive instead of $\mathbb{Z}^d$ additive. Then the following theorem can be deduced from the results of [**24**] and [**49**] :

THEOREM 2.12. *Let $X : \mathbb{R}^d \to \mathbb{R}^d$ be additive and assume that $\rho_X = X\#dx$ has a density in $L^1([0,1]^d)$.*
Then there exists an a.e. unique decomposition

$$X = \nabla\Phi_X \circ \pi_X$$

such that Φ_X *is a convex function and* $\Phi_X(x) - |x|^2/2$ *is* $\mathbb{Z}^d$ *periodic,*
$\pi_X : \mathbb{R}^d \to \mathbb{R}^d$ *is additive and* $\hat{\pi}_X$ *is (Lebesgue) measure preserving in* $\mathbb{T}^d$*. Moreover we have*

$$\|X - \pi_X\|_{L^2([0,1]^d)} = \|\hat{X} - \hat{\pi}_X\|_{L^2(\mathbb{T}^d)}$$

and, Ψ_X *denoting the Legendre transform of* Φ_X*, we have*

$$\pi_X = \nabla\Psi_X \circ X.$$

Remark : The pair Φ_X, Ψ_X is uniquely defined by the density $\rho_X = X\#dx$.

Notice that the periodicity of $\Phi_X(x) - |x|^2/2$ implies that $\nabla\Phi_X$ and $\nabla\Psi_X$ are $\mathbb{Z}^d$ additive, and that $\Psi_X - |x|^2/2$ is also $\mathbb{Z}^d$ periodic. As in the previous case, we introduce the following notation :

Definition 2.13. *Let* ρ *be a probability measure on* $\mathbb{T}^d$ *then we denote* $\Phi[\rho]$ *(resp.* $\Psi[\rho]$*) the unique up to a constant convex function such that*

$$\Phi[\rho] - |x|^2/2 \text{ is } \mathbb{Z}^d \text{ periodic }, \tag{75}$$

$$\forall f \in C^0(\mathbb{T}^d),\ \int_{\mathbb{T}^d} f(\hat{\nabla\Phi}[\rho](x))dx = \int f d\rho \tag{76}$$

(resp. its Legendre fenchel transform).

$\Psi[\rho]$ will thus be a generalized solution of the Monge-Ampère equation $\det D^2\Psi = \rho$. Next the results of Caffarelli in [**18**],[**20**],[**21**] on the regularity of solutions to Monge-Ampère equation yield the following theorem :

Theorem 2.14. *Let* $\rho > 0$ *be a* $C^\alpha(\mathbb{T}^d)$ *probability density on* $\mathbb{T}^d$*.*
Then $\Psi = \Psi[\rho]$ *(see definition 2.13) is a classical solution of*

$$\det D^2\Psi = \rho \tag{77}$$

and satisfies :

$$\|\nabla\Psi(x) - x\|_{L^\infty} \le C(d) = \sqrt{d}/2 \tag{78}$$

$$\|D^2\Psi\|_{C^\alpha} \le K(m, M, \|\rho\|_{C^\alpha}) \tag{79}$$

where $m = \inf\rho$ *and* $M = \sup\rho$*.*

This theorem is an adaptation of the regularity results stated above, whose complete proof is given in appendix.

4.2. Existence of local strong solutions. Let $(x,\xi) \to f_0(x,\xi)$ be in $L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$, positive, compactly supported in x and ξ satisfying the compatibility condition :

$$\int f_0(x,\xi)dxd\xi = 1. \tag{80}$$

We look for $f(t,x,\xi) : [0,T] \times \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}+$ satisfying the system (VMA_p) :

$$\frac{\partial f}{\partial t} + \nabla_x \cdot (\xi f) + \frac{1}{\epsilon^2}\nabla_\xi \cdot ((\nabla\Psi[\rho](x) - x)f) = 0 \tag{81}$$

$$f(0,\cdot,\cdot) = f^0 \tag{82}$$

The macroscopic density ρ is still related to f by equation (30). We mention first that the proof of existence of global weak solutions adapts with minor changes to the periodic case.

Our result in this section is the following :

THEOREM 2.15. *Let $f_0 \in W^{1,\infty}(\mathbb{T}^d \times \mathbb{R}^d)$, be such that :*

$$\exists C_0 > 0 : \ f_0 \equiv 0 \text{ for } |\xi| \geq C_0, \tag{83}$$

$$\exists \alpha > 0 : \ \rho_0(x) = \int_{\mathbb{R}^d} f_0(x,\xi) d\xi \geq \alpha \ \forall x \in \mathbb{T}^d, \tag{84}$$

then there exists $T > 0$ and a solution f to the VMA_p system (81,82), in the space $W^{1,\infty}([0,T] \times \mathbb{T}^d \times \mathbb{R}^d)$.

Proof of theorem 2.15 : First we deduce from theorem 2.14 :

COROLLARY 2.16. *Let $\rho, \Psi = \Psi[\rho]$ be as in theorem 2.14. Then, we have*

$$\|D^2\Psi\|_{L^\infty(\mathbb{T}^d)} \leq C(m, M, \|\nabla\rho\|_{L^\infty(\mathbb{T}^d)}),$$

and we can define

$$K(m, M, l) = \sup\{\|D^2\Psi[\rho]\|_{L^\infty(\mathbb{T}^d)}; \ \ \|\nabla\rho\|_{L^\infty(\mathbb{T}^d)} \leq l, \quad m \leq \rho \leq M\} < \infty.$$

We see that in order to use theorem 2.14 we need ρ to be bounded away from below. In the following lemma, we show that under suitable assumptions on the initial data, it is possible to enforce locally in time the condition $0 < m \leq \rho \leq M$.

LEMMA 2.17. *Let $f \in L^\infty([0,T] \times \mathbb{T}^d \times \mathbb{R}^d)$ satisfy*

$$\frac{\partial f}{\partial t} + \nabla_x \cdot (\xi f) + \nabla_\xi \cdot (E(t,x) f) = 0 \tag{85}$$

$$f(0,.,.) = f^0 \tag{86}$$

with

$$\|E\|_{L^\infty_t(L^\infty_x \cap BV_x)} \leq F, \tag{87}$$

let the initial condition f_0 be such that :

$$a(x,\xi) \leq f(0,x,\xi) \leq b(x,\xi),$$

with $\rho_a(x) = \int a(x,\xi)d\xi \geq m > 0$ and $\rho_b(x) = \int b(x,\xi)d\xi \leq M < \infty$ and a, b satisfying

$$|\nabla_{x,\xi} a, b| \leq \frac{c}{1 + |\xi|^{d+2}} \tag{88}$$

Then there exists a constant $R > 0$ depending on m, M, c, F, such that

$$(\rho_a(x) - Rt) \leq \rho(t,x) \leq (\rho_b(x) + Rt).$$

The proof of the lemma is given in appendix.

4.2.1. *Construction of approximate solutions.* Let us consider $(t,x) \to E(t,x)$ a smooth vector-field on $\mathbb{T}^d$, and write

$$T_E(f) = \frac{\partial f}{\partial t} + \nabla_x \cdot (\xi f) + \nabla_\xi \cdot (E\, f)$$

then, if $T_E(f) = 0$ then we have :

$$\begin{aligned} T_E \nabla_x f &= -(\nabla_x E) \cdot \nabla_\xi f \\ T_E \nabla_\xi f &= -\nabla_x f \\ T_E \partial_t f &= -\partial_t E \cdot \nabla_\xi f, \end{aligned}$$

thus if $T_E(f) = 0$ we have

$$\frac{d}{dt} \|\nabla_{x,\xi} f\|_{L^\infty} \le \|\nabla_{x,\xi} f\|_{L^\infty} (1 + \|\nabla_x E\|_{L^\infty}) \tag{89}$$

and

$$\|\nabla_{x,\xi} f(t)\|_{L^\infty} \le \|\nabla_{x,\xi} f(t=0)\|_{L^\infty} \exp\left(\int_0^t (1 + \|\nabla_x E(s)\|_{L^\infty}) ds \right).$$

Now let f_0 be given as in theorem 2.15, satisfying (83,84). Thanks to lemma 2.17 it is possible to find t_1, m, M such that for any f satisfying

$$\begin{aligned} T_E(f) &= 0 \\ f(t=0) &= f_0, \end{aligned}$$

for any field $E \in L^1([0,t_1], BV(\mathbb{T}^d))$ satisfying $\|E\|_{L^\infty([0,t_1]\times\mathbb{T}^d)} \le C(d)$, we have

$$m \le \rho(t, \cdot) \le M, \quad \forall t \in [0, t_1] \tag{90}$$

$$\|\xi\|_{max} \le C_1 = 10 C_0, \tag{91}$$

with f supported in $\{|\xi| \le \|\xi\|_{max}\}$ and with C_0 as in theorem 2.15, so that we have for $0 \le t \le t_1$:

$$\|\nabla \rho\|_{L^\infty} \le \omega_d C_1^d \|\nabla_x f\|_{L^\infty}, \tag{92}$$

ω_d being the volume of the unit ball of $\mathbb{R}^d$. Then we construct a family of approximate solutions (f_h, Ψ_h) to (81), in the same spirit as we did for weak solutions, by solving

$$\begin{aligned} &\frac{\partial f_h}{\partial t} + \xi \cdot \nabla_x f_h + \frac{\nabla \Psi_h(x) - x}{\epsilon^2} \cdot \nabla_\xi f_h = 0 \\ &f_h(t=0) = f_0 \\ &\Psi_h(t) = \Psi(\rho(t=nh)) \text{ for } t \in [nh, (n+1)h[. \end{aligned}$$

Note that we have neither mollified the term $\nabla \Psi$ nor the initial condition and that $\|\nabla \Psi_h\|_{L^\infty} \le C(d)$. Now choose $l = 10 \|\nabla_{x,\xi} f_0\|_{L^\infty} \omega_d C_1^d$. If for some $t = nh \le t_1 - h$ we have

$$\|\nabla_{x,\xi} f^h(t=nh)\|_{L^\infty} \le \frac{l}{\omega_d C_1^d}$$

this implies, thanks to (92), that

$$\|\nabla_x \rho^h(t=nh)\|_{L^\infty} \le l$$

and conditions (90,91) are satisfied because $t \leq t_1$. Then if we denote $K = K(m, M, l)$ as in corollary 2.16, we have for $nh \leq t < nh + h$,

$$\frac{d}{dt}\|\nabla_{x,\xi} f^h\|_{L^\infty} \leq (K+1)\|\nabla_{x,\xi} f^h\|_{L^\infty},$$

and then

$$\|\nabla_{x,\xi} f^h(t = nh + h)\|_{L^\infty} \leq \|\nabla_{x,\xi} f^h(t = nh + h)\|_{L^\infty} \exp{(K+1)h}.$$

So if we define T as

$$T = \min\{t_1, t_2\},$$

with $\exp((K+1)t_2) = 10$, then we have, for $0 \leq t \leq T$,

$$\begin{aligned}
&\|\nabla_{x,\xi} f^h\|_{L^\infty} \leq 10\|\nabla_{x,\xi} f_0\|_{L^\infty} \\
&\|\nabla \rho^h\|_{L^\infty} \leq l \\
&m \leq \rho \leq M \\
&\|D^2 \Psi^h\|_{L^\infty} \leq K.
\end{aligned}$$

Thus we can extract a subsequence converging to a strong solution of (81,82). Then we argue as in section 2 to show that all terms converge to the correct limit.

5. Asymptotic analysis

5.1. Convergence to the Euler equation. In this section we justify that the Vlasov-Monge-Ampère system describes approximate geodesics on volume preserving transforms : indeed we will show that weak solutions of this system converge to a solution of the incompressible Euler equations (E) as the parameter ϵ goes to 0, at least for well prepared initial data. We restrict ourselves to the space periodic case. In this section,the macroscopic density ρ is still defined by (30) and the convex potentials $\Phi[\rho], \Psi[\rho]$ are still as in definition 2.13.

For sake of simplicity, we slightly modify our notations and introduce the following rescaled potentials :

$$\tilde{\varphi}[\rho] = \frac{|x|^2/2 - \Psi[\rho]}{\epsilon} \tag{93}$$

$$\varphi[\rho] = \frac{\Phi[\rho] - |x|^2/2}{\epsilon} \tag{94}$$

so that

$$\nabla\varphi[\rho] = \nabla\tilde{\varphi}[\rho] \circ \nabla\Phi[\rho], \tag{95}$$

and the (VMA_p) (p stands for periodic) system takes the following form :

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla_x f - \frac{\nabla\tilde{\varphi}[\rho]}{\epsilon} \cdot \nabla_\xi f = 0. \tag{96}$$

$$f(0, \cdot, \cdot) = f_0 \tag{97}$$

The energy is given by :

$$E(t) = \frac{1}{2}\int f(t,x,\xi)|\xi|^2 dx d\xi + \frac{1}{2}\int |\nabla\varphi|^2 dx \tag{98}$$

$$= \frac{1}{2}\int f(t,x,\xi)|\xi|^2 dx d\xi + \frac{1}{2}\int \rho|\nabla\tilde{\varphi}|^2 dx \tag{99}$$

It has been shown in section 3.2 that the energy is conserved. The Euler equations for incompressible fluids read (E) :

$$\partial_t \bar{v} + \bar{v}\cdot\nabla\bar{v} = -\nabla p \tag{100}$$

$$\nabla\cdot\bar{v} = 0. \tag{101}$$

THEOREM 2.18. *Let f be a weak solution of (96, 97) with finite energy, let $(t,x)\to\bar{v}(t,x)$ be a smooth $C^2([0,T]\times\mathbb{T}^d)$ solution of (100,101) for $t\in[0,T]$, and $p(t,x)$ the corresponding pressure, let*

$$H_\epsilon(t) = \frac{1}{2}\int f(t,x,\xi)|\xi-\bar{v}(t,x)|^2 dx d\xi + \frac{1}{2}\int |\nabla\varphi(t,x)|^2 dx,$$

then

$$H_\epsilon(t) \le C\exp(Ct)(H_\epsilon(0)+\epsilon^2),\ \forall t\in[0,T].$$

The constant C depends only on $\sup_{0\le s\le T}\left\{\|\bar{v}(s,.),p(s,.),\partial_t p(s,.),\nabla p(s,.)\|_{W^{1,\infty}(\mathbb{T}^d)}\right\}$.

Remark : This estimate is enough to compare the weak solutions f to the VMA system (for well prepared initial datas) and the smooth solutions $\bar{v}$ of the Euler equations. For instance, $\int f(t=0,x,\xi)d\xi \equiv 1$ implies $\varphi(t=0,x)\equiv 0$ and, therefore,

$$\int |\xi - v(t=0,x)|^2 f(t=0,x,\xi) dx d\xi \le C_0\epsilon^2$$

implies

$$\sup_{t\in[0,T]}\int |\xi - v(t,x)|^2 f(t,x,\xi) dx d\xi \le C_T\epsilon^2,$$

where C_T depends only on C_0, T and $\bar{v}$.

Proof of Theorem 2.18. We shall show that

$$\begin{aligned}\frac{d}{dt}H_\epsilon = & -\int f(t,x,\xi)(\xi-\bar{v})\nabla\bar{v}(\xi-\bar{v}) \\ & +\int f(t,x,\xi)\frac{1}{\epsilon}\bar{v}\cdot\nabla\tilde{\varphi} \\ & -\int f(t,x,\xi)(\bar{v}-\xi)\cdot\nabla p,\end{aligned} \tag{102}$$

where we will use the notation

$$u\ \nabla\bar{v}\ w = \sum_{i,j=1}^{d} u^i\partial_i\bar{v}^j w^j.$$

The proof of this identity is postponed to the end of the section.

Now we look at all terms of the right hand side. All the constants that we call C are controlled as in theorem 2.18. We set

$$T_1 = -\int f(t,x,\xi)(\xi-\bar{v})\nabla\bar{v}(\xi-\bar{v})$$
$$T_2 = \int f(t,x,\xi)\frac{1}{\epsilon}\bar{v}\cdot\nabla\tilde{\varphi}$$
$$T_3 = -\int f(t,x,\xi)(\bar{v}-\xi)\cdot\nabla p$$

First we have $T_1 \le CH_\epsilon$. For T_2 we have :

$$\begin{aligned} T_2 = \frac{1}{\epsilon}\int \rho\bar{v}\cdot\nabla\tilde{\varphi} = & \quad \frac{1}{\epsilon}\int \bar{v}(\nabla\Phi[\rho])\cdot\nabla\tilde{\varphi}(\nabla\Phi[\rho]) \\ = & \quad \frac{1}{\epsilon}\int \bar{v}(x+\epsilon\nabla\varphi)\cdot\nabla\varphi \\ = & \quad \frac{1}{\epsilon}\int \bar{v}\cdot\nabla\varphi + (\bar{v}(x+\epsilon\nabla\varphi)-\bar{v}(x))\cdot\nabla\varphi \\ \le & \quad 0 + C\int |\nabla\varphi|^2 \le CH_\epsilon, \end{aligned}$$

we have used that $\bar{v}$ is divergence-free thus its integral against any gradient is zero. Next we have the following lemma :

Lemma 2.19. *Let $G : \mathbb{T}^d \to R$ be Lipschitz continuous such that $\int_{\mathbb{T}^d} G = 0$ then for all $R > 0$ one has*

$$|\int \rho G| \le \frac{1}{2}\|\nabla G\|_{L^\infty}(\frac{1}{R}\epsilon^2 + RH_\epsilon).$$

Proof :

$$\begin{aligned} & \left|\int(\rho-1)G\right| = \left|\int(G(x+\epsilon\nabla\varphi)-G(x)\right| \\ \le \quad & \epsilon\|\nabla G\|_{L^\infty}\|\nabla\varphi\|_{L^1} \le \epsilon\|\nabla G\|_{L^\infty}H_\epsilon^{1/2} \le \frac{1}{2}\|\nabla G\|_{L^\infty}(\frac{1}{R}\epsilon^2 + RH_\epsilon). \end{aligned}$$

□

Again, since $\bar{v}$ is divergence-free $\int \bar{v}\cdot\nabla p = 0$, thus from lemma 2.19 we have :

$$-\int \rho\bar{v}\cdot\nabla p \le C(\epsilon^2 + H_\epsilon).$$

We remind that

$$\partial_t\rho(t,x) = -\nabla_x\cdot\int f(t,x,\xi)\xi d\xi.$$

Since it costs no generality to suppose that for all $t \in [0,T]$, $\int p(t,x)dx \equiv 0$, we get :

$$\begin{aligned} & \int f(t,x,\xi)\xi \cdot \nabla p \\ = & \int \frac{\partial \rho}{\partial t} p \\ = & \frac{d}{dt}\int \rho p - \int \rho \frac{\partial p}{\partial t} \\ \leq & C(\epsilon^2 + H_\epsilon) - \frac{dQ}{dt} \end{aligned}$$

again using lemma 2.19, where $Q(t) = -\displaystyle\int \rho p$. Thus

$$T_3 \leq C(H_\epsilon + \epsilon^2) - \frac{dQ}{dt}.$$

Thus we have the following inequality :

$$\frac{d}{dt}(H_\epsilon + Q) \leq CH_\epsilon + O(\epsilon^2). \tag{103}$$

Moreover, using lemma 2.19

$$|Q(t)| \leq C\epsilon^2 + H_\epsilon(t)/2, \tag{104}$$

thus

$$H_\epsilon + Q \geq H_\epsilon/2 - C\epsilon^2, \tag{105}$$

and we can transform (103) in

$$\frac{d}{dt}(H_\epsilon + Q) \leq C(H_\epsilon + Q) + C\epsilon^2, \tag{106}$$

and thus Gronwall's lemma yields

$$H_\epsilon(t) + Q(t) \leq (H_\epsilon(0) + Q(0) + \epsilon^2)\exp(Ct).$$

Using again (104) we obtain

$$H_\epsilon(t) \leq C(H_\epsilon(0) + C\epsilon^2)\exp(Ct), \tag{107}$$

which achieves the proof of Theorem 2.18. □

Proof of identity (102) :
We first notice that, for all $g \in C^1(\mathbb{R} \times \mathbb{T}^d)$, we have :

$$\frac{d}{dt}\int \rho(t,x)g(t,x)dx = \int\int f(t,x,\xi)(\partial_t g(t,x) + \xi \cdot \nabla g(t,x))d\xi dx.$$

We also use the conservation of energy defined by (98). Then we get

$$\begin{aligned}\frac{d}{dt}H_\epsilon = & \frac{d}{dt}\frac{1}{2}\int f(t,x,\xi)(|\bar{v}|^2 - 2\xi\cdot\bar{v})dxd\xi \\ = & \int f(t,x,\xi)(\partial_t\bar{v}\cdot\bar{v} - \partial_t\bar{v}\cdot\xi) - \frac{1}{2}\int\nabla_x\cdot(f(t,x,\xi)\xi)(|\bar{v}|^2 - 2\xi\cdot\bar{v})dxd\xi \\ & +\frac{1}{2}\int\nabla_\xi\cdot(\frac{1}{\epsilon}\nabla\tilde{\varphi}f(t,x,\xi))(|\bar{v}|^2 - 2\xi\cdot\bar{v})\end{aligned}$$

integrating by part, we get :

$$\begin{aligned}\frac{d}{dt}H_\epsilon = & \int f(t,x,\xi)(\partial_t\bar{v}\cdot\bar{v} - \partial_t\bar{v}\cdot\xi) + \int f(t,x,\xi)\xi\nabla\bar{v}(\bar{v}-\xi) \\ & +\int f(t,x,\xi)\frac{1}{\epsilon}\nabla\tilde{\varphi}\cdot\bar{v}\end{aligned}$$

the first two terms can be rewritten as

$$\begin{aligned}& \int f(t,x,\xi)(\partial_t\bar{v}\cdot\bar{v} - \partial_t\bar{v}\cdot\xi) + \int f(t,x,\xi)\xi\nabla\bar{v}(\bar{v}-\xi) \\ = & -\int f(t,x,\xi)(\bar{v}-\xi)\nabla\bar{v}(\bar{v}-\xi) + \int f(t,x,\xi)\partial_t\bar{v}\cdot(\bar{v}-\xi) \\ & +\int f(t,x,\xi)\bar{v}\nabla\bar{v}(\bar{v}-\xi) \\ = & -\int f(t,x,\xi)(\bar{v}-\xi)\nabla\bar{v}(\bar{v}-\xi) + \int f(t,x,\xi)(\bar{v}-\xi)\cdot(\partial_t\bar{v}+\bar{v}\cdot\nabla\bar{v})\end{aligned}$$

and finally using equation (100) we conclude. □

5.2. Comparison with the Euler-Poisson system. Here we show that, as mentioned in the introduction, the Euler-Poisson (EP) system is a more accurate approximation to the Vlasov Monge-Ampère system than the Euler equations, as ϵ goes to zero. For notational simplicity, we drop most ϵ's.

The EP system Let us recall that the (pressureless) Euler-Poisson system describes the motion of a continuum of electrons on a neutralizing background of ions through electrostatic interaction. Let $\bar{v}$ and $\bar{\rho}$ be the velocity and density of electrons. Let $\bar{\varphi}$ be the (rescaled) electric potential. Under proper scaling, these functions of $x\in\mathbb{R}^d$ and $t>0$ satisfy the Euler-Poisson system which will be referred to as (EP) :

$$\partial_t\bar{v} + \bar{v}\cdot\nabla\bar{v} = -\frac{1}{\epsilon}\nabla\bar{\varphi} \tag{108}$$

$$\partial_t\bar{\rho} + \nabla\cdot(\bar{\rho}\bar{v}) = 0 \tag{109}$$

$$1 - \epsilon\Delta\bar{\varphi} = \bar{\rho}. \tag{110}$$

The so-called 'quasi-neutral' limit $\epsilon \to 0$ of similar systems has been studied for example in [**41**] and [**25**] and convergence results have been established using pseudo-differentials energy estimates. For well-prepared initial datas, we expect solutions of (EP) to behave as solutions of Euler-Incompressible equations. This is actually proved by the second author in the third chapter of this thesis.

THEOREM 2.20. *Let v be a smooth solution of (100,101) on $[0,T]\times\mathbb{T}^d$, with initial data v_0 and satisfying $v\in L^\infty([0,T],H^s(\mathbb{T}^d))$ for some $s\geq s_0=\frac{d}{2}+2$. Let $\bar{v}_0^\epsilon,\bar{\rho}_0^\epsilon$ be a sequence of initial data such that*

$$\frac{\bar{v}_0^\epsilon - v_0}{\epsilon} \text{ and } \frac{\bar{\rho}_0^\epsilon - 1}{\epsilon^2} \tag{111}$$

are bounded in $H^s(\mathbb{T}^d)$. Then there exists a sequence of solutions $\bar{v}^\epsilon,\bar{\rho}^\epsilon$ to the Euler-Poisson system, with initial data $\bar{v}_0^\epsilon,\bar{\rho}_0^\epsilon$, on $[0,T_\epsilon[$ with $\liminf_{\epsilon\to 0}T_\epsilon\geq T$. Moreover, for $T'<T$ and ϵ small enough $\epsilon^{-1}(\bar{v}^\epsilon-v)$ and $\epsilon^{-2}(\bar{\rho}^\epsilon-1)$ are bounded in $L^\infty([0,T'],H^{s'}(\mathbb{T}^d))$ for some $s'<s$, s' going to $+\infty$ as s goes to $+\infty$.

Here we consider v a smooth (at least $C^2([0,T]\times\mathbb{T}^d)$) solution to (100, 101) with initial data v_0, a sequence f of solutions of (96,97) , and a sequence $\bar{v},\bar{\rho}$ solutions of (EP) satisfying the assumptions of Theorem 2.20 We still define H_ϵ as in Theorem 2.18 :

$$H_\epsilon(t)=\frac{1}{2}\int f^\epsilon(t,x,\xi)|\xi-v(t,x)|^2dxd\xi+\frac{1}{2}\int|\nabla\varphi^\epsilon|^2dx \tag{112}$$

with v,p as above. We choose the initial data f_0^ϵ such that

H1 : f_0^ϵ satisfies $H_\epsilon(0)\leq C\epsilon^2$ for some $C>0$ fixed.

From Theorem 2.18, this implies that there exists a constant that we still denote C such that

$$H_\epsilon(t)\leq C\epsilon^2 \text{ for } t\in[0,T]. \tag{113}$$

Then we fix s' so large that $H^{s'}(\mathbb{T}^d)$ is continuously embedded in $W^{2,\infty}(\mathbb{T}^d)$, and we make the following assumption on the solutions of (EP) :

H2 : The sequence $(\bar{v}_0^\epsilon,\bar{\rho}_0^\epsilon)$ of initial data of (EP) is chosen such that $\epsilon^{-1}(\bar{v}^\epsilon-v),\epsilon^{-2}(\bar{\rho}^\epsilon-1)$ is a bounded sequence in $L^\infty([0,T],H^{s'}(\mathbb{T}^d))$.

This condition is non-empty from Theorem 2.20 and take s large enough. H2 implies then that the sequence $\epsilon^{-1}(\bar{v}^\epsilon-v),\epsilon^{-2}(\bar{\rho}^\epsilon-1)$ remains bounded in $L^\infty([0,T],W^{2,\infty}(\mathbb{T}^d))$.

We are now ready to prove

THEOREM 2.21. *Let $f_0^\epsilon,\bar{v}_0^\epsilon,\bar{\rho}_0^\epsilon,v,T$ be as above, satisfying assumptions H1 and H2. Define*

$$G_\epsilon(t)=\frac{1}{2}\int f^\epsilon(t,x,\xi)|\xi-\bar{v}^\epsilon(x)|^2dxd\xi+\frac{1}{2}\int|\nabla\varphi^\epsilon-\nabla\bar{\varphi}^\epsilon|^2dx.$$

Then there exists $C>0$ such that

$$G_\epsilon(t)\leq C\exp(Ct)(G_\epsilon(0)+\epsilon^3),\ \forall t\in[0,T].$$

where C does not depend on ϵ.

Remark : the theorem shows that the distance between solutions of the EP system and the VMA system measured with G_ϵ is like $O(\epsilon^3)$ whereas Theorem 2.18 stated that the distance between the solution of the Euler equation and the VMA system was like $O(\epsilon^2)$. Note also that G_ϵ and H_ϵ can be interpreted as the square if a distance.

Proof of theorem 2.21 : Proceeding as in (102) and noticing that :

$$\frac{d}{dt}\int_{\mathbb{T}^d} |\nabla\bar{\varphi}|^2 = \frac{1}{\epsilon}\int_{\mathbb{T}^d} \bar{\rho}\bar{v}\cdot\nabla\bar{\varphi}$$

we obtain the following identity :

$$\begin{aligned}\frac{d}{dt}G_\epsilon = \quad & -\int f(t,x,\xi)(\xi-\bar{v})\nabla\bar{v}(\xi-\bar{v}) \\ & +\int f(t,x,\xi)\frac{1}{\epsilon}\bar{v}\cdot\nabla\tilde{\varphi} - \int f(t,x,\xi)\frac{1}{\epsilon}\bar{v}\cdot\nabla\bar{\varphi} \\ & +\int f(t,x,\xi)\frac{1}{\epsilon}\xi\cdot\nabla\bar{\varphi} + \int\frac{1}{\epsilon}\bar{\rho}\bar{v}\cdot\nabla\bar{\varphi} \\ & -\frac{d}{dt}\int\nabla\bar{\varphi}\cdot\nabla\varphi \end{aligned} \tag{114}$$

Then we notice

$$\int f(t,x,\xi)\frac{1}{\epsilon}\xi\cdot\nabla\bar{\varphi} = \frac{d}{dt}\left(\int\frac{1}{\epsilon}\rho\bar{\varphi}\right) - \frac{1}{\epsilon}\int\rho\partial_t\bar{\varphi}.$$

Next, we state

LEMMA 2.22. *Let us introduce notations*

$$\begin{aligned} < \nabla\theta > (x) &= \int_0^1 \nabla\theta(x+s\epsilon\nabla\varphi(x))ds \\ < \nabla^2\theta > (x) &= \int_0^1 (1-s)\nabla^2\theta(x+s\epsilon\nabla\varphi(x))ds, \end{aligned}$$

for any $\theta\in C^2(\mathbb{T}^d)$. Then,

$$\begin{aligned}\int\rho\theta &= \int\theta + \epsilon\int < \nabla\theta > \nabla\varphi \\ &= \int\theta + \epsilon\int\nabla\theta\cdot\nabla\varphi + \epsilon^2\int < \nabla^2\theta > \nabla\varphi\nabla\varphi.\end{aligned}$$

Proof : the proof just uses the Taylor expansion and the identity $\int\rho\theta = \int\theta(x+\epsilon\nabla\varphi)$. □

Using Lemma 2.22, we get

$$\begin{aligned} & \frac{1}{\epsilon}\int\rho\partial_t\bar{\varphi} \\ = \quad & \frac{1}{\epsilon}\int\partial_t\bar{\varphi} + \int\partial_t\nabla\bar{\varphi}\cdot\nabla\varphi + \epsilon\int < \partial_t\nabla^2\bar{\varphi} > \nabla\varphi\nabla\varphi.\end{aligned}$$

Using the assumptions of Theorem 2.21 we have

$$\|\partial_t \nabla^2 \bar{\varphi}\|_{L^\infty([0,T']\times\mathbb{T}^d)} \leq C.$$

Proof : from equation (109), we know $\partial_t \bar{\rho} = -\bar{\rho}\nabla \cdot \bar{v} - \bar{v} \cdot \nabla \bar{\rho}$. Thanks to H2, this implies that $\|\partial_t \bar{\rho}\|_{H^{s'-1}} \leq C\epsilon$. Since $H^{s'}(\mathbb{T}^d)$ is continuously embedded in $W^{1,\infty}(\mathbb{T}^d)$, $H^{s'-1}(\mathbb{T}^d)$ is continuously embedded in $L^\infty(\mathbb{T}^d)$. Then, using equation (110) and classical elliptic regularity, we have

$$\epsilon\|\partial_t \nabla^2 \bar{\varphi}\|_{H^{s'-1}} \leq C\|\partial_t \bar{\rho}\|_{H^{s'-1}},$$

and the desired result follows.

So, from assumption H1 and (113), we have

$$\left|\epsilon \int < \partial_t \nabla^2 \bar{\varphi} > \nabla\varphi\nabla\varphi\right| \leq C\epsilon^3.$$

Next,

$$\int \partial_t \nabla\bar{\varphi}\cdot\nabla\varphi = -\int \partial_t \Delta\bar{\varphi}\varphi$$
$$= \frac{1}{\epsilon}\int \partial_t\bar{\rho}\varphi = \frac{1}{\epsilon}\int \bar{\rho}\bar{v}\cdot\nabla\varphi.$$

Using again lemma 2.22, we get

$$\frac{d}{dt}\int \nabla\bar{\varphi}\cdot\nabla\varphi$$
$$= \quad \frac{1}{\epsilon}\frac{d}{dt}\left(\int \rho\bar{\varphi} - \epsilon^2\int < \nabla^2\bar{\varphi} > \nabla\varphi\nabla\varphi\right)$$

and for the same reasons we have $\|\nabla^2\bar{\varphi}\|_{L^\infty([0,T]\times\mathbb{T}^d))} \leq C\epsilon$ and this yields

$$Q(t) = \epsilon\int < \nabla^2\bar{\varphi} > \nabla\varphi\nabla\varphi = O(\epsilon^4) \quad .$$

Moreover, it does not cost to set $\int \bar{\varphi} \equiv 0$ and deduce

$$\int f(t,x,\xi)\frac{1}{\epsilon}\xi\cdot\nabla\bar{\varphi} - \frac{d}{dt}\int \nabla\bar{\varphi}\cdot\nabla\varphi = -\frac{1}{\epsilon}\int \bar{\rho}\bar{v}\cdot\nabla\varphi + O(\epsilon^3) + \frac{d}{dt}Q \quad .$$

Thus the remaining terms are

$$R = \frac{1}{\epsilon}\int [\rho\nabla\tilde{\varphi} - \rho\nabla\bar{\varphi} + \bar{\rho}\nabla\bar{\varphi} - \bar{\rho}\nabla\varphi]\cdot\bar{v} \quad .$$

Calculations that we postpone to the end of the proof show that

$$R \leq \quad \int(\nabla\varphi - \nabla\bar{\varphi})\nabla\bar{v}(\nabla\varphi - \nabla\bar{\varphi}) + C\int|\nabla\varphi - \nabla\bar{\varphi}|^2$$
$$-\frac{1}{2}\int \nabla\cdot\bar{v}(|\nabla\bar{\varphi}|^2 - 2\nabla\varphi\cdot\nabla\bar{\varphi}) + O(\epsilon^3) \quad . \tag{115}$$

with C controlled by $\|\nabla^2\bar{v}\|_{L^\infty([0,T]\times\mathbb{T}^d)}$. So finally we have

$$\begin{aligned}\frac{d}{dt}G_\epsilon \leq \quad & -\int f(t,x,\xi)(\xi-\bar{v})\nabla\bar{v}(\xi-\bar{v}) + (\nabla\varphi-\nabla\bar{\varphi})\nabla\bar{v}(\nabla\varphi-\nabla\bar{\varphi}) \\ & -\frac{1}{2}\int(\nabla\cdot\bar{v})(|\nabla\bar{\varphi}|^2 - 2\nabla\bar{\varphi}\cdot\nabla\varphi) + C\int|\nabla\varphi-\nabla\bar{\varphi}|^2 \\ & +C\epsilon^3 + \frac{d}{dt}Q\end{aligned}$$

with $|Q(t)| \leq C\epsilon^4$ for $t \in [0,T]$. If we assume Theorem 2.20 and H2 we have $\|\nabla\cdot v\|_{L^\infty([0,T]\times\mathbb{T}^d)} \leq C\epsilon$ and $\|\nabla\bar{\varphi}\|_{L^\infty([0,T]\times\mathbb{T}^d)} \leq C\epsilon$, whereas H1 yields $\int|\nabla\varphi|^2 \leq C\epsilon^2$ Note that we also have

$$\begin{aligned}& -\int f(t,x,\xi)(\xi-\bar{v})\nabla\bar{v}(\xi-\bar{v}) + (\nabla\varphi-\nabla\bar{\varphi})\nabla\bar{v}(\nabla\varphi-\nabla\bar{\varphi}) \\ & +C\int|\nabla\varphi-\nabla\bar{\varphi}|^2 \leq CG_\epsilon\end{aligned}$$

so we have :

$$\frac{d}{dt}(G_\epsilon - Q) \leq C((G_\epsilon - Q) + \epsilon^3)$$

Then the Theorem follows by Gronwall's lemma.

□

Proof of identity (115) : Here we have to compute :

$$R = \frac{1}{\epsilon}\int \bar{v}(x+\epsilon\nabla\varphi)\cdot\nabla\varphi - (\bar{v}\nabla\bar{\varphi})(x+\epsilon\nabla\varphi) + (1-\epsilon\Delta\bar{\varphi})(\bar{v}\cdot\nabla\bar{\varphi} - \bar{v}\cdot\nabla\varphi)$$

Using lemma 2.22 we have :

$$\begin{aligned}R = \quad & \frac{1}{\epsilon}\int \bar{v}\cdot\nabla\varphi - \bar{v}\cdot\nabla\bar{\varphi} + \bar{v}\cdot\nabla\bar{\varphi} - \bar{v}\cdot\nabla\varphi \\ & +\int \nabla\bar{v}\cdot\nabla\varphi\nabla\varphi - \nabla(\bar{v}\nabla\bar{\varphi})\nabla\varphi - \bar{v}\nabla\bar{\varphi}\Delta\bar{\varphi} + \bar{v}\nabla\varphi\Delta\bar{\varphi} \\ & +\int(<\nabla\bar{v}> - \nabla\bar{v})\nabla\varphi\nabla\varphi - \epsilon <\nabla^2(\bar{v}\nabla\bar{\varphi})> \nabla\varphi\nabla\varphi\end{aligned}$$

We see that the first line cancels. Then we show that the last line is bounded by $C\epsilon^3$. This is obvious for the last term since from Theorem 2.20 we have $\|\nabla\bar{\varphi},\bar{v}\|_{W^{2,\infty}} \leq C\epsilon$. Then for the first term we show the

LEMMA 2.23. *We define*

$$\Delta = \int(<\nabla\bar{v}>(x) - \nabla\bar{v}(x))\nabla\varphi\nabla\varphi dx,$$

then one has :

$$|\Delta| \leq C\epsilon^{10/3} + C\int|\nabla\varphi-\nabla\bar{\varphi}|^2 dx.$$

Proof : First we show that if $\Theta(R) = \int_{\{|\nabla\varphi|\geq R\}} |\nabla\varphi|^2$,

$$\Theta(R) \leq C \int |\nabla\varphi - \nabla\bar{\varphi}|^2 + \frac{C\epsilon^4}{R^2}.$$

Proof : $\int |\nabla\varphi|^2 \leq C\epsilon^2$, implies that

$$\mathrm{meas}\{|\nabla\varphi| \geq R\} \leq C(\frac{\epsilon}{R})^2.$$

Since $|\nabla\bar{\varphi}(t,x)| \leq \epsilon$ for $(t,x) \in [0, T'x] \times \mathbb{T}^d$

$$\begin{aligned}\Theta(R) &\leq \int_{\{|\nabla\varphi|\geq R\}} |\nabla\bar{\varphi}|^2 + \int_{\{|\nabla\varphi|\geq R\}} |\nabla\varphi - \nabla\bar{\varphi}|^2 \\ &\leq \frac{C\epsilon^4}{R^2} + \int |\nabla\varphi - \nabla\bar{\varphi}|^2 .\end{aligned}$$

Then we have

$$\begin{aligned} & \Delta \leq C\Theta(R) + \int_{|\nabla\varphi|\leq R} |< \nabla\bar{v} > (x) - \nabla\bar{v}(x)| \, \nabla\varphi\nabla\varphi \\ \text{with} \quad & |< \nabla\bar{v} > (x) - \nabla\bar{v}(x)| \leq C\epsilon \, |\nabla\varphi| \\ \text{thus} \quad & \Delta \leq C\epsilon \int_{|\nabla\varphi|\leq R} |\nabla\varphi|^3 + C\Theta(R) \\ & \leq C\left(\epsilon R \int |\nabla\varphi|^2 + \frac{\epsilon^4}{R^2} + \int |\nabla\varphi - \nabla\bar{\varphi}|^2\right) \\ & \leq C\left(\epsilon^3 R + + \frac{\epsilon^4}{R^2} + \int |\nabla\varphi - \nabla\bar{\varphi}|^2\right)\end{aligned}$$

for all R, so for $R = \epsilon^{(1/3)}$ one obtains :

$$\Delta \leq C\epsilon^{10/3} + C \int |\nabla\varphi - \nabla\bar{\varphi}|^2 .$$

□

Thus we have shown that $R = S + O(\epsilon^4)$, and $S = \Sigma_{k=1}^6 T_k$ where each T_k is given by :

$$\begin{aligned} T_1 &= \partial_j\bar{v}_i\partial_j\varphi\partial_i\varphi \\ T_2 &= -\partial_j\bar{v}_i\partial_j\varphi\partial_i\bar{\varphi} \\ T_3 &= -\bar{v}_i\partial_{ij}\bar{\varphi}\partial_j\varphi \\ T_4 &= \partial_j\bar{v}_i\partial_j\bar{\varphi}\partial_i\bar{\varphi} \\ T_5 &= \bar{v}_i\partial_{ij}\bar{\varphi}\partial_j\bar{\varphi} \\ T_6 &= \bar{v}_i\partial_{jj}\bar{\varphi}\partial_i\varphi \end{aligned}$$

where we have used Einstein's convention for repeated indices. First we have

$$T_5 = -\frac{1}{2}\int (\nabla\cdot\bar{v})\,|\nabla\bar{\varphi}|^2$$

$$T_1 + T_2 + T_4 = \int \partial_j \bar{v}_i (\partial_j \varphi - \partial_j \bar{\varphi})(\partial_i \varphi - \partial_i \bar{\varphi}) + T_7$$

with $T_7 = \int \partial_j \bar{v}_i \partial_j \bar{\varphi} \partial_i \varphi$.

$$T_6 = -\int \partial_i \bar{v}_i \partial_{jj} \bar{\varphi} \varphi + \bar{v}_i \partial_{ijj} \bar{\varphi} \varphi$$

and

$$-\int \bar{v}_i \partial_{ijj} \bar{\varphi} \varphi = \int \partial_j \bar{v}_i \partial_{ij} \bar{\varphi} \varphi + \bar{v}_i \partial_{ij} \bar{\varphi} \partial_j \varphi$$

thus

$$T_6 = \int -(\nabla \cdot \bar{v}) \Delta \bar{\varphi} \ \varphi + T_8 - T_3$$

with $T_8 = \int \partial_j \bar{v}_i \partial_{ij} \bar{\varphi} \varphi$. Then

$$\begin{aligned} T_8 = \quad & -\int \partial_j \bar{v}_i \partial_j \bar{\varphi} \partial_i \varphi + \partial_{ij} \bar{v}_i \partial_j \bar{\varphi} \varphi \\ & = -T_7 + \int \nabla \cdot \bar{v} (\Delta \bar{\varphi} \varphi + \nabla \bar{\varphi} \nabla \varphi) \end{aligned}$$

and finally we obtain

$$\begin{aligned} S(t) = \quad & \int \nabla \bar{v} (\nabla \bar{\varphi} - \nabla \varphi)(\nabla \bar{\varphi} - \nabla \varphi) - \frac{1}{2} (\nabla \cdot \bar{v}) \left| \nabla \bar{\varphi} - \nabla \varphi \right|^2 \\ & + \frac{1}{2} \int (\nabla \cdot \bar{v}) \left| \nabla \varphi \right|^2 \end{aligned}$$

and the identity (115) is proved.

$\square$

6. Appendix

6.1. Existence and uniqueness of solutions to second order ODE's with BV field. In this section we prove existence and a.e. uniqueness for ordinary differential equations of the form :

$$\frac{d}{dt}\begin{pmatrix} X \\ V \end{pmatrix} = \begin{pmatrix} V \\ E(t,X) \end{pmatrix} \tag{116}$$

for $X \in \mathbb{T}^d$, $Y \in \mathbb{T}^d$, and where the field E belongs to $L^\infty(]0,T[\times\mathbb{T}^d) \cap L^1(]0,T[,BV(\mathbb{T}^d))$. We work in the flat torus for simplicity, but our results are still valid in an open subset of $\mathbb{R}^d$. This result is an adaptation of the proof of [**30**] that uses the result of [**9**] on renormalized solutions of transport equations.

Renormalized solutions for Vlasov equation with BV field. Theorem 3.4 in [**9**] adapted to the periodic case sates that if $f \in L^\infty(]0,T[\times\mathbb{T}^d \times \mathbb{R}^d)$ satisfies :

$$\frac{\partial f}{\partial t} + \nabla_x \cdot (\xi f) + \nabla_\xi \cdot (E(t,x)f) = 0 \tag{117}$$

with $E(t,x) \in L^1(]0,T[\times\mathbb{T}^d) \cap L^1(]0,T[,BV(\mathbb{T}^d))$, then for all g Lipschitz continuous we have

$$\frac{\partial g(f)}{\partial t} + \nabla_x \cdot (\xi g(f)) + \nabla_\xi \cdot (E(t,x)g(f)) = 0$$

The property of renormalization implies that

- solutions to (117) with initial data in $L^\infty_{loc}(\mathbb{T}^d \times \mathbb{R}^d)$ belong to $C(]0,T[,L^p_{loc}(\mathbb{T}^d \times \mathbb{R}^d))$ for any $1 \leq p < \infty$,
- solutions to (117) with prescribed initial data in $L^\infty(]0,T[\times\mathbb{T}^d \times \mathbb{R}^d)$ are a.e. unique,
- if E_n converges to E in $L^1(]0,T[\times\mathbb{T}^d)$ then the solutions of (117) with E_n instead of E converge to the solution of (117).

We notice that equation (81) satisfies the hypothesis of the Theorem, and thus will have the renormalization property. This renormalization property enabled DiPerna and Lions in [**30**] to obtain a.e. uniqueness for solutions of the corresponding ODE's. Indeed, the ODE

$$\partial_t X(t,s,x) = b(t,X)$$
$$X(s,s,x) = x$$

is associated to the transport equation :

$$\partial_t u + b(t,x).\nabla u = 0$$

whose solutions satisfy for all $t,s \in]0,T[$

$$u(t,X(t,s,x)) = u(s,x).$$

We extend this consequence to the case of second order equation, with BV acceleration field. To the kinetic equation

$$\partial_t f + \xi \cdot \nabla_x f + E(t,x) \cdot \nabla_\xi f = 0 \tag{118}$$

we associate the second order ODE (116) which can rewritten as $\partial_{tt} X = E(t,X)$. The result is then the following :

THEOREM 2.24. *Let $E(t,x) \in L^\infty(]0,T[\times\mathbb{T}^d) \cap L^1(]0,T[,BV(\mathbb{T}^d))$, then the ODE*

$$\partial_{tt}X(t,s,x,\xi) = E(t,X) \tag{119}$$

$$(X(s,s,x,\xi),\partial_t X(s,s,x,\xi)) = (x,\xi) \tag{120}$$

admits an a.e. unique solution.

Proof of Theorem 2.24 : We know that through equation (116) equation (119) can be considered as a first order differential equation. Let us first consider the case where E is smooth. Note $Y \in \mathbb{T}^d \times \mathbb{R}^d$ (resp. y) for (X,V) (resp. for (x,ξ)) and $B \in \mathbb{R}^d \times \mathbb{R}^d$ for (ξ,E). Then for all $s \in]0,T[$, Y solves :

$$\partial_t Y(t,s,y) = B(t,Y(t,s,y)) \tag{121}$$

$$Y(s,s,y) = y \tag{122}$$

Then for all $t,t_1,t_2,t_3 \in]0,T[$ we have the following :

$$\begin{aligned}
&Y(t_3,t_2,Y(t_2,t_1,y)) = Y(t_3,t_1,y)\\
&Y(t,t,y) = y\\
&Y(t_1,t_2,Y(t_2,t_1,y)) = y
\end{aligned}$$

differentiating the last equation with respect to t_2 yields :

$$\partial_s Y(t,s,y) + \nabla_y Y(t,s,y)\cdot B(s,y) = 0 \tag{123}$$

$$Y(t,t,y) = y \tag{124}$$

$Y_t(s,y) = Y(t,s,y)$ thus solves a transport equation which is nothing but equation (118). Using Theorem 2.10 we know that for all $g : \mathbb{R} \to \mathbb{R}$ Lipschitz continuous, $g(t,s,y) = g_0(Y(t,s,y))$ is the unique solution of

$$\partial_s g(t,s,y) + \nabla_y g(t,s,y)\cdot B(s,y) = 0 \tag{125}$$

$$g(t,t,y) = g_0(y). \tag{126}$$

Now we show existence and uniqueness for solutions of (121,122). Let t and s be fixed. Let us consider a regularization E_n of the the field E and set $B_n = (\xi,E_n)$. We have

- $t \to Y_{1,n}(t,s,y)$ that satisfies (121,122)
- $s \to Y_{2,n}(t,s,y)$ that satisfies (123,124).

From the stability Theorem 2.4 in [**30**] we know that the whole sequence $t \to Y_{2,n}(t,s,.)$ converges in $C(]0,T[,L^p_{loc}(\mathbb{R}^d\times\mathbb{T}^d))$ to $t \to Y_2(t,s,.)$, the unique renormalized solution of (123,124). Thus for fixed t the whole sequence $Y_{2,n}(t,s,.)$ converges strongly in $L^p_{loc}(\mathbb{R}^d\times\mathbb{T}^d)$. Now since for every n we have $Y_{1,n}(t,s,y) = Y_{2,n}(t,s,y)$ the same property holds for $Y_{1,n}(s,t,.)$. Now we can pass to the limit in the term $B_n(t,Y_{1,n}(t,s,y))$. Indeed, by density of C_c^∞ functions in L^1, if we have $E_s \in C_c^\infty$ approximating E then

$$\begin{aligned}
&\|B(t,Y_n(t,s,y)) - B(t,Y(t,s,y))\|_{L^1}\\
\leq\ &\|B(t,Y_n(t,s,y)) - B_s(t,Y_n(t,s,y))\|_{L^1}\\
&+\|B_s(t,Y_n(t,s,y)) - B_s(t,Y(t,s,y))\|_{L^1}\\
&+\|B(t,Y(t,s,y)) - B_s(t,Y(t,s,y))\|_{L^1}
\end{aligned}$$

The second term goes to 0 because of the strong convergence of Y_n, the first and the third go to 0 because Y and Y_n are measure preserving mappings, and so for example $\|B(t, Y(t, s, y)) - B_s(t, Y(t, s, y))\|_{L^1} = \|B(t, y) - B_s(t, y)\|_{L^1}$. So finally we have

$$\begin{array}{rl} & \|B_n(t, Y_n(t, s, y)) - B(t, Y(t, s, y))\|_{L^1} \\ \leq & \|B_n(t, Y_n(t, s, y)) - B(t, Y_n(t, s, y))\|_{L^1} \\ & + \|B(t, Y_n(t, s, y)) - B(t, Y(t, s, y))\|_{L^1} \end{array}$$

that goes to 0 and we can pass to the limit in the equation (121,122) and the existence of a solution to (121,122) is proved.

To obtain uniqueness, we argue as in DiPerna-Lions. Any function of the form $g_0(Y(t, s, y))$ is a solution of (125,126), thus by uniqueness of the solution of the transport equation we obtain uniqueness of the ODE.

A remark on ODE's of second order. In this section, we want to solve the Cauchy problem for :

$$\begin{array}{l} \partial_{tt} X(t, x) = E(t, X) \\ (X(0, x)\ , \partial_t X(0, x)) = (x, v(x)) \end{array}$$

with E as above. We are thus interested in monokinetic initial data.

THEOREM 2.25. *for all $v^0(x)$ vector field on $\mathbb{T}^d$, and for Lebesgue almost every $\delta v \in \mathbb{R}^d$, there exists an a.e. unique solution to*

$$\begin{array}{l} \partial_{tt} X(t, x) = E(t, X(t, x)) \\ (X(0, x)\ , \partial_t X(0, x)) = (x, v^0(x) + \delta v) \end{array}$$

Proof : Let $g(x, \xi)$ be the indicator function of the set of those $(x; \xi)$ such that the trajectory coming from x is not well defined. We just have to prove that for a.e. $\delta v \in \mathbb{R}^d$ we have $\int g(x, v^0(x) + \delta v) dx = 0$, which is true because

$$\int g(x, v^0(x) + \xi) dx d\xi = \int g(x, \xi) dx d\xi = 0.$$

Stability. Using the fact that for E_n converging to E in L^1 with $E \in L^1(]0, T[, BV(\mathbb{T}^d))$, we have $X_n(t, x, v) \to X(t, x, v)$ in $C([0, T], L^p)$, we have then, for all t, for almost every δv, $X_n(t, x, v^0(x) + \delta v) \to X(t, x, v^0(x) + \delta v)$ in L^p. Thus we have

THEOREM 2.26. *If E_n converges to E in L^1 let X_n be solution of*

$$\begin{array}{l} \partial_{tt} X_n(t, x) = E_n(t, X_n(t, x)) \\ (X_n(0, x), \partial_t X_n(0, x)) = (x, v^0(x) + \delta v) \end{array}$$

then for all t, for almost every δv, X_n converges in $L^p(\mathbb{R}^3) - s$ to a solution (unique for almost every δv) of

$$\begin{array}{l} \partial_{tt} X(t, x) = E(t, X) \\ (X(0, x), \partial_t X(0, x)) = (x, v^0(x) + \delta v) \end{array}$$

6.2. Control of macroscopic density in kinetic equations. Here we are going to prove the lemma 2.17 :

LEMMA 2.27. *Let $f \in L^\infty([0,T] \times \mathbb{T}^d \times \mathbb{R}^d)$ satisfy*

$$\frac{\partial f}{\partial t} + \nabla_x \cdot (\xi f) + \nabla_\xi \cdot (E(t,x)f) = 0 \tag{127}$$

$$f(0,.,.) = f^0 \tag{128}$$

with

$$\|E\|_{L^\infty_t(L^\infty_x \cap BV_x)} \leq F. \tag{129}$$

Let the initial condition f_0 be such that :

$$a(x,\xi) \leq f(0,x,\xi) \leq b(x,\xi),$$

with $\rho_a(x) = \int a(x,\xi)d\xi \geq m > 0$ and $\rho_b(x) = \int b(x,\xi)d\xi \leq M < \infty$ and a, b satisfying

$$|\nabla_{x,\xi} a, b| \leq \frac{c}{1+|\xi|^{d+2}} \tag{130}$$

Then there exists a constant $R > 0$ such that

$$(\rho_a(x) - Rt) \leq \rho(t,x) \leq (\rho_b(x) + Rt).$$

Proof : First suppose that the force field and the initial data are smooth. For equation (127,128) we can exhibit characteristics $(x,\xi)(t;t_0,x_0,\xi_0)$, giving the evolution of the particles in the phase space. We have $f(t,x,\xi) = f(t_0,x_0,\xi_0)$. Since the initial data is compactly supported and the force field is bounded in the L^∞ norm, we have

$$|\xi - \xi_0| \leq F|t - t_0|,$$

$$|x - x_0| \leq (|\xi_0| + \frac{F}{2}|t-t_0|)|t - t_0|.$$

If for $t = 0$ we have $a(x,\xi) \leq f(0,x,\xi) \leq b(x,\xi)$ then

$$\underline{A}(t,x,\xi) \leq f(t,x,\xi) \leq \overline{B}(t,x,\xi)$$

$$\underline{A}(t,x,\xi) = \inf_{|\sigma_1|,|\sigma_2|\leq 1} a(x + |t-t_0|(\xi + \frac{F}{2}|t-t_0|)\sigma_1, \xi + F|t-t_0|\sigma_2)$$

$$\overline{B}(t,x,\xi) = \sup_{|\sigma_1|,|\sigma_2|\leq 1} b(x + |t-t_0|(\xi + \frac{F}{2}|t-t_0|)\sigma_1, \xi + F|t-t_0|\sigma_2)$$

Using (130) and integrating in ξ we find thus a constant $R = R(F,C,d)$ such that for $t - t_0 \leq 1$ we have :

$$\rho_a(x) - R|t-t_0| \leq \rho(t,x) \leq \rho_b(x) + R|t-t_0|.$$

Next we need to show that the solution of the regularized equation converges to the solution we are studying : this result comes from the uniqueness of the solution to (127,128) which is a consequence of the renormalization property. Indeed since E is bounded in BV the system (127,128) admits a unique renormalized solution and the sequence of approximate solutions converge in $C([0,T],L^p_{x,\xi})$ for $1 \leq p < \infty$ thus the bounds obtained above are preserved. We notice that the uniqueness of the limit is crucial to conclude. □

6.3. Regularity of the polar factorization on the flat torus. Here we deduce from [**49**], [**24**] and [**18**], [**20**], [**21**] the Theorem 2.14.

THEOREM 2.28. *If* $\rho \in C^\alpha(\mathbb{T}^d)$ *with* $0 < m \leq \rho \leq M$ *is a probability measure on* $\mathbb{T}^d$ *then* $\Psi = \Psi[\rho]$ *(see definition 2.13) is a classical solution of*

$$\det D^2\Psi = \rho \tag{131}$$

and satisfies :

$$\|\nabla\Psi(x) - x\|_{L^\infty} \leq C(d) = \sqrt{d}/2 \tag{132}$$

$$\|D^2\Psi\|_{C^\alpha} \leq K(m, M, \|\rho\|_{C^\alpha}) \tag{133}$$

Proof of Theorem 2.28 : Consider ρ a $\mathbb{Z}^d$ periodic probability measure, satisfying

$$0 < m \leq \rho \leq M, \tag{134}$$

and $\Phi[\rho]$ as in Definition 2.13. First it is shown in [**24**] that

$$|\nabla\Phi[\rho](x) - x| \leq C(d). \tag{135}$$

It follows that the strict convexity argument of [**18**] applies : indeed if $\Phi = \Phi[\rho]$ is not strictly convex its graph contains a line and this contradicts (135). Moreover since $\Phi - |x|^2/2$ is globally Lipschitz and periodic there exists $N(d)$ such that $\|\Phi - |x|^2/2\|_{L^\infty} \leq N(d)$. It follows then that there exists $0 < r(d) \leq R(d)$ and $M(d)$ such that

$$B(r(d)) \subset \{\Phi - \Phi(0) \leq M(d)\} \subset B(R(d)) \tag{136}$$

It remains to show that our solution " à la Brenier" is a solution in the Aleksandrov sense of the Monge-Ampère equation

$$m \leq \det D^2\Phi \leq M.$$

This is a direct consequence with minor changes (to adapt to the periodic case) of Lemma 2 of [**21**]. Then, normalizing Φ to $\tilde\Phi = \Phi - \Phi(0) - M(d)$ it follows that $\tilde\Phi$ is a solution of

$$\rho(\nabla\tilde\Phi)\det D^2\tilde\Phi = 1$$
$$\tilde\Phi = 0 \quad \text{on } \partial\Omega$$
$$B(r(d)) \subset \Omega \subset B(R(d))$$

Thus the interior regularity results of [**20**] apply uniformly to all $\Phi[\rho]$ with ρ satisfying (134) and $\|\rho\|_{C^\alpha(\mathbb{T}^d)}$ bounded and Theorem 2.14 follows.

□

Acknowledgments. The authors acknowledge the support of the European IHP network "HYKE" HPRN-CT-2002-00282 and the LRC CEA-Cadarache/UNSA.

CHAPITRE 3

The Euler-Monge-Ampère equation

G. Loeper[1]

RÉSUMÉ. This paper studies the Euler-Poisson system and its fully non-linear counterpart, the Euler-Monge-Ampère system, where the Monge-Ampère equation substitutes for the Poisson equation. Using energy estimates, long term existence and convergence of both systems to the Euler equations is proved.

1. Introduction

In this paper we consider a model of a collisionless plasma where the ions are supposed to be at rest and create a neutralizing background field. The motion of the electrons can then be described by using either the kinetic formalism or the hydrodynamic equations of conservation of mass and momentum as we do here. The self-induced electric field is the gradient of a potential given either by the linear Poisson equation $\Delta\phi = \frac{1}{\epsilon}(\rho - 1)$ or by the fully non-linear Monge-Ampère equation $\det(I + \epsilon\partial_{ij}\phi) = \rho$. This gives the Euler-Poisson ($(E-P)$) system and Euler-Monge-Ampère ($(E-MA)$) system. The systems are thus the following :

$$\begin{aligned}
&\partial_t\rho + \nabla\cdot(\rho v) = 0\\
&\partial_t v + v\cdot\nabla v = \frac{1}{\epsilon}\nabla\phi\\
&\Delta\phi = \frac{1}{\epsilon}(\rho - 1) \text{ in the Poisson case}\\
&\det(I + \epsilon\partial_{ij}\phi) = \rho \text{ in the Monge-Ampère case.}
\end{aligned}$$

Note that the systems are pressureless, and the only force is due to electrostatic interaction. The asymptotic we look at consists in considering large scales compared to the Debye length (ϵ). At those scales the plasma appears to be electrically neutral. In this limit the plasma behaves like an incompressible fluid thus governed by the incompressible Euler equation (E). We intend to rigorously justify those limits in this paper.

Whereas the Euler-Poisson system relies on a well known physical model, the Euler-Monge-Ampère system, less famous, is a fully non-linear (but asymptotically close in the quasi neutral regime) version of the Euler-Poisson system, which also has the geometric interpretation to describe approximate geodesics on the group of measure preserving diffeomorphisms, an interpretation that will be developed more accurately in the sections 3.1, 3.1.2 devoted to the Euler-Monge-Ampère equation. (See also chapter 2 where the closely related Vlasov-Monge-Ampère system is introduced). The reader can refer to [**42**] and [**55**] where different regimes of the Euler-Poisson system are studied.

To see why both systems should be asymptotically close in the quasi-neutral regime, notice that if ρ is close to 1 then $\epsilon\partial_{ij}\phi$ should be small and thus $\det(I + \epsilon\partial_{ij}\phi) = 1 + \epsilon\Delta\phi + O(\epsilon^2)$ and one recovers the Poisson equation. For this reason the proof of the convergence of both systems will be very close and this is why we present them altogether.

This work can be seen as the continuation of the study begun in the chapter 2 where the quasi neutral limit of the Vlasov-Monge-Ampère system was studied. It also provides the proof of the uniform bounds of solutions to $(E-P)$ used in the section 5 of chapter 2. In a broader perspective, it concerns the motion of slightly compressible fluids seen as singular perturbations of the Euler incompressible equation : this field as been investigated using different techniques :

[1]Laboratoire J.A.Dieudonné, Université de Nice-Sophia-Antipolis, Parc Valrose, 06108 NICE Cedex 2.

- Traditional analysis and the geometrical property of Euler equation (see sections 3.1 and 3.1.2) as done in [**31**] where the convergence holds in say L^∞ for the velocity and in the case of barotropic fluids (i.e. when the pressure is a function of the density ρ, a case different from the one studied here.)

- Energy estimates as done in [**45**] again for the case of barotropic fluids, where convergence holds in all H^s norms for well prepared initial data. The result has been also extended to the non-isentropic case by [**50**].

- Pseudo-differential estimates as done in [**41**] which can be seen as a pseudo-differential generalization of [**45**] and where the same convergence holds for a broader class of singular perturbations.

- Modulated energy techniques for convergence of the Vlasov-Poisson system to the so-called dissipative solutions of the Euler equation, as done in [**13**], and for the convergence of Vlasov-Monge-Ampère system (a kinetic version of the Euler-Monge-Ampère system) to Euler as done in chapter 2. The result obtained there says roughly that the convergence holds in "L^2" norms and does not require smoothness of the initial data.

Here we obtained by energy estimates a convergence to the Euler incompressible system in $L^\infty_t H^s_x$ norm for any $s \geq s_0 = \mathbf{E}(d/2) + 2$ where $\mathbf{E}$ will denote the integer part. The convergence of both systems holds on the range of time on which the solution of Euler is smooth enough. We obtain also that both systems are closer to each other than they are close to the Euler incompressible system. $(E-P)_\epsilon$ is thus a corrector in the convergence of $(E-MA)_\epsilon$ to (E). Although the operators that define the acceleration from the density are differential operators, (and even fully non-linear in the second case) our proof doesn't use the pseudo-differential formalism. Actually the general theorem obtained by Grenier for singular perturbations does not apply directly to the Euler-Poisson system.

We split the rest of the paper in two sections : the first one devoted to the study of the Euler-Poisson system, and the second devoted to the study of the Euler-Monge-Ampère system.

2. The Euler-Poisson system

Here $x \in \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and $t \in \mathbb{R}^+$. $v(t,x) \in \mathbb{R}^d$ stands for the velocity and $\rho(t,x) \in \mathbb{R}^+$ is the macroscopic density of electrons. $\phi(t,x) \in \mathbb{R}$ is the electrostatic potential. $d = 2$ or 3. We consider the following Euler-Poisson system denoted by $(E-P_\epsilon)$:

$$\partial_t \rho + \nabla \cdot (\rho v) = 0 \tag{137}$$

$$\partial_t v + v \cdot \nabla v = \frac{\nabla \phi}{\epsilon} \tag{138}$$

$$\epsilon \Delta \phi = \rho - 1 \tag{139}$$

and consider the limit ϵ going to 0. We recall also the incompressible Euler equation (E) :

$$\begin{aligned} &\partial_t v + v \cdot \nabla v = \nabla p \\ &\nabla \cdot v = 0. \end{aligned} \tag{140}$$

We will then prove the following :

2.1. Result.

THEOREM 3.1. *Let $(\bar{v}, p)$ be a smooth solution of the Euler incompressible system (140) on $[0,T] \times \mathbb{T}^d$, with initial data $\bar{v}_0$, satisfying $\bar{v} \in L^\infty([0,T], H^s(\mathbb{T}^d))$ for some $s \geq s_0 = \mathbf{E}(d/2)+2$, ($\mathbf{E}$ the integer part), let $v_0^\epsilon, \rho_0^\epsilon$ be a sequence of initial data such that $\epsilon^{-1}(v_0^\epsilon - \bar{v}_0)$ and $\epsilon^{-2}(\rho_0^\epsilon - 1)$ are bounded in $H^s(\mathbb{T}^d)$. Then there exists a sequence $(v^\epsilon, \rho^\epsilon)$ of solutions to $(E-P_\epsilon)$ with initial data $v_0^\epsilon, \rho_0^\epsilon$ belonging* *to $L^\infty([0,T_\epsilon],H^s(\mathbb{T}^d))$ with $\liminf_{\epsilon\to 0} T_\epsilon \geq T$. Moreover for $T'<T$ and ϵ small enough $\epsilon^{-1}(v^\epsilon-\bar{v})$ and $\epsilon^{-2}(\rho^\epsilon-1)$ are bounded in $L^\infty([0,T'],H^{s'}(\mathbb{R}^2))$ for some $s'<s$, s' going to $+\infty$ as s goes to $+\infty$. Finally when $T=+\infty$, T_ϵ goes to infinity.*

2.2. Proof of Theorem 3.1.

2.2.1. *Heuristics.* Let us introduce $\bar{v},p$ the solution of the Euler incompressible system (140) and corresponding pressure. Note that by taking the divergence of (140) the pressure is given by the following :

$$\Delta p = \sum_{i,j=1}^{d} \partial_i \bar{v}^j \partial_j \bar{v}^i.$$

We will all along the paper use the following notation : for two vector fields u,v

$$\nabla u : \nabla v = \sum_{i,j=1}^{d} \partial_i u^j \partial_j v^i.$$

If v is solution to $(E-P_\epsilon)$ we note also

$$v = \bar{v} + \epsilon v_1,$$
$$\rho = 1 + \epsilon^2 \rho_1.$$

In terms of the unknowns $v_1, E_1 = \nabla\Delta^{-1}(\rho_1)$ the system $(E-P_\epsilon)$ takes the following form :

$$\begin{aligned} &\partial_t v_1 + v_1\cdot\nabla\bar{v} + \bar{v}\cdot\nabla v_1 + \epsilon v_1\cdot\nabla v_1 = \frac{E_1-\nabla p}{\epsilon} \\ &\partial_t E_1 + \nabla\cdot E_1\ \bar{v} + \epsilon\nabla\cdot E_1\ v_1 = -\frac{v_1}{\epsilon} + \nabla\times F \\ &\nabla\times E_1 = 0 \end{aligned} \tag{141}$$

where $\nabla\times F$ is just a Lagrange multiplier for the irrotationality constraint on E_1. Keeping only the leading order terms of (141), one gets the following system

$$\partial_t v_1 = \frac{E_1-\nabla P}{\epsilon}$$
$$\partial_t(E_1-\nabla P) = -\frac{v_1}{\epsilon}.$$

that describes the oscillation of the electric field. Setting $\tilde{E}_1 := E_1 - \nabla P$ this yields

$$\partial_{tt}\begin{pmatrix} v_1 \\ \tilde{E}_1 \end{pmatrix}(t,x) = -\frac{1}{\epsilon^2}\begin{pmatrix} v_1 \\ \tilde{E}_1 \end{pmatrix}(t,x).$$

The solution is thus with complex notations

$$\mathbf{u}^\epsilon = \mathbf{u}_0^\epsilon \exp i(\frac{t}{\epsilon}+\varphi)$$

where $\mathbf{u}^\epsilon$ is the unknown vector of $\mathbb{R}^{2d}$, given by $u^\epsilon = (v_1^1,...,v_1^d,\tilde{E}_1^1,...,\tilde{E}_1^d)$ with the compatibility condition $\tilde{E}_1^j = \frac{i}{\epsilon}v_1^j$ and the energy of the perturbation is given by $\mathcal{E} = \frac{1}{2}(|v_1|^2+\epsilon^2|\tilde{E}_1|^2)$ and conserved.

2.2.2. *Reformulation of the system with new unknowns.* We define the new unknowns $\omega_1, \beta_1, \rho_1$ as follows :

$$\nabla \cdot v = \epsilon \beta_1$$
$$\rho = 1 + \epsilon^2 \rho_1 = 1 + \epsilon \Delta \phi$$
$$\text{curl } v = \omega = \bar{\omega} + \epsilon \omega_1$$

with $\bar{v}, p, \bar{\omega} = \text{curl } \bar{v}$ as before. We note also

$$v = \bar{v} + \epsilon v_1$$

but v_1 is not really an unknown since it can be obtained from ω_1 and β_1 : Indeed when $d = 2$ we have

$$\partial_1 \beta_1 + \partial_2 \omega_1 = \Delta v_1, \tag{142}$$
$$\partial_2 \beta_1 - \partial_1 \omega_1 = \Delta v_2. \tag{143}$$

In the 3 dimensional case we have equations (142, 143) replaced by

$$(\nabla \times \nabla \times v) + \nabla(\nabla \cdot v) = \Delta v$$

and thus

$$(\nabla \times \omega_1)^i + \partial_i \beta_1 = \Delta u_1^i. \tag{144}$$

Note that when $d = 2$ the vorticity is scalar and when $d \neq 2$ it is a vector field of $\mathbb{T}^d$. Taking the curl of equation (138) we recall the following identities :

$$\partial_t \omega + (v \cdot \nabla)\omega + (\nabla \cdot v)\omega = 0 \text{ when } d = 2, \tag{145}$$
$$\partial_t \omega + (v \cdot \nabla)\omega + (\nabla \cdot v)\omega + (\omega \cdot \nabla) v = 0 \text{ when } d = 3. \tag{146}$$

When $d = 2$ the EP system then becomes :

$$\partial_t(\bar{\omega} + \epsilon \omega_1) + v \cdot \nabla(\bar{\omega} + \epsilon \omega_1) = -(\bar{\omega} + \epsilon \omega_1)\epsilon \beta \tag{147}$$
$$\partial_t \epsilon \beta_1 + v \cdot \nabla \epsilon \beta_1 + 2\epsilon \partial_i \bar{v}^j \partial_j v_1^i + \epsilon^2 \partial_i v_1^j \partial_j v_1^i = \frac{\Delta \phi}{\epsilon} - \partial_i \bar{v}^j \partial_j \bar{v}^i \tag{148}$$
$$\partial_t \epsilon^2 \rho_1 + v \cdot \nabla \epsilon^2 \rho_1 = -(1 + \epsilon^2 \rho_1)\epsilon \beta_1 \tag{149}$$

whereas when $d = 3$ one would have to replace equation (147) by

$$\partial_t(\bar{\omega} + \epsilon \omega_1) + v \cdot \nabla(\bar{\omega} + \epsilon \omega_1) = -(\bar{\omega} + \epsilon \omega_1)\epsilon \beta - \omega \cdot \nabla v \tag{150}$$

Noticing that $\Delta \phi = \epsilon \rho_1$, if we set

$$\tilde{\rho}_1 = \rho_1 - \Delta p \tag{151}$$

we get the following system in two dimensions :

$$\begin{cases} \partial_t \omega_1 + v \cdot \nabla \omega_1 = -\bar{\omega}\beta_1 - \epsilon \omega_1 \beta_1 - v_1 \cdot \nabla \bar{\omega} \\ \partial_t \beta_1 + v \cdot \nabla \beta_1 = \dfrac{\tilde{\rho}_1}{\epsilon} - 2\nabla v_1 : \nabla \bar{v} - \epsilon \nabla v_1 : \nabla v_1 \\ \partial_t \tilde{\rho}_1 + v \cdot \nabla \tilde{\rho}_1 = -\dfrac{\beta_1}{\epsilon} - \epsilon(\tilde{\rho}_1 + \Delta p)\beta_1 - (\partial_t \Delta p + v \cdot \nabla \Delta p) \end{cases} \tag{152}$$

In 3 dimensions one would replace the first equation by

$$\partial_t \omega_1 + v \cdot \nabla \omega_1 = \tag{153}$$
$$-\bar{\omega}\beta_1 - v_1 \cdot \nabla \bar{\omega} - \bar{\omega} \cdot \nabla v_1 - \omega \cdot \nabla \bar{v} - \epsilon \omega_1 \beta_1 - \epsilon \omega_1 \cdot \nabla v_1 \tag{154}$$

Remark : For the pair $\beta_1, \tilde\rho_1$ we have at the leading order the equation

$$\partial_t \beta_1 = \frac{\tilde\rho_1}{\epsilon}$$
$$\partial_t \tilde\rho_1 = -\frac{\beta_1}{\epsilon}$$

We for which the solutions are

$$\beta_1(t) = \mathcal{R}\ [\beta_1^0 \exp(\frac{it}{\epsilon})]$$
$$\tilde\rho_1(t) = \mathcal{R}\ [\tilde\rho_1^0 \exp(\frac{it}{\epsilon})]$$

with $\beta_1^0, \tilde\rho_1^0 = i\beta_1^0 \in \mathbb{C}$. The remainder of the right hand side consists in terms that are like $O(|\omega_1, \beta_1, \tilde\rho_1|) + \epsilon\ O(|\omega_1, \beta_1, \tilde\rho_1|^2) + O(1)$ as long as the solution of Euler is smooth.

2.2.3. *Energy estimates.* We handle the energy estimates when $d = 2$ but the same result would hold when $d = 3$, just with more terms. The system can be written in the following way :

$$\partial_t \mathbf{u}^\epsilon + \sum_i v^i \partial_i \mathbf{u}^\epsilon + R^\epsilon \mathbf{u}^\epsilon = S^\epsilon(\mathbf{u}^\epsilon) \tag{155}$$

$$\mathbf{u}^\epsilon(0) = \mathbf{u}_0^\epsilon \tag{156}$$

where v is still the velocity, and where

$$\mathbf{u} = \begin{pmatrix} \omega_1 \\ \beta_1 \\ \tilde\rho_1 \end{pmatrix}, R_\epsilon = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\epsilon} \\ 0 & -\frac{1}{\epsilon} & 0 \end{pmatrix}$$

S_ϵ is given by :

$$S_\epsilon = \begin{pmatrix} -\bar\omega\beta_1 - \epsilon\omega_1\beta_1 - v_1 \cdot \nabla\bar\omega \\ -2\nabla v_1 : \nabla\bar v - \epsilon\nabla v_1 : \nabla v_1 \\ -\epsilon(\tilde\rho_1 + \Delta p)\beta_1 - (\partial_t \Delta p + v \cdot \nabla\Delta p) \end{pmatrix}$$

We apply ∂^γ to this equation, with $\gamma = (\gamma^1, ..., \gamma^d)$ and ∂^γ means $\frac{\partial^{|\gamma|}}{\partial x_1^{\gamma^1} ... \partial x_d^{\gamma^d}}$ and where $|\gamma| = \sum_{i=1}^d \gamma^i$. One gets

$$\partial_t \partial^\gamma \mathbf{u} + v^i \partial_i \partial^\gamma \mathbf{u} + \Sigma + R_\epsilon \partial^\gamma \mathbf{u} = \partial^\gamma S$$

where

$$\Sigma = \sum_{i=1}^d \sum_{|\mu| \geq 1, \gamma \geq \mu} \partial^\mu v^i \partial^{\gamma-\mu} \partial_i \mathbf{u}.$$

Then we have

LEMMA 3.2. *If $\gamma > d/2$, then for Σ and S defined as above we have :*

$$\|\Sigma(t, \cdot)\|_{L^2(\mathbb{T}^d)} \leq C(1 + \|\mathbf{u}(t, \cdot)\|_{H^\gamma(\mathbb{T}^d)} + \epsilon\|\mathbf{u}(t, \cdot)\|^2_{H^\gamma(\mathbb{T}^d)})$$

and

$$\|\partial^\gamma S(t, \cdot)\|_{L^2(\mathbb{T}^d)} \leq C(1 + \|\mathbf{u}(t, \cdot)\|_{H^\gamma(\mathbb{T}^d)} + \epsilon\|\mathbf{u}(t, \cdot)\|^2_{H^\gamma(\mathbb{T}^d)}).$$

Proof : Point 1 : the proof is a straightforward adaptation of the proof of [**1**] p.151. It is based on the following estimate : [**1**] Proposition 2.1.2 p. 100 :

PROPOSITION 3.3. *If* $u, v \in L^\infty \cap H^s$ $s \in \mathbb{N}$, *then for any* $\delta, \eta, |\delta| + |\eta| = s$, *one has*

$$\|(\partial^\delta u)(\partial^\eta v)\|_{L^2} \leq C(\|u\|_{L^\infty}\|v\|_{H^s} + \|u\|_{H^s}\|v\|_{L^\infty}).$$

Applying this result to $\partial^\mu v^i \partial^{\gamma-\mu}\partial_i \mathbf{u}$ $|\mu| \geq 1$ we obtain

$$\|\partial^\mu v^i \partial^{\gamma-\mu}\partial_i \mathbf{u}\|_{L^2(\mathbb{T}^d)} \leq C\left(\|\nabla v^i\|_{L^\infty(\mathbb{T}^d)}\|\mathbf{u}\|_{H^\gamma(\mathbb{T}^d)} + \|\nabla v^i\|_{H^\gamma(\mathbb{T}^d)}\|\mathbf{u}\|_{L^\infty(\mathbb{T}^d)}\right).$$

We know that $\|\cdot\|_{L^\infty(\mathbb{T}^d)} \leq C\|\cdot\|_{H^\gamma(\mathbb{T}^d)}$ if $\gamma > d/2$. Thanks to (142, 143) we have for any s

$$\|\nabla v_1\|_{H^s(\mathbb{T}^d)} \leq C(\|\omega_1\|_{H^s(\mathbb{T}^d)} + \|\beta_1\|_{H^s(\mathbb{T}^d)}). \tag{157}$$

thus $\|\nabla v_1\|_{H^\gamma(\mathbb{T}^d)} \leq C\|\mathbf{u}\|_{H^\gamma(\mathbb{T}^d)}$. Then using that $v = \bar{v} + \epsilon v_1$ we conclude.

Point 2 : We also know thanks to proposition 3.3 that for $s > d/2$

$$\|\nabla v_1 : \nabla v_1\|_{H^s} \leq C\|\nabla v_1\|_{L^\infty}\|\nabla v_1\|_{H^s}.$$

It follows that for $s > d/2$, we have

$$\|S^\epsilon\|_{H^s} \leq C(1 + \|u^\epsilon\|_{H^s} + \epsilon\|u^\epsilon\|^2_{H^s})$$

where C depends on the smoothness of the solution of (140).

□

Then after having applied ∂^γ to (141) and multiplied by $\partial^\gamma \mathbf{u}$ and noticing that for any $w \in \mathbb{R}^{d+2}$ one has $(w, R_\epsilon w) = 0$ one gets finally

$$\begin{aligned} &\partial_t |\partial^\gamma \mathbf{u}|^2 + \sum_{i=1}^d \partial_i (v^i |\partial^\gamma \mathbf{u}|^2) \\ = \quad &\nabla \cdot v\ |\partial^\gamma \mathbf{u}|^2 + (\partial_\gamma S + \Sigma)\partial^\gamma \mathbf{u} \end{aligned}$$

Since $\|\nabla \cdot v\|_{L^\infty} = \epsilon\|\beta_1\|_{L^\infty} \leq C\epsilon\|\mathbf{u}\|_{H^\gamma}$ if $\gamma > d/2$, using lemma 3.2, and integrating over $\mathbb{T}^d$ we have, for any $\gamma > d/2$:

$$\frac{d}{dt}\|\partial_\gamma \mathbf{u}(t,\cdot)\|^2_{L^2(\mathbb{T}^d)} \leq C\left(1 + \|\partial_\gamma \mathbf{u}(t,\cdot)\|^2_{L^2(\mathbb{T}^d)} + \epsilon\|\partial_\gamma \mathbf{u}(t,\cdot)\|^3_{L^2(\mathbb{T}^d)}\right)$$

and we can conclude using a standard Gronwall lemma that if the smooth solution $\bar{v}, p$ of Euler exists on a time interval $[0,T]$, for any $T' < T$ the sequence $\mathbf{u}^\epsilon$ for $\epsilon < \epsilon_0$ is bounded in $L^\infty([0,T'], H^\gamma(\mathbb{R}^2))$ for some ϵ_0.

Minimal regularity for Euler. In order to perform our computations, we need to have at least $\beta_1, \tilde{\rho}_1, \omega_1$ in $L^\infty([0,T] \times \mathbb{T}^d)$ and thus to have an estimate on their norms in $L^\infty([0,T], H^{c_0}(\mathbb{T}^d))$, with c_0 strictly grater that $d/2$. We thus have to differentiate $\mathbf{E}(d/2) + 1$ times the equation. It can be checked that this requires that the $L^\infty([0,T], H^{s_0}(\mathbb{T}^d))$ of $\bar{v}$ is bounded with s_0 equal to $\mathbf{E}(d/2) + 2$.

This achieves the proof of Theorem 3.1. □.

3. The Euler-Monge-Ampère system

We consider here the following Euler-Monge-Ampère system denoted by $(E - MA_\epsilon)$:

$$\partial_t \rho + \nabla \cdot (\rho v) = 0 \tag{158}$$

$$\partial_t v + v \cdot \nabla v = \frac{\nabla \psi - x}{\epsilon^2} \tag{159}$$

$$\det D^2 \psi = \rho \tag{160}$$

The last equation is to be understood in the following weak sense : ψ is the only (up to a constant) convex function with $\psi - |x|^2/2$ being $\mathbb{Z}^d$ periodic such that

$$\forall f \in C^0(\mathbb{T}^d), \int_{\mathbb{T}^d} f(\nabla \psi) d\rho = \int_{\mathbb{T}^d} f(x) dx. \tag{161}$$

A more precise sense will be given to the definition of ψ, in Theorem 3.4 and definition 3.5.

3.1. Geometric derivation of the Euler-Monge-Ampère system. This derivation has been introduced in chapter 2 and [**13**] . We reproduce it for sake of completeness. The reader only interested in the convergence proof may therefore skip this section.

3.1.1. *The Euler equations of incompressible fluid mechanics.* The motion of an incompressible fluid in a domain $\Omega \subset \mathbb{R}^d$ is described by the Euler incompressible equation (E) that we recall here :

$$\partial_t v + (v \cdot \nabla) v = \nabla p,$$
$$\nabla \cdot v = 0,$$

Following Arnold (see [**2**]), we have a formal interpretation of the Euler incompressible equations : introducing $G(\Omega)$ the group of all volume preserving diffeomorphisms of Ω with jacobian determinant equal to 1, the Euler equations describe the geodesics of $G(\Omega)$ with length measured in the L^2 sense.

3.1.2. *Approximate geodesics.* A general strategy to define approximate geodesics along a manifold M (in our case $M = G(\Omega)$) embedded in a Hilbert space H (here $H = L^2(\Omega, \mathbb{R}^d)$) is to introduce a penalty parameter $\epsilon > 0$ and the following *unconstrained* dynamical system in H

$$\partial_{tt} X + \frac{1}{2\epsilon^2} \nabla_X \left(d^2(X, M) \right) = 0. \tag{162}$$

In this equation, the unknown $t \to X(t)$ is a curve in H, $d(X, M)$ is the distance (in H) of X to the manifold M, i.e. in our case when $M = G(\Omega)$,

$$d(X, G(\Omega)) = \inf_{g \in G(\Omega)} \|X - g\|_H, \tag{163}$$

finally, ∇_X denotes the gradient operator in H. This penalty approach has been introduced for the Euler equations by Brenier in [**13**]. It is similar-but not identical- to Ebin's slightly compressible flow theory [**31**], and is a natural extension of the theory of constrained finite dimensional mechanical systems [**58**]. Actually if $G(\Omega)$ were a smooth manifold, the result would be exactly the one of [**31**], Theorem 2.7, but this is not the case, here because the L^2 metric is too weak. The penalized system is formally hamiltonian in variables $(X, \partial_t X)$ with hamiltonian (or energy) given by :

$$E = \frac{1}{2} \|\partial_t X\|_H^2 + \frac{1}{2\epsilon^2} d^2(X, G(\Omega))$$

Multiplying equation (162) by $\partial_t X$, we get immediately that the energy is formally conserved. Therefore it is plausible that the map $X(t)$ will remain close to $G(\Omega)$ if it is close at $t = 0$. A formal computation shows that, given a point X for which there is a unique closest point π_X to X in the H closure of $G(\Omega)$, we have :

(164)
$$\nabla_X\left(d(X,G)\right) = \frac{1}{d(X,G)}(X-\pi_X).$$

Thus the equation (162) formally becomes :

$$\partial_{tt}X + \frac{1}{\epsilon^2}(X-\pi_X) = 0.$$

To understand why solutions to such a system may approach geodesics along $G(\Omega)$ as ϵ goes to 0, just recall that, in the simple framework of a surface S embedded in the 3 dimensional Euclidean space, a geodesic $t \to s(t)$ along S is characterized by the fact that for every t, the plane defined by $\{\dot{s}(t), \ddot{s}(t)\}$ is orthogonal to S. In our case, $\partial_{tt}X(t)$ is orthogonal to $G(\Omega)$ thanks to (163) and $X(t)$ remains close to $G(\Omega)$.
The approximate geodesic equation was introduced in [**13**] in order to allow a spatial discretization of $G(\Omega)$ by the group of permutations of N points A_j chosen to form a grid of Ω. On such a discrete group, the concept of geodesics becomes unclear meanwhile approximate geodesics still make sense. They can be interpreted as trajectories of a cloud of N particles X_i moving in the Euclidean space $\mathbb{R}^{dN}$, which substitutes for H. These particles solve the following coupled system of springs

$$\epsilon^2\frac{d^2X_i}{dt^2} + X_i - A_{\sigma_i} = 0,$$

where σ is a time dependent permutation minimizing, at each fixed time t, $\Sigma\left|X_i - A_{\sigma(i)}\right|^2$ among all other permutations of the first N integers. The convergence of this discrete model to the incompressible Euler equations for well prepared initial data was proved in [**13**]. In order to study the continuous version (162), a specific study of the problem (163) is needed.
Notation
Since we intend to work on the flat torus $\mathbb{T}^d$ we might consider $\mathbb{Z}^d$ additive mappings (i.e. such that

$$\forall\vec{p}\in\mathbb{Z}^d, X(.+\vec{p}) = X(.)+\vec{p}),$$

as well as periodic mappings (i.e. mappings from $\mathbb{T}^d$ into itself).
Then given m an additive mapping,we denote by $\hat{m}$ the naturally associated mapping on $\mathbb{T}^d$. The following polar factorization Theorem is a periodic version of [**10**], it has been discovered independently by [**49**] and [**24**].

THEOREM 3.4. *Let $X:\mathbb{R}^d\to\mathbb{R}^d$ be $\mathbb{Z}^d$ additive suppose that $\rho_X = X\#dx$ has a density in $L^1([0,1]^d)$ then there exits an a.e. unique pair $\nabla\phi_X, \pi_X$ satisfying*

$$X = \nabla\phi_X\circ\pi_X$$

with ϕ_X a convex function such that $\phi_X(x)-|x|^2/2$ is $\mathbb{Z}^d$ periodic, and $\pi_X:\mathbb{R}^d\to\mathbb{R}^d$ additive such that $\hat{\pi}_X$ is measure preserving in $\mathbb{T}^d$. Moreover we have

$$\|X-\pi_X\|_{L^2([0,1]^d)} = \|\hat{X}-\hat{\pi}_X\|_{L^2(\mathbb{T}^d)} = d(\hat{X}, G(\mathbb{T}^d))$$

and if ψ_X is the Legendre transform of ϕ_X then

$$\pi_X = \nabla\psi_X\circ X.$$

Remark : The pair ϕ_X, ψ_X is uniquely defined by the density $\rho_X = X\#dx$.

Important properties of the optimal potential : The periodicity of $\phi_X(x) - |x|^2/2$ implies that $\nabla\phi_X$ and $\nabla\psi_X$ are $\mathbb{Z}^d$ additive, and that $\psi_X - |x|^2/2$ is also $\mathbb{Z}^d$ periodic. This allows the following definition :

Definition 3.5. *Let ρ be a probability measure on $\mathbb{T}^d$ then we denote $\phi[\rho]$ (resp. $\psi[\rho]$) the unique up to a constant convex function such that*

$$\phi[\rho] - |\cdot|^2/2 \text{ is } \mathbb{Z}^d \text{ periodic }, \tag{165}$$

$$\forall f \in C^0(\mathbb{T}^d), \int_{\mathbb{T}^d} f(\hat{\nabla}\phi[\rho](x))dx = \int_{\mathbb{T}^d} f(x)d\rho(x) \tag{166}$$

(resp. its Legendre fenchel transform).

Remark : We recover thus that $\psi[\rho], \phi[\rho]$ will be generalized solutions of the following Monge-Ampère equations

$$\det D^2\psi = \rho$$
$$\rho(\nabla\phi)\det D^2\phi = 1$$

3.1.3. *The Vlasov-Monge-Ampère equation.* As mentioned in the introduction, in chapter 2 a kinetic version of this model has been studied :

$$\frac{\partial f}{\partial t} + \nabla_x(\xi f) + \frac{1}{\epsilon^2}\nabla_\xi(\nabla\psi(x) - x)f) = 0 \tag{167}$$

$$\det D^2\psi = \rho \tag{168}$$

$$f(0,\cdot,\cdot) = f^0 \tag{169}$$

where the following was proved :

Theorem 3.6. *Let f be a weak solution of (167, 168,169) with finite energy, let $(t,x) \to \bar{v}(t,x)$ be a smooth solution of (140) for $t \in [0,T]$, and $p(t,x)$ the corresponding pressure, let*

$$H_\epsilon(t) = \frac{1}{2}\int f(t,x,\xi)|\xi - \bar{v}(x)|^2 dxd\xi + \frac{1}{2\epsilon^2}\int \rho|\nabla\psi(x) - x|^2 dx,$$

then

$$H_\epsilon(t) \le C\exp(Ct)(H_\epsilon(0) + C\epsilon^2),\ \forall t \in [0,T].$$

The constant C depends only of the $W_x^{1,\infty}$ norm of the quantities
$\{\bar{v}(s,.), p(s,.), \partial_t p(s,.), \nabla p(s,.)\, s \in [0,T]\}$.

Thus a "L^2" convergence was obtained for the VMA system to the incompressible Euler System. What we intend to obtain here is a convergence in any H^s norm for well prepared initial data provided that s is larger that $d/2+2$.

3.2. Result.

Theorem 3.7. *Let $\bar{v}, p$ be a smooth solution of the Euler incompressible system (140) on $[0,T] \times \mathbb{T}^d$, with initial data $\bar{v}_0$ and satisfying $\bar{v} \in L^\infty([0,T], H^s(\mathbb{T}^d))$ for some $s \ge s_0 = \mathbf{E}(d/2) + 2(d/2) + 2$, let $v_0^\epsilon, \rho_0^\epsilon$ be a sequence of initial data such that $\epsilon^{-1}(v_0^\epsilon - \bar{v}_0)$ and $\epsilon^{-2}(\rho_0^\epsilon - 1)$ are bounded in $H^s(\mathbb{T}^d)$. Then there exists a sequence $(v^\epsilon, \rho^\epsilon)$ of solutions to $(E - MA_\epsilon)$ with initial data $v_0^\epsilon, \rho_0^\epsilon$ belonging to $L^\infty([0,T_\epsilon], H^s(\mathbb{T}^d))$ with $\liminf_{\epsilon\to 0} T_\epsilon \ge T$. Moreover for $T' < T$ and ϵ small enough $\epsilon^{-1}(v^\epsilon - \bar{v})$ and*

$\epsilon^{-2}(\rho^\epsilon - 1)$ *are bounded in* $L^\infty([0,T'], H^{s'}(\mathbb{R}^2))$ *for some* $s' < s$*,* s' *going to* $+\infty$ *as* s *goes to* $+\infty$*. Finally when* $T = +\infty$*,* T_ϵ *goes to infinity.*

3.3. Proof of Theorem 3.7. The proof is much inspired from the proof of Theorem 3.1 for the following reason : by taking the divergence of equation (159) one gets :

$$\partial_t(\nabla \cdot v) + v \cdot \nabla(\nabla \cdot v) + \partial_i v^j \partial_j v^i = \frac{\Delta\psi - d}{\epsilon^2}$$

Suppose that ρ is close to 1, we guess that we have the following :

$$\psi = |x|^2/2 + \epsilon^2\varphi,$$
$$\rho = \det D^2\psi = 1 + \epsilon^2\Delta\varphi + O(\epsilon^4)$$

and thus $\Delta\psi = d + \epsilon^2\Delta\varphi = d + \rho - 1 + O(\epsilon^4)$. Therefore we expect that

$$\partial_t(\nabla \cdot v) + v \cdot \nabla(\nabla \cdot v) + \partial_i v^j \partial_j v^i = \frac{\rho - 1}{\epsilon^2} + O(\epsilon^2),$$

and that the technique of Theorem 3.1 will apply. We will justify the previous expansion in the next subsection :

3.3.1. *Linearization of the Monge-Ampère operator in* H^s *norm.* This section is devoted to the proof of the following proposition :

THEOREM 3.8. *let* ρ *be a probability measure on* $\mathbb{T}^d$*,* $d \le 3$*, let* ψ *satisfy*

$$\det D^2\psi = \rho$$

in the sense of definition (3.5). Then, there exists ϵ_0 *such that if* $\|\rho - 1\|_{H^2(\mathbb{T}^d)} \le \epsilon_0$ *then for any* $s > d/2$*, there exists* $C(s)$ *that satisfies*

$$\|D^2\psi - I\|_{H^s(\mathbb{T}^d)} \le C(s)\|\rho - 1\|_{H^s(\mathbb{T}^d)} \tag{170}$$

$$\|(\Delta\psi - d) - (\rho - 1)\|_{H^s(\mathbb{T}^d)} \le C(s)\|(\rho - 1)\|^2_{H^s(\mathbb{T}^d)}. \tag{171}$$

We first state the following result that can be obtained from [**18**] on the regularity of solutions to Monge-Ampère equation adapted to the periodic case.

THEOREM 3.9. *Let* $\rho \in C^\alpha(\mathbb{T}^d)$ *for some* $\alpha > 0$*, with* $0 < m \le \rho \le M$ *be a probability measure on* $\mathbb{T}^d$*, let then* $\psi = \psi(\rho)$ *in the sense of definition 3.5. Then* ψ *is a classical solution of*

$$\det D^2\psi = \rho$$

and satisfies for any $\alpha' < \alpha$ *:*

$$\|\nabla\psi(x) - x\|_{L^\infty} \le C(d) = \sqrt{d}/2 \tag{172}$$

$$\|D^2\psi\|_{C^{\alpha'}} \le K(m, M, \|\rho\|_{C^\alpha}, \alpha, \alpha') \tag{173}$$

Then we state a classical result of elliptic regularity that we will need to use in the course of the the proof. It can be found in [**39**], Theorem 9.11.

THEOREM 3.10. *Let* Ω *be an open set in* $\mathbb{R}^d$*,* $u \in W^{2,p}_{loc}(\Omega) \cap L^p(\Omega), 1 < p < \infty$*, be a strong solution of the equation*

$$\sum_{i,j=1}^{d} a^{ij}\partial_{ij}u = f$$

in Ω *where the coefficients* a^{ij} *satisfy*

$$a^{ij} \in C^0(\Omega), f \in L^p(\Omega);$$
$$\lambda|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq \Lambda|\xi|^2,$$

for $i,j = 1..d$. *Then for any* $\Omega' \subset\subset \Omega$,

$$\|u\|_{W^{2,p}(\Omega')} \leq C(\|u\|_{L^p(\Omega')} + \|f\|_{L^p(\Omega')}),$$

where C *depends on* $d, p, \lambda, \Lambda, \Omega', \Omega$ *and the moduli of continuity of the coefficients* a^{ij} *on* Ω'.

Now we are ready to prove Proposition 3.8. We recall that ψ satisfies

$$\det D^2\psi = \rho \tag{174}$$

$$\|\rho - 1\|_{H^2} \leq \epsilon_0 \tag{175}$$

We suppose $d = 3$ and the proof can be reproduced in the case $d = 2$ with minor changes. The proof is in 3 steps :
1-In the first step we prove that under condition (175) for any $N \geq 0$ there exit $\epsilon_N, C_N > 0$ such that for $|\gamma| \leq N$ if $\|\rho - 1\|_{H^{|\gamma|}} \leq \epsilon_N$ we have $\|D^2\psi\|_{H^{|\gamma|}} \leq C_N$.
2-Then we show that we have indeed the control (170).
3-Finally we prove (171).

We first prove by induction that if $\gamma \in \mathbb{N}^d$ then $\rho \in H^{|\gamma|}$ implies $D^2\psi \in H^{|\gamma|}$. It can be checked during the proof that this bound will be uniform under the condition (175) for ϵ_0 small enough.

First notice that this condition implies that there exists $0 < \lambda_1 < 1 < \lambda_2$ with $\rho(x) \in [\lambda_1, \lambda_2]$ and that ρ is in C^α for some $\alpha = \frac{1}{2}$. Then from Theorem 3.9, $D^2\psi \in C^{\alpha'}$ with $\alpha' < \alpha$. Note also that since $\rho \in [\lambda_1, \lambda_2]$ and using equation (174), $D^2\psi \in C^{\alpha'}$ implies that $[D^2\psi]^{-1} \in C^{\alpha'}$, and thus M^{ij} the comatrix of $D^2\psi$ is uniformly elliptic and in $C^{\alpha'}$.

The case $\gamma = 0$ is a consequence of Theorem 3.9.

For $|\gamma| = 1$ we differentiate (174) with respect to x_ν, this gives :

$$M^{ij}\partial_{ij}(\partial_\nu\psi) = \partial_\nu\rho \tag{176}$$

with M^{ij} the comatrix of $\partial_{ij}\psi$. Then if $\partial_\nu\rho \in L^2$, by Theorem 3.10, $\partial_\nu\psi \in W^{2,2}$. If $\partial_\nu\rho \in L^6$ we also get that $\partial_\nu\psi \in W^{2,6}$.

For $|\gamma| = 2$ differentiating once more one with respect to x_β one gets :

$$M^{ij}\partial_{ij}(\partial_{\nu\beta}\psi) + (\partial_\beta M^{ij})\partial_{ij}(\partial_\nu\psi) = \partial_{\nu\beta}\rho \tag{177}$$

still with M^{ij} the comatrix of $\partial_{ij}\psi$. Suppose that $\rho \in H^2$, then $W^{2,2} \subset W^{1,\frac{2d}{d-2}} = W^{1,6}$ if $d = 3$, and $\partial_\nu\psi \in W^{2,6}$. $\partial_\beta M^{ij}$ is a sum of terms of the form $\partial_{ij}(\partial_\beta\psi)\partial_{kl}\psi$ and thus the second term of the left hand side of (177) is bounded in L^2. Then once again by Theorem 3.10 one gets that $\partial_{ij}\partial_{\nu\beta}\psi \in L^2$ if $\partial_{\nu\beta}\rho \in L^2$.

Moreover if $\partial_{\nu\beta}\rho \in L^6$ then $\partial_\nu\rho \in C^\alpha$ for some $\alpha > 0$ and $D^2\partial_\nu\psi \in C^{\alpha'}$ from (176) and Schauder interior estimates (see [**39**], Theorem 6.2.). Thus $(\partial_\beta M^{ij})\partial_{ij}(\partial_\nu\psi) \in C^{\alpha'}$. From (177) and Theorem 3.10 we obtain $\partial_{\nu\beta}D^2\psi \in L^6$.

We now make the following induction assumption :
(Hn) : Under assumption (175), for any $\gamma \in \mathbb{N}^d$, $|\gamma| \leq n$, $\rho \in H^{|\gamma|}$ implies that $\partial^\gamma D^2\psi \in L^2$. If moreover $\rho \in W^{|\gamma|,6}$ then $\partial^\gamma D^2\psi \in L^6$.
As we just saw, this assertion is true for $n = 1, 2$. Take $|\gamma| = n + 1 \geq 3$, $\rho \in H^{|\gamma|}$, and apply ∂^γ to (174) :

$$M^{ij}\partial_{ij}\partial^\gamma\psi + \sum_{\substack{\gamma_1+\gamma_2+\gamma_3=\gamma \\ |\gamma|-1\geq|\gamma_1|\geq|\gamma_2|\geq|\gamma_3|}} * \,\partial_{ij}\partial^{\gamma_1}\psi\partial_{kl}\partial^{\gamma_2}\psi\partial_{mn}\partial^{\gamma_3}\psi = \partial^\gamma\rho \tag{178}$$

with $*$ some constant coefficients. We call $\mathcal{T}$ the second term of the left hand side of (178). Since $\rho \in H^{|\gamma|}$, $\rho \in W^{|\gamma|-1,6}$ we have $\partial^\alpha D^2\psi \in L^6(\mathbb{T}^d)$ for any $|\alpha| \leq n$ using **(Hn)**. Therefore $\mathcal{T} \in L^2$ and since $\partial^\gamma\rho \in L^2$ we obtain $M^{ij}\partial_{ij}\partial^\gamma\psi \in L^2$. Using Theorem 3.10 it follows that $\partial^\gamma D^2\psi \in L^2$.
Remember that $|\gamma| \geq 3$ thus $|\gamma_3| \leq \frac{1}{3}|\gamma| \leq \gamma - 2$, and $|\gamma_2| \leq \frac{1}{2}|\gamma| \leq \gamma - 2$. Since $d = 3$, we have $H^2 \subset C^\alpha$ for some $\alpha > 0$ and thus $\partial_{kl}\partial^{\gamma_2}\psi$, $\partial_{mn}\partial^{\gamma_3}\psi$ are in C^α, moreover $H^1 \subset L^6$ and since $|\gamma_1| \leq |\gamma| - 1$, $\partial_{ij}\partial^{\gamma_1}\psi$ is in L^6. Therefore $\mathcal{T}$ is in L^6. If $\partial^\gamma\rho \in L^6$ we have $\partial^\gamma D^2\psi$ in L^6. So far we have achieved the first step of the proof.

Now by induction on $|\gamma|$ we prove (170) and (171). From (175) we have $\|\rho - 1\|_{L^2} \leq \epsilon_0$ small. Take $\psi = |x|^2/2 + \varphi$ solution of (160) with φ periodic and $\int_{\mathbb{T}^d}\varphi = 0$. If ρ is close enough to 1 in L^2 then the Wasserstein distance from ρ to 1 given by

$$W_2^2(\rho, 1) = \int_{\mathbb{T}^d} \frac{1}{2}\rho(x)|\nabla\varphi(x)|^2 \, dx$$

will be small. See ([**62**]) for references about the Wasserstein distance. Remember that $0 < \lambda_1 \leq \rho \leq \lambda_2$. Then φ will be small in in H^1 norm, and $D^2\varphi$ is bounded in C^α thus by interpolation, $D^2\varphi$ will be small in C^0 norm. Then we have

$$\det(I + D^2\varphi) = 1 + \Delta\varphi + R_{ij}\partial_{ij}\varphi$$

where R is a symmetric matrix whose coefficients are polynomials in $\partial_{ij}\varphi$ of degree greater or equal than 1. $|R_{ij}|$ is small and belongs to C^α uniformly for $\|\rho - 1\|_{C^\alpha} \leq \epsilon_0$, thus the matrix $\delta_{ij} + R_{ij}$ are uniformly bounded, elliptic, and C^α continuous. Since φ satisfies

$$(\delta_{ij} + R_{ij})\partial_{ij}\varphi = \rho - 1 \tag{179}$$

it follows from Theorem 3.10 that $\|\partial_{ij}\varphi\|_{L^2} \leq C\|\rho - 1\|_{L^2}$ and this proves (170) for $\gamma = 0$. If $|\gamma| = 1$, we have

$$(M^{ij})\partial_{ij}\partial_\nu\varphi = \partial_\nu\rho$$

with M uniformly bounded, elliptic and C^α continuous. For the same reasons we have $\|\partial_{ij}\partial_\nu\varphi\|_{L^2} \leq C\|\partial_\nu\rho\|_{L^2}$.
Now if $|\gamma| \geq 2$, $\mathcal{T}$ is a sum of terms which contain all a product of at least two derivatives of ψ of degree higher than 3. Since $D^3\psi = D^3\varphi$ this means that

$$\begin{aligned} & \partial_{ij}\partial^{\gamma_1}\psi\partial_{kl}\partial^{\gamma_2}\psi\partial_{mn}\partial^{\gamma_3}\psi \\ = \; & \partial_{ij}\partial^{\gamma_1}\varphi\partial_{kl}\partial^{\gamma_2}\varphi\partial_{mn}\partial^{\gamma_3}\psi \end{aligned}$$

Then by induction we know that $\|\partial_{ij}\varphi\|_{H^{|\gamma|-1}} \leq C\|\rho - 1\|_{H^{|\gamma|-1}}$. Since $\rho - 1 \in H^{|\gamma|}$ we also have that $D^2\psi \in H^{|\gamma|}$. Therefore $\partial_{kl}\partial^{\gamma_2}\varphi$ and $\partial_{mn}\partial^{\gamma_3}\psi$ are bounded in L^∞. We obtain that

$$\|\partial_{ij}\partial^{\gamma_1}\psi\partial_{kl}\partial^{\gamma_2}\psi\partial_{mn}\partial^{\gamma_3}\psi\|_{L^2} \leq C\|\rho - 1\|_{H^{|\gamma|-1}}$$

$\partial^\gamma\varphi$ satisfies

$$(M^{ij})\partial_{ij}\partial^\gamma\varphi = \partial^\gamma\rho - \mathcal{T}$$

thus from Theorem 3.10

$$\begin{aligned}\|\partial^\gamma D^2\varphi\|_{L^2} &\leq C(\|\rho - 1\|_{H^{|\gamma|-1}} + \|\partial^\gamma\rho\|_{L^2}) \\ &\leq C(\|\rho - 1\|_{H^{|\gamma|}})\end{aligned}$$

and we conclude that

$$\|D^2\psi - I\|_{H^s} \leq C(s)\|\rho - 1\|_{H^s}$$

for $s \in \mathbb{N}, s \geq 2$ and under condition (175) ; thus (171) is obtained.

Using Proposition 3.3 and the fact that $\partial_{mn}\partial^{\gamma_3}\psi$ is bounded in L^∞, we can also obtain that

$$\partial_{ij}\partial^{\gamma_1}\varphi\partial_{kl}\partial^{\gamma_2}\varphi\partial_{mn}\partial^{\gamma_3}\psi \leq C\|D^2\varphi\|^2_{H^{|\gamma|}} \leq C\|\rho - 1\|^2_{H^{|\gamma|}}$$

for $|\gamma| \geq 2$. To obtain (171) note that formula (178) can be written in the following way :

$$\Delta\partial^\gamma\psi + S^{ij}\partial_{ij}\partial^\gamma\psi + \mathcal{T} = \partial^\gamma\rho$$

where $S^{ij} = M^{ij} - I$. The components of S are polynomials of degree greater or equal to 1 of $\partial_{ij}\varphi$, thus $\|S^{ij}\|_{C^\alpha} \leq C\|D^2\varphi\|_{C^\alpha} \leq C\|\rho - 1\|_{H^2}$ for some $\alpha > 0$ and thus $\|S^{ij}\partial_{ij}\partial^\gamma\varphi\|_{L^2} \leq C\|\rho - 1\|^2_{H^{|\gamma|}}$. It follows that

$$\|\Delta\partial^\gamma\varphi - \partial^\gamma\rho\|_{L^2} \leq C\|\partial^\gamma\rho\|^2_{L^2}.$$

Since $\Delta\varphi$ and $\rho - 1$ are periodic with zero mean value (171) follows and Proposition 3.8 is proved. □

3.3.2. *Estimates.* Doing the same change of variables as in the proof of Theorem 3.1

$$\begin{aligned}&\nabla \cdot v = \epsilon\beta_1 \\ &\rho = 1 + \epsilon^2\rho_1 \\ &\text{curl} v = \omega = \bar{\omega} + \epsilon\omega_1\end{aligned}$$

we obtain :

$$\partial_t(\bar{\omega} + \epsilon\omega_1) + v \cdot \nabla(\bar{\omega} + \epsilon\omega_1) = -(\bar{\omega} + \epsilon\omega_1)\epsilon\beta \tag{180}$$

$$\partial_t\epsilon\beta_1 + v \cdot \nabla\epsilon\beta_1 + 2\epsilon\partial_i\bar{v}^j\partial_j v_1^i + \epsilon\partial_i v_1^j\partial_j v_1^i = \frac{\Delta\psi - d}{\epsilon^2} - \partial_i\bar{v}^j\partial_j\bar{v}^i \tag{181}$$

$$\partial_t\epsilon^2\rho_1 + v \cdot \nabla\epsilon^2\rho_1 = -(1 + \epsilon^2\rho_1)\epsilon\beta_1 \tag{182}$$

Now we define Ξ by

$$\Delta\psi - d = \epsilon^2\rho_1 + \epsilon^4\Xi,$$

and from Proposition 3.8 inequality (171), we have $\|\Xi\|_{H^s} \leq C\|\rho_1\|^2_{H^s}$. The system can here be written in the following way :

$$\begin{aligned}&\partial_t\mathbf{u}^\epsilon + \sum_i v^i\partial_i\mathbf{u}^\epsilon + R^\epsilon\mathbf{u}^\epsilon = S^\epsilon(\mathbf{u}^\epsilon) + V^\epsilon \\ &\mathbf{u}^\epsilon(0) = \mathbf{u}_0^\epsilon\end{aligned}$$

still with

$$\mathbf{u} = \begin{pmatrix} \omega_1 \\ \beta_1 \\ \tilde{\rho}_1 \end{pmatrix}, R_\epsilon = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\epsilon} \\ 0 & -\frac{1}{\epsilon} & 0 \end{pmatrix}$$

with the same R^ϵ, S^ϵ as in the Euler-Poisson case and with

$$V^\epsilon = \begin{pmatrix} 0 \\ \epsilon \Xi \\ 0 \end{pmatrix},$$

and thus $\|V^\epsilon\|_{H^s} \leq \epsilon \|\mathbf{u}\|^2_{H^s}$. Then the energy estimates are the same as in the first proof, the solution $\mathbf{u}$ satisfying a control of the form :

$$\frac{d}{dt}\|\mathbf{u}\|_{H^s} \leq C\|\mathbf{u}\|_{H^s} + \epsilon\|\mathbf{u}\|^2_{H^s}$$

and the same conclusion holds true. □

4. Higher order approximation

Here we prove that the the Euler-Poisson system and the Euler-Monge-Ampère system are closer as ϵ goes to 0 than Euler-Poisson and Euler. We consider $\bar{v}, p$ a solution of the Euler incompressible system (140) smooth in $[0, T]$. For any $\epsilon > 0$, we consider also $v^\epsilon_{ep}, \rho^\epsilon_{ep}$ a smooth solution of the $(E-P)_\epsilon$ system with initial conditions $v^\epsilon_{ep,0}, \rho^\epsilon_{ep,0}$ satisfying the assumptions of Theorem (3.1). Thus $\epsilon^{-1}(v^\epsilon_{ep} - \bar{v})$ and $\epsilon^{-2}(\rho^\epsilon_{ep} - 1)$ are bounded in $L^\infty([0, T'], H^s(\mathbb{T}^d))$ for some $s > s_0$ for any $0 < T' < T$.

THEOREM 3.11. *Let $\bar{v}, v^\epsilon_{ep}, \rho^\epsilon_{ep}$ be as above. Let $v^\epsilon_0, \rho^\epsilon_0$ be a sequence of initial data such that $\epsilon^{-2}(v^\epsilon_0 - v^\epsilon_{ep,0})$ and $\epsilon^{-3}(\rho^\epsilon_0 - \rho^\epsilon_{ep,0})$ are bounded in $H^s(\mathbb{T}^d)$ for $s \geq s_0 = \mathbf{E}(d/2) + 2$. Then there exists a sequence $(v^\epsilon, \rho^\epsilon)$ of solutions to $(E - MA_\epsilon)$ with initial data $v^\epsilon_0, \rho^\epsilon_0$ belonging to $L^\infty([0, T_\epsilon], H^s(\mathbb{T}^d))$ with $\liminf_{\epsilon \to 0} T_\epsilon \geq T$. Moreover for $T' < T$ and ϵ small enough $\epsilon^{-2}(v^\epsilon - v^\epsilon_{ep})$ and $\epsilon^{-3}(\rho^\epsilon - \rho^\epsilon_{ep})$ are bounded in $L^\infty([0, T'], H^{s'}(\mathbb{R}^2))$ for some $s' < s$, s' going to $+\infty$ as s goes to $+\infty$.*

Remark : We see here that the difference between solutions of $(E-P)_\epsilon$ and $(E-MA)_\epsilon$ is of order ϵ^3 for the density and of order ϵ^2 for the velocity whereas the difference between solutions of $(E-P)_\epsilon$ (or $(E-MA)_\epsilon$) and Euler was of order ϵ^2 for the density and of order ϵ for the velocity.
Proof : We introduce $v_{ep} = \bar{v} + \epsilon v_1, \rho_{ep} = 1 + \epsilon^2 \rho_1$ solution to $(E-P)_\epsilon$ with $\beta_1 = \nabla \cdot v_1, \omega_1 = \text{curl} v_1$. Then we set

$$\begin{aligned} v &= \bar{v} + \epsilon v_1 + \epsilon^2 v_2 \\ \nabla \cdot v &= \epsilon \beta_1 + \epsilon^2 \beta_2 \\ \rho &= 1 + \epsilon^2 \rho_1 + \epsilon^3 \rho_2 \\ \text{curl} v &= \omega = \bar{\omega} + \epsilon \omega_1 + \epsilon^2 \omega_2. \end{aligned}$$

The system $(E-MA)_\epsilon$ now reads :

$$\partial_t(\bar\omega+\epsilon\omega_1+\epsilon^2\omega_2)+v\cdot\nabla(\bar\omega+\epsilon\omega_1+\epsilon^2\omega_2)$$
$$= -(\bar\omega+\epsilon\omega_1+\epsilon^2\omega_2)(\epsilon\beta_1+\epsilon^2\beta_2) \tag{183}$$
$$\partial_t(\epsilon\beta_1+\epsilon^2\beta_2)+v\cdot\nabla(\epsilon\beta_1+\epsilon^2\beta_2)$$
$$+\nabla(v_{ep}+\epsilon^2 v_2):\nabla(v_{ep}+\epsilon^2 v_2)$$
$$= \frac{\Delta\psi-d}{\epsilon^2}-\partial_i\bar v^j\partial_j\bar v^i \tag{184}$$
$$\partial_t(\epsilon^2\rho_1+\epsilon^3\rho_2)+v\cdot\nabla(\epsilon^2\rho_1+\epsilon^3\rho_2)$$
$$= -(1+\epsilon^2\rho_1+\epsilon^3\rho_2)(\epsilon\beta_1+\epsilon^2\beta_2) \tag{185}$$

Here we will define Ξ by

$$\Delta\psi-d=\epsilon^2\rho_1+\epsilon^3\rho_2+\epsilon^4\Xi,$$

and thus from Proposition 3.8 we will have

$$\|\Xi\|_{H^s(\mathbb{T}^d)}\le C\|\rho_1+\epsilon\rho_2\|^2_{H^s(\mathbb{T}^d)}\le C(\|\rho_1\|^2_{H^s(\mathbb{T}^d)}+\epsilon^2\|\rho_2\|^2_{H^s(\mathbb{T}^d)})$$

Setting

$$\mathbf{u}=\begin{pmatrix}\omega_2\\ \beta_2\\ \rho_2\end{pmatrix}$$

and using that $\bar v+\epsilon v_1, 1+\epsilon^2\rho_1$ is a smooth solution to $(E-P)_\epsilon$ we obtain that

$$\partial_t\mathbf{u}+v\cdot\nabla\mathbf{u}=R_\epsilon\mathbf{u}+T_\epsilon$$

with R_ϵ as before and T_ϵ defined by

$$T_\epsilon=\begin{pmatrix}-v_2\cdot\nabla w_{ep}-\epsilon\beta_1\omega_2-\beta_2\omega_{ep}-\epsilon^2\omega_2\beta_2\\ -\epsilon v_2\cdot\nabla\beta_1-2\nabla v_{ep}:\nabla v_2-\epsilon^2\nabla v_2:\nabla v_2+\Xi\\ -\epsilon v_2\cdot\nabla\rho_1-\epsilon\beta_1\rho_2-\epsilon\beta_2\rho_1-\epsilon^2\beta_2\rho_2\end{pmatrix}$$

Using again Proposition 3.3 as in lemma 3.2 we obtain that

$$\|T_\epsilon\|_{H^s(\mathbb{T}^d)}\le C(1+\|\mathbf{u}\|_{H^s(\mathbb{T}^d)}+\epsilon\|\mathbf{u}\|^2_{H^s(\mathbb{T}^d)})$$

for s large enough where the constant C depends on the smoothness of $\bar v, p, v_1, \rho_1$ which is controlled for $t\le T'<T$ with T the time on which the solution of (140) is smooth. Then ρ_2, v_2 remain bounded in $L^\infty([0,T''],H^s(\mathbb{T}^d))$ for any $T''<T'$ and for $\epsilon<\epsilon_0$ small enough. It follows that $\frac{1}{\epsilon^3}(\rho_{ep}-\rho_{ema})$ and $\frac{1}{\epsilon^2}(v_{ep}-v_{ema})$ remain bounded in $L^\infty([0,T''],H^s(\mathbb{T}^d))$. This achieves the proof of Theorem 3.11.
□

CHAPITRE 4

Regularity of the polar factorization for time dependent maps

G. Loeper[1]

RÉSUMÉ. We consider the polar factorization of vector valued mappings introduced in [**10**] in the case of a family of mappings depending on a parameter. We investigate the regularity with respect to this parameter of the terms of the polar factorization by constructing some a priori bounds. To do so, we consider the linearization of the associated Monge-Ampère equation which we view as a conservation law.

1. Introduction

Brenier in [**10**] showed that given Ω an open set of $\mathbb{R}^d$ such that $|\partial\Omega| = 0$ with $|.|$ the Lebesgue measure of $\mathbb{R}^d$, every Lebesgue measurable mapping $X \in L^2(\Omega, \mathbb{R}^d)$ satisfying the non-degeneracy condition

$$\forall B \subset \mathbb{R}^d \text{ measurable, } |B| = 0 \Rightarrow |X^{-1}(B)| = 0 \tag{186}$$

can be factorized in the following (unique) way :

$$X = \nabla\Phi \circ g, \tag{187}$$

where Φ is a convex function and g is Lebesgue-measure preserving for Ω, i.e.

$$\forall f \in C^0(\Omega), \int_\Omega f(g(x))\, dx = \int_\Omega f(x)\, dx \tag{188}$$

If the measure ρ is defined by

$$\forall f \in C^0(\mathbb{R}^d), \int_{\mathbb{R}^d} f d\rho = \int_\Omega f(X(a))da \tag{189}$$

one sees first that the condition (186) is equivalent to the fact that ρ is absolutely continuous with respect to the Lebesgue measure, or has a density in $L^1(\mathbb{R}^d, dx)$. Then Φ satisfies on Ω in the weak sense of (191) below the following Monge-Ampère equation :

$$\rho(\nabla\Phi(x)) \det D^2\Phi(x) = 1,$$

and Ψ the Legendre transform of Φ, defined by

$$\Psi(y) = \sup_{x\in\mathbb{R}^d}\{x \cdot y - \Phi(x)\} \tag{190}$$

satisfies in the weak sense of (192) the equation

$$\det D^2\Psi(x) = \rho(x).$$

In this paper we are interested in the following problem : given a "time" dependent family of mappings $t \to X(t, .)$, look at the regularity of the curve $t \to g(t, .), \Phi(t, .), \Psi(t, .)$. We will obtain our results by linearizing the factorization (187). One of the interests of such a study is its application to the semi-geostrophic equation, a system arising in meteorology to model frontogenesis. This application is discussed in section 7.

[1]Laboratoire J.A.Dieudonné, Université de Nice-Sophia-Antipolis, Parc Valrose, 06108 NICE Cedex 2.
email : loeper@math.unice.fr

1.1. Heuristics. Suppose that Ω is bounded, and for any t we denote by $d\rho(t,\cdot) = X(t,\cdot)\#da$ (with da the Lebesgue measure on Ω) the measure defined by (189). Then if Φ is as in (187) and Ψ is its Legendre transform, we have

$$\forall g \in C^0(\mathbb{R}^d), \int_\Omega g(\nabla\Phi(y))dy = \int_{\mathbb{R}^d} g(x)d\rho(x) \tag{191}$$

$$\forall f \in C^0(\Omega), \int_{\mathbb{R}^d} f(\nabla\Psi(x))d\rho(x) = \int_\Omega f(y)dy. \tag{192}$$

Note that the existence and uniqueness of the pair $\nabla\Phi, \nabla\Psi$ is not subject to the condition (186). We suppose that X is chosen such that $d\rho(t,\cdot)$ is absolutely continuous with respect to the Lebesgue measure of $\mathbb{R}^d$ thus $d\rho(x) = \rho(x)dx$ with $\rho \in L^1(\mathbb{R}^d)$. Considering formally that all the terms are smooth, if we differentiate with respect to time, we obtain :

$$\begin{aligned} \forall f \in C^1(\mathbb{R}^d), \qquad & \int_\Omega \nabla f(X(t,a)) \cdot \dot{X}(t,a)da = \int_{\mathbb{R}^d} f(x)\partial_t\rho(t,x)dx \\ = \quad & \int_\Omega \nabla f(\nabla\Phi(t,x)) \cdot \partial_t\nabla\Phi(t,x)dx. \end{aligned} \tag{193}$$

1.1.1. *Parallel with the Hodge decomposition of vector fields.* By differentiating (187) with respect to time one finds

$$\partial_t X(t,a) = \partial_t\nabla\Phi(t,g(t,a)) + D^2\Phi(t,g(t,a))\partial_t g(t,a).$$

Thus if X is invertible, one can write write $\partial_t X(t,a) = v(t,X(t,a))$ for some "Eulerian" vector field $v(t,x)$. g will then also be invertible and composing with g^{-1} one gets :

$$v(t,\nabla\Phi(t,x)) = \partial_t\nabla\Phi(t,x) + D^2\Phi(t,x)w(t,x) \tag{194}$$

with $w = \partial_t g(t,g^{-1}(t,x))$. Then w is divergence free on Ω : indeed (188) is valid for all time, and differentiating w.r.t time one gets formally :

$$\forall f \in C^0(\mathbb{R}^d), \int_\Omega \nabla f(g(t,x))\partial_t g(t,x)\; dx = 0.$$

By composing with g^{-1} we find that $\partial_t g(t,g^{-1}(t,x))$ is divergence free. This gives rise to a decomposition of vector fields : if $d\rho = da$ the Lebesgue measure of Ω then $\nabla\Phi(x) = x$ and one recovers the usual "div-curl" or Hodge decomposition of vector fields. Then for a generic ρ, the Lagrangian velocity $v(t,X(t,a)) = \partial_t X(t,a)$ is decomposed as the sum of two terms : the first $T_1 = \partial_t\nabla\Phi(t,g(t,a))$ is a gradient computed at g, and the second $T_2 = D^2\Phi(t,g(t,a))w(t,g(t,a))$ doesn't move any mass, i.e in Eulerian coordinates, $T_2 = \bar{w}(t,X(t,a))$ such that $\nabla\cdot(\rho\bar{w}) = 0$.

1.1.2. *The associated elliptic problem.* Since Ψ solves the equation $\det D^2\Psi = \rho$ then one retrieves $\partial_t\Psi$ by solving the following elliptic problem :

$$\begin{aligned} &\rho[D^2\Psi]^{-1}_{ij}\partial_{ij}\partial_t\Psi = \partial_t\rho \\ &\nabla\partial_t\Psi \cdot \vec{n_1}(\nabla\Psi) = 0 \text{ on } (\nabla\Psi)^{-1}(\partial\Omega) \end{aligned} \tag{195}$$

where $\rho[D^2\Psi]^{-1}$ is the comatrix of $D^2\Psi$. This is what we intend to do here, the difficulty coming from the lack of regularity and ellipticity of this equation since we do not make any smoothness assumption on ρ.

Let us now state our results :

2. Results

In the remainder of the paper Ω will be kept fixed once for all and we will assume for simplicity (although one may possibly remove this assumption through approximation) that it is smooth and strictly convex. For compatibility ρ will be a probability measure on $\mathbb{R}^d$ and Ω of Lebesgue measure one. $\mathcal{M}(\Omega)$ will design the set of (possibly vector valued) bounded measures on Ω, with norm $\|.\|_{\mathcal{M}(\Omega)}$. I will be an non-empty open interval of $\mathbb{R}$.

THEOREM 4.1. *Let Ω, I be as above, let*

$$X : I \times \Omega \to \mathbb{R}^d.$$

Let for any $t \in I$ $\rho(t,\cdot)$ be the measure defined by $d\rho(t,\cdot) = X(t,\cdot)\#da$ as in (189). Suppose that $\rho(t,\cdot)$ is supported for any $t \in I$ in $B_R = B(0,R)$ that $\partial_t X \in L^\infty(I \times \Omega)$ and $\rho \in L^\infty(I \times \mathbb{R}^d)$. Take

$$X(t) = \nabla\Phi(t) \circ g(t)$$
$$g(t) = \nabla\Psi(t) \circ X(t)$$

to be the polar factorization of X as in (187) where for some x_0 in Ω we impose $\forall t \in I$ $\int_\Omega \Phi(t,x)\,dx = 0$, then
1 - For a.e. $t \in I$ $\partial_t \nabla\Phi(t,\cdot)$ is a bounded measure on Ω with

$$\|\partial_t \nabla\Phi\|_{L^\infty(I,\mathcal{M}(\Omega))} \leq C(R,d)\|\rho\|^{\frac{1}{2}}_{L^\infty(I\times B_R)}\|\partial_t X\|_{L^\infty(I\times B_r)}$$

and thus $\partial_t\Phi \in L^\infty(I, L^{1*}(\Omega))$ with $1 = d/(d-1)$.*
2 - $\Phi(t,x) \in C^\alpha(I, C^0(\bar{\Omega}))$ for some $\alpha \in]0,1[$,
3 - $\partial_t g$ is a bounded measure on Ω with

$$\|\partial_t g\|_{L^\infty(I,\mathcal{M}(\Omega))} \leq C(R,d)\|\rho\|_{L^\infty(I\times B_R)}\|\partial_t X\|_{L^\infty(I\times\Omega)}$$

4- If for any $t \in I$, ρ is supported in some open set Ω' and $\rho(\cdot,\cdot) \geq C > 0$ on Ω' then there exists $1 > \beta > 0$ such that for any $\omega' \subset\subset \Omega'$, $\nabla\Psi \in C^\beta(I \times \omega')$.
4'-If moreover Ω' is convex, then for any $\omega \subset\subset \Omega$, $\nabla\Phi \in C^\beta(I \times \omega)$

Remark1 : The $C^{1,\alpha}$ regularity in the space variable in point $4, 4'$ are due to [**18**],[**20**]. The precise result is given in Proposition 4.4.
Remark2 : The condition $\int_\Omega \Phi(t,x)\,dx = 0$ is necessary, since the function Φ is only defined up to a constant.
The Theorem 4.1 will be deduced through approximation from the following theorem :

THEOREM 4.2. *let I be as above, for any $t \in I$, let $\rho(t,\cdot)$ be a smooth probability density of $\mathbb{R}^d$ such that $\rho(t,\cdot)$ is supported in $B(0,R)$ and positive in $\overline{B(0,R)}$, let $v(t,x) \in \mathbb{R}^d$ be a smooth vector field on $\mathbb{R}^d$ and satisfy on $I \times \mathbb{R}^d$*

$$\partial_t\rho + \nabla\cdot(\rho v) = 0, \tag{196}$$

Let Ω be as above let $\Psi(t,\cdot), \Phi(t,\cdot)$ be as in (192,191), then for any $t \in I$, for any $1 \leq p, r \leq \infty$,

$$\|\partial_t\nabla\Phi\|_{L^q(\Omega)} \leq \left(\|\rho|v|^2\|_{L^r}\|D^2\Psi\|_{L^{r'}(B_R)}\|D^2\Phi\|_{L^p(\Omega)}\right)^{1/2} \tag{197}$$

with $q = \frac{2p}{1+p}$, and in particular

$$\|\partial_t\nabla\Phi\|_{\mathcal{M}(\Omega)} \leq C(R,d)\left(\|\rho|v|^2\|_{L^\infty}\right)^{1/2}.$$

$$\left[\int \rho|\partial_t \nabla\Psi|^q\right]^{1/q} \leq \left(\|\rho|v|^2\|_{L^r}\|D^2\Psi\|_{L^{r'}(B_R)}\left[\int \rho|D^2\Psi|^p\right]^{1/p}\right)^{1/2}, \tag{198}$$

and in particular

$$\int_{\mathbb{R}^d} \rho|\partial_t\nabla\Psi| \leq C(R,d)\|\rho\|^{\frac{1}{2}}_{L^\infty(\mathbb{R}^d)}\|\rho|v|^2\|^{\frac{1}{2}}_{L^\infty(\mathbb{R}^d)}.$$

Remark 1 : Those theorems can be almost immediately adapted to the case of polar factorization for mappings of the d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, see [**49**] and [**24**].
Remark 2 : It follows that for a pair (ρ, v) satisfying the conditions of Theorem 4.1 one can define the decomposition as in (194). The uniqueness comes from the uniqueness of the polar factorization.

3. Related results

The linearization of the Monge-Ampère equation is a well known subject since it is used to carry out the continuity method, in order to obtain solutions of the Monge-Ampère equation. However for this purpose this is always made in the case where the densities and the domains considered are smooth.

In [**22**] the authors proved Harnack inequality for solutions of

$$a_{ij}\partial_{ij}u = 0$$

with a the comatrix of $D^2\Psi$, Ψ convex, under the assumption that the measure $\rho = \det D^2\Psi$ satisfies the following absolute continuity condition :
C : For any $0 < \delta_1 < 1$ *there exists* $0 < \delta_2 < 1$ *such that for any section* S *and any measurable set* $E \subset S$

$$\text{if } \frac{|E|}{|S|} \leq \delta_2 \text{ then } \frac{\rho(E)}{\rho(S)} \leq \delta_1, \tag{199}$$

(a section is a set of the form

$$S_t(x_0) = \{x|u(x) - u(x_0) \leq p\cdot(x - x_0) + t, p \in \partial u(x_0)\})$$

they showed that the solution of $(\det D^2\Psi)(D^2\Psi)^{-1}_{ij}D_{ij}u = 0$ satisfies a Harnack inequality on the sections of Ψ and subsequently is C^α. The assumptions required for this result are thus non-satisfied here since
1- we don't know if the condition (199) is satisfied, and in fact this condition is sufficient to carry out the arguments of [**19**], and then to prove $C^{1,\alpha}$ regularity. Here the solution of the Monge-Ampère equation is not even required to be C^1.
2- our right hand side would be the time derivative of ρ and is not supposed to be in any L^p. It will actually be only in $W^{-1,\infty}$ the dual of $W^{1,1}$.

4. Proof of Theorem 4.2

4.1. Preliminary results. First we need the following regularity results concerning smooth solutions of Monge-Ampère equation :

Proposition 4.3. *Let Ω be as above, let $\rho(t,x)$ be a probability measure supported in B_R for every $t \in I$, belong to $C^{1,\alpha}(I \times B_R)$ and satisfy $0 < \lambda \leq \rho(t,x) \leq \Lambda$, then there exists a unique solution to*

$$\det D^2\Psi(t,x) = \rho(t,x) \tag{200}$$

$$\nabla\Psi(t,.) \text{ maps } B_R \text{ into } \Omega \tag{201}$$

moreover , $D^2\Psi \in C^{1,\alpha'}(I \times B_R)$ for any $\alpha' < \alpha$.

Remark : The system (200, 201) is understood in the sense of (192).
Proof : The regularity in the space variable follows from the regularity theory of the Monge-Ampère equation, for which the reader can refer to [**18**]-[**21**], [**29**], [**61**]. Now if we consider $u = \partial_t\Psi$, $[a_{ij}](t,x) = \rho(t,x)\,[D^2\Psi(t,x)]^{-1}$ then $\forall t$, $u(t,\cdot)$ is solution of :

$$\partial_t\rho(t,x) = \sum_{i,j} a_{ij}\partial_{ij}u(t,x) \text{ in } B_R \tag{202}$$

$$\nabla u(x)\cdot \vec{n_1}(\nabla\Psi(x)) = 0 \;\; x \in \partial B_R \tag{203}$$

with $\vec{n}_1$ the outer unit normal to $\partial\Omega$. this problem is uniformly elliptic with coefficients in C^α and right hand side in C^α. Let us say a word about the boundary condition : if $\vec{n}_2$ is the outer unit normal to ∂B_R, we know that uniformly we have then $\vec{n}_2 \cdot \vec{n}_1(\nabla\Psi) \geq C > 0$, (see [**21**] and [**61**], [**29**]) and thus the boundary condition is strictly oblique.
Thus we conclude that $\partial_t\Psi$ belongs to $C^{2,\alpha}(B_R)$.

□

The next proposition can be found in [**18**], [**20**].

Proposition 4.4. *Let ρ be supported in Ω', $0 < \lambda \leq \rho \leq \Lambda$, and Ψ satisfy*

$$\det D^2\Psi = \rho$$

in the sense of (192) with Ω convex. Then for some $\alpha \in]0,1[$, $\Psi \in C^{1,\alpha}_{loc}(\Omega')$ moreover if Ω' is also convex then Ψ (resp. its Legendre transform Φ) is in $C^{1,\alpha}(\bar{\Omega}')$ (resp. in $C^{1,\alpha}(\bar{\Omega})$).

Then we need the following classical lemma :

Lemma 4.5. *Let φ be a convex function from $\mathbb{R}^d$ to $\mathbb{R}$, globally Lipschitz with Lipschitz constant L. Then we have*

$$\|D^2\varphi\|_{\mathcal{M}(B_R)} \leq C(R,d)L.$$

Proof : we have

$$\begin{aligned} \|D^2\varphi\|_{\mathcal{M}(B_R)} &\leq C\int_{B_R}\Delta\varphi \\ &= \int_{\partial B_R}\nabla\varphi\cdot n \\ &\leq \mathcal{H}^{d-1}(\partial B_R)L \end{aligned}$$

□

Finally we need three useful identities :

LEMMA 4.6. *Let $\Phi(t,\cdot),\Psi(t,\cdot)$ be for any t smooth convex functions Legendre transforms of each other then :*

$$\Phi(t,x)+\Psi(t,\nabla\Phi(t,x))=x\cdot\nabla\Phi(t,x) \tag{204}$$

$$\partial_t\Phi+\partial_t\Psi(\nabla\Phi)=0 \tag{205}$$

$$\partial_t\nabla\Phi+D^2\Phi\partial_t\nabla\Psi(\nabla\Phi)=0 \tag{206}$$

Proof : the first identity expresses just the fact that $\Phi(t,\cdot),\Psi(t,\cdot)$ are Legendre transforms of each other (see (190)), then the two other come by differentiating with respect to time and then to space, and noticing that $\nabla\Phi(\nabla\Psi(x))=x$. □

4.1.1. *Idea of the proof.* The key point of the estimates comes from the fact that the equation (195) is satisfied in divergence form, i.e.

$$\partial_t\rho=\partial_i\left(a_{ij}\partial_j\partial_t\Psi\right)$$

with summation over repeated indices. This comes from a property of the comatrix, and is hidden in the identity (193). Then using that $\partial_t\rho=-\nabla\cdot(\rho v)$ and using some integrability on a_{ij} (the comatrix of $D^2\Psi$) we will obtain Theorem 4.2.

4.2. Estimates and proof of Theorem 4.2. Now from Proposition 4.3 we can perform the following computations. We have from (191)

$$\int_{\mathbb{R}^d}\partial_t\Psi\rho=\int_\Omega\partial_t\Psi(\nabla\Phi)$$

Then we use the continuity equation :

$$\partial_t\rho+\nabla\cdot(\rho v)=0$$

which implies for any smooth f

$$\int_{\mathbb{R}^d}f\partial_t\rho=\int_{\mathbb{R}^d}\rho v\cdot\nabla f.$$

We thus obtain

$$\begin{aligned}\int_{\mathbb{R}^d}\partial_t\Psi\partial_t\rho &= \int_{\mathbb{R}^d}\partial_t\nabla\Psi\cdot\rho v\\ &= \int_\Omega\partial_t\nabla\Psi(\nabla\Phi)\cdot\partial_t\nabla\Phi\\ &= -\int_\Omega\partial_t\nabla^t\Psi(\nabla\Phi)\cdot D^2\Phi\cdot\partial_t\nabla\Psi(\nabla\Phi)\end{aligned}$$

where we have used (206). Since we can write $\sqrt{D^2\Phi}$ because this is a positive symmetric matrix, we have

$$\begin{aligned}\|\sqrt{D^2\Phi}\;\partial_t\nabla\Psi(\nabla\Phi)\|^2_{L^2(\Omega)} &= -\int_{\mathbb{R}^d}\rho\partial_t\nabla\Psi\cdot v\\ &= -\int_\Omega\partial_t\nabla\Psi(\nabla\Phi)\cdot v(\nabla\Phi)\\ &= -\int_\Omega\partial_t\nabla^t\Psi(\nabla\Phi)\cdot\sqrt{D^2\Phi}\sqrt{D^2\Phi}^{-1}\cdot v(\nabla\Phi)\end{aligned}$$

using this with (206) we get

$$\begin{aligned}\|\sqrt{D^2\Phi}^{-1}\partial_t\nabla\Phi\|_{L^2(\Omega)} &= \|\sqrt{D^2\Phi}\partial_t\nabla\Psi(\nabla\Phi)\|_{L^2(\Omega)} \\ &\le \|\sqrt{D^2\Phi}^{-1}v(\nabla\Phi)\|_{L^2(\Omega)}.\end{aligned}$$

The right hand side of this inequality satisfies

$$\begin{aligned}\left(\int_\Omega v^t(\nabla\Phi)\cdot(D^2\Phi)^{-1}\cdot v(\nabla\Phi)\right)^{1/2} &= \left(\int_{\mathbb{R}^d}\rho v^t\cdot(D^2\Phi(\nabla\Psi))^{-1}\cdot v\right)^{1/2} \\ &= \left(\int_{\mathbb{R}^d}\rho v^t\cdot D^2\Psi\cdot v\right)^{1/2} \\ &\le \left(\|D^2\Psi\|_{L^r(B_R)}\|\rho v^2\|_{L^{r'}(B_R)}\right)^{1/2}.\end{aligned}$$

In the second line we have used $D^2\Phi(\nabla\Psi) = (D^2\Psi)^{-1}$. Writing

$$\partial_t\nabla\Phi = \sqrt{D^2\Phi}^{-1}\sqrt{D^2\Phi}\partial_t\nabla\Phi,$$

we obtain

$$\begin{aligned}\|\partial_t\nabla\Phi\|_{L^q(\Omega)} &\le \|\sqrt{D^2\Phi}^{-1}\partial_t\nabla\Phi\|_{L^2(\Omega)}\|\sqrt{D^2\Phi}\|_{L^s(\Omega)} \\ &\le \left(\|\rho v^2\|_{L^r(B_R)}\|D^2\Psi\|_{L^{r'}(B_R)}\|D^2\Phi\|_{L^{s/2}(\Omega)}\right)^{1/2}\end{aligned}$$

with $q = \frac{2s}{2+s}$. By taking $p := s/2$ we have

$$\|\partial_t\nabla\Phi\|_{L^q(\Omega)} \le \left(\|\rho v^2\|_{L^r(B_R)}\|D^2\Psi\|_{L^{r'}(B_R)}\|D^2\Phi\|_{L^p(\Omega)}\right)^{1/2}$$

and $q = \frac{2p}{1+p}$. The first part of Theorem 4.2 is obtained.

To obtain a bound on $\partial_t\Psi$ we proceed as follows : from what has been done above, we have

$$\begin{aligned}\int_{\mathbb{R}^d}\rho\left|\sqrt{D^2\Phi(\nabla\Psi)}\partial_t\nabla\Psi\right|^2 &= \int_{\mathbb{R}^d}\rho\partial_t\nabla^t\Psi\cdot D^2\Phi(\nabla\Psi)\cdot\partial_t\nabla\Psi \\ &= \int_\Omega\partial_t\nabla^t\Psi(\nabla\Phi)\cdot D^2\Phi\cdot\partial_t\nabla\Psi(\nabla\Phi) \\ &\le \left[\|D^2\Psi\|_{L^r(B_R)}\|\rho v^2\|_{L^{r'}(B_R)}\right]^{1/2}\end{aligned}$$

then using Hölder's inequality, with $q = \frac{2p}{2+p}$

$$\left[\int_{\mathbb{R}^d}\rho|\partial_t\nabla\Psi|^q\right]^{1/q} \le$$

$$\left[\int_{\mathbb{R}^d}\rho\left|\sqrt{D^2\Phi(\nabla\Psi)}\partial_t\nabla\Psi\right|^2\right]^{1/2}\left[\int_{\mathbb{R}^d}\rho\left|[D^2\Phi(\nabla\Psi)]^{-1}\right|^{p/2}\right]^{1/p}$$

the first factor of the right hand product has been treated above, and the second is equal to $[\int \rho |D^2\Psi|^{p/2}]^{1/p}$ and we conclude that

$$\left[\int_{\mathbb{R}^d} \rho |\partial_t \nabla \Psi|^q\right]^{1/q} \leq \left[\|D^2\Psi\|_{L^r(B_R)} \|\rho v^2\|_{L^{r'}(B_R)}\right]^{1/2} \left[\int_{\mathbb{R}^d} \rho |D^2\Psi|^{s/2}\right]^{1/s}.$$

Taking again $p := s/2$, Theorem 4.2 is proved.

5. Proof of Theorem 4.1

5.0.1. *Construction of smooth solutions.* The density ρ and $\partial_t\rho$ are constructed from $X, \partial_t X$ respectively by the following procedure :

$$\forall f \in C^1(\mathbb{R}^d), \qquad \int_{\mathbb{R}^d} \rho(t,x) f(x) dx = \int_\Omega f(X(t,a))\, da$$
$$\int_{\mathbb{R}^d} \partial_t\rho(t,x) f(x) dx = \int_\Omega \nabla f(X(t,a)) \cdot \partial_t X(t,a)\, da.$$

To define v such that $\partial_t\rho + \nabla \cdot (\rho v) = 0$, we define the product ρv as follows

$$\forall \phi \in C^0(I \times \mathbb{R}^d, R^d), \int_{I\times\mathbb{R}^d} \rho v \cdot \phi \, dtdx = \int_{I\times\Omega} \phi(X(t,a)) \cdot \partial_t X(t,a)\, dtda.$$

One sees that for any $1 \leq p \leq \infty$,

$$\|v\|_{L^p(\mathbb{R}^d, d\rho(t))} = \|\partial_t X(t,.)\|_{L^p(\Omega)}.$$

Now we construct ρ_n, v_n a smooth approximating sequence for ρ, v as follows : (remember that we have taken $\rho(t,\cdot)$ to be supported in B_R at any time $t \in I$). We take η a standard mollifier, of integral 1, supported in $B(0,1)$ and positive. $\eta_n = n^d \eta(nx)$. We also note $\chi_{R+1/n}$ the characteristic function of the ball $B(0, R+1/n)$. Let

$$\rho_n = (\frac{1}{n}\chi_{R+1/n} + \eta_n * \rho) c_n$$
$$v_n = c_n \frac{\eta_n * (\rho v)}{\rho_n}$$

with c_n chosen such that ρ_n remains a probability measure. (Note that c_n is close to 1 for n large). The purpose of this construction is to have the following properties :

(1) $\|\rho_n, v_n\|_{L^\infty} \leq \|\rho, v\|_{L^\infty}$,

(2) ρ_n, v_n are smooth and still satisfy the continuity equation (196),

(3) ρ_n is supported and strictly positive in $B(0, R+1/n)$.

We can therefore apply Theorem 4.2 to ρ_n, v_n.

5.1. Proof of the bounds. Proof of the bound on $\partial_t \nabla \Phi$

First we take the estimate (197) in the case where $r' = 1, p = 2$ and combine it with Lemma 4.5. To see why we indeed have $\liminf \|\rho_n |v_n|^2\|_{L^\infty} \leq \|\rho |v|^2\|_{L^\infty}$ notice that $F(\rho, v) = \rho |v|^2/2 = \frac{(\rho|v|)^2}{2\rho}$ is a convex functional in $\rho v, \rho$ since it is expressed as :

$$\frac{(\rho|v|)^2}{2\rho} = \sup_{c+|m|^2/2\leq 0} \{\rho c + \rho v \cdot m\}.$$

Then since $\rho_n v_n = c_n \eta_n * (\rho v))$, $\rho_n = c_n(\frac{1}{n} + \eta_n * \rho)$ we get that

$$F(\rho_n, \rho_n v_n) \leq c_n \eta_n * F(\rho, \rho v) \leq \|\rho \frac{|v|^2}{2}\|_{L^\infty}$$

and letting $n \to \infty$:

$$\begin{aligned} \|\partial_t \nabla \Phi\|_{\mathcal{M}(\Omega)} &\leq \left(\|\rho |v|^2\|_{L^\infty}\right)^{\frac{1}{2}} C(R, d) \\ &\leq \|\rho\|^{\frac{1}{2}}_{L^\infty(B_R)} \|v\|_{L^\infty(B_r)} C(R, d). \end{aligned}$$

By Gagliardo-Nirenberg inequality we get also a bound on $\|\partial_t \Phi_n\|_{L^{1*}(\Omega)}$. Then we obtain that $\Phi \in C^\alpha(I, C^0(\Omega))$ from the following lemma :

Lemma 4.7. *Let ϕ_1 and ϕ_2 be two $R-$Lipschitz convex functions on Ω convex. Then there exists C, β depending on Ω, R, d such that*

$$\|\phi_1 - \phi_2\|_{L^\infty(\Omega)} \leq C \|\phi_1 - \phi_2\|^\beta_{L^p(\Omega)}.$$

Moreover if $\phi_1 \in C^{1,\alpha}$ for some $0 < \alpha < 1$ then there exists $0 < \beta' < 1$ depending also on α such that if $\Omega_\delta = \{x \in \Omega, d(x, \partial\Omega) \geq \delta\}$ with δ going to 0 with $\|\phi_1 - \phi_2\|_{L^p(\Omega)}$,

$$\|\nabla \phi_1 - \nabla \phi_2\|_{L^\infty(\Omega_\delta)} \leq C \|\phi_1 - \phi_2\|^{\beta'}_{L^p(\Omega)}.$$

Proof : Suppose that $\int_\Omega |\phi_1 - \phi_2|^p \leq \epsilon^p$ then pick a point inside Ω (say 0) such that $|\phi_1(0) - \phi_2(0)| = M$. ϕ_1 and ϕ_2 are globally Lipschitz with Lipschitz constant bounded by R. On $B_{M/2R}(x) \cap \Omega$ we have $|\phi_1 - \phi_2|(x) \geq M/2$ and thus

$$\int_{B_r} |\phi_1 - \phi_2|^p \geq \text{vol}(\Omega \cap B_{M/2R}(x))(M/2)^p.$$

Next note that for Ω convex, M small enough, for any $x \in \Omega$, $\text{vol}(\Omega \cap B_{M/2R}(x)) \geq C_\Omega \text{vol}(B_{M/2R}(x))$. Finally we have

$$\epsilon^p \geq \int_\Omega |\phi_1 - \phi_2|^p \geq C(\Omega, R, d) M^{p+d}, \tag{207}$$

and thus

$$M \leq C'(\Omega, R, d) \left[\int_{B_r} |\phi_1 - \phi_2|^p\right]^{\frac{1}{p}\frac{p}{p+d}},$$

which gives the first part of the lemma.

Now suppose that $|\nabla \phi_1(0) - \nabla \phi_2(0)| = M$. One can also set $\phi_1(0) = 0, \nabla \phi_1(0) = 0$. We know that ϕ_1 is $C^{1,\alpha}$ thus $\phi_1(x) \leq C|x|^{1+\alpha}$. It follows that going in the direction of $\nabla \phi_2$ one will have

$$\phi_2(x) - \phi_1(x) \geq M|x| - C|x|^{1+\alpha} + \phi_2(0).$$

Keeping in mind that $|\phi_1(x) - \phi_2(x)| \leq C\epsilon^\beta$ yields $M| - C|x|^{1+\alpha} \leq C\epsilon^\beta$. The maximum of the right hand side is attained for $|x| = \left(\frac{M}{(1+\alpha)C}\right)^{1/\alpha}$, and is equal to $\left(\frac{M}{(1+\alpha)C}\right)^{1/\alpha} \frac{\alpha}{1+\alpha} M$. Therefore we have

$$M \leq C\epsilon^{\beta'}$$

in Ω_δ with $\delta = \delta(\epsilon)$ going to 0 as ϵ goes to 0 and with $\beta' = \frac{\alpha\beta}{1+\alpha}$. □

Remark : Suppose, as it is the case for Ψ, that we only know that $\int \rho|\Psi_1 - \Psi_2|^p \leq \epsilon^p$ then we have instead of (207),

$$\epsilon^p \geq \int_{B_r} \rho|\Psi_1 - \Psi_2|^p \geq \rho(B_{M/2R}(x))M^{p+d}.$$

Thus if $\rho \geq C > 0$ on some open set Ω' the same conclusion holds. Now the results combined with Proposition 4.4 give the point 4 and 4' of Theorem 4.1.

Proof of the bound on $\partial_t g$

Recall from Theorem 4.2 :

$$\int_{\mathbb{R}^d} \rho_n|\partial_t \nabla \Psi_n| \leq C(d,R)\|\rho\|^{\frac{1}{2}}_{L^\infty(B_R)}\|\rho v^2\|^{\frac{1}{2}}_{L^\infty(B_R)}$$

We have $g(t,a) = \nabla\Psi(t, X(t,a))$ and thus formally

$$\partial_t g(t,a) = \partial_t \nabla\Psi(t, X(t,a)) + D^2\Psi(t, X(t,a))\partial_t X(t,a).$$

Since ρ_n converges strongly (actually weakly would be enough) to ρ, we know that $\nabla\Psi_n$ converges almost everywhere to $\nabla\Psi$. (See [**10**] for a proof of this fact, which relies on the convexity of Ψ_n and on the uniqueness of the polar factorization). Now consider

$$g_n(t,a) = \int_{\mathbb{R}^d} \nabla\Psi_n(t,y)\eta_n(y - X(t,a))dy = (\eta_n * \nabla\Psi_n)(t, X(t,a))$$

then g_n converges almost everywhere to g. For $f \in C^0(I \times \Omega, R^d)$, let us compute

$$\int_I \int_\Omega \partial_t g_n(t,a) \cdot f(t,a)\, dtda = T_1 + T_2,$$

with

$$\begin{aligned} T_1 &= \int_I \int_\Omega \int_{\mathbb{R}^d} \eta_n(y - X(t,a))\partial_t \nabla\Psi_n(t,y) \cdot f(t,a)\, dydadt \\ T_2 &= -\int_I \int_\Omega \int_{\mathbb{R}^d} \nabla\Psi_n(t,y) \cdot f(t,a)\ \partial_t X(T,a) \cdot \nabla\eta_n(y - X(t,a))\, dydadt \end{aligned}$$

Let us evaluate T_1 and T_2.

$$\begin{aligned} |T_1| &\leq \int_I \|f(t,.)\|_{L^\infty(\Omega)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \rho(x)\eta_n(y-x)|\partial_t \nabla\Psi_n(t,y)|\, dxdydt \\ &\leq \int_I \|f(t,.)\|_{L^\infty(\Omega)} d_n \int_{\mathbb{R}^d \times \mathbb{R}^d} \rho_n(y)|\partial_t \nabla\Psi_n(t,y)|\, dxdy \\ &\leq \int_I \|f(t,.)\|_{L^\infty(\Omega)} C(R,d)\|\rho_n\|_{L^\infty(I\times\mathbb{R}^d)}\|v_n\|_{L^\infty(I\times\mathbb{R}^d)} \end{aligned}$$

with $d_n = 1/c_n$ and from Theorem 4.2. For T_2 we have :

$$\begin{aligned} |T_2| &= \left|\int_I\int_\Omega\int_{\mathbb{R}^d} \nabla\Psi_n(t,y)\cdot f(t,a)\ \partial_t X(T,a)\cdot\nabla\eta_n(y-X(t,a))\,dydadt\right| \\ &= \left|\int_I\int_\Omega\int_{\mathbb{R}^d} \partial_t X^t(T,a)\cdot(D^2\Psi_n*\eta_n)(t,X(t,a))\cdot f(t,a)\,dydadt\right| \\ &\leq \int_I \||f(t,.)||\partial_t X(t,.)|\|_{L^\infty(\Omega)}\int_{\mathbb{R}^d}\rho(t,x)(\Delta\Psi_n*\eta_n)(x)dx\ dt \\ &\leq \int_I \||f(t,.)||\partial_t X(t,.)|\|_{L^\infty(\Omega)}\|\rho(t,.)\|_{L^\infty(\mathbb{R}^d)}C(R,d)\ dt \end{aligned}$$

thus we conclude that

$$\|\partial_t g\|_{L^\infty(I,\mathcal{M}(\Omega))} \leq C(R,d)\|\rho\|_{L^\infty(I\times B_R)}\|\partial_t X\|_{L^\infty(I\times\Omega)}$$

6. Counter-examples

Here we show through some examples that the bounds obtained in Theorem 4.1 are sharp under our present assumptions.

Example 1 : $\partial_t\nabla\Phi \notin L^1_{loc}$ and $\partial_t\Phi \notin C^0$.
Consider in $\Omega = B(0,1)$ in $\mathbb{R}^2$, and $X(t,\cdot) : B(0,1) \to \mathbb{R}^2$ defined with complex notations $X = x+iy$ by

on $y > 0$,

$$X(t,(x,y)) = e^{it}(x+iy) + it,$$

on $y < 0$,

$$X(t,(x,y)) = e^{it}(x+iy) + t^2.$$

We check that $X_\# dx$ has a density bounded by 1, that $\partial_t X \in L^\infty(\Omega\times\mathbb{R}^+)$. If $X = \nabla\Phi\circ g$ is the polar factorization of X then up to a constant, Φ is defined for $t > 0, (x,y) \in \Omega$ by :

$$\Phi(t,(x,y)) = \sup\{\frac{1}{2}(x^2+y^2) + t^2x, \frac{1}{2}(x^2+y^2) + ty\}$$

On $\{y > tx\}$ we have

$$\begin{aligned} \Phi(t,(x,y)) &= \frac{1}{2}(x^2+y^2) + ty \\ \nabla\Phi(t,(x,y)) &= (x,y) + (0,t) \end{aligned}$$

and on $\{y < tx\}$

$$\begin{aligned} \Phi(t,(x,y)) &= \frac{1}{2}(x^2+y^2) + t^2x \\ \nabla\Phi(t,(x,y)) &= (x,y) + (t^2,0). \end{aligned}$$

Thus

$$\begin{aligned} &\partial_t\Phi(t,(x,y)) = y\chi_{\{y>tx\}} + 2tx\chi_{\{y<tx\}} \notin C^0, \\ &\partial_t\nabla\Phi(t,(x,y)) = (0,1)\chi_{\{y>tx\}} + (2t,0)\chi_{\{y,tx\}} + (t^2,-t)\mathcal{H}^{d-1}\{y=tx\} \notin L^1 \end{aligned}$$

Example 2 : Here we adapt a counterexample of Wang to build an example of a solution where $\partial_t \Psi \notin C^0$.
In $\mathbb{R}^d$, let $x = (x_i)_{1 \leq i \leq d}$ and

$$X(0, x) = \nabla \Phi_0(x)$$

$\Phi_0(x)$ convex Lipschitz on Ω, $\Phi = +\infty$ outside, such that $\rho = \nabla \Phi_0(x)_{\#} dx$ has a density in $L^\infty(\mathbb{R}^2)$. Let

$$X(t, x) = \nabla \Phi_0(x) + tv$$

for some fixed $v \in \mathbb{R}^d$. X is Lipschitz with respect to time. Then

$$\Phi(t, x) = \Phi(x) + tx \cdot v$$
$$\nabla \Phi(t, x) = \nabla \Phi_0(x) + tv$$

Then if Ψ_0 is the Legendre transform of Φ_0, the Legendre transform of $\Phi(t, \cdot)$ is given by

$$\Psi(t, x) = \Psi_0(x - tv)$$
$$\nabla \Psi(t, x) = \nabla \Psi_0(x - tv)$$

thus

$$\partial_t \Psi(t, x) = v \cdot \nabla \Psi_0(x - tv)$$
$$\partial_t \nabla \Psi(t, x) = D^2 \Psi_0(x - tv) \cdot v$$

Wang has shown in [**63**] some counterexamples to the regularity of solutions of Monge-Ampère equations : namely, for $d \geq 3$ he has exhibited a solution u of

$$\det D^2 u = f$$

with f only bounded by above, such that $u \notin C^1$. By taking $\Phi_0 = u^*$ one has an example of time dependent map such that

$$\partial_t \Psi(t, x) = v \cdot \nabla \Psi_0(x - tv) \notin C^0$$

Note that this shows that the C^α result for $\partial_t \Phi$ obtained by [**22**] does not extend to our less restrictive assumptions.

7. Application : the semi-geostrophic equations.

The semi-geostrophic equations are used in meteorology to model frontogenesis, see [**27**]. Their formulation is the following : We look for a time dependent probability measure on $\mathbb{R}^2$ satisfying the following evolution equation :

$$\partial_t \rho + \nabla \cdot (\rho v) = 0 \tag{208}$$

$$v(t, x) = (\nabla \Psi(t, x) - x)^\perp \tag{209}$$

$$\det D^2 \Psi(t, x) = \rho(t, x) \tag{210}$$

where equation (210) is understood in the sense of (192), where an open set Ω of total mass 1 has been fixed before. The set Ω is here called the physical space. Existence of global weak solutions with initial data in L^1 has been proved in [**6**], [**25**] and [**47**]. See also the chapter 5 for the existence of measure valued solutions and smooth solutions for finite time. Nevertheless uniqueness is still unproven. If one considers for simplicity the periodic case, one can construct using the procedure of [**6**] weak solutions that satisfy the condition $0 < \lambda_1 \leq \rho \leq \lambda_2$ for all time. The problem being two-dimensional, it follows that $\Psi \in W^{2,p}$ for some $p > 1$ depending on λ_1/λ_2, (see [**53**]). The velocity field is divergence free, and then Di-Perna Lions ([**30**]) theory applies : the transport equation admits renormalized solutions and the trajectories are well defined. Moreover the flow being incompressible, all the L^p norms of ρ are conserved.

7.1. The associated ODE. The characteristics of the equation (208) are given by the following ODE :

$$\partial_t X(t,a) = (\nabla\Psi(t, X(t,a)) - X(t,a))^{\perp} \tag{211}$$

and if at $t = 0$, $X(t,a)$ satisfies $X_{|t=0}\#da = \rho_{|t=0}$ one has $d\rho(t,\cdot) = X(t,\cdot)\#da$. Moreover if Ω is bounded and ρ is compactly supported, the velocity is bounded. Thus the hypothesis of Theorem 4.1 are satisfied. If $\rho \geq \alpha > 0$ (this can be easily enforced in the periodic case) we can have a uniform $C^{1,\alpha}$ bound in space for Ψ. The C^{β} bound with respect to time for $\nabla\Psi$ thus follows. Note also that the velocity along characteristics defined by (211) is thus C^{β} with respect to time.

Finally let us notice that if $g = \nabla\Psi(X)$ then g defines the trajectories in the physical space and the Theorem 4.1 tells us that their velocity remains bounded as a measure.

CHAPITRE 5

Mesure valued and classical solutions to the semi-geostrophic equations

Measure-valued and classical solutions for the semi-geostrophic equations.

Grégoire Loeper[1]

RÉSUMÉ. We show existence of solutions to the semi-geostrophic equations in the case of measure valued initial density, and existence of a continuous solution when the initial density is continuous. We also show uniqueness of solutions with Lipschitz continuous density.

1. Introduction

The semi-geostrophic equations (212, 213, 214) are used in meteorology to model frontogenesis. Their Eulerian formulation is

$$\begin{aligned}&\partial_t\rho + \nabla\cdot(\rho v) = 0\\ &v = \nabla\phi(x)^\perp\\ &\det(I + D^2\phi) = \rho\\ &\rho(t=0) = \rho_0\end{aligned}$$

where ρ_0 is a probability measure on $\mathbb{R}^2$ and the velocity field is given at each time by solving a Monge-Ampère equation in a sense that will be precised later on.

This formulation can be viewed as a non-linear version of the 2-d incompressible Euler equation : indeed if one replaces the equation

$$\det(I + D^2\phi) = \rho$$

by the equation

$$\Delta\phi = \rho - 1$$

one obtains the vorticity formulation of the 2d incompressible Euler equation.

This model arises in a hierarchy of approximations of the equation of motion in meteorology. We shall give a brief idea of the derivation of the model, and the reader can refer to [**27**] for a more accurate derivation.

1.1. The semi-geostrophic equations.

1.1.1. *Their lagrangian formulation.* We start from the 2d incompressible Euler equations with the Coriolis force in a domain Ω :

$$\begin{aligned}&\frac{Dv}{Dt} + fv^\perp = \nabla p\\ &\nabla\cdot v = 0\\ &v\cdot\partial\Omega = 0\end{aligned}$$

where $\frac{D\cdot}{Dt}$ stands for $\partial_t + v\cdot\nabla$ and the suffix $\perp$ means "rotated of $\pi/2$". $fv^\perp$ is the Coriolis force due to rotation of the Earth, it turns out that for large scale flows, this term dominates the term $\frac{Dv}{Dt}$. Thus the geostrophic approximation consists in stating that the geostrophic balance holds :

$$v = v_g = -f^{-1}\nabla^\perp p.$$

[1]Laboratoire J.A.Dieudonné, Université de Nice-Sophia-Antipolis, Parc Valrose, 06108 NICE Cedex 2.

The semi-geostrophic equation consists in saying that if one decomposes $v = v_g + v_{ag}$ where the second component means small departures from the geostrophic balance, then one has

$$\begin{aligned}&\frac{Dv_g}{Dt} + fv^{\perp} = \nabla p\\&\nabla \cdot v = 0\end{aligned}$$

Setting the new variables

$$\begin{aligned}&X = x + f^{-1}\nabla p = \nabla P(x)\\&P = \frac{1}{2}|x|^2 + f^{-1}p\end{aligned}$$

one obtains for $X(t)$ the following

$$\frac{DX(t)}{Dt} = (x(t) - X(t))^{\perp}$$

Here $X(t)$ and $x(t)$ have to be seen as functions of the time and the initial position of x, say x_0. $x \to x(t, x_0)$ will be the flow corresponding to the velocity field $v(t, x)$.

As stated the system looks under determined : indeed P is unknown ; however we have the condition $X(t) = \nabla P(x(t))$. Now we must remember that the dynamic in the x space being incompressible and contained in Ω, the map $x(t, \cdot)$ must be measure preserving in Ω for each t, i.e.

$$\mathcal{L}(x(t)^{-1}(B)) = \mathcal{L}(B)$$

for each $B \subset \Omega$ measurable with $\mathcal{L}$ the Lebesgue measure. Then Hoskins stability criteria asserts on physical basis that P should be a convex function for the system to be stable to small displacements of particles in the x space. Thus for each t, P must be a convex function such that

$$X(t, \cdot) = \nabla P(t, x(t, \cdot))$$

with $x(t, \cdot)$ a measure preserving mapping on Ω. We shall see under very mild assumptions on X that this can only happen for a unique choice of x and ∇P. Now if P^* is the Legendre transform of P, then ∇P and ∇P^* are inverse maps of each other, and the semi-geostrophic system then reads

$$\frac{DX}{Dt} = (\nabla P^*(X(t)) - X(t))^{\perp}.$$

The existence and uniqueness of the gradients $\nabla P, \nabla P^*$ is given by the polar factorization theorem.

1.2. The polar factorization of vector valued maps. The polar factorization of maps has been discovered by Brenier in [**10**]. It has later been extended to the case of general Riemannian manifolds by [**49**]. Let us state the results :

1.2.1. *The Euclidean case.* Let Ω be a fixed bounded domain of $\mathbb{R}^d$ (for our purpose we will only consider $d = 2$) of Lebesgue measure 1 and satisfying the condition $\mathcal{L}(\partial\Omega) = 0$. We consider a mapping $X \in L^2(\Omega, \mathbb{R}^d)$. We will also consider the push-forward of the Lebesgue measure of Ω (referred to as dx) by X, that we will denote by $X_{\#}dx = d\rho$ and which is defined by

$$\forall f \in C^0(\mathbb{R}^d), \int_{\mathbb{R}^d} f(x)\, d\rho(x) = \int_{\Omega} f(X(x))\, dx.$$

Let $\mathcal{P}_a^2$ be the set of probability measures ρ on $\mathbb{R}^d$ that are absolutely continuous with respect to the Lebesgue measure (or equivalently that have a density in $L^1(\mathbb{R}^d)$), and with finite second moment. (i.e. such that

$$\int_{\mathbb{R}^d} |x|^2 d\rho(x) < +\infty.)$$

Then the condition $X \in L^2(\Omega, \mathbb{R}^d)$ is equivalent to the condition $\rho = X_\# dx$ has a finite second moment. For any $\rho \in \mathcal{P}_a^2$ we have the following definition :

Definition 5.1. *Given ρ such that $\rho \in \mathcal{P}_a^2$, there exists a unique up to a constant convex function that we denote $\Phi[\rho]$ satisfying :*

$$\forall f \in C^0(\mathbb{R}^d), \ \int_\Omega f(\nabla\Phi[\rho](x))\ dx = \int_{\mathbb{R}^d} f(x) d\rho(x).$$

$\Psi[\rho]$, the Legendre transform of $\Phi[\rho]$, is the unique up to a constant convex function satisfying

$$\forall f \in C^0(\mathbb{R}^d), \ \int_{\mathbb{R}^d} f(\nabla\Psi[\rho](x))\ d\rho(x) = \int_\Omega f(x) dx.$$

Finally if $X : \Omega \to \mathbb{R}^d$ is such that $\rho = X_\# dx \in \mathcal{P}_a^2$ then X admits the following unique polar factorization :

$$X = \nabla\Phi[\rho] \circ g$$

with g measure preserving in Ω.

Then $\Psi[\rho], \Phi[\rho]$ are weak solutions (in some special sense) respectively on $\mathbb{R}^d$ and Ω of

$$\det D^2\Psi = \rho$$
$$\rho(\nabla\Phi)\ \det D^2\Phi = 1.$$

When Ψ and Φ are not in C^2_{loc} these equations can be understood in the viscosity (or Alexandrov) sense or in the the sense of Definition 5.1. For the regularity of those solutions and the consistency of the different weak formulations the reader can refer to [**21**].

1.2.2. *The periodic case.* In the case of a probability measure defined on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and still in $L^1(\mathbb{T}^d)$ we define $\Psi[\rho], \Phi[\rho]$ by the following :

Definition 5.2. $\Phi[\rho]$ *is the unique up to a constant convex function over $\mathbb{R}^d$ satisfying : $\Phi[\rho](x) - x^2/2$ is $\mathbb{Z}^d$ periodic (and thus $\nabla\Phi[\rho](x) - x$ is $\mathbb{Z}^d$ periodic), and*

$$\forall f \in C^0(\mathbb{T}^d), \ \int_{\mathbb{T}^d} f(\nabla\Phi[\rho](x))\ dx = \int_{\mathbb{T}^d} f(x)\ d\rho(x).$$

$\Psi[\rho]$ is the Legendre transform of $\Phi[\rho]$, the unique up to a constant convex function satisfying $\Psi[\rho](x) - x^2/2$ is $\mathbb{Z}^d$ periodic (and thus $\nabla\Psi[\rho](x) - x$ is $\mathbb{Z}^d$ periodic), and

$$\forall f \in C^0(\mathbb{T}^d), \ \int_{\mathbb{T}^d} f(\nabla\Psi[\rho](x))\ d\rho(x) = \int_{\mathbb{T}^d} f(x)\ dx.$$

If $X : \mathbb{T}^d \to \mathbb{T}^d$ is such that $\rho = X_\# dx \in L^1(\mathbb{T}^d)$ then X admits the following unique polar factorization :

$$X = \nabla\Phi[\rho] \circ g$$

with g measure preserving from $\mathbb{T}^d$ into itself, and $\Phi[\rho]$ convex, $\Phi[\rho] - |x|^2/2$ periodic.

Remark 1 : from the periodicity of $\nabla\Phi[\rho](x)-x$, $\nabla\Psi[\rho](x)-x$, for every f $\mathbb{Z}^d$ periodic, $f(\nabla\Psi[\rho])$, $f(\nabla\Phi[\rho])$ are well defined on $\mathbb{R}^d/\mathbb{Z}^d$.
Remark 2 : the definitions of $\Psi[\rho]$ and $\Phi[\rho]$ make sense if ρ is absolutely continuous with respect to the Lebesgue measure. If not, the definition and uniqueness of $\Phi[\rho]$ is still valid, and the definition of $\Psi[\rho]$ as the Legendre transform of $\Phi[\rho]$ is still valid also, but then the product $\int f(\nabla\Psi[\rho](x))\, d\rho(x)$ does not necessarily make sense since $\nabla\Psi$ is not necessarily continuous. Moreover the polar factorization does not hold any more.
Remark 3 : The Lagrangian equation becomes

$$\frac{DX}{Dt} = [\nabla\Psi(t,X) - X]^\perp$$
$$\Psi = \Psi[\rho(t)]$$
$$\rho(t) = X_\# dx$$

where $\Psi[\rho]$ is meant as in the definition above.

1.3. Eulerian formulation in dual variables. The reason of these manipulations is that things should be easier in the dual space (the space where X lives), where we look for the equation followed by the measure $\rho = X_\# dx$. In both cases (periodic and non periodic) we thus investigate the following system that will be referred to as (S-G) : we look for a time dependent probability measure $t \to \rho(t,\cdot)$ satisfying

$$\partial_t\rho + \nabla\cdot(\rho v) = 0 \tag{212}$$
$$v(t,x) = (\nabla\Psi[\rho(t)](x) - x)^\perp \tag{213}$$
$$\rho(t=0) = \rho_0 \tag{214}$$

with ρ_0 a probability measure. Weak solutions (which are defined below) of this system with L^p initial data for $p \geq 1$ have been found, see [**6**], [**26**], [**47**].

2. Results

We will show (Theorem 5.3) using an appropriate weak formulation that measure-valued solutions exist, are weakly stable and consistent with L^1 solutions.
In a second part (Theorems 5.4, 5.5, 5.6)of the paper we will also show that classical solutions exist locally in time at least for the spatially periodic case, and are unique. More precisely we will show existence of solutions that are only continuous, of strong solutions (i.e.with $\rho \in W^{1,p}, p > 2$), and existence and uniqueness of solutions with ρ Lipschitz.

THEOREM 5.3. *Let ρ_0 be a probability measure compactly supported. There exists a weak measure solution to the system (S-G) in the sense of Definition 5.7. Moreover, if ρ_0^n is a sequence of probability measures uniformly compactly supported that converge weakly-$*$ to ρ_0, if ρ^n are weak measure solutions of (S-G) with initial datas ρ_0^n, also uniformly compactly supported, then any converging subsequence ρ_{n_k} converges in $C([0,T],\mathcal{P}-w*)$ to a a weak measure solution of (S-G) with ρ_0 as initial data.*

Remark : the uniqueness of weak solutions is still an open question.
We state the existence of strong solutions on the periodic case :

THEOREM 5.4. *If $\rho_0 \in W^{1,p}(\mathbb{T}^2)$ for $2 < p \leq \infty$ and satisfies $0 < \alpha \leq \rho_0 \leq \beta$. then there exists $T > 0$ such that there is on $[0,T]$ a solution ρ to (S-G) periodic such that $(\partial_t\rho, \nabla_x\rho) \in L^\infty([0,T], L^p(\mathbb{T}^2))$.*

THEOREM 5.5. *Let ρ_0 be strictly positive and satisfy the continuity condition (219). Then there exists $T > 0$ such that on $[0,T]$ $w(t,r)$ the modulus of continuity of $\rho(t,.)$ solution of (S-G) satisfies (219) and the velocity field $v(t,x) = (\nabla\Psi(t,x) - x)^\perp$ remains Lipschitz.*

Condition (219) : w the modulus of continuity of ρ satisfies

$$\int_0^1 \frac{w(r)}{r}\, dr < +\infty.$$

THEOREM 5.6. *Suppose that $\rho^0 \in C^{0,1}(\mathbb{T}^2)$ with $0 < \alpha \leq \rho_0 \leq \beta$. From Theorem 5.4, for some $T > 0$ there exists $\bar\rho$ in $L^\infty([0,T], C^{0,1}(\mathbb{T}^2))$ solution to SG . Then every solution of SG in $L^\infty([0,T'], C^{0,1}(\mathbb{T}^2))$ for $T' > 0$ with same initial data coincides with $\bar\rho$ on $[0, \inf\{T, T'\}]$.*

3. Measure solutions

3.0.1. *Definition of weak solutions.* We have first the following classical weak formulation of equation (212) :

$$\forall\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^n), \int \partial_t\varphi\, \rho + \nabla\varphi \cdot \rho v\; dtdx = \int \varphi(0,x)\rho(0,x)dx$$

where v is given by (213). The problematic part in the case of measure valued solutions is to give sense to the product $\rho\nabla\Psi[\rho]$ since at a point where ρ is singular $\nabla\Psi[\rho]$ is unlikely to be continuous. Therefore we use the definition (5.1) to write for any $\rho \in \mathcal{P}_a(\mathbb{R}^2)$

$$\forall\varphi \in C_c^\infty(\mathbb{R}^2), \int_{\mathbb{R}^2} \rho\nabla\Psi[\rho]^\perp \cdot \nabla\varphi = \int_\Omega x^\perp \cdot \nabla\varphi(\nabla\Phi[\rho])$$

(the integrals would be performed over $\mathbb{T}^2$ in the periodic case). We see then that the right hand side formulation extends unambiguously to the case where $\rho \notin L^1(\mathbb{R}^2)$.

3.0.2. *Geometric interpretation.* This weak formulation allows a nice geometric interpretation : at a point where $\Psi[\rho]$ is not differentiable, and thus where $\partial\Psi[\rho]$ is not reduced to a single point, $\nabla\Psi[\rho]$ should be replaced by $\bar{\partial\Psi}[\rho]$ the center of mass of the (convex) set $\partial\Psi[\rho]$. This new vector field is defined everywhere and one can check that if ρ_n converges weakly to ρ, then for all $F \in C^0(\mathbb{R}^d \times \mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} F(\nabla\Psi[\rho_n], x)\; d\rho_n(x) \to \int_{\mathbb{R}^d} F(\bar{\partial\Psi}[\rho], x)\; d\rho(x).$$

This motivates the following definition of weak solutions

DEFINITION 5.7. *ρ is said to be a weak solution to (S-G) if*

(1) *ρ belongs to $C([0,T], \mathcal{P} - w*)$*
(2) *there exists $T \to R(T)$ non decreasing and finite for every $T \geq 0$ such that for all $t \in [0,T]$, $\rho(t,\cdot)$ is supported in $B(0, R(T))$*

(3) *for all $T > 0$ we have*

$$
\begin{aligned}
& \forall \varphi \in C_c^\infty([0,T]\times\mathbb{R}^2), \\
& \int_{[0,T]\times\mathbb{R}^2} \partial_t \varphi(t,x)\ d\rho(dt,x) \\
+ & \int_{[0,T]\times\Omega} \nabla\varphi(t,\nabla\Phi[\rho(t)](x))\cdot x^\perp\ dtdx - \int_{[0,T]\times\mathbb{R}^2} \nabla\varphi(t,x)\cdot x^\perp\ d\rho(dt,x) \\
= & \int \varphi(T,x) d\rho(T,x)\ dx - \int \varphi(0,x) d\rho(0,x)\ dx
\end{aligned} \tag{215}
$$

This definition is consistent with the classical definition of weak solutions if for all t $\rho(t,\cdot)$ is absolutely continuous w.r.t. the Lebesgue measure.

3.1. Proof of Theorem 5.3.

3.1.1. *Weak stability of solutions.* Let us check *a priori* the weak stability of such solutions. Therefore we consider a sequence ρ_n of solutions of (S-G) in the sense of Definition 5.7. First note that Ω is bounded by hypothesis and that also by the Definition of weak solutions, the sequence we consider is uniformly compactly supported, thus

$$
\begin{aligned}
& \left| -\int_{[0,T]\times\mathbb{R}^2} \nabla\varphi(t,x)\cdot x^\perp\ d\rho(dt,x) + \int_{[0,T]\times\Omega} \nabla\varphi(t,\nabla\Phi[\rho(t)](x))\cdot x^\perp\ dtdx \right| \\
\leq\ & C(T)\|\varphi\|_{L^\infty([0,T],C^1(B_{R(T)})}
\end{aligned}
$$

Thus from Definition 5.7 equation (215) we know that for any time $t \geq 0$, $\partial_t \rho_n(t,\cdot)$ is bounded in the dual of $L^\infty([0,T],C^1(\mathbb{R}^2))$ and thus in the dual of $L^\infty([0,T],W^{2,p}(\mathbb{R}^2))$ for p large enough by Sobolev embedding Theorem. Thus for some $p > 1$ we have

$$
\partial_t \rho_n \in L^\infty([0,T],W^{-2,p}(\mathbb{R}^2))
$$

Moreover by hypothesis for every $0 < t < T$ the sequence of probability measures $\rho_n(t,\cdot)$ is compactly supported in some ball $B(0,R(T))$. Thus it is weakly-$*$ precompact. Using classical arguments, it follows that the sequence is relatively compact in $C([0,T],\mathcal{P}-w*)$ and one can thus extract a converging subsequence that we still denote by ρ_n. Since for every t, $\rho_n(t,\cdot)$ converges to $\rho(t,\cdot)$ weakly-$*$, using Theorem 5.8 cited below , we obtain that $\nabla\Phi[\rho_n]$ converge strongly in $L^1_{loc}(\Omega)$ to $\nabla\Phi[\rho]$. Note also that since ρ_n is uniformly compactly supported, $\nabla\Phi[\rho_n]$ is uniformly bounded in $L^\infty(\Omega)$. Thus one can pass to the limit in the formulation of Definition 5.7 and the limit of the sequence ρ_n is still solution of (S-G) in the sense of Definition 5.7.

Here we cite the following stability Theorem taken from [**10**] that was used in the proof above :

Theorem 5.8. *Let ρ_n be a sequence of bounded positive measures on $\mathbb{R}^d$, of total mass $|\Omega|$ such that $\forall n$, $\int(1+|x|^2)d\rho_n \leq C$, let $\Phi_n = \Phi[\rho_n]$ and $\Psi_n = \Psi[\rho_n]$ be as in definition 5.1. If for any $f \in C^0(\mathbb{R}^d)$ such that $|f(x)| \leq C(1+|x|^2)$, $\int f\rho_n \to \int \rho f$,then $\Phi_n \to \Phi[\rho]$ uniformly on each compact set of Ω and strongly in $W^{1,1}(\Omega;\mathbb{R}^d)$, and $\Psi_n \to \Psi[\rho]$ uniformly on each compact set of $\mathbb{R}^d$ and strongly in $W^{1,1}_{loc}(\mathbb{R}^d)$.*

3.2. Existence of solutions. We show briefly the existence of a solution to the Cauchy problem in the sense of Definition 5.7. Indeed given ρ^0 the initial data for the problem that we want to solve, we can take a sequence ρ_n^0 of initial datas belonging to $L^1(\mathbb{R}^2)$, uniformly compactly supported and converging weakly to ρ^0.

We know already from [**6**], [**26**], [**47**] that for every such initial data we can build a solution of (212, 213, 214), that will be uniformly compactly supported on $[0,T]$ for all $T \geq 0$. This sequence will also be solution in the sense of Definition 5.7. We then use the stability result.
This achieves the proof of Theorem 5.3. $\square$.

4. Classical solutions

We prove here existence of classical solutions. We do this in the periodic case in order to be able to guarantee the condition $\rho \geq \alpha$ for all time. Our proof here is much inspired of the one in [**16**]. It uses the following regularity result for solutions of the Monge-Ampère equation which is an adaptation of [**18**] and uses the result of [**24**] and [**49**] :

THEOREM 5.9. *Let ρ be the density of a probability measure on $\mathbb{T}^d$ such that $\rho \in C^\alpha(\mathbb{T}^d)$ and $0 < m \leq \rho \leq M$ for some numbers m, M. Then $\Psi = \Psi[\rho]$ is a classical solution of*

$$\det D^2\Psi = \rho \tag{216}$$

and satisfies :

$$\|\nabla\Psi(x) - x\|_{L^\infty} \leq C(d) = \sqrt{d}/2 \tag{217}$$

$$\|D^2\Psi\|_{C^\alpha} \leq K(\alpha, m, M, \|\rho\|_{C^\alpha}) \tag{218}$$

Then if $\rho \in C^{k,\alpha}$ for $k \in \mathbb{N}$, $\Psi \in C^{k+2,\alpha}$.

4.1. A priori estimates. Here we suppose that we have a smooth solution, and get a bound on the time evolution of its derivatives. Then the existence of such a classical solution follows by constructing approximate solution as in ([**16**]) or even ([**6**]), and using the uniform bounds that provide convergence of the sequence.

Recall first that the flow being incompressible, the condition $0 < m \leq \rho \leq M$ is preserved for all times. Then by differentiating (212) one gets :

$$\partial_t \nabla\rho + [(\nabla\Psi(x) - x)^\perp \cdot \nabla]\ \nabla\rho - [D^2\Psi - I] \cdot \nabla\rho^\perp = 0$$

Noticing that $(\nabla\Psi(x) - x)^\perp$ is divergence-free, this implies that

$$\frac{d}{dt}\|\nabla\rho(t,\cdot)\|_{L^p(\mathbb{T}^2)} \leq C\|D^2\Psi - I\|_{L^\infty(\mathbb{T}^2)}\|\nabla\rho\|_{L^p(\mathbb{T}^2)}.$$

If $\nabla\rho \in L^p(\mathbb{T}^2)$ for $p > 2$, from the Sobolev embedding Theorem, $\rho \in C^\alpha(\mathbb{T}^2)$ and thus $D^2\Psi \in C^\alpha(\mathbb{T}^2)$. Take G large , and $K = K(\alpha, m, M, G)$. As long as $\|\rho^0\|_{C^\alpha} \leq G$ which is true as long as $\|\nabla\rho\|_{L^p} \leq G'$ for some G', one has

$$\frac{d}{dt}\|\nabla\rho\|_{L^p} \leq K\|\nabla\rho\|_{L^p}.$$

Thus as long as $\|\nabla\rho_0\|_{L^p(\mathbb{T}^2)} \exp(Kt) \leq G'$, the condition $\|\nabla\rho\|_{L^p} \leq G'$ is preserved. This gives the a priori bound sufficient to conclude the existence of local strong solutions. Note that we could have taken $p = +\infty$ and thus ρ Lipschitz.

4.2. Continuous initial data. Wang in [**63**] has shown the following :

THEOREM 5.10. *Let u be a strictly convex solution of*

$$\det D^2 u = \rho$$

If $w(r)$ *the modulus of continuity of* ρ *satisfies*

$$\int_0^1 \frac{w(r)}{r} dr < \infty \tag{219}$$

then u *is in* C^2_{loc}.

From [**21**] we know that a solution of $0 < m \le \det D^2 u \le M$ in all of $\mathbb{R}^d$ has to be strictly convex for if not it would contain a whole line and thus be affine.

Proof of Theorem 5.5 :

The proof is quite simple : if $\Psi \in C^2$, then the flow $t \to X(t,x)$ generated by the velocity field $[\nabla\Psi(x) - x]^\perp$ is Lipschitz in space. Since the flow is incompressible, we have $\rho(t,x) = \rho_0(X^{-1}(t,x))$. Now we use the following property : If two functions f, g have modulus of continuity respectively w_f, w_g then $g \circ f$ has modulus $w_g \circ w_f$.

Thus if $X^{-1}(t)$ is Lipschitz, we have $w_{\rho_0 \circ X^{-1}(t)} \le w_{\rho_0}(L\ \cdot)$ with L the Lipschitz constant of $X^{-1}(t)$ and condition (219) remains satisfied. □

Remark 1 : Note that Hölder continuous functions satisfy the condition (219).

Remark 2 : Note also that we do not need any information on $\nabla\rho$ and the solution has still to be understood in the distribution sense.

5. Uniqueness of strong solutions

Here we prove Theorem 5.6. Take two solutions of SG periodic ρ_1, ρ_2 that coincide at time 0, and such that ρ_1, ρ_2 satisfy the regularity hypothesis of the Theorem5.6 . Then

$$\partial_t(\rho_1 - \rho_2) + (\nabla\Psi_2 - x)^\perp \cdot \nabla(\rho_1 - \rho_2) = \nabla\rho_1 \cdot (\nabla\Psi_2 - \nabla\Psi_1)^\perp$$

From the assumptions of Theorem 5.6, we have $\nabla\rho_1 \in L^\infty([0,T] \times \mathbb{T}^2)$, thus we see that we can conclude our argument if we can show that

$$\|\nabla\Psi_1 - \nabla\Psi_2\|(t)_{L^\infty(\mathbb{T}^2)} \le C\|\rho_1 - \rho_2\|(t)_{L^\infty(\mathbb{T}^2)}.$$

We will show this using the a-priori $C^{0,1}(\mathbb{T}^2))$ bound on ρ_1, ρ_2.

For this we consider for $t \in [0,T[$, $\theta \in [0,1]$ Ψ_θ solution of

$$\det D^2\Psi_\theta = \theta\rho_2(t) + (1-\theta)\rho_1(t)$$

we have supposed that ρ_1 and ρ_2 are $C^{0,1}$. Then $D^2\Psi_\theta \in C^\alpha$ for some $\alpha > 0$ thanks to Theorem 5.9. Moreover using that $0 < m \le \rho_1, \rho_2 \le M$ for all times we get that $m^* I \le D^2\Psi_\theta \le M^* I$ for some strictly positive constants m^*, M^*.

Differentiating with respect to θ we get the following equation :

$$\sum_{i,j} a_{ij}\partial_{x_i,x_j}\partial_\theta\Psi = \rho_2 - \rho_1$$

$$\partial_\theta\Psi \text{ is periodic over } \mathbb{T}^2.$$

where the matrix a_{ij} is the comatrix of $D^2\Psi_\theta$ given by

$$a_{ij} = (\theta\rho_2 + (1-\theta)\rho_1)\left[D^2\Psi_\theta\right]^{-1}$$

Since we a priori know that $D^2\Psi_\theta \in C^\alpha$ uniformly in θ, and that $D^2\Psi_\theta$ is uniformly elliptic, its comatrix has the same properties. Using standard elliptic regularity and the periodic boundary condition, we obtain that

$$\|\partial_\theta \Psi\|_{W^{2,p}(\mathbb{T}^2)} \leq C\|\rho_1 - \rho_2\|(t)_{L^p(\mathbb{T}^2)}$$

and this implies using Sobolev embedding Theorem that

$$\|\nabla\partial_\theta \Psi\|_{L^\infty(\mathbb{T}^2)} \leq C\|\rho_1 - \rho_2\|_{L^\infty(\mathbb{T}^2)}$$

where the constant C depends on $m, M, \|\rho_1, \rho_2\|_{C^{0,1}}$. Thus we conclude that

$$\|\nabla\Psi_2 - \nabla\Psi_1\|(t)_{L^\infty(\mathbb{T}^2)} \leq C\|\rho_1 - \rho_2\|_{L^\infty(\mathbb{T}^2)}$$

uniformly on $[0, T]$.

$\delta = \rho_1 - \rho_2$ satisfies an inequation of the form :

$$|\partial_t \delta + u \cdot \nabla\delta| \leq C|\delta|$$
$$\delta(0, \cdot) = 0$$

thus $\delta \equiv 0$ and the uniqueness is proved. □.

CHAPITRE 6

The inverse problem for the Euler-Poisson system in cosmology

Grégoire Loeper[1]

RÉSUMÉ. The motion of a continuum of matter subject to gravitational interaction is classically described by the Euler-Poisson system. Prescribing the density of matter at initial and final times, we are able to obtain weak solutions for this equation by minimizing the action of the Lagrangian which is a convex functional. Then we see that such minimizing solutions are consistent with smooth solutions of the Euler-Poisson system and enjoy some special regularity properties.

1. Introduction

The Euler-Poisson system describes the motion of a self-gravitating fluid. It is used in cosmology, to model the evolution of the primitive universe. In the classical (non-relativistic) description, the gravitational field generated by a continuum of matter with density ρ, is the gradient of a potential p satisfying the Poisson equation

$$\Delta p = \rho - \rho_m$$

with ρ_m the uniform average background mass density. In this paper we will restrict ourselves to the flat torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ with $\rho_m = 1$. In this framework the Euler-Poisson system hereafter referred to as $(E-P)$ takes the following form :

$$\partial_t \rho + \nabla \cdot (\rho v) = 0 \tag{220}$$

$$\partial_t(\rho v) + \nabla \cdot (\rho v \otimes v) = -\rho \nabla p \tag{221}$$

$$\Delta p = \rho - 1 \tag{222}$$

with the additional constraint

$$\int_{\mathbb{T}^d} \rho(.,x)dx \equiv 1.$$

Note that this form is the cosmological one, i.e. that the potential is attractive. In the case of a repulsive potential (used for the description of a plasma) the associated Poisson equation would be $\Delta p = -[\rho - \rho_m]$.

1.1. Definition of the two point boundary problem. Given the $(E-P)$ system , one can try to solve the Cauchy problem, i.e. given ρ and v at time $t = 0$ find a solution to (220, 221, 222) on a time interval $[0, T[$. Another approach is to look for a solution over the time interval $[0, T]$ satisfying the two conditions :

$$\rho|_{t=0} = \rho_0 \tag{223}$$

$$\rho|_{t=T} = \rho_T. \tag{224}$$

This approach has been used by Brenier in [**11**],[**12**] for the incompressible Euler equation and allows to introduce variational techniques. Indeed the system (220, 221, 222) is hamiltonian, with hamiltonian (or energy) given by :

$$H(\rho, v) = \frac{1}{2}\int_{\mathbb{T}^d} \rho(t,x)|v(t,x)|^2 - |\nabla p(t,x)|^2 dx.$$

[1]Laboratoire J.A.Dieudonné, Université de Nice-Sophia-Antipolis, Parc Valrose, 06108 NICE Cedex 2.

Solutions of hamiltonian systems are critical points for the action of the Lagrangian here defined by

$$I = I(\rho, v, p) = \frac{1}{2}\int_0^T \int_{\mathbb{T}^d} \rho(t,x)|v(t,x)|^2 + |\nabla p(t,x)|^2 \, dxdt.$$

under the constraints (220,222,223,224).

In this particular case, it will be shown that the Lagrangian is a convex functional in some new variables, and this allows to use duality techniques to find the critical point (which will necessarily be a minimum of I). Our goal in then to solve the following minimization problem :

PROBLEM 6.1. *Find $\bar\rho, \bar v, \bar p$ such that*

$$I(\bar\rho, \bar v, \bar p) = \inf I(\rho, v, p)$$

over all the ρ, v, p satisfying

$$\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho v) = 0,\\
&\Delta p = \rho - 1\\
&\rho|_{t=0} = \rho_0\\
&\rho|_{t=T} = \rho_T.
\end{aligned}$$

The problem is here formulated in a very vague way : we do not mention in what space lie ρ and v. This will be precised in the next subsection.

1.2. Motivations. The practical interest of studying this boundary problem is twofold. First it is studied in cosmology for the reconstruction of the early universe. For this we refer for example to the PhD thesis of J.Bec ([**4**], p.11,12) and to [**37**] and [**15**]. On the other hand it has been observed first by Brenier in [**12**] in the case of the Euler incompressible equation and by Evans and Gomes in [**33**] in the smooth finite dimensional case that solutions of Hamiltonian flows that minimize the action of the Lagrangian are somehow better than other solutions. Some aspects of the present work can be seen as a continuation of their contribution in one special case of infinite dimensional Hamiltonian system. Another interest of this study is to generalize the approach developed in [**7**], in which the authors gave a continuum mechanics interpretation of the Monge-Kantorovitch problem involving the concept of interpolation between two measures, induced by the Wasserstein distance. This interpolation was introduced earlier by [**48**] to develop the useful concept of displacement convexity. [**54**] also used it to endow the set of probability measures with a formal Riemannian metric, in which the interpolation plays the role of geodesic, then allowing rich interpretations of some dissipative equations in terms of gradient flows. Here our variational problem induces an interpolation that has more regularity than the one of [**7**] where the Lagrangian is only

$$\frac{1}{2}\int_0^T \int_{\mathbb{T}^d} \rho(t,x)|v(t,x)|^2 \; dtdx.$$

Indeed we will see that the additional Dirichlet term forces the intermediate densities to be in $L^\infty(\mathbb{T}^d)$ independently of the initial and final densities. Some interesting displacement convexity properties will also appear.

We finally mention that the first steps of this study had been already done in E. Camalet's PhD [**23**] under supervision of Y.Brenier.

1.3. Precise definition of Problem 6.1. We introduce the domain $D = [0,T] \times \mathbb{T}^d$. We also define the flux of matter J by $J = \rho v$. Given $J \in \mathbb{R}^d, \rho \in \mathbb{R}^+$ we use the fact that

$$\sup_{c\in\mathbb{R}, m\in\mathbb{R}^d, c+|m|^2/2\leq 0} \{\rho c + J\cdot m\} = \begin{cases} +\infty \text{ if } \rho = 0, J \neq 0 \\ 0 \text{ if } J = 0 \\ J^2/2\rho \text{ if } \rho > 0 \end{cases}$$

Notice that as a supremum of affine functions, this is a (possibly infinite) convex functional in (ρ, J). Given ρ_0, ρ_T as in Theorem 6.4, the functional I can thus be formulated as

$$\tilde{I}(\rho, J, p) =$$
$$\sup_{c+|m|^2/2\leq 0} \{\int_D c(t,x)d\rho(t,x) + m(t,x)\cdot dJ(t,x)\} + \frac{1}{2}\int_D |\nabla p(t,x)|^2\, dtdx$$

where the supremum is taken over all $(c,m) \in C(D) \times C(D)^d$. This formulation is consistent with the former one in the case where $v \in L^2([0,T], L^2(\mathbb{T}^d, d\rho))$ and well defined (although leading to possibly infinite value) for $\rho \in C([0,T], \mathcal{P}(\mathbb{T}^d) - w*)$, $J \in (\mathcal{M}(D))^d$, $\nabla p \in L^2(D)$ where $\mathcal{M}(D)$ denotes the set of bounded measures on D and $\mathcal{P}(\mathbb{T}^d)$ the set of probability measures on $\mathbb{T}^d$.
The new formulation of the Problem 6.1 is then :

Problem 6.2. *Minimize*

$$\begin{aligned} \tilde{I}(\rho, J, p) &= \sup_{c+|m|^2/2\leq 0} \{\int_D c(t,x)d\rho(t,x) + m(t,x)\cdot dJ(t,x)\} \\ &+ \frac{1}{2}\int_D |\nabla p(t,x)|^2\, dtdx \end{aligned}$$

among all ρ, J, p that satisfy $\rho \in C([0,T], \mathcal{P}(\mathbb{T}^d) - w*)$, $J \in (\mathcal{M}(D))^d$, $\nabla p \in L^2(D)$ and satisfy in the distribution sense

$$\partial_t \rho + \nabla \cdot J = 0 \tag{225}$$
$$\Delta p = \rho - 1 \tag{226}$$
$$\rho(t=0) = \rho_0 \tag{227}$$
$$\rho(t=T) = \rho_T. \tag{228}$$

We denote

$$K = \inf_{\rho,J,p} \tilde{I}(\rho, J, p)$$

among all such ρ, J, p.

1.4. Cosmological form of the problem. In cosmology, new variables are used in order to take into account the expansion of the universe. The velocity is decomposed into a sum of two terms : $\mathbf{v} = u + v$ where the first describes the global expansion and the second called the peculiar velocity describes fluctuations around the global expansion. The density μ is also decomposed as $\mu = r + \rho$ where r is the isotropic background density and depends only on time. With some change of variables the system can then be put under the form

$$\partial_t \rho + \nabla\cdot(\rho v) = 0$$
$$\partial_t(\rho v) + \nabla\cdot(\rho v\otimes v) = -2\frac{\dot a}{a}\rho v - \frac{1}{a}\rho\nabla p$$
$$\Delta p = \frac{4\pi\mathcal{G}c_0}{a}(\rho - 1)$$

The function of time $a(t)$ is called the expansion factor and is supposed to be known as well as the constant c_0. The solutions of such a system can be sought as critical points for the following action :

$$I_a = \frac{1}{2}\int_0^T \int_{\mathbb{T}^d} a^2(t)\left[\rho(t,x)|v(t,x)|^2 + \frac{4\pi\mathcal{G}c_0}{a(t)}|\nabla p(t,x)|^2\right]\, dxdt.$$

Then the same technique as in the following gives the same results. For more details on the cosmological aspects of the problem the reader can refer to [**15**]. [1]

[1]If we look at the Lagrangian

$$\tilde I = \frac{1}{2}\int_{t_0}^T \int_{\mathbb{T}^d} \rho|v|^2 + \frac{1}{2}K(t,x,\nabla p)(\nabla p,\nabla p)\, dxdt$$

where K is a symmetric matrix such that $\int K(t,x,\nabla p)(\nabla p,\nabla p)\, dxdt$ is a convex functional in p then we may obtain weak solutions of

$$\partial_t v + (v\cdot\nabla)v = -K\nabla p - \frac{1}{2}\frac{\partial K}{\partial \nabla p}(\nabla p,\nabla p)$$

and thus solve non-linear Euler-Poisson systems.

2. Results

We give here the definition of a weak solution of $(E-P)$:

DEFINITION 6.3. *(ρ, v, p) is said to be a weak solution of $(E-P)$ if :*

1- $\rho \in L^2([0,T], H^{-1}(\mathbb{T}^d)) \cap C([0,T], \mathcal{P}(\mathbb{T}^d) - w*)$, $v \in L^2([0,T], L^2(\mathbb{T}^d, d\rho))$,

2- for any $\varphi \in \left(C_c^\infty(]0,T[\times\mathbb{T}^d)\right)^d$ one has

$$\int_{[0,T]\times\mathbb{T}^d} \partial_t\varphi \cdot v \; d\rho + D\varphi : v \otimes v \; d\rho - \varphi \cdot \nabla p \tag{229}$$

$$+D\varphi : \nabla p \otimes \nabla p - \frac{1}{2}(\nabla \cdot \varphi)|\nabla p|^2 = 0, \tag{230}$$

3- for any $\varphi \in C^\infty([0,T]\times\mathbb{T}^d)$:

$$\int_{[0,T]\times\mathbb{T}^d} \partial_t\varphi d\rho + \nabla\varphi \cdot v d\rho = \int_{\mathbb{T}^d} \rho_T \varphi|_{t=T} - \int_{\mathbb{T}^d} \rho_0 \varphi|_{t=0}$$

$$\int_{[0,T]\times\mathbb{T}^d} (d\rho - 1)\varphi + \nabla p \cdot \nabla\varphi = 0.$$

Equation (229) is equivalent to equation (221) for smooth p using the fact that for any smooth p

$$(1+\Delta p)\nabla p = \nabla \cdot (\nabla p \otimes \nabla p) - \frac{1}{2}\nabla|\nabla p|^2 + \nabla p.$$

The right hand side of this identity is well defined in the sense of distribution if we only know that $\nabla p \in L^2([0,T]\times\mathbb{T}^d)$.

Our results are the following :

THEOREM 6.4. *Let ρ_0, ρ_1 be two probability measures in $L^{\frac{2d}{d+2}}(\mathbb{T}^d)$, then there exists a unique $(\rho, J, p) \in (\mathcal{M} \times \mathcal{M}^d \times L^2([0,T], H^1(\mathbb{T}^d)))$ with $\Delta p = \rho - 1 in \mathcal{D}'$ minimizer of the Problem 6.2. J has a density v with respect to ρ, (ρ, v, p) is a weak solution of the Euler Poisson system $(E-P)$ in the sense of Definition 6.3 and coincides with any smooth solution of $(E-P)$ satisfying (227,228) which therefore must be unique. Moreover*

1- There exists $\phi \in L^2_{loc}(]0,T[, H^1(\mathbb{T}^d))$ such that $v = \nabla\phi \; d\rho$ a.e. and we can thus extend the definition of v to all of $\mathbb{T}^d$ as a function belonging to $L^2_{loc}(]0,T[, L^2(\mathbb{T}^d))$,

2- any such extension satisfies

$$\int_{\mathbb{T}^d}\int_\tau^{T-\tau} \rho(t,x)\,|v(t,x+y) - v(t,x)|^2\, dtdx \le C_\tau |y|^2 \tag{231}$$

for all τ in $]0,T/2]$, y in $\mathbb{R}^d$,

3- ϕ satisfies in the sense of distribution $\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + p \le 0$, and $\phi \in L^\infty_{loc}(]0,T[, \mathbb{T}^d)$,

4- for any $\epsilon > 0$ small enough, $\rho \in L^2_{loc}(]0,T[, L^2(\mathbb{T}^d)) \cap C(]0,T[, L^{3/2-\epsilon})$.

Then we have the regularity result :

THEOREM 6.5. *If ρ_0 and ρ_T are in $L^{\frac{2d}{d+2}}$ then there exists a unique solution ρ, v, p of Problem 6.2 that has the following regularity properties :*

1- The density ρ is is in $L^\infty_{loc}(]0,T[, L^\infty(\mathbb{T}^d)) \cap C(]0,T[, L^k(\mathbb{T}^d))$ for every $1 \le k < \infty$. For every $\tau \in]0, T/2[$ there exists C_τ such that for every t in $[\tau, T-\tau]$

$$\|\rho(t,\cdot)\|_{L^\infty(\mathbb{T}^d)} \le C_\tau.$$

There exists C such that

$$-C(1+\frac{1}{t}) \le \frac{d}{dt}\|\rho(t,\cdot)\|_{L^k(\mathbb{T}^d)} \le C(1+\frac{1}{T-t}).$$

and the constants C_τ, C are independent of the choice of ρ_0 and ρ_T.

2- The velocity $v = \nabla\phi$ can be chosen in $L^\infty_{loc}(]0,T[\times\mathbb{T}^d)$, this bound is also independent of the choice of ρ_0 and ρ_T.

3- The functions $\int_{\mathbb{T}^d}[\rho]^k(t,x)dx$, $k \ge 1$, $\int_{\mathbb{T}^d}[\rho\log\rho](t,x)dx$ are convex with respect to time.

4- ϕ can be chosen in $W^{1,\infty}_{loc}(]0,T[\times\mathbb{T}^d)$ and to be viscosity solution of $\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + p = 0$ on every $[s,t] \subset]0,T[$.

5- If ρ_T is in $L^p(\mathbb{T}^d)$ with $p > d$ then point 4 extends up to $t = T$.

6- One can also choose ϕ such that $(\psi, q)(t,x) = -\phi(T-t,x), q(T-t,x)$ is a viscosity solution of $\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + p = 0$ on every $[s,t] \subset]0,T[$ and point 5 applies.

The reader can refer to the books of Evans [**32**] and Barles [**3**] for the definition of viscosity solution of $\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + p = 0$.

Remark 1 : the assumption that the final and initial densities are in $L^{\frac{2d}{d+2}}(\mathbb{T}^d)$ is technical : it allows us to show that there exists at least one admissible flow with finite action transporting ρ_0 on ρ_T, see section 3.0.1. Actually all the results are true assuming that there exists a (ρ, v, p) satisfying all the constraints (225,..,228) such that $\tilde{I}(\rho, v, p)$ is finite.
Remark 2 : (231) is a finite difference version of the formal (but non rigorous since ρ has no regularity) assertion $\int_{[\tau,T-\tau]\times\mathbb{T}^d}\rho|\nabla v|^2 dtdx < +\infty$. See [**8**] where the authors look at an appropriate definition of the tangent space related to a measure.
Remark 3 : one may observe that the time regularity obtained for ρ is stronger in the second theorem than in the first. The third point of Theorem 6.4 is obtained just by using the regularity result of v (point 2 of Theorem 6.4), while in the second theorem we use in a crucial way the fact that $\Delta p = \rho - 1 \ge -1$. The techniques are thus different.
Remark 4 : the consistency with smooth solutions is detailed in Theorem 6.14.
Remark 5 : in many assertions we only state that "ϕ can be chosen in such a way that..."; this is because ϕ is uniquely determined only in the $d\rho$ a.e. sense.
The paper is organized as follows : in section 3 we introduce the variational techniques and obtain existence and uniqueness of the solution of Problem 6.2. In section 4 we show that this minimizer is a weak solution of $(E-P)$. In section 5 we show some regularity properties of this solution. This is the first theorem. In the section 6 we introduce a time discretization that enables us to obtain additional regularity properties, which give the second theorem.

3. Existence and uniqueness for Problem 6.2

This section is devoted to the proof of

PROPOSITION 6.6. *Under the assumption that ρ_0 and ρ_T are in $L^{\frac{2d}{d+2}}$ there exists a unique minimizer $(\rho, J, \nabla p)$ in $C([0,T], \mathcal{P}(\mathbb{T}^d) - w*) \times \mathcal{M}(D) \times L^2(D)$ for the Problem 6.2 under the constraints (225,226,227,228).*

3.0.1. *Existence of an admissible solution.* We are now going to prove that the infimum of Problem 6.2 is finite.

LEMMA 6.7. *Under the assumption that ρ_0, ρ_T are in $L^{\frac{2d}{d+2}}(\mathbb{T}^d)$ there exists $(\rho, J = \rho v, p)$ satisfying (225, 226, 227, 228) and such that $\tilde{I}(\rho, J, p)$ is finite. Moreover $\rho \in L^\infty([0,T], L^{\frac{2d}{d+2}}(\mathbb{T}^d))$ and $v \in L^\infty(D, d\rho dt)$.*

Proof : We use the following result that combines [**7**] and [**48**] (See also [**54**]).

PROPOSITION 6.8. *Let ρ_0 and ρ_1 belong to $\mathcal{P}(\mathbb{T}^d) \cap L^k$ for some $1 \leq k \leq \infty$. There exists a unique pair $(\bar\rho(t,x), \bar J = \bar\rho\bar v(t,x))$ with $\bar v \in L^\infty([0,T] \times \mathbb{T}^d, d\bar\rho dt))$ that minimizes the action*

$$A(\rho, \rho v) = \int_{[0,1]\times\mathbb{T}^d} \rho(t,x)|v(t,x)|^2 dtdx$$

among all (ρ, J) that satisfy $\rho \in C([0,1], \mathcal{P}(\mathbb{T}^d) - w*)$, $J \in (\mathcal{M}([0,1]\times\mathbb{T}^d))^d$ and

$$\partial_t \rho + \nabla\cdot J = 0$$
$$\rho(t=0) = \rho_0$$
$$\rho(t=1) = \rho_1.$$

$A(\bar\rho, \bar\rho\bar v)$ is finite and for $k \geq 1 - 1/n$ the function $t \to \|\bar\rho(t,.)\|_{L^k}^k$ is convex on $[0,1]$ thus bounded by $\max\{\|\rho_0\|_{L^k}^k, \|\rho_1\|_{L^k}^k\}$. Finally $\bar v \in L^\infty([0,1]\times\mathbb{T}^d, d\bar\rho dt)$.

Using the Sobolev imbedding theorem and classical elliptic regularity, if $k = \frac{2d}{d+2}$ we have

$$\|\nabla\Delta^{-1}(\bar\rho(t,\cdot))\|_{L^2(\mathbb{T}^d)} \leq C\|\bar\rho(t,\cdot)\|_{L^k(\mathbb{T}^d)}.$$

Then considering $(\rho(s,\cdot), v(s,\cdot)) := (\bar\rho(s/T,\cdot), \frac{1}{T}\bar v(s/T,\cdot))$ we get an admissible pair for which I is finite. This completes the proof of lemma 6.7. □

3.1. Proof of Proposition 6.6. We use here standard convex analysis arguments that can be found in [**17**] and the proof is an adaptation of the one found in [**12**]. As we will show later on the optimal density ρ is in $L^\infty_{loc}(]0,T[\times\mathbb{T}^d)$. Thus to simplify the notations we will often denote $\rho\ dxdt$ instead of $d\rho$ even if our optimization covers a broader class of probability measures than those that are in L^1. The constraints (225,226,227,228) can be formulated in the following weak way :

$$\forall\phi \in C^\infty(D),\ \int_D \partial_t\phi(\rho - \bar\rho) + \nabla\phi\cdot(J - \bar J)\ dtdx = 0 \tag{232}$$

$$\forall p \in C^\infty(D),\ \int_D (\rho - \bar\rho)q\ dtdx = -\int_D (\nabla p - \nabla\bar p)\cdot\nabla q\ dtdx \tag{233}$$

where $(\bar{\rho}, \bar{J})$ is the admissible solution of subsection 3.0.1. Minimize $\tilde{I}$ under the constraints (225,226,227,228) is thus equivalent to find

$$K = \inf_{\rho,J,p} \sup_{\phi,q,c,m} \int_D \rho c + J \cdot m - \partial_t \phi(\rho - \bar{\rho}) - \nabla\phi \cdot (J - \bar{J}) + \frac{1}{2}|\nabla p|^2 - \nabla q \cdot (\nabla p - \nabla \bar{p}) - q(\rho - \bar{\rho}) \; dtdx$$

with the supremum taken over all the continuous functions c, m with $c : D \to \mathbb{R}$ and $m : D \to \mathbb{R}^d$ satisfying $c + |m|^2/2 \leq 0$.

$C(D)$ is the space of continuous functions on D and $C_\#(D)$ is defined by the additional constraint that the integral over $\mathbb{T}^d$ vanishes for all $t \in [0,T]$. On $C(D)$ we have the usual duality bracket $< f, g >$ denoted by $\int_D f \, dg$ with $g \in \mathcal{M}(D)$ the set of bounded measures on D. The dual space of $C_\#$ is reduced to the set of bounded measures g on D whose total mass at any time is zero (i.e. for all $g \in C'_\#$, for all $z \in C^0[0,T]$, $\int_D z(t) dg = 0$) and denoted by $\mathcal{M}_\#(D)$. We introduce the functionals α and β defined on $(c, m, r) \in C(D) \times (C(D))^d \times C_\#(D)$. It will be convenient to denote $r = \Delta q$, and this is possible since the mean value of r is zero.

$$\alpha(c,m,r) = \frac{1}{2}\int_D |\nabla \Delta^{-1} r|^2 \; dtdx = \frac{1}{2}\int_D |\nabla q|^2 \; dtdx$$

if $c + |m|^2/2 \leq 0$,

$\alpha(c,m,r) = +\infty$ otherwise;

$$\beta(c,m,r) = \int_D \bar{\rho} c + \bar{J} \cdot m + \bar{p} r \; dtdx$$

if $\exists \, \phi \in C^1(D)$ such that $c + \partial_t \phi + q = 0, \; m + \nabla_x \phi = 0$,

$\beta(c,m,r) = +\infty$ otherwise

with $(\bar{\rho}, \bar{J}, \bar{p})$ as above.

We compute α^* and β^* the Legendre-Fenchel transform (see [**17**] for definition) of respectively α and β, They are defined on $(\rho, J, p) \in \mathcal{M}(D) \times \mathcal{M}(D)^d \times \mathcal{M}_\#(D)$ the dual space of $C(D) \times (C(D))^d \times C_\#(D)$

$$\alpha^*(\rho, J, p) = \sup_{c + |m|^2/2 \leq 0, \; r = \Delta q} \{ \int_D \rho c + J \cdot m + rp - |\nabla q|^2/2 \; dtdx \}$$

We have then

$$\alpha^*(\rho, J, p) = \frac{1}{2}\int_D \frac{|J|^2}{\rho} + |\nabla p|^2 \; dtdx.$$

Note that this can possibly be $+\infty$. Then for β we have :

$$\beta^*(\rho, J, p) = \sup_{c,m,r} \{ \int_D (\rho - \bar{\rho})c + (J - \bar{J}) \cdot m + (p - \bar{p}) r \; dtdx \}$$

the supremum being restricted to all the $c, m, r = \Delta q$ such that there exists ϕ satisfying :

$$c + \partial_t \phi + q = 0$$
$$m + \nabla_x \phi = 0$$

Thus in terms of ϕ, q we have

$$\beta^*(\rho, J, p) =$$
$$\sup_{\phi,q}\{\int_D (\rho-\bar\rho)(-\partial_t\phi - q) - (J-\bar J)\cdot\nabla_x\phi - \nabla q\cdot(\nabla p - \nabla\bar p)\ dtdx\}.$$

Using the fact that $\bar\rho, \bar p$ satisfy (225,226) we find that $\beta^*(\rho, J, p) = 0$ if ρ, J, p satisfies (232,233) and $\beta^*(\rho, J, p) = +\infty$ otherwise. It follows then that

$$I = \inf_{\rho,J,p}\{\alpha^*(\rho, J, p) + \beta^*(\rho, J, p)\}$$

where we now compute the infimum over all ρ, J, p. We have just relaxed the constraints (225,226,227,228) by adding the functional β^* which is $+\infty$ if they are not satisfied and 0 if they are satisfied.

3.1.1. *The duality theorem.* Functions α, β are convex with values in $]-\infty, +\infty]$ At point $c = -1, m = 0, r = 0$, $\alpha(-1, 0, 0) = 0$, α is continuous with respect to the norm of $C(D)\times(C(D))^d\times C_\#(D)$, and $\beta(-1, 0, 0) = -\int_D\bar\rho = -1$ is finite. The conditions to apply Fenchel-Rockafellar duality Theorem (see [**17**] ch. 1) are thus fulfilled that

$$\begin{aligned} & \inf\{\alpha^*(\rho, J, \varphi) + \beta^*(\rho, J, \varphi)\} \\ = \quad & \sup\{-\alpha(-c, -m, -r) - \beta(c, m, r)\} \\ = \quad & K \end{aligned}$$

and the infimum is attained. So we have

$$\begin{aligned} & K = \sup_{c,m,r=\Delta q}\{\int_D -|\nabla q|^2/2 - \bar\rho c - \bar J\cdot m + \nabla\bar p\cdot\nabla q\ dtdx\} \\ & c = -\partial_t\phi - q, \\ & m = -\nabla\phi, \\ & -c + |m|^2/2 \le 0 \end{aligned}$$

which is also

$$K = \sup_{\phi,q}\{\int_D -|\nabla q|^2/2 + \bar\rho(\partial_t\phi + q) + \bar J\cdot\nabla\phi + \nabla\bar p\cdot\nabla q\ dtdx\}$$

with

$$\partial_t\phi + q + |\nabla\phi|^2/2 \le 0$$

If we choose $\bar\rho = \rho = 1 + \Delta p$ and $\bar J = \rho v$ to be any optimal solution (i.e. any minimizing solution). Note that necessarily J has a density v with respect to ρ and $v \in L^2(D, d\rho\ dt)$. This justifies the notation $J = \rho v$. Then for all $\epsilon > 0$ there exists $\phi_\epsilon, p_\epsilon \in C^1(D)$ with $\partial_t\phi_\epsilon + p_\epsilon + |\nabla\phi_\epsilon|^2/2 \le 0$ such that :

$$I = \frac{1}{2}\int_D \rho|v|^2 + |\nabla p|^2\ dtdx$$

$$\le \int_D -|\nabla p_\epsilon|^2/2 + \rho(\partial_t\phi_\epsilon + p_\epsilon) + J\cdot\nabla\phi_\epsilon + \nabla p\cdot\nabla p_\epsilon\ dtdx + \epsilon^2, \tag{234}$$

thus

$$\begin{aligned} & \frac{1}{2}\int_D \rho|v-\nabla\phi_\epsilon|^2+|\nabla p-\nabla p_\epsilon|^2\ dtdx \\ \leq \quad & \int_D \rho(\partial_t\phi_\epsilon+p_\epsilon+|\nabla\phi|^2/2)\ dtdx+\epsilon^2, \end{aligned}$$

and we obtain

$$\int_D \frac{1}{2}\rho|v-\nabla\phi_\epsilon|^2+\frac{1}{2}|\nabla p-\nabla p_\epsilon|^2+\rho\left|\partial_t\phi_\epsilon+p_\epsilon+|\nabla\phi_\epsilon|^2/2\right|\ dtdx\leq\epsilon^2.$$

It follows that as $\epsilon\to 0$,

– $\nabla\phi_\epsilon$ converges to v in $L^2(D,dtd\rho)$,

– ∇p_ϵ converges to ∇p in $L^2(D,dtdx)$,

– $\partial_t\phi_\epsilon+p_\epsilon+|\nabla\phi|^2/2$ converges to 0 in $L^1(D,dtd\rho)$.

3.1.2. *Uniqueness of the minimizer.* Notice that the sequence $(\phi_\epsilon,p_\epsilon)$ does not depend on the optimal solution (ρ,v) we have chosen, thus if we have ∇p_1 and ∇p_2 two optimal solutions, then ∇p_ϵ converges to both ∇p_1 and ∇p_2, and they are equal Lebesgue a.e. It follows then that two optimal solutions have the same ρ. Then since $\nabla\phi_\epsilon$ converges to both v_1 and v_2 in $L^2(d\rho)$, v_1 and v_2 are equal $d\rho$ a.e. and the uniqueness of the optimal solution is proved. This ends the proof of Proposition 6.6. □

4. Optimality equation.

In this section we prove the following :

PROPOSITION 6.9. *The solution of Problem 6.2 is a weak solution of the Euler-Poisson system $(E-P)$ in the sense of definition 6.3. The energy of the system defined for a.e. $t\in[0,T]$ by*

$$E(t)=\frac{1}{2}\int_{\mathbb{T}^d}\rho(t,x)|v(t,x)|^2-|\nabla p(t,x)|^2dx$$

does not depend on time.

Remark : the energy is a priori well defined in $L^1(0,T)$ since $I(\rho,v,p)$ is finite.

4.1. Proof of Proposition 6.9.

4.1.1. *Derivation of equation (221).* Let δ and η be two small parameters and take $\tau\in[0,\frac{T}{2}[$. Let $\zeta(t)$ be a smooth function compactly supported for $0<t<T$. We choose η small enough such that $t\to t+\eta\zeta(t)$ is a diffeomorphism from $[0,T]$ to $[0,T]$. Let $x\to w(x)$ be a smooth vector field and $x\to e^{sw}(x)$ the flow associated to $w(x)$ defined by

$$\partial_s e^{sw}(x)=w(e^{sw}(x))\text{ and }e^{0w}(x)=x,$$

we can thus define $e^{\delta\zeta(t)w}(x)$. We introduce, as in [**12**], the following measures :

$$\rho^\eta(t,x)=\rho(t+\eta\zeta(t),x),\ \ v^\eta(t,x)=v(t+\eta\zeta(t),x)(1+\eta\dot{\zeta}(t)).$$

We check that the pair $(\rho^\eta,\rho^\eta v^\eta)$ satisfies the continuity equation (232). Then we define the measures $(\rho^{\eta,\delta},J^{\eta,\delta})$ so that for every $f\in C(D)$ and $g\in(C(D))^d$ we have

$$\int_D f(t,x)\rho^{\eta,\delta}(t,x)\ dtdx=\int_D f(t,e^{\delta\zeta(t)w}(x))\rho^\eta(t,x)\ dtdx,$$

and

$$\begin{aligned} & \int_D g(t,x)\cdot J^{\eta,\delta}(t,x)\ dtdx \\ = & \int_D g(t, e^{\delta\zeta(t)w}(x))\cdot[(\partial_t + v^\eta(t,x)\cdot\nabla)e^{\delta\zeta(t)w}(x)]\rho^\eta(t,x)\ dtdx. \end{aligned}$$

We check that the pair $(\rho^{\eta,\delta}, J^{\eta,\delta})$ satisfies also the continuity equation (232). η, δ being fixed we will use the following notation :

$$v^{\eta,\delta}(t,x) = (\partial_t + v^\eta(t,x)\cdot\nabla_x)\cdot e^{\delta\zeta(t)w}(x).$$

We have

$$\rho^\eta(t,x)(\partial_t\phi_\epsilon + \frac{|\nabla\phi_\epsilon|^2}{2} + p_\epsilon)(t, e^{\delta\zeta(t)w}(x)) \le 0,$$

and using (234) we can write :

$$\begin{aligned} & \frac{1}{2}\int_D \rho|v|^2 + |\nabla p|^2 \\ \le\ & \epsilon^2 + \int_D \rho(\partial_t\phi_\epsilon + p_\epsilon) + \rho v\cdot\nabla\phi_\epsilon + \nabla p\cdot\nabla p_\epsilon - |\nabla p_\epsilon|^2/2 \\ & - \int_D \rho^{\eta,\delta}(\partial_t\phi_\epsilon + \frac{|\nabla\phi_\epsilon|^2}{2} + p_\epsilon). \end{aligned}$$

Then using (232) we have

$$\begin{aligned} \frac{1}{2}\int_D \rho|v|^2 + |\nabla p|^2 \le\ & \epsilon^2 + \int_D \rho^\eta\left(v^{\eta,\delta}\cdot\nabla\phi_\epsilon(e^{\delta\zeta w}) - \frac{1}{2}|\nabla\phi_\epsilon|^2(e^{\delta\zeta w})\right) \\ + & \int_D \rho p_\epsilon - \rho^\eta p_\epsilon(e^{\delta\zeta w}) + \nabla p\cdot\nabla p_\epsilon - |\nabla p_\epsilon|^2/2, \end{aligned}$$

which yields

$$\begin{aligned} \frac{1}{2}\int_D \rho|v|^2 + |\nabla p|^2 \le\ & \epsilon^2 - \frac{1}{2}\int_D \rho^\eta\left|v^{\eta,\delta} - \nabla\phi_\epsilon(e^{\delta\zeta w})\right|^2 + \frac{1}{2}\int_D \rho^\eta|v^{\eta,\delta}|^2 \\ + & \int_D p_\epsilon - p_\epsilon(e^{\delta\zeta w}) \\ + & \int_D D(e^{\delta\zeta w}) : \nabla p^\eta\otimes\nabla p_\epsilon(e^{\delta\zeta w}) - |\nabla p_\epsilon|^2/2, \end{aligned}$$

and we obtain the complete formula :

$$
\begin{aligned}
& \frac{1}{2}\int_D \rho^\eta \left|v^{\eta,\delta} - \nabla\phi_\epsilon(e^{\delta\zeta w})\right|^2 + \frac{1}{2}\int_D \left|\nabla p^\eta - D(e^{\delta\zeta w})\nabla p_\epsilon(e^{\delta\zeta w})\right|^2 \\
\leq \quad & \epsilon^2 + \frac{1}{2}\int_D \rho^\eta |v^{\eta,\delta}|^2 - \frac{1}{2}\int_D \rho|v|^2 \\
+ \quad & \frac{1}{2}\int_D |D(e^{\delta\zeta w})\nabla p_\epsilon(e^{\delta\zeta w})|^2 - \frac{1}{2}\int_D |\nabla p_\epsilon|^2 \\
+ \quad & \frac{1}{2}\int_D |\nabla p^\eta|^2 - \frac{1}{2}\int_D |\nabla p|^2 \\
+ \quad & \int_D p_\epsilon - p_\epsilon(e^{\delta\zeta w}).
\end{aligned}
\tag{235}
$$

First taking $\eta = 0$, bounding the L.H.S. from below by 0 and letting ϵ go to 0 we get :

$$
\begin{aligned}
0 \leq \quad & \frac{1}{2}\int_D \rho|v^{0,\delta}|^2 - \frac{1}{2}\int_D \rho|v|^2 \\
+ \quad & \frac{1}{2}\int_D |D(e^{\delta\zeta w})\nabla p(e^{\delta\zeta w})|^2 - \frac{1}{2}\int_D |\nabla p|^2 \\
+ \quad & \int_D p - p(e^{\delta\zeta w})
\end{aligned}
$$

Taking the first order terms in δ, with $e^{\delta\zeta(t)w}(x) = x + \delta\zeta(t)w(x) + O(\delta^2)$ we get

$$
\begin{aligned}
& \frac{1}{2}\int_D |D(e^{\delta\zeta w})\nabla p(e^{\delta\zeta w})|^2 \\
= \quad & \frac{1}{2}\int_D \left|(I + \delta\zeta Dw)(e^{-\delta\zeta w})\nabla p\right|^2 \mathbf{J}(e^{-\delta\zeta w}) + O(\delta^2)
\end{aligned}
$$

with I the identity matrix of order d and $\mathbf{J}(e^{-\delta\zeta(t)w})$ the jacobian determinant of the mapping $x \to e^{-\delta\zeta(t)w}(x)$. Using that $\mathbf{J}(e^{-\delta\zeta(t)w}) = 1 - \delta\zeta(t)\nabla\cdot w + O(\delta^2)$, this is equal to

$$
\frac{1}{2}\int_D (|\nabla p|^2 + 2\delta\zeta Dw : \nabla p \otimes \nabla p - \delta\zeta|\nabla p|^2 \nabla\cdot w) + O(\delta^2).
$$

Then for v we have

$$
\begin{aligned}
& \frac{1}{2}\int_D \rho|v^\delta|^2 - \frac{1}{2}\int_D \rho|v|^2 \\
= \quad & \int_D \rho v \cdot (\partial_t + v\cdot\nabla)\delta\zeta w + O(\delta^2)
\end{aligned}
$$

and

$$
\int_D -p(e^{\delta\zeta w}) = -\int_D p - \int_D \nabla p \cdot \delta\zeta w + O(\delta^2).
$$

This yields finally

$$0 \leq \quad \delta \left[\int_D \rho v \cdot (\partial_t + v \cdot \nabla)\zeta w + \zeta Dw : \nabla p \otimes \nabla p \right.$$
$$\left. -\frac{1}{2}|\nabla p|^2 \nabla \cdot \zeta w - \nabla p \cdot \zeta w \; dtdx \right] + O(\delta^2)$$

Thus for every w smooth vector field on $\mathbb{T}^d$ we have

$$0 = \quad \int_D \rho v \cdot (\partial_t + v \cdot \nabla)(\zeta w) + \zeta Dw : \nabla p \otimes \nabla p$$
$$-\frac{1}{2}|\nabla p|^2 \nabla \cdot \zeta w - \nabla p \cdot \zeta w \; dtdx$$

and thus (ρ, v, p) is a weak solution of the Euler-Poisson system in the sense of Definition 6.3. □

4.1.2. *Conservation of energy.* Here using inequality (235) we are going to show conservation of energy ; we take $\delta = 0$ in (235), minorize the LHS by 0 and let ϵ go to 0 to obtain

$$0 \leq \quad \frac{1}{2} \int_D \rho(t + \eta\zeta(t), x)(1 + \eta\dot{\zeta}(t))^2 |v(t + \eta\zeta(t), x))|^2 \; dtdx$$
$$- \quad \frac{1}{2} \int_D |\nabla p(t + \eta\zeta(t), x)|^2$$
$$+ \quad \frac{1}{2} \int_D \rho(t, x)|v(t, x)|^2 \; dtdx - \frac{1}{2} \int_D |\nabla p(t, x)|^2 \; dtdx$$

Changing variable in time $t := t + \eta\zeta(t)$, $dt := dt(1 + \eta\dot{\zeta}(t))$ we get

$$0 \leq \quad \frac{1}{2} \int_D \rho(t, x)|v(t, x)|^2 \eta\dot{\zeta}(t) + |\nabla p(t, x)|^2 (\frac{1}{1 + \eta\dot{\zeta}(t)} - 1)$$

Taking the first order term in η we get

$$\frac{1}{2} \int_D \left[\rho(t, x)|v(t, x)|^2 - |\nabla p(t, x)|^2 \right] \dot{\zeta}(t) \; dtdx = 0$$

for any $\zeta \in C_c^\infty(0, T)$ which gives the conservation of energy, with

$$E = \frac{1}{2} \int_{\mathbb{T}^d} \rho(t, x)|v(t, x)|^2 - |\nabla p(t, x)|^2 \; dx.$$

this ends the proof of Proposition 6.9. □

5. Regularity

In this section we obtain several regularity properties of solutions of Problem 6.1. Those properties come from the fact that the solution is a minimizer of the action of the Lagrangian and not only a critical point. For points 1 and 2 we follow closely the method of [**12**] where similar results were obtained in the case of the Euler incompressible equation. For the third point we use the first two points and a method close to the one used in [**30**]. Similar results have also been obtained in [**33**] for finite dimensional hamiltonian systems, using properties of a special Hamilton-Jacobi equation related to the Hamiltonian flow.

In this section we prove the following :

PROPOSITION 6.10. *The optimal solution $\rho, \rho v$ of Problem 6.2 satisfies :*
- $\rho \in L^2_{loc}(]0,T[, L^2(\mathbb{T}^d))$,
- v *can be extended to all of* $\mathbb{T}^d$ *to a function of* $L^2_{loc}(]0,T[, L^2(\mathbb{T}^d))$, *in such a way that for all* τ *in* $]0,T/2]$, *for all* y *in* $\mathbb{R}^d$,
$$\int_{\mathbb{T}^d}\int_\tau^{T-\tau} \rho(t,x)\,|v(t,x+y)-v(t,x)|^2\,dtdx \le C_\tau |y|^2.$$
- ϕ *can be chosen in* $L^\infty_{loc}(]0,T[\times\mathbb{T}^d)$.
- ρ *belongs to* $C(]0,T[, L^p(\mathbb{T}^d))$ *for any* $p \in [1, 3/2[$.

5.1. Proof of Proposition 6.10.

5.1.1. *Spatial regularity.* We are going to use inequality (235) in the special case where $\eta = 0$, $\zeta \equiv 1$ in $[\tau, T-\tau]$, $w(x) = y$ fixed. In this case $v^\delta(t,x) = v(t,x) + \delta\dot\zeta(t)y$, and $D(e^{\delta\zeta w}) = I$. Since the first order terms of the R.H.S of inequality (235) cancel, we now obtain :
$$\begin{aligned}&\frac{1}{2}\int_D \rho(t,x)|v^\delta(t,x) - \nabla\phi_\epsilon(t,x+\delta\zeta(t)y)|^2\,dtdx\\ +\;&\frac{1}{2}\int_D |\nabla p(t,x) - \nabla p_\epsilon(t,x+\delta\zeta(t)y)|^2\,dtdx\\ \le\;&\epsilon^2 + \int_D \rho|\delta\dot\zeta y|^2.\end{aligned}$$
Then we have
$$\int_D \rho|v^\delta - v|^2 = \int_D \rho|\delta\dot\zeta y|^2 \le \frac{C}{\tau}\delta^2|y|^2$$
and thus
$$\begin{aligned}&\int_{\mathbb{T}^d}\int_\tau^{T-\tau} \rho(t,x)\,|\nabla\phi_\epsilon(t,x+y) - v(t,x)|^2\\ +\;& |\nabla p_\epsilon(t,x+y) - \nabla p(t,x)|^2\;dtdx\\ &\le \epsilon^2 + \frac{C}{\tau}|y|^2.\end{aligned} \tag{236}$$
We let ϵ go to 0 and obtain
$$\int_{\mathbb{T}^d}\int_\tau^{T-\tau} |\nabla p(t,x+y) - \nabla p(t,x)|^2\,dtdx \le \frac{C}{\tau}|y|^2$$
thus D^2p, and $\rho = 1 + \Delta p$ are in $L^2_{loc}(]0,T[, L^2(\mathbb{T}^d))$.
We will also obtain that $\nabla\phi_\epsilon$ is bounded in $L^2_{loc}(]0,T[, L^2(\mathbb{T}^d))$. Indeed, we get first from (236) that
$$\int_{\mathbb{T}^d}\int_\tau^{T-\tau} \rho(t,x)\,|\nabla\phi_\epsilon(t,x+y)|^2\;dxdt \le C(1+\frac{1}{\tau})$$
for $\epsilon \le 1$ and then integrating this over $y \in \mathbb{T}^d$ we get
$$\begin{aligned}&\int_{y\in\mathbb{T}^d}\int_{x\in\mathbb{T}^d}\int_\tau^{T-\tau} \rho(t,x+y)\,|\nabla\phi_\epsilon(t,x)|^2\,dxdydt\\ =\;&\int_{x\in\mathbb{T}^d}\int_\tau^{T-\tau} |\nabla\phi_\epsilon(t,x)|^2\,dxdt \le C(1+\frac{1}{\tau})\end{aligned}$$

thus we can up to extraction of a subsequence define a limit, as ϵ goes to 0, $v = \nabla\phi \in L^2_{loc}(]0,T[, L^2(dx)\cap L^2(d\rho))$. However v will be uniquely defined only in the $d\rho$ a.e. sense. Note also that $(t,x) \to v(t,x+y)$ will be in $L^2_{loc}(]0,T[, L^2(dx)\cap L^2(d\rho))$ for any $y \in \mathbb{T}^d$. Then we obtain

$$\int_{\mathbb{T}^d}\int_{\tau}^{T-\tau} \rho(t,x)\left|v(t,x+y) - v(t,x)\right|^2 \, dtdx \tag{237}$$

This proves the first two points of Proposition 6.10.

5.2. L^∞ bound for the potential ϕ. The bound we obtain here on the potential will allow us in the next section to obtain an unconditional L^∞ bound for the velocity. We assume here that $d \le 3$. Then since $\rho \in L^2_{loc}(]0,T[, L^2(\mathbb{T}^d))$ we have $p \in L^2_{loc}(]0,T[, H^2(\mathbb{T}^d))$ which is continuously embedded in $L^2_{loc}(]0,T[, C^{\frac{1}{2}}(\mathbb{T}^d))$ and thus $\|p(t,.)\|_{L^\infty} \in L^2_{loc}(]0,T[)$. (We will see after that ρ is in $L^\infty_{loc}(]0,T[\times\mathbb{T}^d)$ and we will be able to remove this assumption on the dimension).
Then take a regularization in t,x of ϕ, p : on $]\tau, T-\tau[$,

$$\begin{aligned}\phi_\epsilon(t,x) &= \eta_\epsilon * \phi,\\ p_\epsilon(t,x) &= \eta_\epsilon * p,\\ \eta_\epsilon(t,x) &= \frac{1}{\epsilon^{d+1}}\eta_1(\frac{t}{\epsilon},\frac{x}{\epsilon}),\end{aligned}$$

with η_1 compactly supported in $[0,1]\times B(0,1)$, and $0<\epsilon<\tau/2$. Check first that

$$\partial_t\phi_\epsilon + \frac{1}{2}|\nabla\phi_\epsilon|^2 + p_\epsilon \le 0. \tag{238}$$

Considering $\sigma \to \phi_\epsilon(\sigma,\gamma(\sigma))$ with $\gamma \in C^1([0,T];\mathbb{T}^d)$ we have

$$\begin{aligned}\frac{d}{d\sigma}(\phi_\epsilon(\sigma,\gamma(\sigma))) &= \partial_t\phi_\epsilon(\sigma,\gamma(\sigma)) + \dot\gamma\cdot\nabla\phi_\epsilon(\sigma,\gamma(\sigma))\\ &\le \partial_t\phi_\epsilon(\sigma,\gamma(\sigma)) + \frac{1}{2}|\nabla\phi_\epsilon|^2(\sigma,\gamma(\sigma)) + \frac{1}{2}|\dot\gamma(\sigma)|^2\\ &\le -p_\epsilon(\sigma,\gamma(\sigma)) + \frac{1}{2}|\dot\gamma(\sigma)|^2\end{aligned}$$

using (238), and we get that

$$\phi_\epsilon(t+s,x) \;\le\; \inf_{\gamma\in\Gamma}\{\phi_\epsilon(t,\gamma(t)) + \int_t^{t+s} -p_\epsilon(\sigma,\gamma(\sigma)) + \frac{1}{2}|\dot\gamma|^2(\sigma)\, d\sigma\}$$

with Γ the set of all continuous paths going from $[t,t+s]$ to $\mathbb{T}^d$ such that $\gamma(t+s)=x$. Then restricting the infimum to paths of the form $\gamma(\sigma) = \gamma(t) + \frac{\sigma-t}{s}(x-\gamma(0))$ and noticing that $\|p_\epsilon(t,.)\|_{L^\infty} \in L^2_{loc}(]0,T[)$ (uniformly in ϵ) implies that $\int_t^{t+s}|p_\epsilon(\sigma,\gamma(\sigma))|d\sigma \le C\sqrt{s}$ one obtains the following upper bound :

$$\phi_\epsilon(t+s,x) \le \inf_{z\in\mathbb{T}^d}\{\phi_\epsilon(t,z)\} + C(\frac{1}{s}+\sqrt{s}). \tag{239}$$

A simple computation shows that

$$\begin{aligned}&\int_{\mathbb{T}^d}\rho(t_2,x)\phi_\epsilon(t_2,x)\, dx - \int_{\mathbb{T}^d}\rho(t_1,x)\phi_\epsilon(t_1,x)\, dx\\ &\to \int_{t_1}^{t_2}\int_{\mathbb{T}^d}\frac{1}{2}\rho|\nabla\phi|^2 + |\nabla p|^2\, dtdx \le 2K\end{aligned}$$

as ϵ goes to 0, with K the infimum of Problem 6.2. Thus if we normalize ϕ so that $\int_{\mathbb{T}^d} \rho(\frac{1}{2}T,x)\phi(\frac{1}{2}T,x)\,dx = 0$, there exists for any ϵ small enough x_ϵ^1, x_ϵ^2 such that

$$\begin{aligned} &\phi_\epsilon(\tau/2, x_\epsilon^1) \leq (2K+1), \\ &\phi_\epsilon(T-\tau/2, x_\epsilon^2) \geq -(2K+1). \end{aligned}$$

Using (239) we get that

$$\|\phi_\epsilon\|_{L^\infty([\tau,T-\tau]\times\mathbb{T}^d)} \leq C(K,\tau). \tag{240}$$

and using the fact that ϕ_ϵ converges $dtdx$ a.e to ϕ we conclude.

5.2.1. *Time regularity.* First let us notice that in inequality (235) if we do not set $\eta = 0$ we obtain the mixed derivative estimate :

$$\begin{aligned} &\int_{t=\tau}^{T-\tau} \rho(t,x)\,|v(t+\eta,x+y) - v(t,x)|^2 \\ + \quad &|\nabla p_\epsilon(t+\eta,x+y) - \nabla p(t,x)|^2\,dtdx \\ &\leq C_\tau(|y|^2 + |\eta|^2) \end{aligned} \tag{241}$$

Setting $y = 0$ we get that $\partial_t \nabla p \in L^2_{loc}(]0,T[, L^2(\mathbb{T}^d))$. Thus we have

LEMMA 6.11. *∇p is in $C^{\frac{1}{2}}_{loc}(]0,T[, L^2(\mathbb{T}^d))$.*

Now we prove the last part of Proposition 6.10. The technique is somehow analog to the one used by DiPerna and Lions in [**30**] : we obtain that the density is strongly continuous with respect to time by showing some renormalization property, that gives the continuity of some L^p norm, with $p > 1$. So we will prove :

LEMMA 6.12. *Let $\alpha \in [1,3/2[$ and $G^\alpha(t) = \int_{\mathbb{T}^d} \rho^\alpha(x)\,dx$.*
Then $G \in C(]0,T[,\mathbb{R})$.

We postpone the proof of the lemma after the proof of proposition 6.10.

Proof of Proposition 6.10 : First we check the weak time continuity of ρ : We have $\int_{\mathbb{T}^d} \rho v^2$ uniformly bounded on $[\tau, T-\tau]$ from the conservation of energy and from lemma 6.11. Thus $\rho|v| = \sqrt{\rho}\sqrt{\rho|v|^2} \in L^\infty([\tau,T-\tau],L^p)$ for some $p > 1$ thanks to lemma 6.12. It follows that from equation (220) $\partial_t\rho$ is bounded in $L^\infty([\tau,T-\tau],H^{-s})$ for some s. Using classical arguments of functional analysis, we can deduce that

$$\rho \in C(]0,T[, L^p - w)$$

for some $p > 1$.

Then lemma 6.12 implies that $\rho \in C(]0,T[,L^p)$ for any $p \in [1,3/2[$: indeed it is a classical fact that for a sequence u_n converging weakly to u in $L^P, 1 < p > \infty$, if $\|u_n\|_{L^p}$ converges to $\|u\|_{L^p}$ then the sequence converges strongly.

The last point of Proposition 6.10 is proved. □

Proof of lemma 6.12 : Let us prove the renormalization property when the density and the velocity field are smooth : we use the identity

$$\partial_t[\rho F(\rho)] + \nabla\cdot[\rho F(\rho)v] = -\rho^2 F'(\rho)\nabla\cdot v$$

integrating over $\mathbb{T}^d$ we get that $\int_{\mathbb{T}^d} \rho F(\rho))$ is continuous with respect to time as long as $\rho^2 F'(\rho)\nabla \cdot v$ is in $L^1_{loc}([0,T]\times\mathbb{T}^d)$. We will see that this is true for $F(\rho)=\rho^\beta$, $\beta \le \frac{1}{2}$ from the regularity property (237).
We introduce $\eta(x) = C\exp(-\frac{|x|^2}{\sqrt{1+|x|^2}})$ with C such that $\int_{\mathbb{R}^d}\eta(x)dx = 1$. Then as usual $\eta_\epsilon(x) = \frac{1}{\epsilon^d}\eta(\frac{x}{\epsilon})$, and $\rho_\epsilon(x) = \rho * \eta_\epsilon(x) = \int_{\mathbb{R}^d}\rho(x-y)\eta_\epsilon(y)dy$, ρ being naturally extended to a $\mathbb{Z}^d$ periodic function on all of $\mathbb{R}^d$. With ρ, v as before, we consider the pair $(\rho_\epsilon, v_\epsilon)$ defined by

$$\begin{aligned} \rho_\epsilon &= \eta_\epsilon \star \rho \\ v_\epsilon &= \eta_\epsilon \star (\rho v)/\rho_\epsilon \end{aligned}$$

We check that the pair $\rho_\epsilon, v_\epsilon$ still satisfies equation (220). Then we have the crucial property :

Lemma 6.13. *For any $\alpha \in [0,2]$*

$$\int_\tau^{T-\tau}\int_{\mathbb{T}^d}\rho_\epsilon|\nabla v_\epsilon|^\alpha \le C(\tau,\alpha)$$

This lemma means that the spatial regularity property (237) is conserved through the regularization. Before proving this lemma, we conclude the proof of lemma 6.12 : we have

$$\begin{aligned} \frac{d}{dt}\int \rho_\epsilon F(\rho_\epsilon) &= -\int \rho_\epsilon^2 F'(\rho_\epsilon)\nabla\cdot v_\epsilon \\ \frac{d}{dt}\int \rho_\epsilon^\alpha &= (1-\alpha)\int \rho_\epsilon^\alpha \nabla\cdot v_\epsilon. \end{aligned}$$

Then using lemma 6.13 with $\alpha = 2$ and the fact that $\rho_\epsilon \in L^2_{loc}(]0,T[\times\mathbb{T}^d)$ we get that

$$\int_\tau^{T-\tau}\rho_\epsilon^{3/2}|\nabla\cdot v_\epsilon| \le C(\tau)$$

and also that for any $\alpha \in [0,3/2[$, the sequence $\rho_\epsilon^\alpha|\nabla\cdot v_\epsilon|$ is equiintegrable on $[\tau, T-\tau]\times\mathbb{T}^d$. Thus the sequence of functions of time $t \to \int_{\mathbb{T}^d}\rho^{3/2-\epsilon}(t,x)dx$ is equicontinuous and the limit function is thus continuous. ☐

Proof of lemma 6.13 :
We have

$$\begin{aligned} \nabla\cdot v_\epsilon = &-\frac{\nabla\rho_\epsilon}{\rho_\epsilon^2}\cdot(\rho v)\star\eta_\epsilon + \frac{1}{\rho_\epsilon}(\rho v)\star\nabla\eta_\epsilon \\ = &-\frac{\nabla\rho_\epsilon}{\rho_\epsilon^2}\cdot\int_{\mathbb{R}^d}\rho(x-y)(v(x-y)-v(x))\eta_\epsilon(y) - \frac{\nabla\rho_\epsilon}{\rho_\epsilon}\cdot v \\ + &\frac{\nabla\rho_\epsilon}{\rho_\epsilon}\cdot v + \frac{1}{\rho_\epsilon}\int_{\mathbb{R}^d}\rho(x-y)(v(x-y)-v(x))\cdot\nabla\eta_\epsilon(y) \end{aligned}$$

and we use the special shape of the regularization kernel : there exists $C>0$ such that for all $x\in\mathbb{R}^d$, $|\nabla\eta(x)| \le C\eta(x)$ and thus we have :

$$\left|\frac{\nabla\rho_\epsilon}{\rho_\epsilon}\right| \le \frac{C}{\epsilon}$$

and we also have $\int |y||\nabla\eta_\epsilon(y)|dy \le C, \int |y|\eta_\epsilon(y)dy \le C\epsilon$. We define

$$A(x) = \frac{\nabla\rho_\epsilon}{\rho_\epsilon^2}\int_{\mathbb{R}^d}\rho(x-y)(v(x-y)-v(x))\eta_\epsilon(y)dy$$

Then for $\alpha \le 2$

$$A^\alpha(x) \le C\left|\int_{\mathbb{R}^d}\frac{\rho(x-y)\eta_\epsilon(y)}{\rho_\epsilon(x)}\frac{|v(x-y)-v(x)|}{\epsilon}dy\right|^\alpha$$

and this by Jensen's inequality is less than

$$C\int_{\mathbb{R}^d}\frac{\rho(x-y)\eta_\epsilon(y)}{\rho_\epsilon(x)}\frac{|v(x-y)-v(x)|^\alpha}{\epsilon^\alpha}dy$$

thus we obtain :

$$\int_{\mathbb{T}^d}\rho_\epsilon(x)A^\alpha(x)dx \le \int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\rho(x)\eta_\epsilon(y)\frac{|v(x+y)-v(x)|^\alpha}{\epsilon^\alpha}dydx$$

Then for the next term

$$B(x) = \frac{1}{\rho_\epsilon}\int\rho(x-y)(v(x-y)-v(x))\nabla\eta_\epsilon(y)\ dy$$

we proceed by the same method. Thus we are able to obtain

$$\begin{aligned} &\int_\tau^{T-\tau}\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\rho_\epsilon|\nabla v_\epsilon|^\alpha(x)\ dxdydt \\ \le\ & C\int_\tau^{T-\tau}\int_{\mathbb{T}^d}\int_{\mathbb{R}^d}\rho(t,x)\frac{1}{|y|^\alpha}|v(t,x+y)-v(t,x)|^\alpha\eta_\epsilon(y)\ dxdydt \le C(\tau)\end{aligned}$$

using the spatial regularity property (237). The proof of lemma 6.13 is thus complete. □

5.3. Consistency with smooth solutions of Euler-Poisson system. Here we show that the solution of the minimization problem coincides with a smooth solution of the Euler-Poisson system.

THEOREM 6.14. *Let $(\nabla\phi, \rho, p)$ be the solution of Problem 6.2, let $(\nabla\psi, r, q)$ be such that $\psi \in W^{1,\infty}([0,T]\times\mathbb{T}^d)$, $r \in L^2([0,T], H^{-1}(\mathbb{T}^d))\cap L^\infty([0,T], L^1(\mathbb{T}^d))$ and $q \in L^\infty([0,T]\times\mathbb{T}^d)$. Suppose that $(\nabla\psi, r, q)$ is solution to*

(242) $$\partial_t\psi + \frac{1}{2}|\nabla\psi|^2 + q \le 0$$

(243) $$r(\partial_t\psi + \frac{1}{2}|\nabla\psi|^2 + q) = 0$$

(244) $$\partial_t r + \nabla\cdot(r\nabla\psi) = 0 \text{ in } \mathcal{D}'$$

(245) $$r|_{t=0} = \rho_0,\ r|_{t=1} = \rho_1$$

(246) $$\Delta q = r - 1 \text{ in } \mathcal{D}'$$

Then $\rho = r$ and $\nabla\phi = \nabla\psi\ d\rho$ a.e.

Proof : Using the fact that $(\rho, \nabla\phi)$ satisfies the continuity equation (220) we get that

$$\int_D\rho\nabla\phi\cdot\nabla\psi + \int_D\rho\partial_t\psi = \int_{\mathbb{T}^d}\rho_T\psi(T) - \int_{\mathbb{T}^d}\rho_0\psi(0)$$

combining with (242) we get

$$\int_D \rho\nabla\phi\cdot\nabla\psi - \int_D \rho(\frac{1}{2}|\nabla\psi|^2+q) \geq \int_{\mathbb{T}^d}\rho_T\psi(T) - \int_{\mathbb{T}^d}\rho_0\psi(0)$$

thus using (222)

$$\begin{aligned} &\frac{1}{2}\int_D \rho(-|\nabla\phi-\nabla\psi|^2+|\nabla\phi|^2) + \int_D \nabla p\cdot\nabla q - q \\ \geq\quad & \int_{\mathbb{T}^d}\rho_T\psi(T) - \int_{\mathbb{T}^d}\rho_0\psi(0) \end{aligned}$$

and using (244, 245)

$$\begin{aligned} &\frac{1}{2}\int_D \rho(-|\nabla\phi-\nabla\psi|^2+|\nabla\phi|^2) + \int_D \nabla p\cdot\nabla q - q \\ \geq\quad & \int_D r\nabla\psi\cdot\nabla\psi + \int_D r\partial_t\psi \\ \geq\quad & -\int_D qr + \frac{1}{2}\int_D r|\nabla\psi|^2 \\ \geq\quad & \int_D |\nabla q|^2 - r + \frac{1}{2}\int_D r|\nabla\psi|^2 \end{aligned}$$

where we have used (243) in the second line and (246) in the third line, and finally we get

$$\begin{aligned} &\int_D \rho|\nabla\phi-\nabla\psi|^2 + \int_D |\nabla p - \nabla q|^2 \\ \leq\quad & \int_D \rho|\nabla\phi|^2 + |\nabla p|^2 - \int_D r|\nabla\psi|^2 + |\nabla q|^2. \end{aligned}$$

Since $\rho, \nabla\phi, p$ is solution of the minimization problem, the RHS is non positive, and thus we obtain the desired result.
Remark 1 : This is true in particular if ψ, q is a $C^1(D)\times C^0(D)$ solution of $\partial_t\psi + \frac{1}{2}|\nabla\psi|^2 + q = 0$ and thus shows the consistency with smooth solutions of the $(E-P)$ system.
Remark 2 : From the results of Theorem 6.5 depending on ρ_0 and ρ_T the assumptions on ψ, r, q can be weakened. For instance if ρ_0 and ρ_T are in L^∞ then $\rho, \nabla\phi, \nabla p$ are in $L^\infty([0,T]\times\mathbb{T}^d)$ and one only needs $(1+r)|\nabla\psi|^2, (1+r)|\partial_t\psi|, |\nabla q|^2$ to be integrable to perform our computation. (Note that these assumptions imply that $\psi\in C([0,T],L^1(\mathbb{T}^d))$ and thus $\int_{\mathbb{T}^d}\rho_T\psi(T) - \int_{\mathbb{T}^d}\rho_0\psi(0)$ is well defined).
This ends the proof of the Theorem 6.14. □

6. Proof of Theorem 6.5

In this section we prove several additional regularity properties for the variational solution. The problems that we will treat are closely related to viscosity solutions of Hamilton Jacobi equation. In the remainder we will denote by HJ1, HJ2 the following operators :

$$\text{HJ1 }(\phi) = \partial_t\phi + \frac{1}{2}|\nabla\phi|^2,$$

$$\text{HJ2 }(\phi,p) = \partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + p,$$

and we will precise in what sense they must be understood. See [**32**] for references about Hamilton Jacobi equations.

6.1. Formal bounds. Solutions of our variational problem satisfy

$$\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + p = 0 \ d\rho \ a.e. \tag{247}$$

$$\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + p \leq 0 \tag{248}$$

$$\partial_t\rho + \nabla(\rho\nabla\phi) = 0 \tag{249}$$

$$\rho = 1 + \Delta p \tag{250}$$

If some $C^2(\mathbb{T}^d)$ function Z satisfies $Z(x_0) = 0$ and $Z \leq 0$ in $\mathbb{T}^d$ then $D^2(Z)(x_0) \leq 0$ in the sense of matrices and in particular this implies that $\Delta Z(x_0) \leq 0$. Using (247,248) and applying this result to $Z = \partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + p$ one formally obtains

$$\partial_t\Delta\phi + (\nabla\phi\cdot\nabla)\Delta\phi + \sum_{ij}|\partial_{ij}\phi|^2 + \Delta p \leq 0 \ d\rho \ a.e.$$

which combined with the inequality $|\Delta\phi|^2 \leq d\sum_{i,j}|\partial_{ij}\phi|^2$ and with (250) gives the inequality

$$\frac{d}{dt}\Delta\phi \leq 1 - \rho - \frac{1}{d}|\Delta\phi|^2 \tag{251}$$

where $\frac{d}{dt} = \partial_t + \sum_{i=1}^d \partial_i\phi\,\partial_i$ denotes the convective derivative along the flow generated by the velocity field $\nabla\phi(t,x)$. We first deduce of this an interior upper bound for $\Delta\phi \ d\rho$ a.e. :

$$\Delta\phi \leq C(d)(1 + \frac{1}{t})$$

obtained by looking at the behavior of the differential inequality

$$\dot{f} \leq 1 - C(d)f^2$$

for large f. This bound is well known for viscosity solutions of HJ2=0 provided that p (as is the case here) satisfies $\Delta p \geq C$. However we don't know a priori that our solution is a viscosity solution, and moreover this bound is true in the sense of distributions. Here a complication is added by the fact that the solution ϕ, p satisfies $\mathrm{HJ2}(\phi,p) = 0$ only $d\rho$ a.e.

Now notice that our variational solution exists on $t \in [0,T]$ and thus $(\psi,q)(t) = (-\phi,p)(T-t)$ is also a solution to equations (247) to (250). Therefore we can obtain by the same way that

$$\Delta\psi(t) \leq C(d)(1 + \frac{1}{t}), \tag{252}$$

$$\Delta\phi(T-t) \geq -C(d)(1 + \frac{1}{t}). \tag{253}$$

This gives the following uniform bound

$$\|\Delta\phi\|_{L^\infty([\tau,1-\tau]\times\mathbb{T}^d,\ d\rho)} \leq C(d)(1 + \frac{1}{\tau}).$$

Here the constant is universal, and it is only supposed that the solution exists from $t = 0$ to $t = T$. This surprising (in the sense that it is not true for viscosity solutions) result comes from the fact that the transformation $(\phi,p) \to (-\phi,p)(T-t)$ does not necessarily transform a viscosity solution of (247) into another viscosity solution but transforms a variational solution into another variational

solution. Actually it will be proved that one can choose the variational solution to be a viscosity solution in one time direction, but it may only be a subsolution when reversing the time.
Now $\Delta\phi$ is the divergence of the velocity field, and thus we have $\Delta\phi = -\frac{d}{dt}\log\rho$ and from (251) we have the following control on the second time derivative of ρ along the flow :

$$\frac{d^2}{dt^2}\log\rho \geq \rho + \frac{1}{d}|\frac{d}{dt}\log\rho|^2 - 1 \tag{254}$$

Following the path of a "particle", the ordinary differential inequation satisfied by $\Theta = \log\rho$ is :

$$\ddot{\Theta} \geq \exp\Theta - 1 + \frac{1}{d}|\dot{\Theta}|^2$$

Now we look for solutions of this equation that do not become infinite before $t = T$. One will check that this condition implies that $\Theta(t) \leq C(\tau)$ for $\tau \leq t \leq T-\tau$, independently of the initial and final values of Θ. Thus we have an interior unconditional bound for the L^∞ norm of ρ, namely that

$$\|\rho\|_{L^\infty([\tau,T-\tau]\times\mathbb{T}^d)} \leq C(\tau).$$

The above differential inequality will also yield that some functionals of ρ are convex along the displacement induced by our variational problem : indeed a formal computation gives the following :

$$\partial_{tt}\int_{\mathbb{T}^d}\rho\log\rho \geq 0$$
$$\partial_{tt}\int_{\mathbb{T}^d}|\rho|^k \geq 0 \text{ for every } k \geq 1$$

Remark : This displacement convexity property is analog to the one found in [**48**] which was true for $k \geq 1 - 1/d$, however, our the analogy stops here since our displacement does not induce a distance : indeed take $\rho_0 = \rho_T \not\equiv 1$ and check that the cost of the transportation of ρ_0 on ρ_T following the Euler-Poisson flow is not 0.

In the next section we give a rigorous sense to the computations made above in order to obtain the Theorem 6.5.

6.2. Rigorous proof of Theorem 6.5. We prove the theorem using a time discretization of the problem.

6.2.1. *Construction of a sequence of approximate solutions.* We introduce the following times $t_i = Ti/N,\ i = 1..N-1$, and we consider the functional

$$I_N(\rho, v) = \frac{1}{2}\int_D \rho(t,x)|v(t,x)|^2\ dtdx + \frac{T}{2N}\sum_{1=1}^{N-1}\int_{\mathbb{T}^d}|\nabla p(t_i,x)|^2\ dx$$

We are now interested in solving the following variational problem :

PROBLEM 6.15. *Minimize*

$$\tilde{I}_N(\rho, J, p) = \frac{T}{2N}\sum_{1=1}^{N-1}\int_{\mathbb{T}^d} |\nabla p(t_i, x)|^2 \, dx$$
$$+ \sup_{\substack{c,m\in C^0(\mathbb{T}^d)\times(C^0(\mathbb{T}^d))^d \\ c+|m|^2/2\leq 0}} \{\frac{1}{2}\int_D c(t,x)\rho(t,x) + m(t,x)\cdot J(t,x)\, dtdx\}$$

among all ρ, J, p *that satisfy* $\rho \in C([0,T], \mathcal{P}(\mathbb{T}^d) - w*)$, $J \in (\mathcal{M}(D)^d)$, $\nabla p(t_i) \in L^2(D)$ *for all* $1 \leq i \leq N-1$ *and*

$$\partial_t\rho + \nabla\cdot J = 0$$
$$\Delta p = \rho - 1$$
$$\rho(t=0) = \rho_0$$
$$\rho(t=T) = \rho_T.$$

We denote K_N *the value of this infimum.*

The interest of studying this problem is both to make rigorous the arguments of the previous section and to give a possible numerical discretization of the Problem 6.1. It will also let appear some interesting links between optimal transportation, viscosity solutions of Hamilton-Jacobi equations, and transport equations.

6.2.2. *Basic facts on optimal transportation.* We first recall the definition of the pushforward of a measure by a mapping :

DEFINITION 6.16. *Let* ρ_0 *and* ρ_1 *be two probability measures on* $\mathbb{T}^d$ *and let* X *be a* $d\rho_0$ *measurable mapping from* $\mathbb{T}^d$ *into itself. We say that* ρ_1 *is the pushforward of* ρ_0 *by* X *and we denote it by* $\rho_1 = X\#\rho_0$ *if we following holds :*

$$\forall f \in C^0(\mathbb{T}^d), \int f(X(x))d\rho_0(x) = \int f(y)d\rho_1(y).$$

The effect of the time discretization is that between two t_i having chosen $\rho(t_i)$ and $\rho(t_{i+1})$ the problem becomes the following :

PROBLEM 6.17. *Minimize*

$$\tilde{C}(\rho, J) = \sup_{\substack{c,m\in C^0(\mathbb{T}^d)\times(C^0(\mathbb{T}^d))^d \\ c+|m|^2/2\leq 0}} \{\frac{1}{2}\int_D \rho c + J\cdot m\, dtdx\}$$

among all ρ, J *that satisfy* $\rho \in C([t_i, t_{i+1}], \mathcal{P}(\mathbb{T}^d) - w*)$, $J \in (\mathcal{M}(D))^d$ *and*

$$\partial_t\rho + \nabla\cdot J = 0$$
$$\rho(t=t_i) = \rho_i$$
$$\rho(t=t_{i+1}) = \rho_{i+1}.$$

The infimum is denoted $\underline{C}(\rho_i, \rho_{i+1}, |t_i - t_{i+1}|)$.

Remark : performing a dilatation in the time variable we see that
$C(\rho_i, \rho_{i+1}, t) = \frac{1}{t} C(\rho_i, \rho_{i+1}, 1)$.
Then the Wasserstein distance between ρ_i and ρ_{i+1} is given by

$$(W_2)^2(\rho_i, \rho_{i+1}) = \underline{C}(\rho_i, \rho_{i+1}, 1)$$

This problem has been solved in [**7**], [**10**] where it is shown that there exists a unique solution $\rho, J = \rho v$ (v is only unique $d\rho$ a.e.) that satisfies :

$$\begin{aligned}
&v(t = t_i, x) = \frac{1}{t_{i+1} - t_i} (\nabla \varphi(x) - x) \ d\rho_i \ a.e. \\
&\partial_t (\rho v) + \nabla \cdot (\rho v \otimes v) = 0 \\
&\det D^2 \varphi(x) \rho_{i+1}(\nabla \varphi(x)) = \rho_i(x)
\end{aligned}$$

with φ a convex function. The second equation means that the particle move with constant speed. The third equation is the Monge-Ampère equation that is verified in the following weak sense :

$$\forall f \in C^0(\mathbb{T}^d), \int f(\nabla \varphi(x)) d\rho_i(x) = \int f(y) d\rho_{i+1}(y).$$

This means that $\nabla \varphi$ pushes forward ρ_i on ρ_{i+1}. The Wasserstein distance between two probability measures can be defined equivalently in the following ways :

$$\begin{aligned}
W_2^2(\rho_0, \rho_1) &= \inf_{\rho, v} \int_{[0,1] \times \mathbb{T}^d} d\rho |v|^2/2 \\
&= \sup_{\phi(x) + \psi(y) \geq x \cdot y} \int_{\mathbb{T}^d} d\rho_1(x)(|x|^2/2 - \phi(x)) + d\rho_2(y)(|y|^2/2 - \psi(y)) \\
&= \inf_{\mathbf{m} \# d\rho_0 = d\rho_1} \int_{\mathbb{T}^d} \frac{1}{2} |x - \mathbf{m}(x)|^2 d\rho_0
\end{aligned}$$

where the first infimum is taken over all the pair (ρ, v) satisfying
$\partial_t \rho + \nabla \cdot [\rho v] = 0$ and $\rho|_{t=0} = \rho_0, \rho|_{t=1} = \rho_1$.

Those three problems have a unique solution. For the first it has been already mentioned above. Then the optimal $\mathbf{m}$ is equal to $\nabla \varphi$, the optimal pair (ϕ, ψ) is equal (up to a constant) to (φ, φ^*) with φ^* the Legendre transform of φ (see the definition (259) below). For additional references about the Wasserstein distance the reader can refer to [**48**], [**54**], or [**62**] .

6.2.3. *Existence of a minimizer for the approximate problem.* Following exactly the same method as in the first problem we can show the existence of a unique minimizer to the Problem 6.15. In this way we obtain the following :

PROPOSITION 6.18. *There exists a unique ρ_N and a $d\rho_N$ a.e. unique $v_N = \nabla \phi_N$ solution of Problem 6.15. Moreover it satisfies :*

1- There exists C such that for any $0 < \tau < T/2$,
$\frac{T}{N} \sum_{\tau \leq t_i \leq T - \tau} \|\rho(t_i, .)\|_{L^2(\mathbb{T}^d)} \leq \frac{C}{\tau}$.

2- There exists C such that for any $0 < \eta < \tau/2$,
$\frac{T}{N} \sum_{\tau \leq t_i \leq T - \tau} \|\nabla p(t_i, .) - \nabla p(t_i + \eta, .)\|^2_{L^2(\mathbb{T}^d)} \leq \frac{C}{\tau} \eta^2$.

4- The solution $\rho_N, v_N = \nabla\phi_N$ is a weak solution of

$$\partial_t(\rho_N v_N) + \nabla\cdot(\rho_N v_N \otimes v_N) = -\rho_N \frac{T}{N}\sum_{i=1}^{N-1} \delta_{t=t_i}\nabla p(t_i)$$
$$\partial_t \rho_N + \nabla\cdot(\rho_N v_N) = 0$$
$$\Delta p_N = \rho_N - 1$$

the pair ϕ_N, p_N satisfies $\partial_t\phi_N + \frac{1}{2}|\nabla\phi_N|^2 + +\frac{T}{N}\sum_{i=1}^{N-1}\delta_{t=t_i} p \leq 0$ and also satisfies for any $t_i, 1 \leq i \leq N-1$

$$\phi(t_i^+, x) - \phi(t_i^-, x) \leq -\frac{T}{N} p(t_i, x)\ dx\ a.e.$$
$$\phi(t_i^+, x) - \phi(t_i^-, x) = -\frac{T}{N} p(t_i, x)\ d\rho(t_i)\ a.e.$$
$$\nabla\phi(t_i^+, x) - \nabla\phi(t_i^-, x) = -\frac{T}{N}\nabla p(t_i, x)\ d\rho(t_i)\ a.e.$$

4- The energy of the solution defined by

$$E_N(t_i) = \frac{1}{2}\int_{\mathbb{T}^d} [\rho_N|v_N|^2 - |\nabla p_N|^2](t_i)$$

is independent of i.

Proof : The proof is the same as the time continuous version therefore we will briefly sketch it. Concerning the existence of an admissible solutions for Problem 6.15 note that we don't need that ρ_0 neither ρ_T are in any L^p since two probability measures on $\mathbb{T}^d$ are always at finite Wasserstein distance and thus one can exhibit an admissible solution by transporting ρ_0 on $\rho(T/N) = 1$ between $t = 0$ and $t = T/N$, then letting $\rho(t_i) = 1$ for $i \leq N-1$ and transporting $\rho(\frac{N-1}{N}T)$ on ρ_T.
Then having chosen an admissible solution $\bar\rho, \bar v = \nabla\bar\phi, \bar p$, to the primal problem of finding the infimum of I_N we associate the following dual problem : find the supremum over all pairs (ψ, q) such that $\partial_t\psi + \frac{1}{2}|\nabla\psi|^2 + \frac{T}{N}\sum_{i=1}^{N-1}\delta_{t=t_i} q(t_i) \leq 0$ of

$$\begin{aligned} D_N(\psi, q) &= \int_{[0,T]\times\mathbb{T}^d} \bar\rho\partial_t\psi + \bar\rho\bar v\cdot\nabla\psi\ dtdx \\ &+ \frac{T}{N}\sum_{i=1}^{N-1}\int_{\mathbb{T}^d} -|\nabla q(t_i)|^2/2 + \bar\rho(t_i)q(t_i) + \nabla\bar p(t_i)\cdot\nabla q(t_i)\ dx \end{aligned}$$

From the Rockafellar duality Theorem we have $\inf I_N = \sup D_N$ and the infimum is attained. Then taking $(\bar\rho, \bar\phi, \bar p) = (\rho_N, \phi_N, p_N)$ the optimal solution, for any $\epsilon > 0$ we find $\psi_\epsilon, q_\epsilon$ such that $D_N(\psi_\epsilon, q_\epsilon) \geq K_N - \epsilon^2$ and we obtain :

$$\begin{aligned} &\frac{1}{2}\int_D \rho_N|\nabla\phi_N - \nabla\psi_\epsilon|^2 + \frac{1}{2}\frac{T}{N}\sum_{i=1}^{N-1}\int_{\mathbb{T}^d}|\nabla p_N(t_i) - \nabla q_\epsilon(t_i)|^2 \\ +\ &\int_D \rho_N\left|\partial_t\psi_\epsilon + \frac{1}{2}|\nabla\psi_\epsilon|^2 + \frac{T}{N}\sum_{i=1}^{N-1}\delta_{t=t_i} q_\epsilon(t_i)\right| \leq \epsilon^2. \end{aligned} \tag{255}$$

Then perturbing ρ_N, v_N as we did in the time continuous problem, we find the optimality equation and the regularity properties.

6.2.4. *Regularity properties of solutions of Problem 6.15.* Here we state the main result of this section, from which will be deduced Theorem 6.5.

PROPOSITION 6.19. *Let $(t,x) \to (\phi_N(t,x), \rho_N(t,x))$ be solution of Problem 6.15.*

1- There exists C depending only on T and on the dimension such that for all t in $[0,T]$, $\phi_N(t)$ is $d\rho_N(t)$ a.e. twice differentiable and satisfies

$$-C(1+\frac{1}{T-t}) \leq \Delta\phi_N(t,.) \leq C(1+\frac{1}{t})\ d\rho(t)\ a.e.$$

2- The density ρ_N is bounded in $L^\infty_{loc}(]0,T[, L^\infty(\mathbb{T}^d))$ uniformly with respect to N and belongs to $\bigcap_{k>1} C(]0,T[, L^k(\mathbb{T}^d))$.

3- There exists C such that for any $1 \leq k \leq \infty$

$$-C(1+\frac{1}{t}) \leq \frac{d}{dt}\|\rho_N(t,\cdot)\|_{L^k(\mathbb{T}^d)} \leq C(1+\frac{1}{T-t})$$

4- The functions $\int_{\mathbb{T}^d}[\rho_N(t,x)]^k\ dx$, $\int_{\mathbb{T}^d}\rho_N \log\rho_N(t,x)\ dx$ are uniformly Lipschitz with respect to time in every interval $[\tau, T-\tau]$, with $0<\tau<T/2$ and converge as $N\to\infty$ to convex functions on $[0,T]$.

5- The velocity $\nabla\phi_N$ can be chosen in $L^\infty_{loc}(]0,T[\times\mathbb{T}^d)$.

6- ϕ_N can be chosen to be the viscosity solution of $\partial_t\phi_N + \frac{1}{2}|\nabla\phi_N|^2 + \sum_{i=1}^{N-1}\delta_{t=t_i}p = 0$ in the sense of (270).

6- All these results and bounds do not depend on ρ_0 neither on ρ_T.

The proof of this proposition is postponed to the end of the paper. First we use it to show the convergence of solutions of Problem 6.15 toward the solution of Problem 6.1 :

6.2.5. *Convergence of the solutions of Problem 6.15 to the solution of Problem 6.2.*

PROPOSITION 6.20. *Let ρ_N, v_N be as before, and ρ, v, p be solution of the minimization Problem 6.1 with same initial and final densities in $L^{\frac{2d}{d+2}}$, then*

$$\lim_{N\to\infty}\frac{1}{2}\int_{t=0}^{T}\int_{\mathbb{T}^d}\left(\rho|v_N - v|^2 + |\nabla p_N - \nabla p|^2\right)(t,x)\ dxdt\ = 0,$$

and $\rho_N\nabla\phi_N$ converges strongly in $L^1([0,T]\times\mathbb{T}^d)$ to $\rho\nabla\phi$.

Then the last two proposition combined will yield the Theorem 6.5 when passing to the limit.

Proof of Proposition 6.20

Here we prove a slightly weaker version of proposition 6.20 that allows us to pass to the limit in

proposition 6.19 and obtain Theorem 6.5. Then we can get the exact proposition 6.20.
We first choose $\tau \in [0, T/2[$ and k such that $0 < t_{k-1} \le \tau \le t_k < .. < t_{N-k} \le T - \tau$. We set

$$F^\tau(\rho, v) = \frac{1}{2}\int_0^T \int_{\mathbb{T}^d} \rho|v|^2 \; dtdx + \frac{1}{2}\int_\tau^{T-\tau} \int_{\mathbb{T}^d} |\nabla p|^2 \; dxdt$$

$$F_N^\tau(\rho, v) = \frac{1}{2}\int_0^T \int_{\mathbb{T}^d} \rho|v|^2 \; dtdx + \frac{T}{2N}\sum_{i=k}^{N-k} \int_{\mathbb{T}^d} |\nabla p(t_i, x)|^2 dx$$

(understood that p satisfies $\Delta p = \rho - 1$). We need to introduce these truncated functionals since we don't know a priori that the potential energy remains bounded near the boundary of the time interval and thus the convergence of the Riemann sum to the integral is not clear. We shall see after having proved the Theorem 6.5 that this is the case when ρ_0 and ρ_T are in $L^{\frac{2d}{d+2}}$.
It follows from proposition 6.18 that there exists a unique minimizer that we will denote $\rho_N, v_N = \nabla\phi_N, p_N$ for the functional F_N^τ under the constraints of Problem 6.15. It can also be checked later in the proof that the regularity results of proposition 6.19 remain uniformly valid for $\tau \le \tau_0$.
We consider ρ, v solution of Problem 6.2. From lemma 6.11, $\|\nabla p(t, \cdot)\|^2_{L^2(\mathbb{T}^d)}$ is continuous in $]0, T[$ thus

$$\forall \tau > 0, \; F_N^\tau(\rho, v) \to F^\tau(\rho, v) \text{ as } N \to \infty.$$

Then $F^\tau(\rho, v) - F(\rho, v) \to 0$ when τ goes to 0, and there exists a sequence τ_p, N_p such that

$$F_{N_p}^{\tau_p}(\rho, v) \to F(\rho, v).$$

In the remainder of this proof set $F_{N_p}^{\tau_p} = F_N$ (therefore τ, k will both depend on N) and ρ_N, v_N the associated minimizer. Thus

$$F_N(\rho, v) \to F(\rho, v).$$

Then we have $F_N(\rho_N, v_N) \le F_N(\rho, v)$ and thus $\liminf F_N(\rho_N, v_N) \le F(\rho, v)$.
Now we claim that $\lim |F_N(\rho_N, v_N) - F(\rho_N, v_N)| = 0$ and this will imply that $\lim F_N(\rho_N, v_N) = F(\rho, v)$. From the second point of proposition 6.18 we get that for any fixed τ',

$$\left| \int_{\tau'}^{T-\tau'} \int_{\mathbb{T}^d} |\nabla p_N(t, x)|^2 \; dtdx - \frac{T}{N} \sum_{\tau' < t_i < T-\tau'} \int_{\mathbb{T}^d} |\nabla p_N(t_i, x)|^2 \; dx \right| \le \frac{C(\tau')}{N}$$

Since ρ_0 and ρ_T are in $L^{\frac{2d}{d+2}}$, point 4 in proposition 6.19 implies that ρ_N is in $L^\infty([0, T], L^{\frac{2d}{d+2}})$. It follows that the sequence ∇p_N is uniformly bounded in $L^\infty([0, T], L^2)$ since $\Delta p_N = \rho_N - 1$ and from Sobolev embedding. Thus

$$\lim_{N \to \infty} \int_0^T \int_{\mathbb{T}^d} |\nabla p_N(t, x)|^2 dtdx - \frac{T}{N}\sum_{1=k}^{N-k} \int_{\mathbb{T}^d} |\nabla p_N(t_i, x)|^2 \; dx = 0. \tag{256}$$

We can conclude that $\lim F_N(\rho_N, v_N) = \lim F_N(\rho, v) = F(\rho, v)$ as $N \to \infty$.
We show now that this implies that ρ_N, v_N and ρ, v are close to each other. Remember that we have shown that $F_N(\rho, v) \to F(\rho, v)$. Using the dual formulation of the Problem 6.15 as in the proof of

proposition 6.18 we have

$$\begin{aligned}
&\int_{[0,T]\times\mathbb{T}^d} \frac{1}{2}\rho_N|v_N|^2 + \frac{T}{2N}\sum_{i=k}^{N-k}\int_{\mathbb{T}^d}|\nabla p_N(t_i)|^2 \\
= \quad & \sup_{\psi,q}\{\int_{[0,T]\times\mathbb{T}^d} \rho\partial_t\psi + \rho v\cdot\nabla\psi \\
+ \quad & \frac{T}{N}\sum_{i=k}^{N-k}\int_{\mathbb{T}^d} -|\nabla q(t_i)|^2/2 + \rho(t_i)q(t_i) + \nabla p(t_i)\cdot\nabla q(t_i)\}
\end{aligned}$$

the supremum being taken over all pairs $(\psi, p) \in C^1(D)$ such that

$$\partial_t\psi + \frac{1}{2}|\nabla\psi|^2 + \frac{T}{N}\sum_{i=k}^{N-k}\delta_{t=t_i}q \leq 0.$$

Note that in the RHS we can use any admissible solution (i.e satisfying (225,226,227,228)), and that we have taken ρ, v, p the solution of Problem 6.2 which is admissible for Problem 6.15, and not ρ_N, v_N, p_N. Using the fact that $F_N(\rho, v)$ is close to $F_N(\rho_N, v_N)$ for N large, for any $\epsilon, \delta > 0$ there exists $N, \psi_N^\epsilon, q_N^\epsilon$ (with $N = N(\delta)$) such that

$$\begin{aligned}
&\frac{1}{2}\int_D \rho|v|^2 + \frac{T}{2N}\sum_{i=k}^{N-k}\int_{\mathbb{T}^d}|\nabla p(t_i)|^2 \\
\leq \quad & \frac{1}{2}\int_D \rho_N|v_N|^2 + \frac{T}{2N}\sum_{i=k}^{N-k}\int_{\mathbb{T}^d}|\nabla p_N(t_i)|^2 + \delta \\
\leq \quad & \epsilon + \delta + \int_D \rho\partial_t\psi_N^\epsilon + \rho v\cdot\nabla\psi_N^\epsilon \\
+ \quad & \frac{T}{N}\sum_{i=k}^{N-k}\int_{\mathbb{T}^d} -|\nabla q_N^\epsilon(t_i)|^2/2 + \rho(t_i)q_N^\epsilon(t_i) + \nabla p(t_i)\cdot\nabla q_N^\epsilon(t_i)
\end{aligned}$$

This eventually yields

$$\int_D \frac{1}{2}\rho|v - \nabla\psi_N^\epsilon|^2 + \frac{T}{2N}\sum_{i=k}^{N-k}\int_{\mathbb{T}^d}\frac{1}{2}|\nabla p(t_i) - \nabla q_N^\epsilon(t_i)|^2 \leq \epsilon + \delta.$$

At N fixed, being a maximizing sequence for the dual problem, $\nabla\psi_N^\epsilon, q_N^\epsilon$ will converge to v_N, p_N as $\epsilon \to 0$ (see (255)), therefore we obtain

$$\frac{1}{2}\int_D \rho|v - v_N|^2 + \frac{T}{2N}\sum_{i=k}^{N-k}\int_{\mathbb{T}^d}|\nabla p(t_i) - \nabla p_N(t_i)|^2/2 \leq \delta$$

and combining with (256) we obtain

$$\frac{1}{2}\int_D \rho|v - v_N|^2 + |\nabla p_N - \nabla p|^2 dtdx \to 0 \text{ as } N \to \infty.$$

Using the same procedure we can also get that

$$\frac{1}{2}\int_D \rho_N|v-v_N|^2+|\nabla p_N-\nabla p|^2 dtdx \to 0 \text{ as } N\to\infty.$$

Now we show that $\rho_N v_N$ converges to ρv : using the equicontinuity property of the sequence $\|\rho_N(t,.)\|_{L^k}$ in $[\tau, T-\tau]$ for any $\tau \le T/2$ and any $1\le k<\infty$, (see proposition 6.19), the sequence ρ_N converges strongly in $L^k([\tau,T-\tau]\times\mathbb{T}^d)$ for any $1\le k<\infty$. Moreover remember that from Theorem 6.4 $v\in L^2([\tau,T-\tau]\times\mathbb{T}^d)$ Then

$$\begin{aligned}&\int_{[\tau,T-\tau]\times\mathbb{T}^d}|\rho v-\rho_N v_N|\\ \le\ &\int_{[\tau,T-\tau]\times\mathbb{T}^d}\rho_N|v_N-v|+v|\rho_N-\rho|\to 0\end{aligned}$$

and $\rho_N v_N$ converges strongly to ρv in $L^p_{loc}(]0,T[\times\mathbb{T}^d)$ for any $1\le p<\infty$. This implies that one can pass to the limit in equation (221).

Proof of Theorem 6.5 : The theorem is obtained passing to the limit in the proposition 6.19. The point 2,3,4,5 remain true when letting N go ∞.
Concerning the fact that we can choose ϕ viscosity solution of HJ2$(\phi,p)=0$ this will be proved at the end of the paper. □
Then from Theorem 6.5 if $\rho_0,\rho_T\in L^{\frac{2d}{d+2}}(\mathbb{T}^d)$ we have $\rho\in L^\infty([0,T],L^{\frac{2d}{d+2}}(\mathbb{T}^d))$ and using $\Delta p=\rho-1$ and Sobolev embeddings $\nabla p\in L^\infty([0,T],L^2(\mathbb{T}^d))$. This bound shows that the Riemann sum $\frac{T}{N}\sum_{i=1}^{N-1}\int_{\mathbb{T}^d}|\nabla p(t_i,x)|^2dx$ converges to $\int_D|\nabla p|^2\ dtdx$ and this allows us to take $\tau=0$ in the previous proof and to conclude the proof of proposition 6.20. □.

6.3. Proof of Proposition 6.19. In this part N is fixed and for sake of simplicity we drop the suffix N. We consider $\rho=\Delta p+1$, $\phi, v=\nabla\phi$ solution of the Problem (6.15).

6.3.1. *Preliminary : Construction of a special solution.* First we begin to show the consistency with Problem 6.17 : let $t_i\le s,t\le t_{i+1}$. We denote

$$\mathbf{\Phi}_{s,t}(x)=(t-s)\phi(s,x)+|x|^2/2. \tag{257}$$

$\mathbf{\Phi}_{s,t}$ is thus a function going from $\mathbb{R}^d$ to $\mathbb{R}$ if we extend ϕ to a periodic function on all of $\mathbb{R}^d$. Note also that for any $\vec{p}\in\mathbb{Z}^d$, $\nabla\mathbf{\Phi}_{s,t}(.+\vec{p})=\nabla\mathbf{\Phi}_{s,t}(.)+\vec{p}$. If $s=t_i$ and $t=t_{i+1}$ we denote $\mathbf{\Phi}_{i,i+1}$ (resp. $\mathbf{\Phi}_{i+1,i}$) instead of $\mathbf{\Phi}_{s,t}$ (resp. $\mathbf{\Phi}_{t,s}$). Note that ϕ is discontinuous at times t_i and thus

$$\begin{aligned}\mathbf{\Phi}_{i,i+1}(x)&=|x|^2/2+\frac{T}{N}\phi(t_i^+,x)\\ \mathbf{\Phi}_{i,i-1}(x)&=|x|^2/2-\frac{T}{N}\phi(t_i^-,x).\end{aligned}$$

We also have

$$v(s,x)=\nabla\phi(s,x)=\frac{1}{t-s}(\nabla\mathbf{\Phi}_{s,t}(x)-x) \tag{258}$$

which is well defined on $\mathbb{R}^d/\mathbb{Z}^d$.
In the first lemma, we will see that $\nabla\mathbf{\Phi}_{s,t}$ pushes forward $\rho(s)$ on $\rho(t)$ minimizing the cost $\int_{\mathbb{T}^d}\rho(s,x)|\mathbf{m}(x)-x|^2dx$ among all $\mathbf{m}$ pushing forward $\rho(s)$ on $\rho(t)$ (that we denote hereafter $\mathbf{m}\#\rho(s)=\rho(t)$) and that $\mathbf{\Phi}_{s,t}$ coincides with its convex hull $d\rho(s)$ a.e.

Then in the second lemma we will show that we can consider a solution for which every $\mathbf{\Phi}_{s,t}$ is convex. This point that may seem to be a direct consequence of optimal transport (the fact that the optimal transport is given by the gradient of a convex function) needs from our point of view a careful proof : indeed we only know that the gradient of $\mathbf{\Phi}_{s,t}$ will coincide $d\rho(s)$ a.e. with the gradient of a convex function, but since the optimality equation links $\mathbf{\Phi}_{i-1,i}$, $\mathbf{\Phi}_{i,i+1}$ and $p(t_i)$ it must be checked that $\mathbf{\Phi}_{i,i+1}$ can consistently be taken convex. The convexity will then allow us to consider the second derivative of $\mathbf{\Phi}_{i,i+1}$ since a convex function is almost everywhere twice differentiable, and then to make rigorous the inequality (251) and its consequences.

First for any $f : \mathbb{R}^d \to \mathbb{R}$ we denote f^* its Legendre transform defined by

$$f^*(y) = \sup_{x\in\mathbb{R}^d} y \cdot x - f(x). \tag{259}$$

then we have the following lemma :

LEMMA 6.21. *Let $t_i \le s, t \le t_{i+1}$ and $\mathbf{\Phi}_{t,s}, \mathbf{\Phi}_{s,t}$ be defined as above. Then*

$$\begin{aligned}
&\mathbf{\Phi}_{s,t} \ge \mathbf{\Phi}^*_{t,s} \text{ with equality } d\rho(s) \text{ a.e.}\\
&\mathbf{\Phi}_{s,t} = \mathbf{\Phi}^{**}_{s,t}\ d\rho(s) \text{ a.e.}\\
&\nabla\mathbf{\Phi}_{s,t} = \nabla\mathbf{\Phi}^{**}_{s,t}\ d\rho(s) \text{ a.e.}\\
&\nabla\mathbf{\Phi}^{**}_{s,t}\#d\rho(s) = d\rho(t).
\end{aligned}$$

Proof : We know that ϕ is the limit of a smooth sequence ϕ_ϵ satisfying the the constraint :

$$\partial_t\phi_\epsilon + \frac{1}{2}|\nabla\phi_\epsilon|^2 + \frac{1}{N}\sum_{i=1}^{N}\delta_{t=t_i}p_\epsilon(t_i) \le 0.$$

Between t_i and t_{i+1} we have

$$\partial_t\phi_\epsilon + \frac{1}{2}|\nabla\phi_\epsilon|^2 \le 0. \tag{260}$$

Thus if $t > s$ considering $\gamma(\sigma) = x + (\sigma - t)\dfrac{y-x}{t-s}$ and using (260) we find

$$\begin{aligned}
\frac{d}{d\sigma}[\phi_\epsilon(\sigma,\gamma(\sigma))] &= (\partial_t\phi_\epsilon(\sigma,\gamma(\sigma)) + \frac{y-x}{t-s}\cdot\nabla\phi_\epsilon(\sigma,\gamma(\sigma)))\\
&\le (\partial_t\phi_\epsilon(\sigma,\gamma(\sigma)) + \frac{1}{2}(\frac{|y-x|^2}{|t-s|^2} + |\nabla\phi_\epsilon(\sigma,\gamma(\sigma))|^2))\\
&\le \frac{1}{2}\frac{|y-x|^2}{|t-s|^2}\ ,
\end{aligned}$$

and integrating from t to s we find

$$\begin{aligned}
&\phi_\epsilon(t,y) \le \inf_x \phi_\epsilon(s,x) + \frac{|y-x|^2}{2(t-s)}\\
&\phi_\epsilon(s,x) \ge \sup_y \phi_\epsilon(t,y) - \frac{|y-x|^2}{2(t-s)}\ .
\end{aligned}$$

Letting ϵ go to 0, it follows from (257) that

$$\mathbf{\Phi}_{t,s}(y) \geq \sup_x x \cdot y - \mathbf{\Phi}_{s,t}(x) = \mathbf{\Phi}^*_{s,t}(y) \tag{261}$$

$$\text{and similarly} \quad \mathbf{\Phi}_{s,t}(x) \geq \mathbf{\Phi}^*_{t,s}(x). \tag{262}$$

This is the first point of the lemma. The crucial point is the following : for $d\rho(t)$ a.e y we have

$$\phi(t,y) = \inf_x \phi(s,x) + \frac{|y-x|^2}{t-s}$$

or equivalently

$$\mathbf{\Phi}_{t,s}(y) = \sup_x x \cdot y - \mathbf{\Phi}_{s,t}(x) \ d\rho(t) \ a.e.$$

Indeed take a smooth sequence $\phi_\epsilon, p_\epsilon$ such that

$$\partial_t \phi_\epsilon + \frac{1}{2}|\nabla \phi_\epsilon|^2 + \frac{1}{N}\sum_{i=1}^{N} \delta_{t=t_i} p_\epsilon(t_i) \leq 0$$

that maximizes the dual problem, i.e. such that

$$\begin{aligned}
& \int_0^T \int_{\mathbb{T}^d} \rho \partial_t \phi_\epsilon + \rho \nabla \phi \cdot \nabla \phi_\epsilon \ dtdx \\
+ \quad & \frac{T}{N} \sum_{i=1}^{N-1} \int_{\mathbb{T}^d} \rho p_\epsilon(t_i,x) + \nabla p \cdot \nabla p_\epsilon(t_i,x) - \frac{1}{2}|\nabla p_\epsilon|^2(t_i,x) dx \\
\geq \quad & K_N - \epsilon^2.
\end{aligned}$$

Being a maximizing sequence of the dual problem implies the following :

$$\int_D \rho |\partial_t \phi_\epsilon + \frac{1}{2}|\nabla \phi_\epsilon|^2 + \frac{1}{N}\sum_{i=1}^{N} \delta_{t=t_i} p_\epsilon(t_i)| \ dtdx \to 0 \text{ as } \epsilon \to 0,$$

which in turn implies that

$$\begin{aligned}
& \int_s^t \int_{\mathbb{T}^d} \rho \partial_t \phi_\epsilon + \rho \nabla \phi \cdot \nabla \phi_\epsilon \ dt' dx \\
= \quad & \int_{\mathbb{T}^d} \rho(t,x)\phi_\epsilon(t,x) - \rho(s,x)\phi_\epsilon(s,x) \ dx \\
\to \quad & \frac{1}{2}\int_s^t \int_{\mathbb{T}^d} \rho |\nabla \phi|^2 \ dt' dx = \frac{1}{t-s} W_2^2(\rho(s), \rho(t)) \text{ as } \epsilon \to 0.
\end{aligned}$$

where the first line comes from the continuity equation (220) and the last identity comes the fact that between t and s the problem coincides with the problem 6.17.

If $\mathbf{m}$ is a mapping realizing the optimal transport of $\rho(s)$ on $\rho(t)$ then $\mathbf{m}\#\rho(s) = \rho(t)$ implies

$$\int_{\mathbb{T}^d} \rho(t,x)\phi_\epsilon(t,x) - \rho(s,x)\phi_\epsilon(s,x) \ dx = \int_{\mathbb{T}^d} \rho(s,x)(\phi_\epsilon(t,\mathbf{m}(x)) - \phi_\epsilon(s,x)) \ dx.$$

Using that

$$\phi_\epsilon(t,x) - \phi_\epsilon(s,y) \leq \frac{|x-y|^2}{2(t-s)}.$$

and that from the optimality of $\mathbf{m}$ we have

$$\frac{1}{2(t-s)}\int_{\mathbb{T}^d}\rho(s,x)|x-\mathbf{m}(x)|^2\,dx = \frac{1}{t-s}W_2^2(\rho(s),\rho(t))$$

we obtain by taking the limit $\epsilon \to 0$ that

$$\phi(t,\mathbf{m}(x)) = \phi(s,x) + \frac{|x-\mathbf{m}(x)|^2}{2(t-s)}\ d\rho(s)\ a.e$$

which is equivalent to

$$\phi(t,y) = \phi(s,x) + \frac{|y-\mathbf{m}^{-1}(y)|^2}{2(t-s)}\ d\rho(t)\ a.e$$

we remind that $\mathbf{m}$ is only invertible $d\rho(t)$ a.e and $\mathbf{m}^{-1}$ can be defined as the ($d\rho(t)$ a.e unique) mapping realizing the optimal transport of $\rho(t)$ on $\rho(s)$.
This implies that if $t > s$, we have :

$$\begin{aligned}\phi(t,x) &= \inf_y \frac{|y-x|^2}{2(t-s)} + \phi(s,y)\ d\rho(t)\ a.e.\\ \mathbf{\Phi}_{t,s} &= (\mathbf{\Phi}_{s,t})^*\ d\rho(t)\ a.e.\end{aligned}$$

the two lines being equivalent through equation (257). Then since any Legendre transform as a supremum of affine functions is convex, $\mathbf{\Phi}_{t,s}$ coincides with a convex function $d\rho(t)$ a.e and is above this function dx a.e. from (261). Thus since $\mathbf{\Phi}_{t,s}^{**}$ is the convex hull of $\mathbf{\Phi}_{t,s}$ it follows that

$$\mathbf{\Phi}_{t,s}^{**} = \mathbf{\Phi}_{t,s}\ d\rho(t)\ a.e.$$

of which can be deduced that

$$\begin{aligned}&\int_{\mathbb{T}^d}(|x|^2/2 - \mathbf{\Phi}_{t,s}^{**})\rho(s,x)\ dx + \int_{\mathbb{T}^d}(|y|^2/2 - \mathbf{\Phi}_{t,s}^{*}(y))\rho(t,y)\ dy\\ =\ & (t-s)\int_{\mathbb{T}^d}\rho(s,x)\phi(s,x) - \rho(t,x)\phi(t,x)\ dx\\ =\ & W_2^2(\rho(s),\rho(t)).\end{aligned}$$

This implies that $\nabla\mathbf{\Phi}_{s,t}^{**}\#\rho(s) = \rho(t)$ (See [**14**]). Then if we set

$$\tilde{\phi}(s,x) = \frac{1}{t-s}((|\cdot|^2/2 + (t-s)\phi(s,\cdot))^{**}(x) - |x|^2/2)$$

we obtain that

$$\int_{\mathbb{T}^d}\rho(t,x)|\nabla\phi - \nabla\tilde{\phi}|^2(t,x)dx = 0$$

for a.e. t and this ends the proof of lemma 6.21. □

We are going to use the previous lemma to construct a new sequence of solutions for which the potentials $\mathbf{\Phi}_{i,i+1}$ are convex. This will allow us to define $d\rho(t_i)$ a.e. the second derivative of $\mathbf{\Phi}_{i,i+1}$.

This special solution will turn out to be the viscosity solution of $\partial_t\psi + \frac{1}{2}|\nabla\psi|^2 + \frac{T}{N}\sum_{i=1}^{N-1}\delta_{t=t_i}p = 0.$
Remind that from proposition 6.18 $\phi(t_i^+)$ is defined from the optimality equations by

$$\phi(t_i^+,x) - \phi(t_i^-,x) \leq -\frac{T}{N}p(t_i,x)\ dx\ a.e. \tag{263}$$

$$\phi(t_i^+,x) - \phi(t_i^-,x) = -\frac{T}{N}p(t_i,x)\ d\rho(t_i)\ a.e. \tag{264}$$

$$\nabla\phi(t_i^+,x) - \nabla\phi(t_i^-,x) = -\frac{T}{N}\nabla p(t_i,x)\ d\rho(t_i)\ a.e. \tag{265}$$

consider the new solution ψ defined by

$$\psi(t=0,x) = \frac{N}{T}\left[\left(\frac{|\cdot|^2}{2} + \frac{T}{N}\phi(t=0,\cdot)\right)^{**}(x) - |x|^2/2\right] \tag{266}$$

$$\text{on }]t_i,t_{i+1}[,\ \psi(t,x) = \inf_y\{\frac{|x-y|^2}{2(t-t_i)} + \psi(t_i^+,y)\} \tag{267}$$

$$\psi(t_i^+,x) = \frac{N}{T}\left[\left(\frac{|\cdot|^2}{2} + \frac{T}{N}\psi(t_i^-,\cdot) - \frac{T^2p(t_i,\cdot)}{N^2}\right)^{**}(x) - |x|^2/2\right] \tag{268}$$

LEMMA 6.22. *1-For almost every $t\in[0,T]$, $\psi(t,.)$ coincides with $\phi(t,.)$ $d\rho(t)$ a.e. and (ψ,ρ,p) is a solution of the Problem 6.15.*

2- $\forall i\in[0..N-1]$, $(t,x)\to\psi(t,x)$ and $(t,x)\to-\psi(t_{i+1}+t_i-t,x)$ are both viscosity solutions (and subsolutions) of $\partial_t\psi+\frac{1}{2}|\nabla\psi|^2=0$ in $]t_i,t_{i+1}[$.

3- Ψ is a viscosity solution of $\partial_t\psi + \frac{1}{2}|\nabla\psi|^2 + \frac{T}{N}\sum_{i=1}^{N-1}\delta_{t=t_i}p = 0$ on $[0,T]$ in the sense of (270).

Proof : We denote

$$\mathbf{\Psi}_{s,t}(x) = |x|^2/2 + (t-s)\psi(s,x) \text{ for } s,t\in[t_{i-1},t_{i+1}] \tag{269}$$

and $\mathbf{\Psi}_{i,i+1}$, $\mathbf{\Psi}_{i+1,i}$ as well.
Let us now prove by induction that for all $1\leq i\leq N-1$, $\psi(t_i^-,x)\geq\phi(t_i^-,x)$ with equality $d\rho(t_i)$ a.e.
(266) implies that $\mathbf{\Psi}_{0,1} = \mathbf{\Phi}_{0,1}^{**}$. Then (267) implies that $\mathbf{\Psi}_{1,0} = \mathbf{\Psi}_{0,1}^{*} = \mathbf{\Phi}_{0,1}^{***} = \mathbf{\Phi}_{0,1}^{*} \leq \mathbf{\Phi}_{1,0}$ with equality $d\rho(t_1)$ a.e. from lemma 6.21. The equality $\mathbf{\Phi}_{0,1}^{***} = \mathbf{\Phi}_{0,1}^{*}$ comes from the fact that $\mathbf{\Phi}_{0,1}^{*}$ is convex as a Legendre transform and that the the Legendre transform is an involution on convex functions. Thus from (269) we get that $\psi(t_1^-)\geq\phi(t_1^-)$ with equality $d\rho(t_1)$ a.e.
Suppose that $\psi(t_i^-)\geq\phi(t_i^-)$ with equality $d\rho(t_i)$ a.e. Then we have

$$|x|^2/2 + \frac{T}{N}\psi(t_i^-,x) - \frac{T^2}{N^2}p(t_i,x) \geq |x|^2/2 + \frac{T}{N}\phi(t_i^-,x) - \frac{T^2}{N^2}p(t_i,x)$$

with equality $d\rho(t_i)$ a.e. Thus $\mathbf{\Psi}_{i,i+1}\geq\mathbf{\Phi}_{i,i+1}^{**}$ with equality $d\rho(t_i)$ a.e. from (263) and (268). Then

$$\mathbf{\Psi}_{i+1,i} = \mathbf{\Psi}_{i,i+1}^{*} \leq \mathbf{\Phi}_{i,i+1}^{***} = \mathbf{\Phi}_{i,i+1}^{*} \leq \mathbf{\Phi}_{i+1,i}$$

where we have used that the Legendre transform is decreasing with respect to functions, is an involution on convex functions and from lemma 6.21 for the last inequality. This implies that $\psi(t_{i+1}^-) \geq \phi(t_{i+1}^-)$. Moreover we check that if $\mathbf{\Psi}_{i,i+1} = \mathbf{\Phi}_{i,i+1}^{**}\ d\rho(t_i)$ a.e., and $\nabla\mathbf{\Psi}_{i,i+1}\#\rho(t_i) = \nabla\mathbf{\Phi}_{i,i+1}^{**}\#\rho(t_i) = \rho(t_{i+1})$ then $\mathbf{\Psi}_{i,i+1}^* = (\mathbf{\Phi}_{i,i+1}^{**})^*\ d\rho(t_{i+1})$ a.e. using the identity

$$c(x) + c^*(\nabla c(x)) = x \cdot \nabla c(x)$$

that holds for any Lipschitz convex function c and the fact that $\nabla\mathbf{\Psi}_{i,i+1}$ and $\nabla\mathbf{\Phi}_{i,i+1}^{**}$ coincide $d\rho(t_i)$ a.e. We thus have proved that $\phi \equiv \psi\ d\rho$ a.e. It follows that $\phi(t_{i+1}^-) = \psi(t_{i+1}^-)\ d\rho(t_{i+1})$ a.e. Moreover we check that

$$\begin{aligned}\psi(t_{i+1}^+, x) &= \frac{N}{T}[(|\cdot|^2/2 + \frac{T}{N}\psi(t_{i+1}^-, \cdot) - \frac{T^2 p(t_i, \cdot)}{N^2})^{**}(x) - |x|^2/2] \\ &\leq \psi(t_{i+1}^-, x) - \frac{T p(t_i, x)}{N}\end{aligned}$$

with equality $\rho_N(t_i+1)$ a.e. and that on $]t_i, t_{i+1}[$ from (267) we have $\partial_t\psi + \frac{1}{2}|\nabla\psi|^2 = 0$ satisfied in the viscosity sense from the Hopf-Lax formula (267). To see that $-\psi(t_{i+1} - t, .)$ is also a viscosity solution to this equation, first note $\mathbf{\Psi}_{i,i+1} = |x|^2/2 + \psi(t_i, x)$ is convex. Then we have

$$\begin{aligned}\psi(t, x) &= \inf_y\{\psi(t_i, y) + \frac{|x-y|^2}{2(t-t_i)}\} \\ &= \inf_y\{\phi(t_i, y) + \frac{|x-y|^2}{2(t_{i+1}-t_i)} + \gamma\frac{|x-y|^2}{2(t_{i+1}-t)}\}\end{aligned}$$

for some $\gamma > 0$. No see that $\alpha(y) = \phi(t_i, y) + \frac{|x-y|^2}{2(t_{i+1}-t_i)}$ is convex as well as $\beta(y)\gamma\frac{|x-y|^2}{2(t_{i+1}-t)}$ and thus we can apply the Rockafeller duality Theorem that says that if α, β are convex continuous we have

$$\inf_y\{\alpha(y) + \beta(y)\} = \sup_x\{-\alpha^*(x) - \beta^*(-x)\}$$

but computing α^* gives $\frac{1}{t_{i+1}-t}\mathbf{\Psi}_{i+1,i}$ and thus one is able to check that :

$$\psi(t, x) = \sup_y\{\psi(t_{i+1}^-, y) - \frac{|x-y|^2}{2(t_{i+1}-t)}\}$$

which says exactly that $t \to -\psi(t_{i+1} + t_i - t)$ is a viscosity solution of $\partial_t\psi + \frac{1}{2}|\nabla\psi|^2 = 0$.
Then to see that ψ is a viscosity solution of

$$\partial_t\psi + \frac{1}{2}|\nabla\psi|^2 + \frac{T}{N}\sum_{i=1}^{N-1}\delta_{t=t_i}p = 0$$

just notice that our definition of ψ is the following :

$$\begin{aligned}|x|^2/2 - \frac{T}{N}\psi(t_{i+1}^-, x) &= (|\cdot|^2/2 + \frac{T}{N}\psi(t_i^+, \cdot))^* \\ &= (|\cdot|^2/2 + \frac{T}{N}\psi(t_i^-, \cdot) - \frac{T^2}{N^2}p(t_i, \cdot))^*\end{aligned}$$

thus

$$\psi(t_{i+1}^-, x) = \inf_y\{\psi(t_i^-, y) - \frac{T}{N}p(t_i, y) + \frac{|x-y|^2}{2\frac{T}{N}}\}$$

and one gets that

$$\psi(t_{i+p}^-, x) =$$
$$\inf_{\gamma\in\Gamma}\{\psi(t_i^-, \gamma(t_i^-)) + \int_{t_i}^{t_{i+p}^-} \left[-\frac{T}{N}\sum_{i=1}^{N-1} \delta_{\sigma=t_i} p(\sigma, \gamma(\sigma)) + \frac{1}{2}|\dot\gamma|^2(\sigma)\right] d\sigma\}$$

where Γ is the set of all continuous paths with $\gamma(t_{i+p}) = x$, and more generally that

$$\psi(t, x) = \tag{270}$$
$$\inf_{\gamma(t)=x}\{\psi(s, \gamma(s)) + \int_{s}^{t} \left[-\frac{T}{N}\sum_{i=1}^{N-1} \delta_{\sigma=t_i} p(\sigma, \gamma(\sigma)) + \frac{1}{2}|\dot\gamma|^2(\sigma)\right] d\sigma\}$$

and this is a definition of viscosity solutions. This achieves the proof of lemma 6.22. □
Remark : the time reversibility property is not valid for any viscosity solution. Actually this is true before occurrence of shocks. This is what one says when we decompose $\phi(0, y) + \frac{|x-y|^2}{2(t-t_i)}$ as the sum of two convex functions : this means that one can continue the rays further without developing shocks. Thus we see that our variational solution can not develop shocks on the support of ρ in the interior of the time interval.

6.3.2. *Proof of the bound on* $\Delta\phi$: ψ satisfies

$$\partial_t\psi + \frac{1}{2}|\nabla\psi|^2 + \frac{T}{N}\sum \delta_{t=t_i} p(t_i) \le 0,$$
$$\partial_t\rho_N + \nabla\cdot(\rho_N\nabla\psi) = 0$$

From now we consider that $\phi := \psi$ and thus $\mathbf{\Phi}_{t,s}$ is convex for any $t_i \le s, t \le t_{i+1}$. We are going to prove the following lemma :

Lemma 6.23. *$\mathbf{\Phi}_{i,i+1}, \mathbf{\Phi}_{i,i-1}$ are $C^{1,1}$ at every density point of $\rho(t_i)$ with $C^{1,1}$ norm bounded by $C(d)$. Moreover there exists a set $\mathcal{E}_i$ of full measure for $\rho(t_i)$ such that everywhere in $\mathcal{E}_i$, $\phi(t_i^+)$ and $\phi(t_i^-)$ are twice differentiable and the following holds*

$$\Delta\phi(t_i^+, x) - \Delta\phi(t_i^-, x) \le \frac{T}{N}(1 - \rho(t_i, x)). \tag{271}$$

Proof : Using (264, 263) we get that

$$\mathbf{\Phi}_{i,i+1}(x) + \mathbf{\Phi}_{i,i-1}(x) = |x|^2 - \frac{T^2}{N^2} p(t_i, x)\ d\rho(t_i)\ a.e. \tag{272}$$

$$\mathbf{\Phi}_{i,i+1}(x) + \mathbf{\Phi}_{i,i-1}(x) \le |x|^2 - \frac{T^2}{N^2} p(t_i, x)\ dx\ a.e. \tag{273}$$

Thus $d\rho(t_i)$ almost everywhere the convex function $\mathbf{\Phi}_{i,i+1}+\mathbf{\Phi}_{i,i-1}$ is tangent by below to $|\cdot|^2 - \frac{T^2}{N^2}p(t_i, \cdot)$. The Poisson equation satisfied by $p(t_i)$ being only true in the distribution sense we need to introduce a finite difference version of the Laplacian :

LEMMA 6.24. *Let c_d be the volume of the unit ball of $\mathbb{R}^d$ and b_d be the $d-1$ dimensional Hausdorff measure of the unit sphere. Let p be continuous and*

$$\Delta_h p(x) = \frac{k_d}{h^2}\left[\frac{1}{b_d h^{d-1}}\int_{\partial B(x,h)} p(y)dy - p(x)\right]$$
$$\Delta_h^* p(x) = \frac{l_d}{h^2}\left[\frac{1}{c_d h^{d}}\int_{B(x,h)} p(y)dy - p(x)\right]$$

with the constants k_d, l_d chosen so that both operators converge to the Laplacian for smooth functions as $h \to 0$. Then
1 - if $\Delta p \geq C$ in $\mathbb{T}^d$ in the distribution sense then $\Delta_h p \geq C$ and $\Delta_h^* p \geq C$ everywhere in $\mathbb{T}^d$.
2 - if $\Delta p \in L^2(\mathbb{T}^d)$, up to extraction of a subsequence in h, then $\Delta_h^* p \to \Delta p$ dx almost everywhere.

Proof of claim 1 : look at f solution of

$$f - p|_{\partial B(x,h)} = 0$$
$$\Delta f = C \leq \Delta p$$

we mention that the boundary condition has a meaning since p is continuous. Then from the maximum principle $f(x) \geq p(x)$ and thus $\Delta_h f \leq \Delta_h p$ since $f = p$ on $\partial B(x,h)$. But $f(x) = C|x|^2/2d + h$ with $\Delta h = 0$ and then using the fact that the average on a sphere of an harmonic function equals its value at the center of the sphere, we get that $\Delta_h f = C$. To obtain the inequality for Δ_h^* just integrate over h.
Proof of claim 2 : If suffices to show that $\Delta_h^* p$ converges strongly in L^1 to Δp if $\Delta p \in L^2(\mathbb{T}^d)$. The Taylor formula gives :

$$\begin{aligned}
\Delta_h^* p(x) &= \frac{1}{c_d h^d}\frac{l_d}{h^2}\left[\int_{|y|\leq h}\int_{\sigma=0}^1 (1-\sigma) y^t \cdot [D^2 p(x+\sigma y) - D^2 p(x)] \cdot y \, d\sigma \, dy \right.\\
&+ \left. \int_{|y|\leq h} \frac{1}{2} y^t \cdot D^2 p(x) \cdot y \, dy\right]\\
&= \Sigma_1(x) + \Sigma_2(x)
\end{aligned}$$

Then $\Sigma_2(x)$ is equal dx almost everywhere to $\Delta p(x)$ and Σ_1 converges strongly to 0 in L^2 since $p \in W^{2,2}$ from elliptic regularity. □

Now since $\Delta p(t_i) = \rho(t_i) - 1 \in L^2(\mathbb{T}^d)$ (from proposition 6.18), up to extraction of a subsequence in h, for almost every $x \in \mathbb{T}^d$ $\Delta_h^* p(t_i, x)$ converges to $\rho(t_i, x) - 1$.
Then applying Δ_h^* to (272, 273) at a point where equality (272) holds we get that

$$\Delta_h^* \mathbf{\Phi}_{i,i+1} + \Delta_h^* \mathbf{\Phi}_{i,i-1} \leq 2d - \frac{T^2}{N^2}\Delta_h^* p(t_i) \tag{274}$$

But since $\Delta p(t_i) = \rho(t_i) - 1 \geq -1$ in the distribution sense, we have that $\Delta_h^* p(t_i) \geq -1$ pointwise. We can thus have for any $h > 0$ and for any point such that (272) holds

$$\Delta_h^* \mathbf{\Phi}_{i,i+1} + \Delta_h^* \mathbf{\Phi}_{i,i-1} \leq 2d + \frac{T}{N^2}.$$

Now for a convex function to have $\Delta_h^*(x)$ bounded as h goes to 0 implies being $C^{1,1}$ at x. The bound on the second derivative comes from the fact that the trace controls the norm of a positive matrix.

To see that this holds at every density point on $\rho(t_i)$, note that when $d \leq 3$ we have $p(t_i)$ in $C^{\frac{1}{2}}(\mathbb{T}^d)$ for every $1 \leq i \leq N-1$. Thus since convex functions are continuous and $p(t_i)$ is continuous equality (272) holds on a closed set of full measure for $\rho(t_i)$ and thus at every density point of $\rho(t_i)$. Now convex functions are derivable almost everywhere and thus almost everywhere $\Delta_h^* \mathbf{\Phi}_{i,i+1} \to \Delta \mathbf{\Phi}_{i,i+1}$ and $\Delta_h^* \mathbf{\Phi}_{i,i-1} \to \Delta \mathbf{\Phi}_{i,i-1}$. We recall that $\rho(t_i)$ is in L^2 which implies that negligible sets for the Lebesgue measure are also negligible for $\rho(t_i)$.
We define $\mathcal{E}_i$ to be the set of all points x such that
1- x is a point of Lebesgue differentiability for $\rho(t_i)$ where $\rho(t_i, x) > 0$,
2- $\Delta_h^* p(t_i, x) \to \rho(t_i, x) - 1$ as $h \to 0$,
3- $\mathbf{\Phi}_{i,i+1}$ and $\mathbf{\Phi}_{i,i-1}$ are twice differentiable,
4- equality (272) holds,
$\mathcal{E}_i$ is a set of full measure for $\rho(t_i)$ and by removing sets of 0 measure to $\mathcal{E}_i$ one can impose the condition $\nabla \mathbf{\Phi}_{i,i+1}(\mathcal{E}_i) = \mathcal{E}_{i+1}$. Then at every point of $\mathcal{E}_i$ we have

$$\Delta \mathbf{\Phi}_{i,i+1}(x) + \Delta \mathbf{\Phi}_{i,i-1}(x) \leq 2d + \frac{T^2}{N^2}(1 - \rho(t_i, x)) \tag{275}$$

$$\Delta \phi(t_i^+, x) - \Delta \phi(t_i^-, x) \leq \frac{T}{N}(1 - \rho(t_i, x)). \tag{276}$$

This implies the following bound for ρ, ϕ :

$$\begin{array}{l} \|\rho(t_i)\|_{L^\infty} \leq C(d) N^2 \\ \text{for every } x \in \mathcal{E}_i,\ \|\phi(t_i, x)\|_{C_x^{1,1}} \leq C(d) N. \end{array}$$

This achieves the proof of lemma 6.23. □

Construction of the characteristics. Remember that $\mathbf{\Phi}_{i,i+1}$ is given by :

$$\mathbf{\Phi}_{i,i+1}(x) = |x|^2/2 + \frac{T}{N}\phi(t_i^+, x),$$

and that for $s \in [t_i, t_{i+1}]$ (resp. $s \in [t_i - 1, t_i]$) , $\mathbf{\Phi}_{t_i,s}$ is given by

$$\begin{array}{rcl} \mathbf{\Phi}_{t_i,s}(x) & = & |x|^2/2 + (s - t_i)\phi(t_i^+, x) \\ (\text{ resp. } \mathbf{\Phi}_{t_i,s}(x) & = & |x|^2/2 + (s - t_i)\phi(t_i^-, x)) \end{array}$$

We know that in $\mathcal{E}_i$, $\mathbf{\Phi}_{i,i+1}, \mathbf{\Phi}_{i,i-1}$ are convex and twice differentiable. Thus for any $s \in]t_{i-1}, t_{i+1}[$, $\mathbf{\Phi}_{t_i,s}$ is twice differentiable and $D^2 \mathbf{\Phi}_{t_i,s}$ is invertible. This implies (see the Appendix on convex functions of [**48**]) that $D^2 \mathbf{\Phi}_{s,t_i}$ exists at point $\nabla \mathbf{\Phi}_{t_i,s}(x)$ for $x \in \mathcal{E}_i$. Thus we can define $\mathcal{E}_s = \nabla \mathbf{\Phi}_{t_i,s}(\mathcal{E}_i)$, this definition makes sense pointwise on $\mathcal{E}_i$ and pointwise in $\mathcal{E}_s$ ϕ_s is twice differentiable. Note also that from lemma 6.22 the definition $\mathcal{E}_s = \nabla \mathbf{\Phi}_{t_i,s}(\mathcal{E}_i)$ and $\mathcal{E}_s = \nabla \mathbf{\Phi}_{t_{i+1},s}(\mathcal{E}_{i+1})$ are consistent since $\nabla \mathbf{\Phi}_{i,i+1}(\mathcal{E}_i) = \mathcal{E}_{i+1}$. Then we can define a trajectory x_s, $s \in]0, T[$ as follows : starting from $X_{s_0} \in \mathcal{E}_{s_0}$ for any $s_0 \in]0, T[$, we define $x_s = \nabla \mathbf{\Phi}_{s_0,s}$ for any s in the same interval $[t_i, t_{i+1}]$ as s_0, and we proceed similarly in other intervals. Thus we define a flow $\Xi(s, t, x)$ that gives at time s the position of the particle located in x at time t. This flow $\Xi(s, t, x)$ is defined everywhere on $\mathcal{E}_t$, and $\Xi(s, t, \mathcal{E}_t) = \mathcal{E}_s$. We may denote x_s, $s \in [0, T]$ a trajectory and it will be understood that $x_t \in \mathcal{E}_t$, $\forall t \in]0, T[$.

Conclusion of the proof. Now we bound $\Delta\phi$ along a trajectory :

$$\begin{aligned} & (t-s)\left(\Delta\phi(t)(x_t) - \Delta\phi(s)(x_s)\right) \\ = \;& (t-s)\left(\Delta\phi(t,\nabla\mathbf{\Phi}_{s,t}(x_s)) - \Delta\phi(s,x_s)\right) \\ = \;& 2d - \Delta\mathbf{\Phi}_{s,t}(x) - \Delta\mathbf{\Phi}_{t,s}(\nabla\mathbf{\Phi}_{s,t}(x_s)) \end{aligned}$$

but this is negative since we have the relation

$$D^2\mathbf{\Phi}_{s,t}(x) = [D^2\mathbf{\Phi}_{t,s}]^{-1}(\nabla\mathbf{\Phi}_{s,t}(x))$$

and therefore

$$2d - \Delta\mathbf{\Phi}_{s,t}(x) - \Delta\mathbf{\Phi}_{t,s}(\nabla\mathbf{\Phi}_{s,t}(x_s)) = 2d - \sum(\lambda_i + 1/\lambda_i)$$

where the λ_i are the eigenvalues of $D^2\mathbf{\Phi}_{s,t}(x_s)$ defined $d\rho(s)$. Thus we conclude first that for every $x_s \in \mathcal{E}_s$ and $x_t = \nabla\mathbf{\Phi}_{s,t}(x_s)$

(277) $$(\Delta\phi(t,x_t) - \Delta\phi(s,x_s))\cdot(t-s) \le 0 \quad \text{for } t_i \le s,t \le t_{i+1}.$$

Then we obtain a quantitative estimate of the decay of $\Delta\phi$, between t_i and t_{i+1} : we take $s = t_i, t = t_{i+1}$ in the previous inequality, from the convexity of $x \to x + 1/x$ we have

$$\begin{aligned} \sum_{i=1}^{d}(\lambda_i + 1/\lambda_i) \;&\ge\; d(\Delta/d + d/\Delta) \text{ where } \Delta = \sum_{i=1}^{d}\lambda_i = \Delta\mathbf{\Phi}_{i,i+1} \\ &=\; \frac{1}{\Delta}(\Delta - d)^2 + 2d \\ &=\; \frac{(\frac{T}{N}\Delta\phi(t_i^+,x_i))^2}{\frac{T}{N}\Delta\phi(t_i^+,x_i) + d} + 2d \end{aligned}$$

since $\frac{T}{N}\Delta\phi(t_i^+) + d = \Delta\mathbf{\Phi}_{i,i+1}$. Thus

$$\Delta\phi(t_{i+1}^-,x_{i+1}) - \Delta\phi(t_i^+,x_i) \le -\frac{T}{N}\frac{(\Delta\phi(t_i^+,x_i))^2}{\frac{T}{N}\Delta\phi(t_i^+,x_i) + d}$$

Using (271) we obtain

$$\Delta\phi(t_{i+1}^+,x_{i+1}) \le \Delta\phi(t_i^+,x_i) + \frac{T}{N}\left(1 - \frac{(\Delta\phi(t_i^+,x_i))^2}{d + \frac{T}{N}\Delta\phi(t_i^+,x_i)}\right).$$

We know from (275) that $\Delta\mathbf{\Phi}_{i,i+1} \le 2d + T^2/N^2$ thus
$\Delta\phi(t_i^+,x_i) \le \frac{N}{T}(d + T^2/N^2)$. It follows that
$d + \frac{T}{N}\Delta\phi(t_i^+,x_i) \le 2d + \frac{T^2}{N^2} \le 3d$ for N large enough. We finally obtain the following for N large enough :

$$\Delta\phi(t_{i+1}^+,x_{i+1}) \le \Delta\phi(t_i^+,x_i) + \frac{T}{N}\left(1 - \frac{(\Delta\phi(t_i^+,x_i))^2}{3d}\right).$$

This is a discrete version of the differential inequation $\dot{\Theta} \le 1 - \frac{1}{3d}\Theta^2$ and we conclude that
$\Delta\phi(t_i) \le C(d)(1+\frac{1}{t_i})$ in $\mathcal{E}_i$ for any $1 \le i \le N-1$. Then using the transformation $\phi(t,x) \to -\phi(-t,x)$ that transforms the solution of Problem 6.15 in another solution of 6.15 interverting ρ_0 and ρ_T we get that
$\Delta\phi_N(t_i) \ge -C(d)(1+\frac{1}{T-t_i})$ in $\mathcal{E}_i$ for any $1 \le i \le N-1$. Since we know from (277) that $t \to \Delta\phi(t,x_t)$

is decreasing between t_i and t_{i+1} we can conclude that there exists for each t a set of full measure for $d\rho(t)$ on which $\phi(t,.)$ is twice differentiable and where the following equality holds :

$$-C(d)(1+\frac{1}{T-t}) \leq \Delta\phi_N(t,x) \leq C(d)(1+\frac{1}{t}). \tag{278}$$

The first part of Proposition 6.19 is proved.

6.3.3. *Proof of the $L^\infty_{loc}(]0,T[\times\mathbb{T}^d)$ bound on ρ.* We begin by writing the Monge-Ampère equation that links $\rho(t_i)$ to $\rho(t_{i+1})$ (which makes sense because of the second differentiability of ϕ proved in lemma 6.23). :

$$\rho(t_{i+2},x_{i+2})\det(I+\frac{T}{N}D^2\phi(t_{i+1}^+,x_{i+1})) = \rho(t_{i+1},x_{i+1}),$$
$$\rho(t_i,x_i)\det(I-\frac{T}{N}D^2\phi(t_{i+1}^-,x_{i+1})) = \rho(t_{i+1},x_{i+1}).$$

Now using the domination of the geometric mean by the arithmetic mean we have

$$\det(I+\frac{T}{N}D^2\phi(t_{i+1}^+)) \leq (1+\frac{T}{dN}\Delta\phi(t_{i+1}^+))^d$$

thus

$$\frac{\rho(t_{i+1},x_{i+1})}{\rho(t_{i+2},x_{i+2})} \leq (1+\frac{T}{dN}\Delta\phi(t_{i+1}^+,x_{i+1}))^d \tag{279}$$

$$\frac{\rho(t_{i+1},x_{i+1})}{\rho(t_i,x_i)} \leq (1-\frac{T}{dN}\Delta\phi(t_{i+1}^-,x_{i+1}))^d. \tag{280}$$

We deduce first the following :

$$\frac{1}{(1+\frac{T}{dN}\Delta\phi(t_i^+,x_i))^d} \leq \frac{\rho(t_{i+1},x_{i+1})}{\rho(t_i,x_i)} \leq (1-\frac{T}{dN}\Delta\phi(t_{i+1}^-,x_{i+1}))^d$$

Note that we also have

$$\frac{1}{(1+\frac{t-s}{d}\Delta\phi(s,x_s))^d} \leq \frac{\rho(t,x_t)}{\rho(s,x_s)} \leq (1-\frac{t-s}{d}\Delta\phi(t,x_t))^d \tag{281}$$

for $t_i < s,t < t_{i+1}$, $x_s \in \mathcal{E}_s$, and this implies using (278) that along a trajectory $s \to x_s$, $\log(\rho(s,x_s))$ is Lipschitz :

$$|\log(\rho(t_1,x_{t_1}) - \log(\rho(t_2,x_{t_2})| \leq \frac{C}{\tau}|t_2-t_1| \tag{282}$$

for $t_1,t_2 \in [\tau, T-\tau]$. Taking the logarithm of (279,280) we get that

$$\begin{aligned}
& \log(\rho(t_{i+2},x_{i+2})) + \log(\rho(t_i,x_i)) - 2\log(\rho(t_{i+1},x_{i+1})) \\
\geq\ & -d\log(1+\frac{T}{dN}\Delta\phi(t_{i+1}^+,x_{i+1})) - d\log(1-\frac{T}{dN}\Delta\phi(t_{i+1}^-,x_{i+1})) \\
\geq\ & -\frac{T}{N}(\Delta\phi(t_{i+1}^+,x_{i+1}) - \Delta\phi(t_{i+1}^-,x_{i+1})) \\
\geq\ & \frac{T^2}{N^2}(\rho(t_{i+1},x_{i+1})-1) \qquad (283)
\end{aligned}$$

where at the third line we have used the concavity of the log and at the last line we have used (271) :

$$\Delta\phi(t_{i+1}^+,x) - \Delta\phi(t_{i+1}^-,x) \leq \frac{T}{N}(1-\rho_{i+1}(x)).$$

We fix $\tau \in]0,T/2[$. For any trajectory $x_s, s\in]0,T[$ with $x_s \in \mathcal{E}_s$ for all $s\in[\tau,T-\tau]$, $\log(\rho(s,x_s))$ is Lipschitz with respect to s in $[\tau,T-\tau]$ from (282) and $\log(\rho(s,x_s))$ remains finite in $[\tau,T-\tau]$. Moreover (283) holds at every time t_i. Using this we claim an unconditional bound for $\rho(s,x_s)$ for $\tau\leq s\leq T-\tau$.
Proof of claim : The sequence $(\log\rho(t_i,x_i))_{1\leq i\leq N-1}$ satisfies a discretization of the differential inequation

$$\ddot{\Theta} \geq \exp\Theta - 1. \tag{284}$$

with the condition $\dot{\Theta}$ bounded by $C(\tau)$ in $[\tau,T-\tau]$. We argue by contradiction : take $0<\tau<T/4$ and suppose $\Theta(\tau)\geq M$. We have $|\dot{\Theta}|\leq C(\tau)$ in $[\tau,T-\tau]$. Then choose M so large that $M-C(\tau)(T-2\tau)\geq M/2$. Thus, on $[\tau,T-\tau]$ we have $\Theta\geq M/2$ and thus $\ddot{\Theta}\geq\exp(M/2)-1$. Thus we have in $[\tau,T-\tau]$ $\dot{\Theta}(t)\geq -C(\tau)+(t-\tau)(\exp(M/2)-1)$. Choosing M large enough will lead to $\dot{\Theta}(T/2)\geq C(\tau)$ and therefore to a contradiction. Note that this proof does not depend on the initial and final values of Θ.

6.3.4. *Time continuity of ρ.* Remember that in Theorem 6.4 we have proved that $\rho\in C(]0,T[,L^p(\mathbb{T}^d))$ for any $p\in[1,\frac{3}{2}[$. Now we have an unconditional bound on ρ in $L^\infty_{loc}(]0,T[,L^\infty(\mathbb{T}^d))$. Thus the strong time continuity in every $L^p(\mathbb{T}^d)$, $1\leq p<\infty$ follows and the point 2 of Proposition 6.19 is proved.

6.3.5. *Lipschitz bound for $\|\rho(t,\cdot)\|_{L^k(\mathbb{T}^d)}$.* Using that

$$\rho(t,x_t)\det(I+(t-s)D^2\phi(s,x_s)) = \rho(s,x_s)$$

using that ϕ is twice differentiable at $x_s\in\mathcal{E}_s$, the following holds

$$\frac{d}{dt}\int_{\mathbb{T}^d}[\rho(t,x)]^k\,dx = -(k-1)\int_{\mathbb{T}^d}[\rho(t,x)]^k\Delta\phi(t,x)\,dx$$

and using point 1 of Proposition 6.19 we get that

$$-C(1+\frac{1}{t}) \leq \frac{d}{dt}\|\rho(t,\cdot)\|_{L^k(\mathbb{T}^d)} \leq C(1+\frac{1}{T-t})$$

and this proves the point 3.

6.3.6. *Displacement convexity of functionals of ρ.* Here we show that given $\rho_N(t,x)$ solution of Problem 6.15, the functions $\int_{\mathbb{T}^d}\rho_N(t,x)\log(\rho_N(t,x))\,dx$, $\int_{\mathbb{T}^d}[\rho_N(t,x)]^k\,dx$, $k\in[1,+\infty[$ converge to convex functions of $t\in[0,T]$. We drop suffix N. Let us denote $\frac{d}{dt}$ the convective derivative $\partial_t.+\nabla\phi\cdot\nabla.$ and $\frac{d^2}{dt^2}=(\frac{d}{dt})^2$. The density ρ satisfies (283) which is the finite difference version of

$$\frac{d^2}{dt^2}\log\rho(t,x) \geq \rho(t,x)-1.$$

Then using this with the following identities :

$$\frac{d^2}{dt^2}\rho(t,x) = \frac{1}{\rho(t,x)}|\frac{d}{dt}\rho(t,x)|^2 + \rho(t,x)\frac{d^2}{dt^2}\log\rho(t,x)$$

$$\frac{d^2}{dt^2}[\rho(t,x)]^k = k(k-1)[\rho(t,x)]^{k-2}|\frac{d}{dt}\rho(t,x)|^2 + k[\rho(t,x)]^{k-1}\frac{d^2}{dt^2}\rho(t,x)$$

we obtain :

$$\frac{d^2}{dt^2}[\rho(t,x)]^k \geq k^2[\rho(t,x)]^{k-2}|\frac{d}{dt}\rho(t,x)|^2 + k[\rho(t,x)]^k(\rho(t,x)-1)$$

for $k \geq 0$. Noticing that

$$\frac{d^2}{dt^2}\left[\int_{\mathbb{T}^d}\rho(t,x)F(\rho(t,x))\ dx\right] = \int_{\mathbb{T}^d}\rho(t,x)\frac{d^2}{dt^2}(F(\rho(t,x)))\ dx$$

and applying this to $F(\rho(t,x)) = [\rho(t,x)]^k$, $k \geq 0$ we get

$$\frac{d^2}{dt^2}\int_{\mathbb{T}^d}[\rho(t,x)]^{k+1}\ dx \geq \int_{\mathbb{T}^d}k([\rho(t,x)]^{k+2} - [\rho(t,x)]^{k+1})\ dx \geq 0.$$

Indeed by Jensen's inequality we have

$$\begin{aligned}\int_{\mathbb{T}^d}[\rho(t,x)]^{k+1}\ dx &\geq \left(\int_{\mathbb{T}^d}[\rho(t,x)]^k\ dx\right)^{\frac{k+1}{k}}\\ &\geq \int_{\mathbb{T}^d}[\rho(t,x)]^k\ dx\left(\int_{\mathbb{T}^d}[\rho(t,x)]^k\ dx\right)^{\frac{1}{k}}.\end{aligned}$$

Using again Jensen's inequality we have also that

$$\int_{\mathbb{T}^d}[\rho(t,x)]^k\ dx \geq \left(\int_{\mathbb{T}^d}\rho(t,x)\ dx\right)^k = 1$$

for $k \geq 1$, and we conclude.

This convexity property combined with the unconditional bound for ρ in $L^\infty([\tau,T-\tau]\times\mathbb{T}^d)$ yields a uniform Lipschitz bound for $\|\rho(t,\cdot)\|_{L^k}$ in $[\tau,T-\tau]$ for any $1 \leq k \leq \infty$. (For the case $k=+\infty$ this is because of (282)).

6.3.7. *Proof of the $W^{1,\infty}([\tau,T-\tau]\times\mathbb{T}^d)$ bound for ϕ.* From lemma 6.22, ϕ can be given by the Hopf-Lax formula (270) that we recall here :

$$\phi(t,x) = \inf_{\gamma(t)=x}\{\phi(s,\gamma(s)) + \int_s^t\left[-\frac{T}{N}\sum_{i=1}^{N-1}\delta_{\sigma=t_i}p(\sigma,\gamma(\sigma)) + \frac{1}{2}|\dot{\gamma}|^2(\sigma)\right]\ d\sigma\}. \tag{285}$$

This formula is valid for any $0 \leq s \leq t \leq T$. We are going to prove the following lemma that will yield the result when letting N go to $+\infty$.

LEMMA 6.25. *Let ϕ be viscosity solution on $[0,T]\times\mathbb{T}^d$ of*

$$\partial_t\phi(t,x) + \frac{1}{2}|\nabla\phi(t,x)|^2 + \frac{T}{N}\sum_{i=1}^{N-1}\delta_{t=t_i}p(t_i,x) = 0$$

with $t_i = \frac{Ti}{N}$. *If* $|\nabla p(t_i, \cdot)| \le l(\tau)$ *for any* $t_i \in [\tau, T-\tau]$ *with* $l(\tau) < +\infty$ *for* $\tau > 0$ *then* $|\nabla\phi(t,\cdot)| \le C(\tau)$ *and* $|\phi(t,x) - \phi(s,x)| \le C(\tau)\frac{T}{N}E(\frac{N(t-s)}{T})$ *for any* $t, s \ge \tau > 0$ *where* E *is the integer part.*

Proof : For a given pair s, t with $0 < \tau \le s \le t$, take $\gamma_k, k \in \mathbb{N}$ be a minimizing sequence in the infimum (285). We can restrict ourselves to the set of path such that $\dot\gamma$ remains bounded in $L^2([s,t])$ by a large constant C_N. Set $\tilde\gamma_k(\sigma) = \gamma_k(\sigma) + \frac{\sigma - s}{t-s}(z - x)$. Then

$$\begin{aligned}
\phi(t,z) &\le \phi(s,\tilde\gamma_k(s)) + \int_s^t \left[-\frac{T}{N}\sum_{i=1}^{N-1} \delta_{\sigma=t_i} p(\sigma, \tilde\gamma_k(\sigma)) + \frac{1}{2}|\dot{\tilde\gamma}_k|^2(\sigma)\right] d\sigma \\
&\le \phi(s,\gamma_k(s)) + \int_s^t \left[-\frac{T}{N}\sum_{i=1}^{N-1} \delta_{\sigma=t_i} p(\sigma, \gamma_k(\sigma)) + \frac{1}{2}\dot\gamma_k{}^2(\sigma)\right] d\sigma \\
&+ \int_s^t |\dot\gamma_k(\sigma)| \cdot \frac{|z-x|}{|t-s|} + \frac{|z-x|^2}{2|t-s|^2}\, d\sigma + \frac{T}{N}E(\frac{N(t-s)}{T}) l|x-z|
\end{aligned}$$

The first line converges to $\phi(t,x)$ and the second line is bounded by $C(\tau)\frac{|z-x|}{t-s}$ and thus we get that

$$\|\nabla\phi\|_{L^\infty([\tau, T-\tau]\times\mathbb{T}^d)} \le C(\tau).$$

Then $\phi(t,x) \le \phi(s,x) + C(\tau)\frac{T}{N}E(\frac{N(t-s)}{T})$ taking $\gamma(\sigma) \equiv x$ in (285). Using that $\nabla\phi$ is bounded we get

$$\begin{aligned}
&\phi(t,x) - \phi(s,x) \\
\ge\ &\inf_{y\in\mathbb{T}^d}\{\phi(s,y) - \phi(s,x) - \frac{T}{N}E(\frac{N(t-s)}{T})\|p\|_{L^\infty([s,t]\times\mathbb{T}^d)} + \frac{|y-x|^2}{2(t-s)}\} \\
\ge\ &-C(\tau)\frac{T}{N}E(\frac{N(t-s)}{T}).
\end{aligned}$$

This proves the lemma. □

We can use this lemma to conclude since we know already that $|\rho(t,x)| \le C(\tau)$ for $t \in [\tau, T-\tau]$ and from Sobolev embeddings we have $\|p\|_{L^\infty([\tau,T-\tau], C^{1,\alpha}(\mathbb{T}^d))} \le C(\tau)$. Then we let N go to $+\infty$ so that $\frac{T}{N}E(\frac{N(t-s)}{T}) \to t-s$.

6.3.8. *Convergence to viscosity solutions.* For a given smooth $\gamma : [s,t] \to \mathbb{T}^d$ compute

$$\phi_N^\gamma(t,x) = \phi_N(s,\gamma(s)) + \int_s^t \left[-\frac{T}{N}\sum_{i=1}^{N-1}\delta_{\sigma=t_i} p_N(\sigma,\gamma(\sigma)) + \frac{1}{2}|\dot\gamma|^2(\sigma)\right] d\sigma,$$

$$\phi_N^\gamma(t,x) = \phi(s,\gamma(s)) + \int_s^t \left[-p(\sigma,\gamma(\sigma)) + \frac{1}{2}|\dot\gamma|^2(\sigma)\right] d\sigma$$

(Here we use again the suffix N for the solution of Problem 6.15 while ϕ is the solution of Problem 6.2). From the continuity equation (220) the bounds on $\rho_N, \nabla\phi_N$, and elliptic regularity we get that for any $d < p < \infty$

$$\partial_t p(t,.) \in W^{1,p}(\mathbb{T}^d) \subset C^\alpha(\mathbb{T}^d),\ \alpha = 1 - \frac{d}{p}$$

uniformly for $t \in [\tau, T-\tau]$. Then we see that $\phi_N^\gamma(t,x) \to \phi^\gamma(t,x)$ since ϕ_N, p_N converge uniformly to ϕ, p in every compact set of $]0,T[\times\mathbb{T}^d$ and from the bound just obtained on p. Since we can choose $\dot\gamma$ to remain bounded in $L^2([s,t])$ and thus γ bounded in $C^{\frac{1}{2}}([s,t])$ we thus conclude that

$$\inf_{\|\dot\gamma\|_{L^2[s,t]}\le C, \gamma(t)=x}\{\phi_N^\gamma(t,x)\} - \inf_{\|\dot\gamma\|_{L^2[s,t]}\le C, \gamma(t)=x}\{\phi^\gamma(t,x)\} \to 0$$

as $N \to \infty$. Thus we get that

$$\phi(t,x) = \inf_{\gamma(t)=x}\{\phi(s,\gamma(s)) + \int_s^t \left[-p(\sigma,\gamma(\sigma)) + \frac{1}{2}|\dot\gamma|^2(\sigma)\right] d\sigma\} \tag{286}$$

and ϕ is the viscosity solution of $\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + p = 0$ on every $[s,t] \subset]0,T[$.

6.3.9. *Estimates up to the boundary.* If $\rho_T \in L^p(\mathbb{T}^d)$ with $p > d$ then p remains Lipschitz in space up to $t = T$. Thus one can choose $\phi \in W^{1,\infty}([\tau,T]\times\mathbb{T}^d)$. Now note that instead of (286) one can have ϕ defined by

$$\phi(s,x) = \sup_{\gamma(s)=x}\{\phi(t,\gamma(t)) - \int_s^t \left[-p(\sigma,\gamma(\sigma)) + \frac{1}{2}|\dot\gamma|^2(\sigma)\right] d\sigma\} \tag{287}$$

for any $0 \le s \le t \le T$. Both definitions will coincide only $d\rho$ a.e. This amounts to take $t \to -\phi(T-t,\cdot)$ the viscosity solution of $\partial_t\varphi + \frac{1}{2}|\nabla\varphi|^2 + q = 0$ with $q(t,\cdot) = p(T-t,\cdot)$. Thus if $\rho_0 \in L^p(\mathbb{T}^d)$, one can choose $\phi \in W^{1,\infty}([0,T-\tau]\times\mathbb{T}^d)$.

If ρ_0 and ρ_T are both in $L^p(\mathbb{T}^d)$, with $p > d$ then, using (285) (or (286) in the times continuous case) one can choose first ϕ such that $\phi(T,\cdot) \in W^{1,\infty}(\mathbb{T}^d)$. Then using (287) we obtain that $\phi \in W^{1,\infty}([0,T]\times\mathbb{T}^d)$ using the following result that can be found in [**32**] :

PROPOSITION 6.26. *Let ϕ be solution on $[0,T]\times\mathbb{T}^d$ of*

$$\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + p = 0$$
$$\phi(t=0,\cdot) = \phi_0$$

with $p \in L^\infty([0,T], W^{1,\infty}(\mathbb{T}^d))$ and $\phi_0 \in W^{1,\infty}(\mathbb{T}^d)$. Then $\phi \in W^{1,\infty}([0,T]\times\mathbb{T}^d)$.

This last result ends the proof of Theorem 6.5.

CHAPITRE 7

Electric turbulence in a plasma subject to a strong magnetic field

G. Loeper[12] A. Vasseur[12]

RÉSUMÉ. We consider in this paper a plasma subject to a strong deterministic magnetic field and we investigate the effect on this plasma of a stochastic electric field. We show that the limit behavior, which corresponds to the transfer of energy from the electric wave to the particles (Landau phenomena), is described by a Spherical Harmonics Expansion (SHE) model.

1. Introduction

This paper is concerned with the effect of a stochastic electric field on a plasma subject to a strong magnetic field. This is motivated by the study of the electric turbulence in a fusion machine as a Tokamak. Tokamaks are used to confine high energy plasmas in order to obtain the conditions needed for nuclear fusion reactions to take place. The plasma evolves in a toroidal reactor and is confined in the heart of the torus by the the mean of a strong magnetic field. A classical approximation is to suppose the ions to be at rest. Then only the electrons are moving. Another classical approximation argument in this type of study is the following : we are here interested only in interactions of particles over short distances of the order of the Larmor radius, moreover we suppose that at this scale the curvature of the magnetic field-lines can be neglected and that the plasma can be considered to be homogeneous along these field-lines. Thus we can restrict ourselves to a bidimensional problem. In this approach, the Vlasov equation describing the evolution of the repartition function f of the electrons is :

$$m\left(\frac{\partial f}{\partial t}+v\cdot\nabla_x f\right)+q\left(Bv^{\perp}+\nabla V^{\text{turb}}(t,x)\right)\cdot\nabla_v f=0, \tag{288}$$

m stands for the electron's mass, q its electric charge, f the distribution function on $(t,x,v)\in\mathbb{R}^+\times\mathbb{R}^2\times\mathbb{R}^2$, with t the time variable, x the space variable and v the velocity variable. B is the (constant) norm of the transverse magnetic field, ∇V^{turb} is the turbulent electric field, $v^{\perp}$ is the velocity vector after a rotation of $\pi/2$. We denote

$$1/\epsilon=\frac{qB}{m}$$

the cyclotronic frequency, and we want to study the effect of ∇V^{turb} in the limit ϵ going to zero. In the deterministic case, the limits of related problems have been studied by several authors. In the case of the Vlasov-Poisson system (when the electric field is coupled with the density $\int f\,dv$), the limit has been studied, even in the 3D framework, by Frénod and Sonnendrucker [**35**]. In the 2D framework, using a slow time scale adapted to the problem, the convergence of the averaged motion, $\int_{\mathbb{R}^2} f(t,x,v)\,dv$ to the 2D Euler system of equations has been performed simultaneously by Brenier [**17**], Golse and Saint-Raymond [**40**] and Frénod and Sonnendrucker [**36**]. A general result in 3D taking into account the two effects has been performed by Saint-Raymond in [**59**].

In our case we neglect the Poisson non linear effect, concentrating on the stochastic behavior of the equation. Hamiltonian chaos method suggest that the modes of the turbulent electric field

[1]Laboratoire J.A.Dieudonné, Université de Nice-Sophia-Antipolis, Parc Valrose, 06108 NICE Cedex 2.
Supported by
[2]Work supported by the European Atomic Energy Community EURATOM in the Research Laboratory Agremented by CEA n^o 01-24 UMR n^o 6621 CNRS-Universite de Nice.

interacting with the electrons are those having a frequency of $\omega_n = 2\pi n\epsilon$ with n an integer. This is roughly speaking the Landau resonance. Then the quasi-linear theory, (see Garbet [**38**]) predicts a diffusive behavior with respect to the velocity variable. The diffusion coefficient obtained by this method being constant, it can not take into account the abnormal diffusion phenomena. In this paper we are interested in turbulent electric fields whose spectrum is spread around the Landau frequency and whose spatial fluctuations are of the scale of the Larmor radius (of order ϵ). We will show that the limit system is then governed by the following equation :

$$\partial_t\rho - \partial_e(a(e)\partial_e\rho) = 0 \tag{289}$$

where $e = |v|^2/2$, ρ is the average of f over a sphere $|v|^2 = 2e$ and the diffusion parameter $a(e)$ is an explicit function of the correlation of V^{turb}, the turbulent electric potential, and of the energy, thus allowing abnormal diffusion. This diffusion parameter is undimensionally defined by (292). This equation is similar to the so-called Spherical Harmonics Expansion (SHE) model in high field limit modeling microelectronics semiconductor devices (see P.Degond [**28**] or Ben Abdallah, Degond, Markowich and Schmeiser [**5**]). It describes the Landau phenomena of transfer of energy from the electric wave to the particles. This work uses the techniques introduced by Poupaud and Vasseur [**57**] to derive diffusive equation from transport in random media. This method works directly on the equation and, for this reason, is different from the method used in previous works (see Kesten and Papanicolaou [**44**], [**43**] and Fannjiang, Ryzhik and Papanicolaou [**34**]). The paper is organized as follows : the precise result is stated in Section 2. In Section 3 we show how we can compute explicitly the diffusion coefficients. Finally we give the proof of the theorem in Section 4.

2. Results

In the remainder of the paper we fix n and we denote

$$\nabla V^\epsilon(t,x) = \sqrt{\epsilon}\nabla V^{\text{turb}}(2\pi n\epsilon t, \epsilon x) \tag{290}$$

for the stochastic potential. Equation (288) takes then the following undimensional form :

$$\frac{\partial f_\epsilon}{\partial t} + v\cdot\nabla_x f_\epsilon + \left(\frac{v^\perp}{\epsilon} + \frac{1}{\sqrt{\epsilon}}\nabla V^\epsilon(\frac{t}{2\pi n\epsilon}, \frac{x}{\epsilon})\right)\cdot\nabla_v f_\epsilon = 0. \tag{291}$$

we denote $\mathbf{E}$ the expectation value of any variable and make the following assumptions on the electrostatic potential :

$$\begin{aligned}
&(H1)\quad V^\epsilon \in L^\infty(\mathbb{R}^+; W^{3,\infty}(\mathbb{R}^2)) \text{ and } N(\epsilon) := \mathbf{E}\left(\left(\|V^\epsilon\|_{L^\infty(W^{3,\infty})}\right)^3\right) \ < \ \infty,\\
&(H2)\quad \mathbf{E}V^\epsilon(t,x) = 0,\ \text{for all } t\in\mathbb{R}^+,\ x\in\mathbb{R}^2,\\
&(H3)\quad V^\epsilon(t,x),\ V^\epsilon(s,y) \text{ are uncorrelated as soon as } |t-s|\geq 1,\\
&(H4)\quad \mathbf{E}(V^\epsilon(t,x)V^\epsilon(s,y)) = A(t-s,x-y) + g^\epsilon(t,s,x,y),\\
&\qquad\quad \text{with :}\\
&\qquad\quad \frac{\partial^{|\alpha|}}{\partial x^\alpha}A \in L^\infty(\mathbb{R}\times\mathbb{R}^2),\ \text{for } \alpha\in\mathbb{N}^2,\ |\alpha|\leq 3,\\
&\qquad\quad \|\ |\nabla^2_{x,y}g^\epsilon| + |\nabla^3_{x,y,y}g^\epsilon|\ \|_{L^\infty((\mathbb{R}^+)^2\times\mathbb{R}^4)} \overset{\epsilon\to 0}{\longrightarrow} 0,
\end{aligned}$$

where $\nabla^2_{x,y}g^\epsilon$ is the matrix $\left(\partial_{x_i}\partial_{y_j}g^\epsilon\right)_{i,j}$ and $\nabla^3_{x,y,y}g^\epsilon$ is $\left(\partial_{x_i}\partial_{y_j}\partial_{y_k}g^\epsilon\right)_{i,j,k}$.

Those assumptions are the same than in [**57**]. Hypothesis (H1) is an assumption on the regularity of V^ϵ for ϵ fixed. Indeed the norm $N(\epsilon)$ can go to infinity when ϵ goes to 0. Hypothesis (H2) fixes the averaged potential at 0 which is not restrictive. In view of (290), Hypothesis (H3) determines $2\pi n\epsilon$ as the decorrelated lapse of time for the turbulent electric field. Namely, it is the bigger lapse of time $t-s$ such that the electric fields at time t and at time s can be dependent on each other. Finally Hypothesis (H4), which is very classical, can be seen as an homogeneity property which takes place at the local scale ϵ, since a quadratic quantity which depends on four variables (t,x,s,y), at the limit, depends only on two variables $(t-s, x-y)$.

We denote $R_s(v)$ the rotation of angle s with center 0 of v. We consider the angular average of A :

$$\tilde{A}(t,x) = \frac{1}{2\pi}\int_0^{2\pi} A(t, R_\theta x)\, d\theta.$$

We then have the following result :

THEOREM 7.1. *Let V^ϵ be a stochastic potential satisfying assumptions (H) and independent of the initial data $f_\epsilon^0 \in L^2(\mathbb{R}^4)$. Let $a(e)$ be the function defined by :*

$$a(e) = \frac{1}{2\pi n^2}\int_0^{+\infty} (-\partial_{tt}^2 \tilde{A})(-\frac{s}{2\pi n}, 2\sqrt{2e}|\sin\frac{s}{2}|)\, ds. \tag{292}$$

This function is non negative. Assume that there is a constant C_0 such that $\|f_\epsilon^0\|_{L^2(\mathbb{R}^4)} \le C_0$ and :

$$\epsilon(1+N(\epsilon)^2) \to 0. \tag{293}$$

Let ρ_ϵ be the gyro-average of f_ϵ defined by :

$$\rho_\epsilon(t,x,e) = \frac{1}{2\pi}\int_0^{2\pi} f_\epsilon(t,x,R_\theta v)\, d\theta, \tag{294}$$

for every v such that $|v|^2/2 = e$. Then up to extraction of a subsequence, $\mathbf{E}f_\epsilon^0$ converges weakly in $L^2(\mathbb{R}^4)$ to a function $f^0 \in L^2(\mathbb{R}^4)$, $\mathbf{E}f_\epsilon$ converges weakly in L^2 to a function $f \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^4))$, $\mathbf{E}\rho_\epsilon$ converges in $C^0([0,T], L^2(\mathbb{R}^2) - w)$ for all $T>0$ toward a function $\rho \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^2\times\mathbb{R}^+)) \cap C^0(\mathbb{R}^+; L^2(\mathbb{R}^2\times\mathbb{R}^+) - W)$ with $\rho(t=0,x,e) = \int_{|v|^2=2e} f^0\, dv$. This function ρ is solution to :

$$\partial_t\rho - \partial_e(a(e)\partial_e\rho) = 0 \qquad t>0, x\in\mathbb{R}^2, e\in\mathbb{R}^+, \tag{295}$$

in the distribution sense. Finally $\mathbf{E}f_\epsilon$ converges weakly in $L^2(\mathbb{R}^+\times\mathbb{R}^4)$ toward $\rho(t,x,|v|^2/2)$.

Remark : depending on the regularity of the function $a(e)$ the solution of the Cauchy problem for equation (295) may be unique. In this case, the whole sequence ρ_ϵ converges to ρ the unique solution of (295).

3. Explicit computation of the diffusion parameter

In order to explicit the behavior of $a(e)$ we must need the correlation function A. We assume that it follows a "Richardson-like" law

$$A(t,x) = f(t)|x|^\alpha.$$

Then we have

$$\begin{aligned} a(e) &= -\frac{1}{2\pi n^2}\int_0^{+\infty} \partial_{tt}^2\tilde{A}(\frac{-s}{2\pi n}, 2\sqrt{2e}|\sin\frac{s}{2}|)\, ds \\ &= -\frac{2^{\frac{3}{2}\alpha}e^{\alpha/2}}{2\pi n^2}\int_0^{+\infty} f''(\frac{-s}{2\pi n})|\sin\frac{s}{2}|^\alpha\, ds \\ &= Ke^{\alpha/2}. \end{aligned}$$

A necessary condition for a function of the form

$$\rho(t,e) = \gamma(t)\rho_0(e/t^\beta)$$

to be an auto-similar solution of (289) is that

$$\beta = \frac{2}{4-\alpha}.$$

we then have an abnormal diffusion in

$$e = t^{\frac{2}{4-\alpha}}.$$

For example if $\alpha = 4/3$ we find $a(e) = Ke^{2/3}$ and an abnormal diffusion in $e = t^{3/4}$.

4. Proof of the result

We denote $\mathcal{S}(\mathbb{R}^4)$ the Schwartz space and $\mathcal{S}'(\mathbb{R}^4)$ its dual. We denote $\langle\cdot;\cdot\rangle$ the duality brackets between those two spaces. We recall that $L^2(\mathbb{R}^4) \subset \mathcal{S}'(\mathbb{R}^4)$ and by extension we will denote $\langle\cdot;\cdot\rangle$ as well for the scalar product on $L^2(\mathbb{R}^4)$. For every linear operator P on $\mathcal{S}(\mathbb{R}^4)$ we will denote in the same way P its extension on $\mathcal{S}'(\mathbb{R}^4)$ defined for every $\psi \in \mathcal{S}(\mathbb{R}^4)$ by :

$$\langle P\psi;\eta\rangle = \langle \psi; P^*\eta\rangle, \quad \eta \in \mathcal{S}(\mathbb{R}^4).$$

Finally we will say that $\psi_n \in \mathcal{S}'(\mathbb{R}^4)$ converges to $\psi \in \mathcal{S}'(\mathbb{R}^4)$ in $\mathcal{S}'(\mathbb{R}^4)$ if for every $\eta \in \mathcal{S}(\mathbb{R}^4)$, $\langle \psi_n;\eta\rangle$ converges to $\langle\psi;\eta\rangle$. (This is the weak convergence for $\mathcal{S}'(\mathbb{R}^4)$.)

Let us rewrite equation (291) in the following way :

$$\begin{cases} \partial_t f_\epsilon + Cf_\epsilon + \dfrac{Bf_\epsilon}{\epsilon} = -\theta_t^\epsilon f_\epsilon \\ f_\epsilon|_{t=0} = f_\epsilon^0 \end{cases} \tag{296}$$

where C, B, θ_t^ϵ are linear operators on $\mathcal{S}(\mathbb{R}^4)$ defined by :

$$\begin{aligned} C &= v\cdot\nabla_x \\ B &= v^\perp\cdot\nabla_v \\ \theta_t^\epsilon &= \frac{1}{\sqrt{\epsilon}}\nabla V^\epsilon\left(\frac{t}{2\pi n\epsilon},\frac{x}{\epsilon}\right)\cdot\nabla_v. \end{aligned}$$

Notice that C and B are deterministic and non dependent on ϵ nor on t unlike θ_t^ϵ.

We introduce the projection operator J defined on $\mathcal{S}(\mathbb{R}^4)$ which averages the values of the function on the spheres $|v|^2/2 = e$. Namely, for $\eta \in \mathcal{S}(\mathbb{R}^4)$:

$$J\eta(x,v) = \frac{1}{2\pi}\int_0^{2\pi} \eta(x, R_\theta v)\,d\theta.$$

We call J the "gyroaverage operator". This operator is self adjoint for the L^2 scalar product. Applying the projection operator J on (296) and taking its expectation value leads to :

$$\partial_t\mathbf{E}(Jf_\epsilon) + \mathbf{E}(JCf_\epsilon) + \frac{\mathbf{E}(JBf_\epsilon)}{\epsilon} = -\mathbf{E}(J\theta_t^\epsilon f_\epsilon).$$

We first study some properties of the operators in order to pass to the limit in the left hand side of this equation. Then we investigate the limit of $\mathbf{E}(J\theta_t^\epsilon f_\epsilon)$ following the procedure of [**57**]. Finally we derive the SHE equation giving the explicit form of the diffusion coefficient $a(e)$.

4.1. Properties of the operators. We have the following properties on the operators C, B, θ_t^ϵ and J :

LEMMA 7.2. *Operators C,B and θ_t^ϵ are skew adjoint for the L^2 scalar product. Operators C, B, and J commute with the expectation operator* $\mathbf{E}$. *The operator J is the restriction on $\mathcal{S}(\mathbb{R}^4)$ of the orthogonal projector of $L^2(\mathbb{R}^4)$ into* Ker$B$. *In particular :*

$$\begin{aligned}&\|J\|_{\mathcal{L}(L^2(\mathbb{R}^4))} = 1,\\ &J^2 = J,\\ &\mathrm{Ker}B = \mathrm{Im}J.\end{aligned}$$

In addition :

$$JB = JCJ = 0.$$

Proof.
–Operators C,B and θ_t^ϵ can be rewritten as $b.D$, where D is a gradient operator and b a regular function verifying $D \cdot b = 0$. For every functions $\eta_1, \eta_2 \in \mathcal{S}(\mathbb{R}^4)$ we have :

$$\begin{aligned}\langle b \cdot D\eta_1; \eta_2\rangle &= -\langle \eta_1; D(b\eta_2)\rangle\\ &= -\langle \eta_1; b \cdot D\eta_2\rangle.\end{aligned}$$

Hence they are skew adjoint operators for the L^2 scalar product.

–The operator J is clearly the L^2 projection on L^2 functions which depends only on $|v|^2/2$ with respect to v. In polar coordinates $v = (r\sin\theta, r\cos\theta)$, we have $B = \partial/\partial\theta$. So J is the projection on Ker$B$.

–Operator J is the projector on Ker$B$, hence $BJ = 0$. Since B is skew adjoint and J is self adjoint, we have $(BJ)^* = -JB = 0$.

–Let us fix $\eta \in \mathcal{S}(\mathbb{R}^4)$. We denote : $J\eta(x,v) = \rho_\eta(x, \frac{|v|^2}{2})$. Hence

$$JCJ\eta(x,v) = \nabla_x \cdot \left(\rho_\eta(x, \frac{|v|^2}{2})\frac{1}{2\pi}\int_0^{2\pi} R_\theta v \, d\theta\right) = 0.$$

Finally $JCJ = 0$.

–Since C, B and J are linear and deterministic, they commute with $\mathbf{E}$. □.

From those properties we deduce the following proposition :

PROPOSITION 7.3. *For every ϵ and every $t \in \mathbb{R}^+$ we have :*

$$\|f_\epsilon(t)\|_{L^2(\mathbb{R}^4)} = \|f_\epsilon^0\|_{L^2(\mathbb{R}^4)}.$$

There exists a function $f^0 \in L^2(\mathbb{R}^4)$ and a function $f \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^4))$such that, up to a subsequence, $\mathbf{E}f_\epsilon^0$ converges weakly in $L^2(\mathbb{R}^4)$ to f^0, $\mathbf{E}f_\epsilon$ converges weakly in $L^2([0,T]\times\mathbb{R}^4)$ to f for every $T > 0$. For every $t > 0$, f verifies $Jf(t) = f(t)$. The function $J\mathbf{E}f_\epsilon$ is solution to :

$$\left\{\begin{aligned}&\partial_t J\mathbf{E}f_\epsilon + \mathbf{E}(J\theta_t^\epsilon f_\epsilon) = w_\epsilon\\ &J\mathbf{E}f_\epsilon|_{t=0} = J\mathbf{E}f_\epsilon^0,\end{aligned}\right. \tag{297}$$

where w_ϵ converges to 0 in $\mathcal{S}'$.

Proof. Since C, B and θ_t^ϵ are skew adjoint operators, we have :

$$\partial_t \langle f_\epsilon(t); f_\epsilon(t)\rangle = 0,$$

which gives the first equality. By weak compactness there exists two functions $f^0 \in L^2(\mathbb{R}^4)$ and $f \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^4))$ such that, up to a subsequence, $\mathbf{E}f_\epsilon^0$ converges weakly in $L^2(\mathbb{R}^4)$ to f^0, $\mathbf{E}f_\epsilon$ converges weakly in $L^2([0,T]\times\mathbb{R}^4)$ to f for every $T>0$. Notice that $\epsilon\theta_t^\epsilon$ converges to 0 in $\mathcal{S}'(\mathbb{R}^4)$. Multiplying Equation (296) by ϵ, taking its expectation value, and letting ϵ go to 0, we find :

$$Bf(t) = 0 \ \text{ on } \ \mathbb{R}^+,$$

since B and $\mathbf{E}$ commute. Thanks to Lemma 7.2 $f(t) \in \mathrm{Im}J$, and since $J^2 = J$, we have $Jf(t) = f(t)$ for almost every $t > 0$. Since $JB = 0$, applying the operator J on equation (296) and taking its expectation value gives :

$$\partial_t J\mathbf{E}f_\epsilon + \mathbf{E}(J\theta_t^\epsilon f_\epsilon) = w_\epsilon,$$

with $w_\epsilon = -\mathbf{E}JCf_\epsilon$. This converges in $\mathcal{S}'$ to $-JCf = -JCJf = 0$, thanks to Lemma 7.2. □

Hence we are now concerned by the limit in $\mathcal{S}'$ of $\mathbf{E}(J\theta_t^\epsilon f_\epsilon)$.

4.2. Computation of $\mathbf{E}(J\theta_t^\epsilon f_\epsilon)$. Let us denote S_t^ϵ $t\in\mathbb{R}$ the group on $\mathcal{S}(\mathbb{R}^4)$ generated by the operator $C + B/\epsilon$. Namely, for every $h \in \mathcal{S}$, $S_t^\epsilon h$ is the unique solution on $\mathbb{R}$ to :

$$\begin{cases} \partial_t g + Cg + \dfrac{Bg}{\epsilon} = 0 \\ g|_{t=0} = h. \end{cases} \tag{298}$$

The operator S_t^ϵ can be explicitly given by :

$$S_t^\epsilon h(x,v) = h(T_\epsilon(t)(x,v)),$$

where

$$T_\epsilon(t)(x,v) = (x + \epsilon v^\perp - \epsilon R_{-t/\epsilon} v^\perp, R_{-t/\epsilon} v).$$

The function $T_\epsilon(t)(x,v)$ gives the position at $-t$ of the particle being in x with speed v at time 0 and moving at constant speed $|v|$ on a circle of radius $\epsilon|v|$. In particular S_t^ϵ is $2\pi\epsilon$ periodic. Notice that the adjoint of S_t^ϵ is S_{-t}^ϵ.

Following the procedure of [**57**], we use a 2 times iterated Duhamel formula. The first iteration gives :

$$f_\epsilon(t) = S_{2\pi n\epsilon}^\epsilon f_\epsilon(t - 2\pi n\epsilon) - \int_0^{2\pi n\epsilon} (S_\sigma^\epsilon \theta_{t-\sigma}^\epsilon S_{-\sigma}^\epsilon) S_\sigma^\epsilon f_\epsilon(t-\sigma)\, d\sigma$$

and then we write the Duhamel formula for the $f_\epsilon(t-\sigma)$ in the integral, and this yields :

$$\begin{aligned} f_\epsilon(t) &= S_{2\pi n\epsilon}^\epsilon f_\epsilon(t - 2\pi n\epsilon) - \int_0^{2\pi n\epsilon} (S_\sigma^\epsilon \theta_{t-\sigma}^\epsilon S_{-\sigma}^\epsilon) S_{4\pi n\epsilon}^\epsilon f_\epsilon(t - 4\pi n\epsilon)\, d\sigma \\ &\quad + \int_0^{2\pi n\epsilon}\int_0^{4\pi n\epsilon - \sigma} (S_\sigma^\epsilon \theta_{t-\sigma}^\epsilon S_{-\sigma}^\epsilon)(S_{s+\sigma}^\epsilon \theta_{t-\sigma-s}^\epsilon S_{-s-\sigma}^\epsilon) S_{s+\sigma}^\epsilon f_\epsilon(t-\sigma-s)\, ds\, d\sigma. \end{aligned}$$

We obtain :

$$\begin{aligned} \mathbf{E}(J\theta_t^\epsilon f_\epsilon(t)) &= J\mathbf{E}\left(\theta_t^\epsilon S_{2\pi n\epsilon}^\epsilon f_\epsilon(t - 2\pi n\epsilon)\right) \\ &\quad - \int_0^{2\pi n\epsilon} \mathbf{E}\left(J\theta_t^\epsilon (S_\sigma^\epsilon \theta_{t-\sigma}^\epsilon S_{-\sigma}^\epsilon) S_{4\pi n\epsilon}^\epsilon f_\epsilon(t - 4\pi n\epsilon)\right) d\sigma + r_t^\epsilon \end{aligned} \tag{299}$$

with

$$r_t^\epsilon = \int_0^{2\pi n\epsilon} \int_0^{4\pi n\epsilon - \sigma} J\mathbf{E}\left(\theta_t^\epsilon (S_\sigma^\epsilon \theta_{t-\sigma}^\epsilon S_{-\sigma}^\epsilon)(S_{s+\sigma}^\epsilon \theta_{t-\sigma-s}^\epsilon S_{-s-\sigma}^\epsilon) S_{s+\sigma}^\epsilon f_\epsilon(t-\sigma-s)\right) ds\, d\sigma.$$

The function f_ϵ^0 is independent of the operators $\theta_t^\epsilon,\ t \in \mathbb{R}$. In particular, in view of the assumption (H3), θ_t^ϵ and $f_\epsilon(t-s)$ are independent as soon as $t \geq s + 2\pi n\epsilon$ and $s \geq 0$.

Combining this fact with (H2), Equation (299) becomes for $t \geq 4\pi n\epsilon$

$$\begin{aligned} \mathbf{E}(J\theta_t^\epsilon f_\epsilon) &= J\mathbf{E}(\theta_t^\epsilon)\mathbf{E}(S_{2\pi n\epsilon}^\epsilon f_\epsilon(t-2\pi n\epsilon)) \\ &\quad - \int_0^{2\pi n\epsilon} \mathbf{E}(J\theta_t^\epsilon (S_\sigma^\epsilon \theta_{t-\sigma}^\epsilon S_{-\sigma}^\epsilon))\mathbf{E}(S_{4\pi n\epsilon}^\epsilon f_\epsilon(t-4\pi n\epsilon))\, d\sigma + r_t^\epsilon, \end{aligned}$$

$$\mathbf{E}(J\theta_t^\epsilon f_\epsilon) = -\int_0^{2\pi n\epsilon} \mathbf{E}(J\theta_t^\epsilon (S_\sigma^\epsilon \theta_{t-\sigma}^\epsilon S_{-\sigma}^\epsilon))\mathbf{E} f_\epsilon(t)\, d\sigma + r_t^\epsilon + e_t^\epsilon, \tag{300}$$

$$\text{with} \quad e_t^\epsilon = -\int_0^{2\pi n\epsilon} \mathbf{E}(J\theta_t^\epsilon (S_\sigma^\epsilon \theta_{t-\sigma}^\epsilon S_{-\sigma}^\epsilon))(\mathbf{E} S_{4\pi n\epsilon}^\epsilon f_\epsilon(t-4\pi n\epsilon) - \mathbf{E} f_\epsilon(t))\, d\sigma.$$

Since S_t^ϵ is $2\pi\epsilon$ periodic, $S_{4\pi n\epsilon}^\epsilon f_\epsilon(t-4\pi n\epsilon) = f_\epsilon(t-4\pi n\epsilon)$. We have :

$$S_s^\epsilon \theta_{t-s}^\epsilon S_{-s}^\epsilon = \frac{1}{\sqrt{\epsilon}}(S_s^\epsilon E^\epsilon(t-s)) \cdot D_s^\epsilon$$

where we denote

$$E^\epsilon(t,x) = \nabla V^\epsilon\left(\frac{t}{2\pi n\epsilon}, \frac{x}{\epsilon}\right),$$

and we define the differential operator D_s^ϵ by

$$D_s^\epsilon = R_{-s/\epsilon}\nabla_v + \epsilon R_{-s/\epsilon}\nabla_x^\perp - \epsilon \nabla_x^\perp.$$

Note that D_s^ϵ is skew adjoint. Let us introduce the operator L_t^ϵ on $\mathcal{S}(\mathbb{R}^4)$ (extended on $\mathcal{S}'(\mathbb{R}^4)$) defined for every $\eta \in \mathcal{S}(\mathbb{R}^4)$ by :

$$L_t^\epsilon \eta = -\int_0^{2\pi n\epsilon} \mathbf{E}(\theta_t^\epsilon (S_\sigma^\epsilon \theta_{t-\sigma}^\epsilon S_{-\sigma}^\epsilon))\eta\, d\sigma.$$

We can gather those results in the following way :

LEMMA 7.4. *We have the following equality :*

$$\mathbf{E}(J\theta_t^\epsilon f_\epsilon) = JL_t^\epsilon \mathbf{E} f_\epsilon + r_t^\epsilon + e_t^\epsilon,$$

where the operator L_t^ϵ is defined for every $\eta \in \mathcal{S}(\mathbb{R}^4)$ by :

$$L_t^\epsilon \eta(x,v) = -\frac{1}{\epsilon}\int_0^{2\pi n\epsilon} \nabla_v \cdot (\mathbf{E}(S_\sigma^\epsilon E^\epsilon(t-\sigma) \otimes E^\epsilon(t)) \cdot D_\sigma^\epsilon \eta(x,v))\, d\sigma, \tag{301}$$

and the remainders are defined by :

$$\begin{aligned} e_t^\epsilon &= JL_t^\epsilon(\mathbf{E} f_\epsilon(t-4\pi n\epsilon) - \mathbf{E} f_\epsilon(t)), \\ r_t^\epsilon &= \frac{1}{\epsilon\sqrt{\epsilon}} \int_0^{2\pi n\epsilon} \int_0^{4\pi n\epsilon-\sigma} J\mathbf{E}\left(E^\epsilon(t) \cdot \nabla_v (S_\sigma^\epsilon E^\epsilon(t-\sigma) \cdot D_\sigma^\epsilon (S_{s+\sigma}^\epsilon E^\epsilon(t-s-\sigma)\right. \\ &\qquad \left. \cdot D_{s+\sigma}^\epsilon (S_{s+\sigma}^\epsilon f_\epsilon(t-\sigma-s))))\right)\, ds\, d\sigma. \end{aligned}$$

We can now show the following lemma :

LEMMA 7.5. *For every $\eta \in \mathcal{S}(\mathbb{R}^4)$, the remainder r_t^ϵ verifies :*

$$|\langle r_t^\epsilon ; \eta\rangle| \leq C(\eta)\sqrt{\epsilon}N(\epsilon),$$

and $(L_t^\epsilon)^*\eta$ converges in $L^2(\mathbb{R}^4)$ to :

$$\int_0^{2\pi n} R_{-s}\nabla_v \cdot \left(\nabla_{xx}^2 A(\frac{-s}{2\pi n}, v^\perp - R_{-s}v^\perp)\nabla_v \eta\right) ds.$$

Proof. We have :

$$\begin{aligned}
(L_t^\epsilon)^*\eta &= -\frac{1}{\epsilon}\int_0^{2\pi n\epsilon} D_\sigma^\epsilon \cdot (\mathbf{E}(S_\sigma^\epsilon E^\epsilon(t-\sigma)\otimes E^\epsilon(t))\cdot \nabla_v \eta)\, d\sigma \\
&= -\int_0^{2\pi n} D_{\epsilon\sigma}^\epsilon \cdot (\mathbf{E}(S_{\epsilon\sigma}^\epsilon E^\epsilon(t-\epsilon\sigma)\otimes E^\epsilon(t))\cdot \nabla_v)\eta\, d\sigma.
\end{aligned}$$

But thanks to the definition to $T_\epsilon(s)$, E^ϵ and Hypothesis (H3), the term

$$\mathbf{E}(S_{\epsilon\sigma}^\epsilon E^\epsilon(t-\epsilon\sigma)\otimes E^\epsilon(t)) = \mathbf{E}(\nabla V^\epsilon(\frac{t-\epsilon\sigma}{2\pi n\epsilon}, x/\epsilon + v^\perp - R_{-\sigma}v^\perp)\otimes \nabla V^\epsilon(\frac{t}{2\pi n\epsilon}, x/\epsilon))$$

converges strongly to $(-\nabla_{xx}^2 A)(-\sigma/(2\pi n), v^\perp - R_{-\sigma}v^\perp)$ in $L^\infty((\mathbb{R}^+)^2; W^{1,\infty}(\mathbb{R}^2))$. Hence thanks to the definition of D_s^ϵ, $(L_t^\epsilon)^*\eta$ converges strongly to :

$$\int_0^{2\pi n} R_{-s}\nabla_v \cdot \left(\nabla_{xx}^2 A(\frac{-s}{2\pi n}, v^\perp - R_{-s}v^\perp)\nabla_v \eta\right) ds$$

in $L^\infty(\mathbb{R}^+\times\mathbb{R}^4)$. We recall that we have $D_{\epsilon s}^\epsilon = R_{-s}\nabla_v + \epsilon R_{-s}\nabla_x^\perp - \epsilon\nabla_x^\perp$ thus

$$\begin{aligned}
&\|D_{\epsilon s'}^\epsilon \Phi\|_{L^2(\mathbb{R}^+\times\mathbb{R}^4)} \leq C\|\Phi\|_{W^{1,2}(\mathbb{R}^+\times\mathbb{R}^4)}, \\
&\|D_{\epsilon s'}^\epsilon D_{\epsilon s}\Phi\|_{L^2(\mathbb{R}^+\times\mathbb{R}^4)} \leq C\|\Phi\|_{W^{2,2}(\mathbb{R}^+\times\mathbb{R}^4)},
\end{aligned}$$

hence

$$\begin{aligned}
|\langle r_t^\epsilon;\eta\rangle| \leq\ & C(n)\sqrt{\epsilon}\|\Phi\|_{W^{3,2}}\|f_\epsilon\|_{L^2} \\
& \sup_{s,s'}\{\mathbf{E}(|E^\epsilon(T(\epsilon s'))|\,(|E^\epsilon(T(\epsilon s))| + |D_{\epsilon s'}^\epsilon[E^\epsilon(T(\epsilon s))]|) \\
& (|E^\epsilon(t,x)| + |D_{\epsilon s}^\epsilon E^\epsilon(t,x)| + |D_{\epsilon s'}^\epsilon E^\epsilon(t,x)| + |D_{\epsilon s'}^\epsilon D_{\epsilon s}^\epsilon E^\epsilon(t,x)|))\}.
\end{aligned}$$

We then use the following bounds : (recall that $E^\epsilon(t,x) = \nabla V^\epsilon(\frac{t}{2\pi n\epsilon}, \frac{x}{\epsilon})$.)

$$\begin{aligned}
&|E^\epsilon| \leq \|\nabla_x V^\epsilon\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^2))} \\
&|D_{\epsilon s}^\epsilon E^\epsilon(t,x)| \leq \|\nabla_{xx}^2 V^\epsilon\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^2))} \\
&|D_{\epsilon s'}^\epsilon D_{\epsilon s}^\epsilon E^\epsilon(t,x)| \leq \|\nabla_{xxx}^3 V^\epsilon\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^2))} \\
&|D_{\epsilon s'}^\epsilon[E^\epsilon(T(\epsilon s))]| \leq C\epsilon|\nabla_x E^\epsilon|(T(\epsilon s)) \leq C\|\nabla_{xx}^2 V^\epsilon\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^2))}.
\end{aligned}$$

Then Hypothesis $(H1)$ ensures that

$$|\langle r_t^\epsilon;\eta\rangle| \leq C\sqrt{\epsilon}\|f_0^\epsilon\|_{L^2}N(\epsilon)\|\Phi\|_{W^{3,2}},$$

which ends the proof of the lemma. □

We can now state the following proposition :

PROPOSITION 7.6. *Assume that $\epsilon(N(\epsilon))^2$ converges to 0 when ϵ goes to 0. Then the convergence (up to a subsequence) of $J\mathbf{E}f_\epsilon$ to f holds in $C^0(\mathbb{R}^+;L^2(\mathbb{R}^4)-w)$, and f is solution to :*

$$\partial_t f + JL_t^0 Jf = 0, \tag{302}$$

where the operator L_t^0 is defined for every $\eta\in\mathcal{S}(\mathbb{R}^4)$ by :

$$(L_t^0)^*\eta = \int_0^{2\pi n} R_{-s}\nabla_v\cdot\left(\nabla_{xx}^2 A(\frac{-s}{2\pi n}, v^\perp - R_{-s}v^\perp)\nabla_v\eta\right) ds.$$

Proof. Thanks to the previous lemma, for every test function $\eta\in\mathcal{S}(\mathbb{R}^4)$:

$$|\langle r_t^\epsilon;\eta\rangle| \overset{\epsilon\to 0}{\longrightarrow} 0,$$

and $(L_t^\epsilon)^*\eta$ converges strongly in $L^2(\mathbb{R}^4)$ to $(L_t^0)^*\eta$. But, thanks to Proposition 7.3, f_ϵ converges weakly to f in L^2-w. So

$$\langle JL_t^\epsilon\mathbf{E}f_\epsilon;\eta\rangle = \langle\mathbf{E}f_\epsilon;(L_t^\epsilon)^*J\eta\rangle$$

converges to :

$$\langle f;(L_t^0)^*J\eta\rangle = \langle JL_t^0 f;\eta\rangle.$$

The function $f_\epsilon(t) - S_{2\pi n\epsilon}f_\epsilon(t-2\pi n\epsilon)$ converges to 0 in L^2-w as well. So e_t^ϵ converges to 0 in $\mathcal{S}'(\mathbb{R}^4)$. Passing to the limit in equation (297) gives equation (302). This shows that $\partial_t f_\epsilon$ is uniformly bounded in time in a negative Sobolev space. Hence f_ϵ converges to f in the space of continuous function in time with values in this Sobolev space. Finally since f_ϵ is bounded in $L^\infty(\mathbb{R}^+;L^2(\mathbb{R}^4))$, the convergence holds in $C^0([0,T];L^2(\mathbb{R}^4)-w)$ for every $T>0$. □

4.3. Convergence to the SHE model. Since $Jf = f$, we can introduce the gyroaverage function defined by :

$$\rho(t,x,e) = f(t,x,v),$$

for every v such that $2e = |v|^2$. This subsection is devoted to the proof of the following lemma :

LEMMA 7.7. *The function ρ lies in $C^0(\mathbb{R}^+;L^2(\mathbb{R}^2\times\mathbb{R}^+)-w)\cap L^\infty(\mathbb{R}^+;L^2(\mathbb{R}^2\times\mathbb{R}^+))$. It is solution to :*

$$\begin{cases} \partial_t\rho - \partial_e(a(e)\partial_e\rho) = 0 \\ \rho|_{t=0} = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f^0(t,x,R_\theta v)\,d\theta \end{cases}$$

where the diffusion parameter is defined by :

$$a(e) = \int_0^{2\pi n}\int_0^{2\pi} R_\theta v\cdot(-\nabla_{xx}^2 A)(-\frac{s}{2\pi n}, R_\theta v^\perp - R_{-s+\theta}v^\perp)\cdot R_{-s}R_\theta v\,ds\,d\theta,$$

for every v such that $e = |v|^2/2$.

Proof. Let us first compute the operator $JL_t^0 J$. Let η_1,η_1 be two test functions in $\mathcal{S}(\mathbb{R}^4)$. We have :

$$\begin{aligned}\langle\eta_1;JL_t^0 J\eta_2\rangle &= \langle(L_t^0)^*J\eta_1;J\eta_2\rangle \\ &= \int_{\mathbb{R}^4}\int_0^{2\pi n}\nabla_v J\eta_1(-\nabla_{xx}^2 A)(\frac{-s}{2\pi n}, v^\perp - R_{-s}v^\perp)R_{-s}\nabla_v J\eta_2\,ds\,dx\,dv.\end{aligned}$$

Let us denote ρ_{η_i} for $i=1,2$ the functions defined by :

$$\rho_{\eta_i}(x,\frac{|v|^2}{2}) = J\eta_i(x,v).$$

Using polar coordinates and noticing that $dv = d\theta\, de$ we find :

$$\begin{aligned}\langle \eta_1; JL_t^0 J\eta_2\rangle &= \int_{\mathbb{R}^2}\int_0^\infty \partial_e\rho_{\eta_1}(x,e)\partial_e\rho_{\eta_2}(x,e)\int_0^{2\pi n}\int_0^{2\pi} R_\theta\vec{e}\cdot \\ &\qquad (-\nabla^2_{xx}A)(\frac{-s}{2\pi n}, -R_{-s+\theta+\frac{\pi}{2}}\vec{e} + R_{\theta+\frac{\pi}{2}}\vec{e})\cdot R_{-s+\theta}\vec{e}\, ds\, d\theta\, de\, dx \\ &= \int_{\mathbb{R}^2}\int_0^\infty \partial_e\rho_{\eta_1}(x,e)\partial_e\rho_{\eta_2}(x,e)a(e)\, de\, dx,\end{aligned}$$

where $\vec{e} = (\sqrt{2e}, 0)$. Hence for every test function ρ_η, let us multiply it by Equation (302) and integrate with respect to x, v. Since $de\, d\theta = dv$ we find :

$$\partial_t \int_{\mathbb{R}^2}\int_0^\infty \rho(t,x,e)\rho_\eta(x,e)\, dx\, de = \int_{\mathbb{R}^2}\int_0^\infty \rho(t,x,e)\partial_e(a(e)\partial_e\rho_\eta(x,e))\, de\, dx.$$

This, with Proposition 7.6 gives the desired result. □

Remark : we have a family of equations parametrized by $x\in\mathbb{R}^2$, and the solutions of two equations at two distinct x do not interact.

4.4. Explicit computation of the diffusion coefficient. We derive in the following a suitable form to the diffusion coefficient $a(e)$. We will show, in particular, that $a(e)$ is non negative. From $(H4)$ the correlation function $A(t,x)$ is even with respect to t and x. This with $(H3)$ gives :

LEMMA 7.8. *The correlation function A satisfies :*

$$\begin{aligned}&\mathrm{Supp}A \subset [-2\pi n, 2\pi n]\times\mathbb{R}^2,\\ &\nabla_x A(0,0) = 0,\\ &\partial_s A(0,0) = 0.\end{aligned}$$

This last subsection is devoted to the following proposition. Theorem 7.1 follows from this proposition, Proposition 7.3 and Proposition 7.6.

PROPOSITION 7.9. *Let us denote*

$$\tilde{A}(t,x) = \frac{1}{2\pi}\int_0^{2\pi} A(R_\theta x, t)\, d\theta.$$

Then $a(e)$ is non negative and equal to :

$$\frac{1}{2\pi n^2}\int_0^\infty (-\partial^2_{tt}\tilde{A})(\frac{-s}{2\pi n}, 2\sqrt{e}\sqrt{1-\cos s})\, ds.$$

Proof. Thanks to Lemma 7.7 and lemma 7.8, we have

$$a(e) = \int_{s=0}^\infty\int_0^{2\pi} R_\theta v\cdot(-\nabla^2_{xx}A)(-\frac{s}{2\pi n}, R_\theta v^\perp - R_{-s}R_\theta v^\perp)\cdot R_{-s}R_\theta v\, ds\, d\theta.$$

Since

$$\begin{aligned}&-\nabla^2_{xx}A(-\frac{s}{2\pi n}, v^\perp - R_{-s}v^\perp)\cdot R_{-s}v\\ &= \frac{1}{2\pi n}\nabla_x\partial_s A(-\frac{s}{2\pi n}, v^\perp - R_{-s}v^\perp) + \partial_s(\nabla_x A(-\frac{s}{2\pi n}, v^\perp - R_{-s}v^\perp)),\end{aligned}$$

we find

$$\begin{aligned} a(e) &= \frac{1}{2\pi n}\int_{s=0}^{\infty}\int_0^{2\pi} R_\theta v\cdot\nabla_x\partial_s A(-\frac{s}{2\pi n}, R_\theta v^\perp - R_{\theta-s}v^\perp)\,ds\,d\theta \\ &\quad -\int_0^{2\pi} R_\theta v\cdot\nabla_x A(0,0)\,d\theta \\ &= \frac{1}{2\pi n}\int_{s=0}^{\infty}\int_0^{2\pi} R_\theta \vec{e}\cdot\nabla_x\partial_s A(-\frac{s}{2\pi n}, R_{\theta+\pi/2}\vec{e} - R_{-s+\theta+\pi/2}\vec{e})\,ds\,d\theta \end{aligned}$$

where $\vec{e} = (\sqrt{2e}, 0)$. Let us do the change of variables $s' = \theta - s$ to get

$$a(e) = \frac{1}{2\pi n}\int_0^{2\pi}\int_{\mathbb{R}} \mathbf{1}_{\{s\le\theta\}} R_\theta\vec{e}\cdot\nabla_x\partial_s A(\frac{s-\theta}{2\pi n}, R_{\theta+\pi/2}\vec{e} - R_{s+\pi/2}\vec{e})\,d\theta\,ds.$$

Next we have

$$\begin{aligned} R_\theta\vec{e}\cdot\nabla_x\partial_s A(\frac{s-\theta}{2\pi n}, R_{\theta+\pi/2}\vec{e} - R_{s+\pi/2}\vec{e}) &= -\frac{1}{2\pi n}\partial^2_{ss}A(\frac{s-\theta}{2\pi n}, R_{\theta+\pi/2}\vec{e} - R_{s+\pi/2}\vec{e}) \\ &\quad -\partial_\theta\left\{\partial_s A(\frac{s-\theta}{2\pi n}, R_{\theta+\pi/2}\vec{e} - R_{s+\pi/2}\vec{e})\right\}. \end{aligned}$$

Integrating by parts the second term of the RHS gives

$$\begin{aligned} &\frac{1}{2\pi n}\int_0^{2\pi}\int_{\mathbb{R}} \mathbf{1}_{\{s\le\theta\}}\partial_\theta\left\{\partial_s A(\frac{s-\theta}{2\pi n}, R_{\theta+\pi/2}\vec{e} - R_{s+\pi/2}\vec{e})\right\}ds\,d\theta \\ &= \frac{1}{2\pi n}\int_0^{2\pi}\partial_\theta\int_{\mathbb{R}} \mathbf{1}_{\{s\le\theta\}}\partial_s A(\frac{s-\theta}{2\pi n}, R_{\theta+\pi/2}\vec{e} - R_{s+\pi/2}\vec{e})\,ds\,d\theta \\ &- \frac{1}{2\pi n}\int_0^{2\pi}\int_{\mathbb{R}} \delta_{s=\theta}\partial_s A(\frac{s-\theta}{2\pi n}, R_{\theta+\pi/2}\vec{e} - R_{s+\pi/2}\vec{e})\,ds\,d\theta \\ &= \frac{1}{2\pi n}\int_{-\infty}^{2\pi}\partial_s A(\frac{s-2\pi}{2\pi n}, R_{\pi/2}\vec{e} - R_{s+\pi/2}\vec{e})\,ds \\ &- \frac{1}{2\pi n}\int_{-\infty}^{0}\partial_s A(\frac{s}{2\pi n}, R_{\pi/2}\vec{e} - R_{s+\pi/2}\vec{e})\,ds \\ &- \frac{1}{2\pi n}\int_0^{2\pi}\partial_s A(0,0)\,ds \end{aligned}$$

The first two lines cancel by doing the change of variables $s' = s - 2\pi$ and the third line vanishes thanks to Lemma 7.8, thus

$$a(e) = \frac{1}{(2\pi n)^2}\int_0^{2\pi}\int_{\mathbb{R}} \mathbf{1}_{\{s\le\theta\}}(-\partial^2_{ss}A)(\frac{s-\theta}{2\pi n}, R_{\theta+\pi/2}\vec{e} - R_{s+\pi/2}\vec{e})\,d\theta\,ds$$

Doing the change of variables $s' = \theta - s$ gives

$$\begin{aligned} a(e) &= \frac{1}{(2\pi n)^2}\int_0^{2\pi}\int_0^{\infty}(-\partial^2_{ss}A)(-\frac{s}{2\pi n}, R_{\theta+\pi/2}((I-R_{-s})\vec{e}))\,d\theta\,ds \\ &= \frac{1}{2\pi n^2}\int_0^{\infty}(-\partial^2_{ss}\tilde{A})(-\frac{s}{2\pi n}, |(I-R_{-s})\vec{e}|)\,ds. \end{aligned}$$

Finally

$$\begin{aligned} |(I-R_s)\vec{e}| &= \sqrt{(|1-\cos s|^2+\sin^2 s)}\sqrt{2e} \\ &= \sqrt{2(1-\cos s)}\sqrt{2e} \\ &= 2\sqrt{e}\sqrt{1-\cos s} \\ &= 2\sqrt{2e}|\sin(s/2)| \end{aligned}$$

which ends the proof of the second assertion. □

Computation of the sign of the diffusion coefficient.
Here we check the non-negativity of the diffusion coefficient by expressing it in another form. Thanks to lemma 7.7 and to hypothesis $(H3),(H4)$ we have

$$a(e) = \frac{1}{2N}\int_{s=0}^{+\infty}\int_{-2\pi N}^{2\pi N} R_\theta v \cdot (-\nabla^2_{xx}A)(-\frac{s}{2\pi n}, R_\theta v^\perp - R_{\theta-s}v^\perp)\cdot R_{\theta-s}v\, ds\, d\theta$$

for all N. Then doing the change of variable $s := \theta - s$ we find :

$$a(e) = \frac{1}{2N}\int_{s\in\mathbb{R}}\int_{-2\pi N}^{2\pi N} \mathbf{1}_{\{\theta\geq s\}} R_\theta v\cdot(-\nabla^2_{xx}A)(\frac{s-\theta}{2\pi n}, R_\theta v^\perp - R_s v^\perp)\cdot R_s v ds d\theta$$

But we remind that thanks to hypothesis $(H4)$

$$-\nabla^2_{xx}A(\frac{s-\theta}{2\pi n}, R_\theta v^\perp - R_s v^\perp) = \lim_{\epsilon\to 0}\mathbf{E}\left(\nabla_x V^\epsilon(-\frac{s}{2\pi n}, -R_s v^\perp)\otimes\nabla_x V^\epsilon(-\frac{\theta}{2\pi n}, -R_\theta v^\perp)\right)$$

Thus

$$a(e) = \lim_{\epsilon,N}\frac{1}{2N}\cdot$$
$$\int\int_{-2\pi N}^{2\pi N}\mathbf{1}_{\{\theta\geq s\}}\mathbf{E}\left([\nabla_x V^\epsilon(-\frac{s}{2\pi n}, -R_s v^\perp)\cdot R_s v][\nabla_x V^\epsilon(-\frac{\theta}{2\pi n}, -R_\theta v^\perp)\cdot R_\theta v]\right) ds d\theta$$

Interverting s and θ we see that we can replace $\mathbf{1}_{\{\theta-s\geq 0\}}$ by $\mathbf{1}_{\{s-\theta\geq 0\}}$ and finally by adding both we obtain :

$$a(e) = \lim_{\epsilon,N}\frac{1}{4N}\mathbf{E}\left(\left[\int_{-2\pi N}^{2\pi N}\nabla_x V^\epsilon(-\frac{s}{2\pi n}, -R_s v^\perp)\cdot R_s v ds\right]^2\right)$$

which is a positive quantity. □

Acknowledgments : The idea of this paper was born as the authors attended a course on plasma's turbulence given by Xavier Garbet at the CEA center of of Cadarache. They are glad to thank him for having introduced them to the subject and for subsequent discussions.

Bibliographie

[1] S. Alinhac and P. Gérard. *Opérateurs pseudo-différentiels et théorème de Nash-Moser*. Savoirs Actuels. [Current Scholarship]. InterEditions, Paris, 1991.

[2] V. I. Arnold and B. A. Khesin. *Topological methods in hydrodynamics*, volume 125 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1998.

[3] G. Barles. *Solutions de viscosité des équations de Hamilton-Jacobi*. Mathématiques & Applications [Mathematics & Applications], 17. Springer-Verlag, Paris, 1994.

[4] J. Bec. *Particules, Singularités et Turbulence*. PhD thesis, Thèse de l'Université Pierre et Marie Curie, 2002.

[5] N. Ben Abdallah, P. Degond, P. Markowich, and C. Schmeiser. High field approximations of the spherical harmonics expansion model for semiconductors. *Z. Angew. Math. Phys.*, 52(2) :201–230, 2001.

[6] J.-D. Benamou and Y. Brenier. Weak existence for the semigeostrophic equations formulated as a coupled Monge-Ampère/transport problem. *SIAM J. Appl. Math.*, 58(5) :1450–1461 (electronic), 1998.

[7] J.-D. Benamou and Y. Brenier. A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem. *Numer. Math.*, 84(3) :375–393, 2000.

[8] G. Bouchitté, G. Buttazzo, and I. Fragalà. Convergence of sobolev spaces on varying manifolds. *J. Geom. Anal.*, 2001.

[9] F. Bouchut. Renormalized solutions to the Vlasov equation with coefficients of bounded variation. *Arch. Ration. Mech. Anal.*, 157(1) :75–90, 2001.

[10] Y. Brenier. Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math.*, 44(4) :375–417, 1991.

[11] Y. Brenier. A homogenized model for vortex sheets. *Arch. Rational Mech. Anal.*, 138(4) :319–353, 1997.

[12] Y. Brenier. Minimal geodesics on groups of volume-preserving maps and generalized solutions of the Euler equations. *Comm. Pure Appl. Math.*, 52(4) :411–452, 1999.

[13] Y. Brenier. Convergence of the Vlasov-Poisson system to the incompressible Euler equations. *Comm. Partial Differential Equations*, 25(3-4) :737–754, 2000.

[14] Y. Brenier. Derivation of the Euler equations from a caricature of Coulomb interaction. *Comm. Math. Phys.*, 212(1) :93–104, 2000.

[15] Y. Brenier, U. Frisch, M. Hénon, G. Loeper, S. Matarrese, R. Mohayaee, and A. Sobolevskii. Reconstruction of the early universe as a convex optimization problem. in preparation, available on http ://www.Arxiv.org.

[16] Y. Brenier and G. Loeper. The Vlasov-Monge-Ampère equation. in preparation.

[17] H. Brézis. *Analyse fonctionnelle*. Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris, 1983. Théorie et applications. [Theory and applications].

[18] L. A. Caffarelli. Interior $W^{2,p}$ estimates for solutions of the Monge-Ampère equation. *Ann. of Math. (2)*, 131(1) :135–150, 1990.

[19] L. A. Caffarelli. A localization property of viscosity solutions to the Monge-Ampère equation and their strict convexity. *Ann. of Math. (2)*, 131(1) :129–134, 1990.

[20] L. A. Caffarelli. Some regularity properties of solutions of Monge Ampère equation. *Comm. Pure Appl. Math.*, 44(8-9) :965–969, 1991.

[21] L. A. Caffarelli. The regularity of mappings with a convex potential. *J. Amer. Math. Soc.*, 5(1) :99–104, 1992.

[22] L. A. Caffarelli and C. E. Gutiérrez. Properties of the solutions of the linearized Monge-Ampère equation. *Amer. J. Math.*, 119(2) :423–465, 1997.

[23] E. Camalet. PhD thesis, Université de Paris 6, 1995.

[24] D. Cordero-Erausquin. Sur le transport de mesures périodiques. *C. R. Acad. Sci. Paris Sér. I Math.*, 329(3) :199–202, 1999.

[25] S. Cordier and E. Grenier. Quasineutral limit of an Euler-Poisson system arising from plasma physics. *Comm. Partial Differential Equations*, 25(5-6) :1099–1113, 2000.

[26] M. Cullen and W. Gangbo. A variational approach for the 2-dimensional semi-geostrophic shallow water equations. *Arch. Ration. Mech. Anal.*, 156(3) :241–273, 2001.

[27] M. J. P. Cullen and R. J. Purser. Properties of the Lagrangian semigeostrophic equations. *J. Atmospheric Sci.*, 46(17) :2684–2697, 1989.

[28] P. Degond. An infinite system of diffusion equations arising in transport theory : the coupled spherical harmonics expansion model. *Math. Models Methods Appl. Sci.*, 11(5) :903–932, 2001.

[29] P. Delanoë. Classical solvability in dimension two of the second boundary-value problem associated with the Monge-Ampère operator. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 8(5) :443–457, 1991.

[30] R. J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, 98(3) :511–547, 1989.

[31] D. G. Ebin. The motion of slightly compressible fluids viewed as a motion with strong constraining force. *Ann. of Math. (2)*, 105(1) :141–200, 1977.

[32] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[33] L. C. Evans and D. Gomes. Effective Hamiltonians and averaging for Hamiltonian dynamics. I. *Arch. Ration. Mech. Anal.*, 157(1) :1–33, 2001.

[34] A. Fannjiang, L. Ryzhik, and G. Papanicolaou. Evolution of trajectory correlations in steady random flows. In *Recent advances in partial differential equations, Venice 1996*, volume 54 of *Proc. Sympos. Appl. Math.*, pages 105–130. Amer. Math. Soc., Providence, RI, 1998.

[35] E. Frénod and E. Sonnendrücker. Homogenization of the Vlasov equation and of the Vlasov-Poisson system with a strong external magnetic field. *Asymptot. Anal.*, 18(3-4) :193–213, 1998.

[36] E. Frénod and E. Sonnendrücker. The finite Larmor radius approximation. *SIAM J. Math. Anal.*, 32(6) :1227–1247 (electronic), 2001.

[37] U. Frisch, S. Matarrese, R. Mohayaee, and A. Sobolevski. A reconstruction of the initial conditions of the universe by optimal mass transportation. *Nature*, 2002.

[38] X. Garbet. Transport et turbulence. *Publication du CEA*.

[39] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*, volume 224 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1983.

[40] F. Golse and L. Saint-Raymond. The Vlasov-Poisson system with strong magnetic field. *J. Math. Pures Appl. (9)*, 78(8) :791–817, 1999.

[41] E. Grenier. Pseudo-differential energy estimates of singular perturbations. *Comm. Pure Appl. Math.*, 50(9) :821–865, 1997.

[42] Y. Guo. Smooth irrotational flows in the large to the Euler-Poisson system in $\mathbf{R}^{3+1}$. *Comm. Math. Phys.*, 195(2) :249–265, 1998.

[43] H. Kesten and G. C. Papanicolaou. A limit theorem for turbulent diffusion. *Comm. Math. Phys.*, 65 :97–128, 1979.

[44] H. Kesten and G. C. Papanicolaou. A limit theorem for stochastic acceleration. *Comm. Math. Phys.*, 78 :19–63, 1980/81.

[45] S. Klainerman and A. Majda. Compressible and incompressible fluids. *Comm. Pure Appl. Math.*, 35(5) :629–651, 1982.

[46] P.-L. Lions and B. Perthame. Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system. *Invent. Math.*, 105(2) :415–430, 1991.

[47] M. C. Lopes Filho and H. J. Nussenzveig Lopes. Existence of a weak solution for the semigeostrophic equation with integrable initial data. *Proc. Roy. Soc. Edinburgh Sect. A*, 132(2) :329–339, 2002.

[48] R. J. McCann. A convexity principle for interacting gases. *Adv. Math.*, 128(1) :153–179, 1997.

[49] R. J. McCann. Polar factorization of maps on Riemannian manifolds. *Geom. Funct. Anal.*, 11(3) :589–608, 2001.

[50] G. Métivier and S. Schochet. The incompressible limit of the non-isentropic Euler equations. *Arch. Ration. Mech. Anal.*, 158(1) :61–90, 2001.

[51] Yu. A. Neretin. Categories of bistochastic measures and representations of some infinite-dimensional groups. *Mat. Sb.*, 183(2) :52–76, 1992.

[52] J. Nieto, F. Poupaud, and J. Soler. High-field limit for the Vlasov-Poisson-Fokker-Planck system. *Arch. Ration. Mech. Anal.*, 158(1) :29–59, 2001.

[53] I. G. Nikolaev and S. Z. Shefel′. Convex surfaces with positive bounded specific curvature, and a priori estimates for Monge-Ampère equations. *Sibirsk. Mat. Zh.*, 26(4) :120–136, 205, 1985.

[54] F. Otto. The geometry of dissipative evolution equations : the porous medium equation. *Comm. Partial Differential Equations*, 26(1-2) :101–174, 2001.

[55] B. Perthame. Nonexistence of global solutions to Euler-Poisson equations for repulsive forces. *Japan J. Appl. Math.*, 7(2) :363–367, 1990.

[56] K. Pfaffelmoser. Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data. *J. Differential Equations*, 95(2) :281–303, 1992.

[57] F. Poupaud and A. Vasseur. Classical and quantum transport in random media. *Journal de mathématiques pures et appliquées*, To appear.

[58] H. Rubin and P. Ungar. Motion under a strong constraining force. *Comm. Pure Appl. Math.*, 10 :65–87, 1957.

[59] L. Saint-Raymond. The gyrokinetic approximation for the Vlasov-Poisson system. *Math. Models Methods Appl. Sci.*, 10(9) :1305–1332, 2000.

[60] J. Schaeffer. Global existence of smooth solutions to the Vlasov-Poisson system in three dimensions. *Comm. Partial Differential Equations*, 16(8-9) :1313–1335, 1991.

[61] J. Urbas. On the second boundary value problem for equations of Monge-Ampère type. *J. Reine Angew. Math.*, 487 :115–124, 1997.

[62] C. Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.

[63] X. J. Wang. Remarks on the regularity of Monge-Ampère equations. In *Proceedings of the International Conference on Nonlinear P.D.E.(Hangzhou, 1992)*.

Reconstruction of the early Universe as a convex optimization problem

Y. Brenier,[1] U. Frisch,[2,3⋆] M. Hénon,[2] G. Loeper,[1] S. Matarrese,[4]
R. Mohayaee,[2] A. Sobolevskiĭ[2,5]

[1] *CNRS, UMR 6621, Université de Nice-Sophia-Antipolis, Parc Valrose, 06108 Nice Cedex 02, France*
[2] *CNRS, UMR 6529, Observatoire de la Côte d'Azur, BP 4229, 06304 Nice Cedex 4, France*
[3] *Institute for Advanced Study, Einstein Drive, Princeton, NJ 08540, USA*
[4] *Dipartimento di Fisica 'G. Galilei' and INFN, Sezione di Padova, via Marzolo 8, 35131-Padova, Italy*
[5] *Department of Physics, M. V. Lomonossov Moscow University, Leninskie Gory, 119992 Moscow, Russia*

10 January 2005

ABSTRACT

We show that the deterministic past history of the Universe can be uniquely reconstructed from the knowledge of the present mass density field, the latter being inferred from the 3D distribution of luminous matter, assumed to be tracing the distribution of dark matter up to a known bias. Reconstruction ceases to be unique below those scales – a few Mpc – where multi-streaming becomes significant. Above $6\,h^{-1}$ Mpc we propose and implement an effective Monge–Ampère–Kantorovich method of unique reconstruction. At such scales the Zel'dovich approximation is well satisfied and reconstruction becomes an instance of optimal mass transportation, a problem which goes back to Monge (1781). After discretization into N point masses one obtains an assignment problem that can be handled by effective algorithms with not more than $O(N^3)$ time complexity and reasonable CPU time requirements. Testing against N-body cosmological simulations gives over 60% of exactly reconstructed points.

We apply several interrelated tools from optimization theory that were not used in cosmological reconstruction before, such as the Monge–Ampère equation, its relation to the mass transportation problem, the Kantorovich duality and the auction algorithm for optimal assignment. Self-contained discussion of relevant notions and techniques is provided.

Key words: cosmology: theory – large-scale structure of the Universe – hydrodynamics

1 INTRODUCTION

Can one follow back in time to initial locations the highly structured present distribution of mass in the Universe, as mapped by redshift catalogues of galaxies? At first this seems an ill-posed problem since little is known about the peculiar velocities of galaxies, so that equations governing the dynamics cannot just be integrated back in time. In fact, it is precisely one of the goals of reconstruction to determine the peculiar velocities. Since the pioneering work of Peebles (1989), a number of reconstruction techniques have been proposed, which frequently provided non-unique answers.[1]

Cosmological reconstruction should however take advantage of our knowledge that the initial mass distribution was quasi-uniform at baryon-photon decoupling, about 14 billion years ago (see, e.g., Susperregi & Binney 1994). In a recent Letter to Nature (Frisch et al. 2002), four of us have shown that, with suitable assumptions, this *a priori* knowledge of the initial density field makes reconstruction a well-posed instance of what is called the optimal mass transportation problem.

A well-known fact is that, in an expanding universe with self-gravitating matter, the initial velocity field is 'slaved' to the initial gravitational field, which is potential; both fields thus depend on a single scalar function. Hence the number of unknowns matches the number of constraints, namely the single density function characterising the present distribution of mass.

This observation alone, of course, does not ensure

⋆ E-mail: uriel@obs-nice.fr

[1] The reader will find a detailed discussion of several important existing techniques in Section 7.

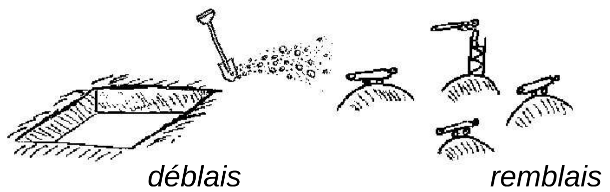


Figure 1. A sketch of Monge's mass transportation problem in which one searches the optimal way of transporting earth from cuts (*déblais*) to fills (*remblais*), each of prescribed shape; the cost of transporting a molecule of earth is a given function of the distance. The MAK method of reconstructing the early Universe described in this paper corresponds to a quadratic cost.

uniqueness of the reconstruction. For this, two restrictions will turn out to be crucial. First, from standard redshift catalogues it is impossible to resolve individual streams of matter with different velocities if they occupy the same space volume. This 'multi-streaming' is typically confined to relatively small scales of a few megaparsecs (Mpc), below which reconstruction is hardly feasible. Second, to reconstruct a given finite patch of the present Universe, we need to know its initial shape at least approximately.

It is our purpose in the present paper to clarify the physical nature of the factors permitting a unique reconstruction and of obstacles limiting it, and to give a detailed account of the way some recent developments in the optimal mass transportation theory are applicable. (Fig. 1 may give the reader some feeling of what mass transportation is about.)

The paper is organized as follows. In Section 2 we formulate the reconstruction problem in an expanding universe and state the main result about uniqueness of the solution.

In the next three sections we devise and test a reconstruction technique called MAK (for Monge–Ampère–Kantorovich) within a restricted framework where the Lagrangian map from initial to present mass locations is taken potential. In Section 3 we discuss the validity of the potentiality assumption and its relation to various approximations used in cosmology; then we derive the Monge–Ampère equation, a simple consequence of mass conservation, introduce its modern reformulation as a Monge–Kantorovich problem of optimal mass transportation and finally discuss different limitations on uniqueness of the reconstruction. In Section 4 we show how discretization turns optimization into an instance of the standard assignment problem; we then present effective algorithms for its solution, foremost the 'auction' algorithm of D. Bertsekas. Section 5 is devoted to testing the MAK reconstruction against N-body cosmological simulations.

In Section 6, we show how the general case, without the potentiality assumption, can also be recast as an optimization problem with a unique solution and indicate a possible numerical strategy for such reconstruction. In Section 7 we compare our reconstruction method with other approaches in the literature. In Section 8 we discuss perspectives and open problems.

A number of topics are left for appendices. In Appendix A we derive the Eulerian and Lagrangian equations in the form used throughout the paper (and provide some background for non-cosmologists). Appendix B is devoted to the history of optimal mass transportation theory, a subject more than two centuries old (Monge 1781), which has undergone significant progress within the last two decades. Appendix C is a brief elementary introduction to the technique of duality in optimization, which we use several times throughout the paper. Appendix D gives details of the uniqueness proof that is only outlined in Section 6.

Finally, a word about notation (see also Appendix A). We are using comoving coordinates denoted by $\boldsymbol{x}$ in a frame following expansion of the Universe. Our time variable is not the cosmic time but the so-called linear growth factor, here denoted by τ, whose use gives to certain equations the same form as for compressible fluid dynamics in a non-expanding medium. The subscript 0 refers to the present time (redshift $z = 0$), while the quantities evaluated at the initial epoch take the subscript or superscript 'in.' Following cosmological usage, the Lagrangian coordinate is denoted $\boldsymbol{q}$.

2 RECONSTRUCTION IN AN EXPANDING UNIVERSE

The most widely accepted explanation of the large-scale structure seen in galaxy surveys is that it results from small primordial fluctuations that grew under gravitational self-interaction of collisionless cold dark matter (CDM) particles in an expanding universe (see, e.g., Bernardeau et al. (2002) and references therein). The relevant equations of motion, derived in Appendix A, are the Euler–Poisson equations[2] written here for a flat, matter-dominated Einstein–de Sitter universe (for more general case see, e.g., Catelan et al. 1995):

$$\partial_\tau \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}})\boldsymbol{v} = -\frac{3}{2\tau}(\boldsymbol{v} + \nabla_{\boldsymbol{x}}\varphi_{\rm g}), \quad (1)$$

$$\partial_\tau \rho + \nabla_{\boldsymbol{x}}\cdot(\rho\boldsymbol{v}) = 0, \quad (2)$$

$$\nabla^2_{\boldsymbol{x}}\varphi_{\rm g} = \frac{\rho - 1}{\tau}. \quad (3)$$

Here $\boldsymbol{v}$ denotes the velocity, ρ denotes the density (normalized by the background density $\bar{\varrho}$) and $\varphi_{\rm g}$ is a rescaled gravitational potential. All quantities are expressed in comoving spatial coordinates $\boldsymbol{x}$ and linear growth factor τ, which is used as the time variable; in particular, $\boldsymbol{v}$ is the Lagrangian τ-time derivative of the comoving coordinate of a fluid element.

2.1 Slaving in early-time dynamics and its fossils

The right-hand sides of the momentum and Poisson equations (1) and (3) contain denominators proportional to τ. Hence, a necessary condition for the problem not to be singular as $\tau \to 0$ is

$$\boldsymbol{v}_{\rm in}(\boldsymbol{x}) + \nabla_{\boldsymbol{x}}\varphi^{\rm in}_{\rm g} = 0, \qquad \rho_{\rm in}(\boldsymbol{x}) = 1. \quad (4)$$

In other words, (i) the initial velocity must be equal to (minus) the gradient of the initial gravitational potential and (ii) the initial normalized mass distribution is uniform. We shall refer to these conditions as *slaving*. Note that the density contrast $\rho - 1$ vanishes initially, but the rescaled gravitational potential and the velocity, as defined here, stay finite

[2] Also often called the Euler equations.

thanks to our choice of the linear growth factor as time variable. Therefore we refer to the initial mass distribution as 'quasi-uniform.'

In the sequel, when we mention the Euler–Poisson *initial-value problem*, it is always understood that we start at $\tau = 0$ and assume slaving. Hence we are extending the Newtonian matter-dominated post-decoupling description back to $\tau = 0$. By examination of the Lagrangian equations for $\boldsymbol{x}(\boldsymbol{q}, \tau)$ near $\tau = 0$, which can be linearized because the displacement $\boldsymbol{x} - \boldsymbol{q}$ is small, it is easily shown that slaving implies the absence of the 'decaying mode,' which behaves as $\tau^{-3/2}$ in an Einstein–de Sitter universe and is thus singular at $\tau = 0$ (for details see Appendix A).

Slaving is also a sufficient condition for the initial problem to be well posed. It is indeed easily shown recursively that (1)–(3) admit a solution in the form of a formal Taylor series in τ (a related expansion involving only potentials may be found in Catelan et al. 1995):

$$\boldsymbol{v}(\boldsymbol{x}, \tau) = \boldsymbol{v}^{(0)}(\boldsymbol{x}) + \tau \boldsymbol{v}^{(1)}(\boldsymbol{x}) + \tau^2 \boldsymbol{v}^{(2)}(\boldsymbol{x}) + \cdots, \quad (5)$$

$$\varphi_{\mathrm{g}}(\boldsymbol{x}, \tau) = \varphi_{\mathrm{g}}^{(0)}(\boldsymbol{x}) + \tau \varphi_{\mathrm{g}}^{(1)}(\boldsymbol{x}) + \tau^2 \varphi_{\mathrm{g}}^{(2)}(\boldsymbol{x}) + \cdots, \quad (6)$$

$$\rho(\boldsymbol{x}, \tau) = 1 + \tau \rho^{(1)}(\boldsymbol{x}) + \tau^2 \rho^{(2)}(\boldsymbol{x}) + \cdots. \quad (7)$$

Furthermore, $\boldsymbol{v}^{(n)}(\boldsymbol{x})$ is easily shown to be curl-free for any n.

We conjecture that a stronger result holds: if the initial gravitational potential is analytic in the space variable (as is likely in the cosmological problem because high-order spatial harmonics are mollified by various damping and dissipation processes), analyticity in both space and τ-time is preserved up to the time of formation of caustics (also called time of first shell-crossing).

Several important consequences of slaving extend to later times as 'fossils' of the earliest dynamics. First, as already stressed in the Introduction, the whole dynamics is determined by only one scalar field (e.g., the initial gravitational potential) which we can hope to determine from the knowledge of the present density field.

Second, slaving trivially rules out multi-streaming up to the time of formation of caustics. Since we are working with collisionless matter, the dynamics should in principle be governed by the Vlassov–Poisson[3] kinetic equation which allows at each $(\boldsymbol{x}, \tau)$ point a non-trivial distribution function $f(\boldsymbol{x}, \boldsymbol{v}, \tau)$. Slaving selects a particular class of solutions for which the distribution function is concentrated on a single-speed manifold, thereby justifying the use of the Euler–Poisson equation without having to invoke any hydrodynamical limit (see, e.g., Vergassola et al. 1994; Catelan et al. 1995).

Third, it is easily checked from (1) that the initial slaved velocity, which is obviously curl-free, remains so for all later times (up to formation of caustics). Note that this vanishing of the curl holds in Eulerian coordinates. A similar property in Lagrangian coordinates can only hold approximately but will play an important role in the sequel (Section 3).

[3] Actually written for the first time by Jeans (1919).

2.2 Formulation of the reconstruction problem

The present Universe is replete with high-density structures: clusters (point-like objects), filaments (line-like objects) and perhaps sheets or walls.[4]

The internal structure of such *mass concentrations* certainly displays multi-streaming and cannot be described in terms of a single-speed solution to the Euler–Poisson equations. In N-body simulations, multi-stream regions are usually found to be of relatively small extension in one or several space directions, typically not more than a few Mpc, and hence have a small volume, although they contain a significant fraction of the total mass (see, e.g. Weinberg & Gunn 1990).

In order not to have to deal with tiny multi-stream regions, we replace the true mass distribution by a 'macroscopic' one which has a regular part and a singular (collapsed) part, the latter concentrated on objects of dimension less than three, such as points or lines.

The general problem of reconstruction is to find as much information as possible on the history of the evolution that carries the initial uniform density into the present macroscopic mass distribution, including the evolution of the velocities. In principle we would like to find a solution of the Euler–Poisson initial-value problem leading to the present density field $\rho_0(\boldsymbol{x})$.

A more restricted problem, which we call the 'displacement reconstruction,' is to find the Lagrangian map $\boldsymbol{q} \mapsto \boldsymbol{x}(\boldsymbol{q})$ and its inverse $\boldsymbol{x} \mapsto \boldsymbol{q}(\boldsymbol{x})$, or in other words to answer the question: where does a given 'Monge molecule'[5] of matter originate from? Of course, the inverse Lagrangian map will not be single-valued on mass concentrations. Furthermore, for practical cosmological applications, we define a 'full reconstruction problem' as (i) displacement reconstruction and (ii) obtaining the initial and present peculiar velocity fields, $\boldsymbol{v}_{\mathrm{in}}(\boldsymbol{q})$ and $\boldsymbol{v}_0(\boldsymbol{x})$.

We shall show in this paper that the displacement reconstruction problem is uniquely solvable and that the full reconstruction problem has a unique solution outside of mass concentrations; as to the latter, they are traced back to *collapsed regions* in the Lagrangian space whose shape and positions are well defined but the inner structure of density and velocity fluctuations is irretrievably lost.

3 POTENTIAL LAGRANGIAN MAPS: THE MAK RECONSTRUCTION

In this and the next two sections we shall assume that the Lagrangian map from initial positions to present ones is potential

$$\boldsymbol{x} = \nabla_{\boldsymbol{q}} \Phi(\boldsymbol{q}), \quad (8)$$

and furthermore that the potential $\Phi(\boldsymbol{q})$ is convex, which is, as we shall see, related to the absence of multi-streaming.

[4] Whether the Great Wall and the Sculptor Wall are sheet-like or filament-like is a moot point (Sathyaprakash et al. 1998).

[5] For Monge and his contemporaries, the word 'molecule' meant a Leibniz infinitesimal element of mass; see Appendix B.

3.1 Approximations leading to maps with convex potentials

The motivation for the potential assumption, first used by Bertschinger & Dekel (1989),[6] comes from the Zel'dovich approximation (Zel'dovich 1970), denoted here by ZA, and its refinements. To recall how the ZA comes about, let us start from the equations for the Lagrangian map $\boldsymbol{x}(\boldsymbol{q},\tau)$, written in the Lagrangian coordinate $\boldsymbol{q}$ (Appendix A)

$$D_\tau^2\boldsymbol{x} = -\frac{3}{2\tau}(D_\tau\boldsymbol{x}+\nabla_{\boldsymbol{x}}\varphi_{\rm g}), \quad (9)$$

$$\nabla_{\boldsymbol{x}}^2\varphi_{\rm g} = \frac{1}{\tau}\left[(\det\nabla_{\boldsymbol{q}}\boldsymbol{x})^{-1}-1\right], \quad (10)$$

where D_τ is the Lagrangian time derivative and $\nabla_{x_i} \equiv (\partial q_j/\partial x_i)\nabla_{q_j}$ is the Eulerian gradient rewritten in Lagrangian coordinates. As shown in Appendix A, in one space dimension the Hubble drag term $D_\tau\boldsymbol{x}$ and the gravitational acceleration term $\nabla_{\boldsymbol{x}}\varphi_{\rm g}$ cancel exactly. Slaving, discussed in Section 2.1, means that the same cancellation holds to leading order in any dimension for small τ. The ZA extends this as an approximation without the restriction of small τ. Within the ZA, the acceleration $D_\tau^2\boldsymbol{x}$ vanishes. Hence the Lagrangian map has the form

$$\begin{aligned}\boldsymbol{x}(\boldsymbol{q},\tau) &= \boldsymbol{q}+\tau(D_\tau\boldsymbol{x})_{\rm in}(\boldsymbol{q}) = \boldsymbol{q}-\tau\nabla_{\boldsymbol{q}}\varphi_{\rm g}^{\rm in}(\boldsymbol{q}) \\ &= \nabla_{\boldsymbol{q}}\Phi(\boldsymbol{q},\tau)\end{aligned} \quad (11)$$

with the potential

$$\Phi(\boldsymbol{q},\tau) \equiv \frac{|\boldsymbol{q}|^2}{2} - \tau\varphi_{\rm g}^{\rm in}(\boldsymbol{q}). \quad (12)$$

Furthermore, taking the time derivative of (11), we see that the velocity $D_\tau\boldsymbol{x}(\boldsymbol{q},\tau)$ is curl-free with respect to the Lagrangian coordinate $\boldsymbol{q}$.

It is noteworthy that the ZA can be formulated as the first order of a systematic Lagrangian perturbation theory (Buchert 1992). Up to second order, the Lagrangian map is still potential under slaving (Moutarde et al. 1991; Buchert & Ehlers 1993; Munshi, Sahni & Starobinsky 1994; Catelan 1995).

It is well known that the ZA map defined by (11) ceases in general to be invertible due to the formation of multi-stream regions bounded by caustics. Since particles move along straight lines in the ZA, the formation of caustics proceeds just as in ordinary optics in a uniform medium in which light rays are also straight.[7] One of the problems with the ZA is that caustics, which start as localized objects, quickly grow in size and give unrealistically large multi-stream regions.

A modification of the ZA that has no multi-streaming at all, but sharp mass concentrations in the form of shocks and other singularities, has been introduced by Gurbatov & Saichev (1984; see also Gurbatov et al. 1989; Shandarin & Zel'dovich 1989). It is known as the *adhesion model*. In Eulerian coordinates it amounts to using a multidimensional Burgers equation (see, e.g., Frisch & Bec 2002)

$$\partial_\tau\boldsymbol{v}+(\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}})\boldsymbol{v} = \nu\nabla_{\boldsymbol{x}}^2\boldsymbol{v}, \qquad \boldsymbol{v}=-\nabla_{\boldsymbol{x}}\varphi_{\rm v}, \quad (13)$$

taken in the limit where the viscosity ν tends to zero. In Lagrangian coordinates, the adhesion model is obtained from the ZA by replacing the velocity potential $\Phi(\boldsymbol{q},t)$ given by (12) by its *convex hull* $\Phi_c(\boldsymbol{q},t)$ in the $\boldsymbol{q}$ variable (Vergassola et al. 1994).

Convexity is a concept which plays an important role in this paper, and a few words on it are in order here (see also Appendix C1). A body in the three-dimensional space is said to be *convex* if, whenever it contains two points, it contains also the whole segment joining them. A function $f(\boldsymbol{q})$ is said to be convex if the set of all points lying above its graph is convex. The convex hull of the function $\Phi(\boldsymbol{q})$ is defined as the largest convex function whose graph lies below that of $\Phi(\boldsymbol{q})$. In two dimensions it can be visualized by wrapping the graph of $\Phi(\boldsymbol{q})$ tightly from below with an elastic sheet.

Note that $\Phi(\boldsymbol{q},\tau)$ given by (12) is obviously convex for small enough τ since it is then very close to the parabolic function $|\boldsymbol{q}|^2/2$. After caustics form, convexity is lost in the ZA but recovered with the adhesion model. It may then be shown that those regions in the Lagrangian space where $\Phi(\boldsymbol{q},t)$ does not coincide with its convex hull will be mapped in the Eulerian space to sheets, lines and points, each of which contains a finite amount of mass. At these locations the Lagrangian map does not have a uniquely defined Lagrangian antecedent but such points form a set of vanishing volume. Everywhere else, there is a unique antecedent and hence no multi-streaming.

Although the adhesion model has a number of known shortcomings, such as non-conservation of momentum in more than one dimension, it has been found to be in better agreement with N-body simulations than the ZA (Weinberg & Gunn 1990). Other single-speed approximations to multi-stream flow, overcoming difficulties of the adhesion model, are given by Buchert & Dominguez (1998). In such models, multi-streaming is completely suppressed by a mechanism of momentum exchange between neighbouring streams with different velocities. This is of course a common phenomenon in ordinary fluids, where it is due to viscous diffusion; dark matter is however essentially collisionless and the usual mechanism for generating viscosity does not operate, so that a non-collisional mechanism must be invoked. A qualitative explanation using the modification of the gravitational forces after the formation of caustics has been proposed by Shandarin & Zel'dovich (1989). In our opinion the mechanism limiting multi-streaming to rather narrow regions is poorly understood and deserves considerable further investigation.

3.2 The Monge–Ampère equation: a consequence of mass conservation

We now show that the assumption that the Lagrangian map is derived from a convex potential leads to a pair of nonlinear partial differential equations, one for this potential and another for its Legendre transform.

Let us first assume that the present distribution of mass has no singular part, an assumption which we shall relax later. Since in our notation the initial quasi-uniform mass distribution has unit density, mass conservation im-

[6] In connection with what was called later the Lagrangian POTENT method (Dekel, Bertschinger & Faber 1990).

[7] Catastrophe theory has been used to classify the different types of singularities thus obtained (Arnol'd, Shandarin & Zel'dovich 1982).

plies $\rho_0(\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{x} = \mathrm{d}^3\boldsymbol{q}$, which can be rewritten in terms of the Jacobian matrix $\nabla_{\boldsymbol{q}}\boldsymbol{x}$ as

$$\det \nabla_{\boldsymbol{q}}\boldsymbol{x} = \frac{1}{\rho_0(\boldsymbol{x}(\boldsymbol{q}))}. \tag{14}$$

Under the potential assumption (8), this takes the form

$$\det(\nabla_{q_i}\nabla_{q_j}\Phi(\boldsymbol{q})) = \frac{1}{\rho_0\big(\nabla_{\boldsymbol{q}}\Phi(\boldsymbol{q})\big)}. \tag{15}$$

A similar equation follows also from Eqs. (1) and (2) of Bertschinger & Dekel (1989).

A simpler equation, in which the unknown appears only in the left-hand side, viz Eq. (19) below, is obtained for the potential of the *inverse Lagrangian map* $\boldsymbol{q}(\boldsymbol{x})$. Key is the observation that the inverse of a map with a convex potential has also a convex potential, and that the two potentials are Legendre transforms of each other.[8] A purely local proof of this statement is to observe that potentiality of $\boldsymbol{q}(\boldsymbol{x})$ is equivalent to the symmetry of the *inverse Jacobian matrix* $\nabla_{\boldsymbol{x}}\boldsymbol{q}$ which follows because it is the inverse of the symmetrical matrix $\nabla_{\boldsymbol{q}}\boldsymbol{x}$; convexity is equivalent to the positive-definiteness of these matrices. Obviously the function

$$\Theta(\boldsymbol{x}) \equiv \boldsymbol{x}\cdot\boldsymbol{q}(\boldsymbol{x}) - \Phi(\boldsymbol{q}(\boldsymbol{x})), \tag{16}$$

which is the Legendre transform of $\Phi(\boldsymbol{q})$, is the potential for the inverse Lagrangian map. The modern definition of the Legendre transformation (see Appendix C1), needed for generalization to non-smooth mass distributions, is

$$\Theta(\boldsymbol{x}) = \max_{\boldsymbol{q}}\ \boldsymbol{x}\cdot\boldsymbol{q} - \Phi(\boldsymbol{q}), \tag{17}$$

$$\Phi(\boldsymbol{q}) = \max_{\boldsymbol{x}}\ \boldsymbol{x}\cdot\boldsymbol{q} - \Theta(\boldsymbol{x}). \tag{18}$$

In terms of the potential Θ, mass conservation is immediately written as

$$\det(\nabla_{x_i}\nabla_{x_j}\Theta(\boldsymbol{x})) = \rho_0(\boldsymbol{x}). \tag{19}$$

This equation, which has the determinant of the second derivatives of the unknown in the left-hand side and a prescribed (positive) function in the right-hand side, is called the (elliptic) Monge–Ampère equation (see Appendix B for a historical perspective).

Notice that our Monge–Ampère equation may be viewed as a non-linear generalization of the Poisson equation (used for reconstruction by Nusser & Dekel (1992); see also Section 7.1), to which it reduces if particles have moved very little from their initial positions.

In actual reconstructions we have to deal with mass concentration in the present distribution of matter. Thus the density in the right-hand side of (19) has a singular component (a Dirac distribution concentrated on sets carrying the concentrated mass) and the potential Θ ceases to be smooth. As we now show, a generalized meaning can nevertheless be given to the Monge–Ampère equation by using the key ingredient in its derivation, namely mass conservation, in integrated form.

For a nonsmooth convex potential Θ, taking the gradient $\nabla_{\boldsymbol{x}}\Theta(\boldsymbol{x})$ still makes sense if one allows it to be multivalued at points where the potential is not differentiable. The gradient at such a point $\boldsymbol{x}$ is then the set of all possible slopes of planes touching the graph of Θ at $(\boldsymbol{x}, \Theta(\boldsymbol{x}))$ (this idea is given a precise mathematical formulation in Appendix C1). As $\boldsymbol{x}$ varies over an arbitrary domain $\mathcal{D}_{\mathrm{E}}$ in the Eulerian space, its image $\boldsymbol{q}(\boldsymbol{x})$ sweeps a domain $\boldsymbol{q}(\mathcal{D}_{\mathrm{E}})$ in the Lagrangian space, and mass conservation requires that

$$\int_{\mathcal{D}_{\mathrm{E}}} \rho_0(\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{x} = \int_{\nabla_{\boldsymbol{x}}\Theta(\mathcal{D}_{\mathrm{E}})} \mathrm{d}^3\boldsymbol{q}, \tag{20}$$

where we take into account that $\boldsymbol{q}(\boldsymbol{x}) = \nabla_{\boldsymbol{x}}\Theta(\boldsymbol{x})$. Eq. (20) must hold for any Eulerian domain $\mathcal{D}_{\mathrm{E}}$; this requirement is known as the *weak formulation* of the Monge–Ampère equation (19). A symmetric formulation may be written for (15) in terms of $\boldsymbol{x}(\boldsymbol{q}) = \nabla_{\boldsymbol{q}}\Phi(\boldsymbol{q})$. For further material on the weak formulation see, e.g., Pogorelov (1978).

Considerable literature has been devoted to the Monge–Ampère equation in recent years (see, e.g., Caffarelli 1999; Caffarelli & Milman 1999). We mention now a few results which are of direct relevance for the reconstruction problem.

In a nutshell, one can prove that when the domains occupied by the mass initially and at present are bounded and convex, the Monge–Ampère equation – in its weak formulation – is guaranteed to have a unique solution, which is smooth unless one or both of the mass distributions is nonsmooth. The actual construction of this solution can be done by a variational method discussed in the next section.

A similar result holds also when the present density field is periodic and the same periodicity is assumed for the map.

Also relevant, as we shall see in Section 3.4, is a recent result of Caffarelli & Li (2001): if the Monge–Ampère equation is considered in the whole space, but the present density contrast $\delta = \rho - 1$ vanishes outside of a bounded set, then the solution $\Theta(\boldsymbol{x})$ is determined uniquely up to prescription of its asymptotic behaviour at infinity, which is specified by a quadratic function of the form

$$\theta(\boldsymbol{x}) \equiv \langle \boldsymbol{x}, A\boldsymbol{x}\rangle + \langle \boldsymbol{b}, \boldsymbol{x}\rangle + c, \tag{21}$$

for some positive definite symmetric matrix A with unit determinant, vector $\boldsymbol{b}$ and constant c.

3.3 Optimal mass transportation

As we are going to see now, the Monge–Ampère equation (19) is equivalent to an instance of what is called the 'optimal mass transportation problem.' Suppose we are given two distributions $\rho_{\mathrm{in}}(\boldsymbol{q})$ and $\rho_0(\boldsymbol{x})$ of the same amount of mass in two three-dimensional convex bounded domains $\mathcal{D}_{\mathrm{in}}$ and $\mathcal{D}_0$. The optimal mass transportation problem is then to find the most cost-effective way of rearranging by a suitable map one distribution into the other, the cost of transporting a unit of mass from a position $\boldsymbol{q} \in \mathcal{D}_{\mathrm{in}}$ to $\boldsymbol{x} \in \mathcal{D}_0$ being a prescribed function $c(\boldsymbol{q}, \boldsymbol{x})$.

Denoting the map by $\boldsymbol{x}(\boldsymbol{q})$ and its inverse $\boldsymbol{q}(\boldsymbol{x})$, we can write the problem as the requirement that the cost

$$I \equiv \int_{\mathcal{D}_{\mathrm{in}}} c(\boldsymbol{q}, \boldsymbol{x}(\boldsymbol{q}))\rho_{\mathrm{in}}(\boldsymbol{q})\,\mathrm{d}^3\boldsymbol{q} = \int_{\mathcal{D}_0} c(\boldsymbol{q}(\boldsymbol{x}), \boldsymbol{x})\rho_0(\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{x} \tag{22}$$

[8] Besides our problem, this fact prominently appears in two other fields of physics: in classical mechanics, the Lagrangian and Hamiltonian functions are Legendre transforms of each other – their gradients relate the generalized velocity and momentum – and so are, in thermodynamics, the internal energy and the Gibbs potential, implying the same relation between extensive and intensive parameters of state.

be minimum, with the constraints of prescribed 'terminal' densities $\rho_{\rm in}$ and ρ_0 and of mass conservation $\rho_{\rm in}(\boldsymbol{q})\,\mathrm{d}^3\boldsymbol{q} = \rho_0(\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{x}$.[9]

This problem goes back to Monge (1781) who considered the case of a linear cost function $c(\boldsymbol{q},\boldsymbol{x}) = |\boldsymbol{x}-\boldsymbol{q}|$ (see Appendix B and Fig. 1).

For our purposes, the central result (Brenier 1987, 1991) is that, when the cost is a quadratic function of the distance, so that

$$I = \int_{\mathcal{D}_{\rm in}} \frac{|\boldsymbol{x}(\boldsymbol{q})-\boldsymbol{q}|^2}{2}\rho_{\rm in}(\boldsymbol{q})\,\mathrm{d}^3\boldsymbol{q} = \int_{\mathcal{D}_0} \frac{|\boldsymbol{x}-\boldsymbol{q}(\boldsymbol{x})|^2}{2}\rho_0(\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{x}, \quad (23)$$

the solution $\boldsymbol{q}(\boldsymbol{x})$ to the optimal mass transportation problem is the gradient of a convex function, which then must satisfy the Monge–Ampère equation (19) by mass conservation.

A particularly simple variational proof can be given for the smooth case, when the two mutually inverse maps $\boldsymbol{x}(\boldsymbol{q})$ and $\boldsymbol{q}(\boldsymbol{x})$ are both well defined.

Performing a variation of the map $\boldsymbol{x}(\boldsymbol{q})$, we cause a mass element in the Eulerian space that was located at $\boldsymbol{x}(\boldsymbol{q})$ to move to $\boldsymbol{x}(\boldsymbol{q})+\delta\boldsymbol{x}(\boldsymbol{q})$. This variation is constrained not to change the density field ρ_0. To express this constraint it is convenient to rewrite the displacement in Eulerian coordinate $\delta\boldsymbol{x}_{\rm E}(\boldsymbol{x}) \equiv \delta\boldsymbol{x}(\boldsymbol{q}(\boldsymbol{x}))$. Noting that the point $\boldsymbol{x}$ gets displaced into $\boldsymbol{y} = \boldsymbol{x}+\delta\boldsymbol{x}$, we thus require that $\rho_0(\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{x} = \rho_0(\boldsymbol{y})\,\mathrm{d}^3\boldsymbol{y}$ or

$$\rho_0(\boldsymbol{x}) = \rho_0(\boldsymbol{x}+\delta\boldsymbol{x}_{\rm E}(\boldsymbol{x}))\,\det\big(\nabla_{\boldsymbol{x}}(\boldsymbol{x}+\delta\boldsymbol{x}_{\rm E}(\boldsymbol{x}))\big). \quad (24)$$

Expanding this equation, we find that, to the leading order,

$$\nabla_{\boldsymbol{x}}\cdot(\rho_0(\boldsymbol{x})\,\delta\boldsymbol{x}_{\rm E}(\boldsymbol{x})) = 0, \quad (25)$$

an equation which just expresses the physically obvious fact that the mass flux $\rho_0(\boldsymbol{x})\,\delta\boldsymbol{x}_{\rm E}(\boldsymbol{x})$ should have zero divergence. Performing the variation on the functional I given by (23), we get

$$\begin{aligned}\delta I &= \int_{\mathcal{D}_{\rm in}} (\boldsymbol{x}(\boldsymbol{q})-\boldsymbol{q})\cdot\delta\boldsymbol{x}(\boldsymbol{q})\,\rho_{\rm in}(\boldsymbol{q})\,\mathrm{d}^3\boldsymbol{q} \\ &= \int_{\mathcal{D}_0} (\boldsymbol{x}-\boldsymbol{q}(\boldsymbol{x}))\cdot\big(\rho_0(\boldsymbol{x})\,\delta\boldsymbol{x}_{\rm E}(\boldsymbol{x})\big)\,\mathrm{d}^3\boldsymbol{x} = 0, \quad (26)\end{aligned}$$

which has to hold under the constraint (25). In other words, the displacement $\boldsymbol{x}-\boldsymbol{q}(\boldsymbol{x})$ has to be orthogonal (in the L_2 functional sense) to all divergence-less vector fields and, thus, must be a gradient. Since $\boldsymbol{x}$ is obviously a gradient, it follows that $\boldsymbol{q}(\boldsymbol{x}) = \nabla_{\boldsymbol{x}}\Theta(\boldsymbol{x})$ for a suitable potential Θ.

It remains to prove the convexity of Θ. First we prove that the map $\boldsymbol{x}\mapsto\boldsymbol{q}(\boldsymbol{x}) = \nabla_{\boldsymbol{x}}\Theta(\boldsymbol{x})$ is *monotone*, i.e., by definition, that for any $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$

$$(\boldsymbol{x}_2-\boldsymbol{x}_1)\cdot(\boldsymbol{q}(\boldsymbol{x}_2)-\boldsymbol{q}(\boldsymbol{x}_1)) \geqslant 0. \quad (27)$$

Indeed, should this inequality be violated for some $\overline{\boldsymbol{x}}_1,\overline{\boldsymbol{x}}_2$, the continuity of $\boldsymbol{q}(\boldsymbol{x})$ would imply that for all $\boldsymbol{x}_1,\boldsymbol{x}_2$ close enough to $\overline{\boldsymbol{x}}_1,\overline{\boldsymbol{x}}_2$

$$\begin{aligned}&|\boldsymbol{q}(\boldsymbol{x}_1)-\boldsymbol{x}_1|^2+|\boldsymbol{q}(\boldsymbol{x}_2)-\boldsymbol{x}_2|^2 \\ &\quad > |\boldsymbol{q}(\boldsymbol{x}_2)-\boldsymbol{x}_1|^2+|\boldsymbol{q}(\boldsymbol{x}_1)-\boldsymbol{x}_2|^2.\end{aligned} \quad (28)$$

This in turn means that if we interchange the destinations of small patches around $\overline{\boldsymbol{x}}_1$ and $\overline{\boldsymbol{x}}_2$, sending them not to the corresponding patches around $\boldsymbol{q}(\overline{\boldsymbol{x}}_1)$ and $\boldsymbol{q}(\overline{\boldsymbol{x}}_2)$ but vice versa, then the value of the functional I will decrease by a small yet positive quantity, and therefore it cannot be minimum for the original map.[10]

To complete the argument, observe that convexity of a smooth function $\Theta(\boldsymbol{x})$ follows if the matrix of its second derivatives $\nabla_{x_i}\nabla_{x_j}\Theta(\boldsymbol{x})$ is positive definite for all $\boldsymbol{x}$. Substituting $\boldsymbol{q}(\boldsymbol{x}) = \nabla_{\boldsymbol{x}}\Theta(\boldsymbol{x})$ into (27), assuming that $\boldsymbol{x}_2$ is close to $\boldsymbol{x}_1$ and Taylor expanding, we find that

$$(\boldsymbol{x}_2-\boldsymbol{x}_1)\cdot(\nabla_{x_i}\nabla_{x_j}\Theta(\boldsymbol{x}_1)\,(\boldsymbol{x}_2-\boldsymbol{x}_1)) \geqslant 0. \quad (29)$$

As $\boldsymbol{x}_2$ is arbitrary, this proves the desired positive definiteness and thus establishes the equivalence of the Monge–Ampère equation (19) and of the mass transportation problem with quadratic cost.

This equivalence is actually proved under much weaker conditions, not requiring any smoothness (Brenier 1987, 1991). The proof makes use of the 'relaxed' reformulation of the mass transportation problem due to Kantorovich (1942). Instead of solving the highly non-linear problem of finding a map $\boldsymbol{q}(\boldsymbol{x})$ minimizing the cost (22) with prescribed terminal densities, Kantorovich considered the *linear programming* problem of minimizing

$$\tilde{I} \equiv \int_{\mathcal{D}_{\rm in}}\int_{\mathcal{D}_0} c(\boldsymbol{q},\boldsymbol{x})\,\rho(\boldsymbol{q},\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{q}\,\mathrm{d}^3\boldsymbol{x}, \quad (30)$$

under the constraint that the joint distribution $\rho(\boldsymbol{q},\boldsymbol{x})$ is nonnegative and has marginals $\rho_{\rm in}(\boldsymbol{q})$ and $\rho_0(\boldsymbol{x})$, the latter being equivalent to

$$\int_{\mathcal{D}_0} \rho(\boldsymbol{q},\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{x} = \rho_{\rm in}(\boldsymbol{q}), \qquad \int_{\mathcal{D}_{\rm in}} \rho(\boldsymbol{q},\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{q} = \rho_0(\boldsymbol{x}). \quad (31)$$

Note that if we assume any of the two following forms for the joint distribution

$$\begin{aligned}&\rho(\boldsymbol{q},\boldsymbol{x}) = \rho_0(\boldsymbol{x})\,\delta\big(\boldsymbol{q}-\boldsymbol{q}(\boldsymbol{x})\big) \\ &\rho(\boldsymbol{q},\boldsymbol{x}) = \rho_{\rm in}(\boldsymbol{q})\,\delta\big(\boldsymbol{x}-\boldsymbol{x}(\boldsymbol{q})\big),\end{aligned} \quad (32)$$

we find that $\tilde{I}$ reduces to the cost I as defined in (22). This relaxed formulation allowed Kantorovich to establish the existence of a mimimizing joint distribution.

The relaxed formulation can be used to show that the minimizing solution actually defines a map, which need not be smooth if one or both of the terminal distribution have a singular component (in our case, when mass concentrations are present). The derivation (Brenier 1987, 1991) makes use of the technique of duality (Appendix C2), which will also appear in discussing algorithms (Section 4.2) and reconstruction beyond the potential hypothesis (Section 6).

We have thus shown that the Monge–Kantorovich optimal mass transportation problem can be applied to solving the Monge–Ampère equation. The actual implementation (Section 4), done for a suitable discretization, will be henceforth called Monge–Ampère–Kantorovich (MAK).

[9] Note that $\boldsymbol{x}(\boldsymbol{q}) = \boldsymbol{q}$ does not solve the above problem as it violates the latter constraint unless the terminal densities are identical.

[10] As we shall see in Section 4.1, the converse is not true: monotonicity alone does not imply that the integral I is a minimum; the minimizing map must also be potential.

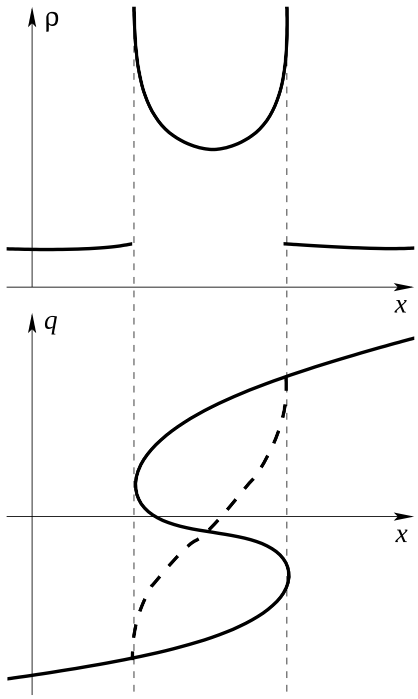


Figure 2. A one-dimensional example of non-unique reconstruction of the Lagrangian map in the presence of multi-streaming. The density distribution (upper graph) is generated by a multi-streaming Lagrangian map (thick line of lower graph) but may also be generated by a spurious single-stream Lagrangian map (dashed line).

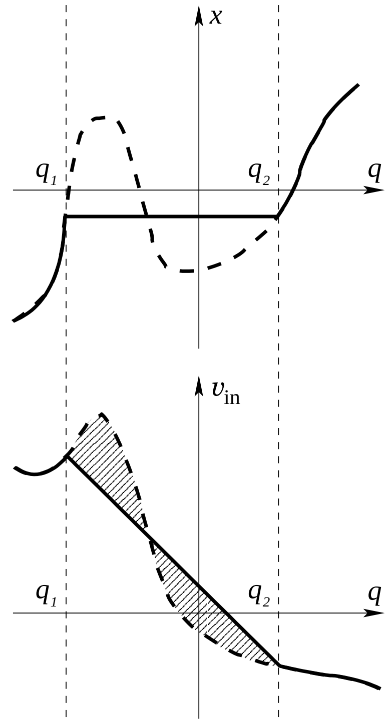


Figure 3. Two initial velocity profiles $v_{\rm in}(q)$ (bottom, solid and dashed lines) leading to the same Lagrangian map $x = q + \tau v_{\rm in}(q)$ (top, solid line) in the adhesion approximation. The Zel'dovich approximation would give multistreaming (top, dashed line). Hatched areas (bottom) are equal in the adhesion dynamics.

3.4 Sources of uncertainty in reconstruction

In this section we discuss various sources of non-uniqueness of the MAK reconstruction: multi-streaming, collapsed regions, reconstruction from a finite patch of the Universe.

We have stated before that our uniqueness result applies only in so far as we can treat present-epoch high-density multi-stream regions as if they were truly collapsed, ignoring their width. We now give a simple one-dimensional example of non-uniqueness in which a thick region of multi-streaming is present. Fig. 2 shows a multi-stream Lagrangian map $x(q)$ and the associated density distribution; the inverse map $q(x)$ is clearly multi-valued. The same density distribution may however be generated by a spurious single-stream Lagrangian map shown on the same figure. There is no way to distinguish between the two inverse Lagrangian maps if the various streams cannot be disentangled.

Suppose now that the present density has a singular part, i.e. there are mass concentrations present which have vanishing (Eulerian) volumes but possess finite masses. Obviously any such object originates from a domain in the Lagrangian space which occupies a finite volume. A one-dimensional example is again helpful. Fig. 3 shows a Lagrangian map in which a whole Lagrangian shock interval $[q_1, q_2]$ has collapsed into a single point of the x axis. Outside of this point the Lagrangian map is uniquely invertible but the point itself has many antecedents. Note that the graph of the Lagrangian map may be inverted by just interchanging the q and x axes, but its inverse contains a piece of vertical line. The position of the Lagrangian shock interval which has collapsed by the present epoch is uniquely defined by the present mass field but the initial velocity fluctuations in this interval cannot be uniquely reconstructed. In particular there is no way to know if collapse has started before the present epoch. We can of course arbitrarily assume that collapse has just happened at the present epoch; if we also suppose that particles have travelled with a constant speed, i.e. use the Zel'dovich/adhesion approximation, then the initial velocity profile within the Lagrangian shock interval will be linear (Fig. 3). Any other smooth velocity profile joining the same end points would have points where its slope (velocity gradient) is more negative than that of the linear profile (Fig. 3) and thus would have started collapse before the present epoch (in one dimension caustics appear at the time which is minus the inverse of the most negative initial velocity gradient).

All this carries over to more than one dimension. The MAK reconstruction gives a unique antecedent for any Eulerian position outside mass concentrations. Each mass concentration in the Eulerian space, taken globally, has a uniquely defined Lagrangian antecedent region but the initial velocity field inside the latter is unknown. In other words, displacement reconstruction is well defined but full reconstruction, based on the Zel'dovich/adhesion approximation for velocities, is possible only outside of mass concentrations (note however that velocities in the Eulerian space are still reconstructed at almost all points). We call the corresponding initial Lagrangian domains *collapsed regions*.

Finally, we consider a uniqueness problem arising from knowing the present mass distribution only over a finite

Eulerian domain $\mathcal{D}_0$, as is necessarily the case when working with a real catalogue. If we also know the corresponding Lagrangian domain $\mathcal{D}_{\rm in}$ and both domains are bounded and convex, then uniqueness is guaranteed (see Section 3.2). What we know for sure about $\mathcal{D}_{\rm in}$ is its volume, which (in our units) is equal to the total mass contained in $\mathcal{D}_0$. Its shape and position may however be constrained by further information. For example, if we know that the typical displacement of mass elements since decoupling is about ten Mpc in comoving coordinates (see Section 5) and our data extend over a patch of typical size one hundred Mpc, then there is not more than a ten percent uncertainty on the shape of $\mathcal{D}_{\rm in}$. Additional information about peculiar velocities may also be used to constrain $\mathcal{D}_{\rm in}$.

Note also that a finite-size patch $\mathcal{D}_0$ with unknown antecedent $\mathcal{D}_{\rm in}$ will give rise to a unique reconstruction (up to a translation) if we assume that it is surrounded by a uniform background extending to infinity. This is a consequence of the result of Caffarelli & Li mentioned at the end of Section 3.2. The arbitrary linear term in (21) corresponds to a translation; as to the quadratic term, it is constrained by the cosmological principle of isotropy to be exactly $|\boldsymbol{q}|^2/2$.

4 THE MAK METHOD: DISCRETIZATION AND ALGORITHMICS

In this section we show how to compute the solution to the Monge–Ampère–Kantorovich (MAK) problem the known present density field. First the problem is discretized into an assignment problem (Section 4.1), then we present some general tools which make the assignment problem computationally tractable (Section 4.2) and finally we present, to the best of our knowledge, the most effective method for solving our particular assignment problem, based on the auction algorithm of D. Bertsekas (Section 4.3), and details of its implementation for the MAK reconstruction (Section 4.4).

4.1 Reduction to an assignment problem

Perhaps the most natural way of discretizing a spatial mass distribution is to approximate it by a finite system of identical Dirac point masses, with possibly more than one mass at a given location. This is compatible both with N-body simulations and with the intrinsically discrete nature of observed luminous matter. Assuming that we have N unit masses both in the Lagrangian and the Eulerian space, we may write

$$\rho_0(\boldsymbol{x}) = \sum_{i=1}^{N} \delta(\boldsymbol{x} - \boldsymbol{x}_i), \qquad \rho_{\rm in}(\boldsymbol{q}) = \sum_{j=1}^{N} \delta(\boldsymbol{q} - \boldsymbol{q}_j). \tag{33}$$

For discrete densities of this form, the mass conservation constraint in the optimal mass transportation problem (Section 3.3) requires that the map $\boldsymbol{q}(\boldsymbol{x})$ induce a one-to-one pairing between positions of the unit masses in the $\boldsymbol{x}$ and $\boldsymbol{q}$ spaces, which may be written as a permutation of indices that sends $\boldsymbol{x}_i$ to $\boldsymbol{q}_{j(i)}$. Substituting this into the quadratic cost functional (23), we get

$$I = \sum_{i=1}^{N} \frac{|\boldsymbol{x}_i - \boldsymbol{q}_{j(i)}|^2}{2}. \tag{34}$$

We thus reduced the problem to the purely combinatorial one of finding a permutation $j(i)$ (or its inverse $i(j)$) that minimizes the quadratic cost function (34).

This problem is an instance of the general *assignment problem* in combinatorial optimization: for a cost matrix c_{ij}, find a permutation $j(i)$ that minimizes the cost function

$$I = \sum_{i=1}^{N} c_{i\,j(i)}. \tag{35}$$

As we shall see in the next sections, there exist effective algorithms for finding minimizing permutations.

Before proceeding with the assignment problem, we should mention an alternative approach in which discretization is performed only in the Eulerian space and the initial mass distribution is kept continuous and uniform. Minimization of the quadratic cost function will then give rise to a tesselation of the Lagrangian space into polyhedric regions which end up collapsed into the discrete Eulerian Dirac masses. Basically, the reason why these regions are polyhedra is that the convex potential $\Phi(\boldsymbol{q})$ of the Lagrangian map has a gradient which takes only finitely many values. This problem, which has been studied by Aleksandrov and Pogorelov (see, e.g., Pogorelov 1978), is closely related to Minkowski's (1897) famous problem of constructing a convex polyhedron with prescribed areas and orientations of its faces (in our setting, areas and orientations correspond to masses and values of the gradient). Uniqueness in the Minkowski problem is guaranteed up to a translation. Starting with Minkowski's own very elegant solution, various methods of constructing solutions to such geometrical questions have been devised. So far, we have not been able to make use of such ideas in a way truly competitive with discretization in both spaces and solving then the assignment problem.

The solution to our assignment problem (with quadratic cost) has the important property that it is monotone: for any two Lagrangian positions $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$, the corresponding Eulerian positions $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are such that

$$(\boldsymbol{x}_1 - \boldsymbol{x}_2) \cdot (\boldsymbol{q}_1 - \boldsymbol{q}_2) \geqslant 0. \tag{36}$$

This is of course the discrete counterpart of (27). In one dimension, when all the Dirac masses are on the same line, monotonicity implies that the leftmost Lagrangian position goes to the leftmost Eulerian position, the second leftmost Lagrangian position to the second leftmost Eulerian position, etc. It is easily checked that this correspondence minimizes the cost (34).

In more than one dimension, a correspondence between Lagrangian and Eulerian positions that is just monotone will usually not minimize the cost (a simple two-dimensional counterexample is given in Fig. 4).[11] Actually, a much stronger condition, called *cyclic monotonicity*, is needed in order to minimize the cost. It requires k-monotonicity for any k between 2 and N; the latter is defined by taking any k Eulerian positions with their corresponding Lagrangian antecedents and requiring that the cost (34) should not decrease under an arbitrary reassignment of the Lagrangian

[11] Note that in one dimension, in the continuous case, any map is a gradient and we have already observed in Section 3.3 that if a gradient map is monotone it is the gradient of a convex function.

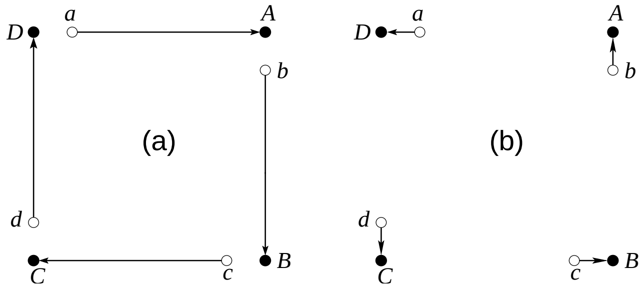


Figure 4. Two monotone assignments sending white points to black ones: (a) an assignment that is vastly non-optimal in terms of quadratic cost but cannot be improved by any pair interchange; (b) the optimal assignment, shown for comparison.

positions within the set of Eulerian positions taken. Note that the usual monotonicity corresponds to 2-monotonicity (stability with respect to pair exchanges).

A strategy called PIZA (Path Interchange Zel'dovich Approximation) for constructing monotone correspondences between Lagrangian and Eulerian positions has been proposed by Croft & Gaztañaga (1997). In PIZA, a randomly chosen tentative correspondence between initial and final positions is successively improved by swapping randomly selected *pairs* of initial particles whenever (36) is not satisfied. After the cost (34) ceases to decrease between iterations, an approximation to a monotone correspondence is established, which is generally neither unique, as already observed by Valentine, Saunders & Taylor (2000) in testing PIZA reconstruction, nor optimal. We shall come back to this in Sections 5 and 7.3.

4.2 Nuts and bolts of solving the assignment problem

For a general set of N unit masses, the assignment problem with the cost function (34) has a single solution which can obviously be found by examining all $N!$ permutations. However, unlike computationally hard problems, such as the travelling salesman's, the assignment problem can be handled in 'polynomial time' – actually in not more than $O(N^3)$ operations. All methods achieving this use a so-called dual formulation of the problem, based on a relaxation similar to that applied by Kantorovich to the optimal mass transportation (Section 3.3; a brief introduction to duality is given in Appendix C2). In this section we explain the basics of this technique, using a variant of a simple mechanical model introduced in a more general setting by Hénon (1995, 2002).

Consider the general assignment problem of minimizing the cost (35) over all permutations $j(i)$. We replace it by a 'relaxed,' linear programming problem of minimizing

$$\tilde{I} = \sum_{i,j=1}^{N} c_{ij} f_{ij}, \tag{37}$$

where auxiliary variables f_{ij} satisfy

$$f_{ij} \geqslant 0, \qquad \sum_{k=1}^{N} f_{kj} = \sum_{k=1}^{N} f_{ik} = 1 \tag{38}$$

for all i, j, an obvious discrete analogue of (31). We show

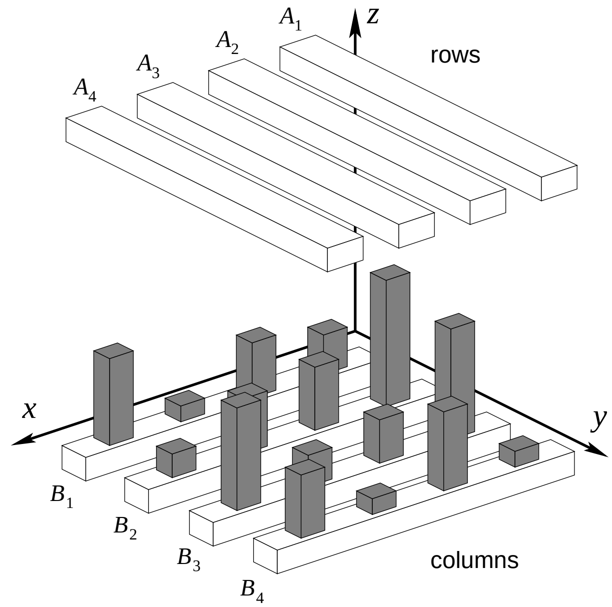


Figure 5. An analogue computer solving the assignment problem for $N = 4$.

now that it is possible to build a simple mechanical device (Fig. 5) which solves this relaxed problem and that the solution will in fact determine a minimizing permutation in the original assignment problem (i.e., for any i or j fixed, only one f_{ij} will be unit and all other zero). The device acts as an *analogue computer*: the numbers involved in the problem are represented by physical quantities, and the equations are replaced by physical laws.

Define coordinate axes x, y, z in space, with the z axis vertical. We take two systems of N horizontal rods, parallel to the x and y axes respectively, and call them *columns* and *rows*, referring to columns and rows of the cost matrix. Each rod is constrained to move in a corresponding vertical plane while preserving the horizontal orientation in space. For a row rod A_i, we denote the z coordinate of its bottom face by α_i and for a column rod B_j, we denote the z coordinate of its top face β_j. Row rods are placed above column rods, therefore $\alpha_i \geqslant \beta_j$ for all i, j (see Fig. 5).

Upper (row) rods are assumed to have unit weight, and lower (column) rods to have negative unit weight, or unit 'buoyancy.' Therefore both groups of rods are subject to gravitational forces pulling them together. However, this movement is obstructed by N^2 small vertical studs of negligible weight put on column rods just below row rods. A stud placed at projected intersection of column B_j and row A_i has length $C - c_{ij}$ with a suitably large positive constant C and thus constrains the quantities α_i and β_j to satisfy the stronger inequality

$$\alpha_i - \beta_j \geqslant C - c_{ij}. \tag{39}$$

The potential energy of the system is, up to a constant,

$$U = \sum_{i=1}^{N} \alpha_i - \sum_{j=1}^{N} \beta_j. \tag{40}$$

In linear programming, the problem of minimizing (40) under the set of constraints given by (39) is called the *dual* *problem* to the 'relaxed' one (37)–(38) (see Appendix C2); the α and β variables are called the *dual variables*.

The analogue computer does in fact solve the dual problem. Indeed, first hold the two groups of rods separated from each other and then release them, so that the system starts to evolve. Rows will go down, columns will come up, and contacts will be made with the studs. Aggregates of rows and columns will be progressively formed and modified as new contacts are made, giving rise to a complex evolution. Eventually the system reaches an equilibrium, in which its potential energy (40) is minimum and all constraints (39) are satisfied (Hénon 2002). Moreover, it may be shown that the solution to the original problem (37)–(38) is expressible in terms of the forces exerted by the rods on each other at equilibrium and is typically a one-to-one correspondence between the A_is and the B_js (for details, see Appendix C3).

The common feature of many existing algorithms for solving the assignment problem, which makes them more effective computationally than the simple enumeration of all $N!$ permutations, is the use of the intrinsically continuous, geometric formulation in terms of the pair of linear programming problems (37)–(38) and (40)–(39). The mechanical device provides a concrete model for this formulation; in fact, assignment algorithms can be regarded as descriptions of specific procedures to make the machine reach its equilibrium state. An introduction into algorithmic aspects of solving the assignment problem, including a proof of the $O(N^3)$ theoretical bound on the number of operations, based on the Hungarian method of Kuhn (1955), may be found in Papadimitriou & Steiglitz (1982).

In spite of the general $O(N^3)$ theoretical bound, various algorithms may show very different performance when applied to a specific optimization problem. During the preparation of the earlier publication (Frisch et al. 2002) the dual simplex method of Balinski (1986) was used, with some modifications inspired by algorithm B of Hénon (2002). Several other algorithms were tried subsequently, including an adaptation of algorithm A of the latter reference and the algorithm of Burkard & Derigs (1980), itself based on the earlier work of Tomizawa (1971). For the time being, the fastest running code by far is based on the auction algorithm of Bertsekas (1992, 2001), arguably the most effective of existing ones, which is discussed in the next section. Needless to say, all these algorithms arrive at the same solution to the assignment problem with given data but can differ by several orders of magnitude in the time it takes to complete the computation.

4.3 The auction algorithm

We explain here the auction algorithm in terms of our mechanical device. Note that the original presentation of this algorithm (Bertsekas 1981, 1992, 2001) is based on a different perspective, that of an *auction*, in which the optimal assignment appears as an economic rather than a mechanical equilibrium; the interested reader will benefit much from reading these papers.

Put initially the column rods at zero height and all row rods well above them, so that no contacts are made and constraints (39) are satisfied. To decrease the potential energy, let now the row rods descend until they all meet studs placed on column rods. Some column rods may then come in contact with multiple row rods.

To distribute row rods between column rods more uniformly, one may try the following procedure. Take a column rod B_j that is in contact with more than one row rod and let it descend while keeping other column rods fixed. As B_j goes down, row rods touching it will follow its motion until they meet studs of other column rods and stay behind. When the last of the row rods touching B_j is about to lose its contact, we stop B_j, which is thus left in contact with only one row rod. Then the next column rod having more than one contact is taken and this procedure is repeated.

This general step can be viewed as an *auction* in which row rods bid for the descending column rod, offering prices equal to decreases in their potential energy as they follow its way down. As the column rod descends, thereby increasing its price, the auction is won by the row rod able to offer the largest *bidding increment*, i.e., to decrease its potential energy by the largest amount while not violating the constraints posed by studs of the rest of column rods. For computational purposes it suffices to compute bidding increments for all competing row rods from the dual α and β variables and assign the descending column rod B_j to the highest bidder A_i, decreasing their heights β_j and α_i correspondingly.

Observe that, at each step, the total potential energy U defined by (40) decreases by the largest amount that can be achieved by moving the descending column rod. Since (40) is obviously nonnegative, the descent cannot proceed indefinitely, and the process may be expected to converge quite fast to a one-to-one pairing that solves the assignment problem.

However, as observed by Bertsekas (1981, 1992, 2001), this 'naive' auction algorithm may end up in an infinite cycle if several row rods bid for a few equally favourable column rods, having thus zero bidding increments. To break such cycles, a perturbation mechanism is introduced in the algorithm. Namely, the constraints (39) are replaced by weaker ones

$$\alpha_i - \beta_j \geqslant C - c_{ij} - \epsilon \tag{41}$$

for a small positive quantity ϵ, and in each auction the descending column rod is pushed down by ϵ in addition to decreasing its height by the bidding increment. It can be shown that this reformulated process terminates in a finite number or rounds; moreover, if all stud lengths are integer and ϵ is smaller than $1/N$, then the algorithm terminates at an assignment that is optimal in the unperturbed problem (Bertsekas 1992).

The third ingredient in the Bertsekas algorithm is the idea of *ϵ-scaling*. When the values of dual variables are already close to the solution of the dual problem, it usually takes relatively few rounds of auction to converge to a solution. Thus one can start with large ϵ to compute a rough approximation for dual variables fast, without worrying about the quality of the assignment, and then proceed reducing ϵ in geometric progression until it passes the $1/N$ threshold, assuring that the assignment thus achieved solves the initial problem.

Bertsekas' algorithm is especially fast for *sparse assignment problems*, in which rods A_i and B_j can be matched only if the pair (i, j) belongs to a given subset $\mathcal{A}$ of the set of N^2 possible pairs. We call such pairs *valid* and define the *filling factor* to be the proportion of valid pairs $f = |\mathcal{A}|/N^2$. When this factor is small, computation can be considerably faster: to find the bidding increment for a rod A_i, we need only to run over the list of rods B_j such that (i, j) is a valid pair.

Note also that the decentralized structure of the algorithm facilitates its parallelization (see references in Bertsekas 1992, 2001).

4.4 The auction algorithm for the MAK reconstruction

We now describe the adaptation of the auction algorithm to the MAK reconstruction. Experiments with various programs contained in Bertsekas' publicly available package (http://web.mit.edu/dimitrib/www/auction.txt) showed that the most effective for our problem is AUCTION_FLP. It assumes integer costs c_{ij}, which in our case requires proper scaling of the cost matrix. To achieve this, the unit of length is adjusted so that the size of the reconstruction patch equals 100, and then the square of the distance between an initial and a final position is rounded off to an integer. In our application, row and column rods correspond to Eulerian and Lagrangian positions, respectively. As the MAK reconstruction is planned for application to catalogues of 10^5 and more galaxies, we do not store the cost matrix, which would require an $O(N^2)$ storage space, but rather compute its elements on demand from the coordinates, which requires only $O(N)$ space.

Our problem is naturally adapted for a sparse description if galaxies travel only a short distance compared to the dimensions of the reconstruction patch. For instance, in the simulation discussed in Section 5, the r.m.s. distance traveled is only about $10\,h^{-1}$ Mpc, or 5% of the size of the simulation box, and the largest distance traveled is about 15% of this size. So we may assume that in the optimal assignment distances between paired positions will be limited. We define then a critical distance $d_{\rm crit}$ and specify that a final position $\boldsymbol{x}_i$ and an initial position $\boldsymbol{q}_j$ form a valid pair only if they are within less than $d_{\rm crit}$ from each other. This critical distance must be adjusted carefully: if it is too small, we risk excluding the optimal assignment; if it is taken too large, the benefit of the sparse description is lost.

However, the saving in computing time achieved by sparse description has to be paid for in storage space: to store the set $\mathcal{A}$ of valid pairs, storage of size $|\mathcal{A}| = fN^2$ is needed, which takes us back to the $O(N^2)$ storage requirement. We have explored two solutions to this problem.

1. Use a *dense* description nevertheless, i.e. the one where all pairs (i, j) are valid and there is no need to store the set A. The auction program is easily adapted to this case (in fact this simplifies the code). However, we forfeit the saving in time provided by the sparse structure.

2. The sparse description can be preserved if the set of valid pairs is computed on demand rather than stored. This is easy if initial positions fill a uniform cubic grid, the simplest discrete approximation to the initial quasi-uniform distribution of matter in the reconstruction problem. Thus, for a given final position $\boldsymbol{x}_i$, the valid pairs correspond to points of the cubic lattice that lie inside a sphere of radius $d_{\rm crit}$ centered at $\boldsymbol{x}_i$, so their list can be generated at run time.

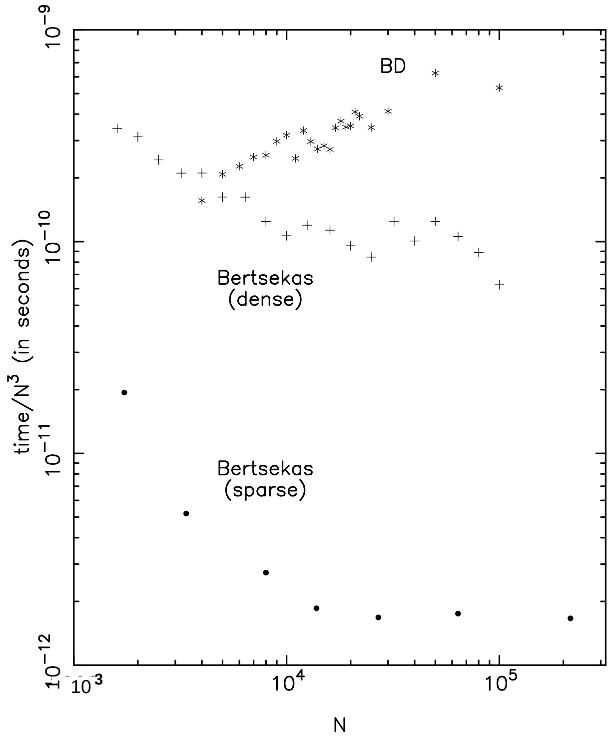


Figure 6. Computing time for different algorithms as a function of the number N of points (divided by N^3 for normalization). Asterisks, the Burkard & Derigs (1980) algorithm (BD); crosses and points, the dense and sparse versions of the auction algorithm (described in the text).

Fig. 6 gives the computing time as a function of the number of points N used in the assignment problem. Shown are the dense and sparse versions of the auction algorithm (in the latter, the critical distance squared was taken equal to 200) and the Burkard & Derigs (1980) algorithm, which ranked the next fastest in our experiments. The N initial and final positions are chosen from the file generated by an N-body simulation described in Section 5; the choice is random except for the sparse algorithm, in which the initial positions are required to fill a cubic lattice. Hence, the performance of the sparse auction algorithm shown in the figure is not completely comparable to that of the two other algorithms.

It is evident that the difference in computing time between the dense auction and the Burkard & Derigs algorithms steadily increases. In the vicinity of $N = 10^5$, the dense auction algorithm is about 10 times faster than the other one. For the sparse version, the decrease in computing time is spectacular: as could be expected, the ratio of computing times for the two versions of the auction algorithm is of the order of f. For large N, the $O(N^3)$ asymptotic of the computing time is quite clear for the sparse auction algorithm. For two other algorithms, similar asymptotic was found for larger N in other experiments (not shown).

In all three cases shown, the initial positions fill a constant volume while N is varied. This is what we call *constant-volume computations*. In the sparse case, this results in a constant filling factor, equal to the ratio of the volume of the sphere with radius $d_{\rm crit}$ to the volume occupied by the initial positions. Here this filling factor is about $f = 0.019$. Another choice, not shown in the figure, is that of *constant-density computations*, when the initial positions are taken from a volume whose size increases with N. In this case the time dependence of algorithms for large N is close to $O(N^{1.5})$.

We finally observe that the sparse auction algorithm applied to the MAK reconstruction requires 5 hours of single-processor CPU time on a 667 MHz COMPAQ/DEC Alpha machine for 216,000 points.

5 TESTING THE MAK RECONSTRUCTION

In this section we present results of our testing the MAK reconstruction against data of cosmological N-body simulations. In a typical simulation of this kind, the dark matter distribution is approximated by N particles of identical mass. Initially the particles are put on a uniform cubic grid and given velocities that form a realization of the primordial velocity field whose statistics is prescribed by a certain cosmological model. Trajectories of particles are then computed according to the Newtonian dynamics in a comoving frame, using periodic boundary conditions. The reconstruction problem is therefore to recover the pairing between the initial (Lagrangian) positions of the particles and their present (Eulerian) positions in the N-body simulation, knowing only the set of computed Eulerian positions in the physical space.

We test our reconstruction against a simulation of 128^3 particles in a box of $200\,h^{-1}$ Mpc size (where h is the Hubble parameter in units of $100\,\mathrm{km\ s^{-1}\ Mpc^{-1}}$) performed using the adaptive P^3M code HYDRA (Couchman, Thomas & Pearce 1995).[12] A ΛCDM cosmological model is used with parameters $\Omega_m = 0.3$, $\Omega_\Lambda = 0.7$, $h = 0.65$, $\sigma_8 = 0.9$.[13] The value of these parameters within the model are determined by fitting the observed cosmic microwave background (CMB) spectrum.[14] The output of the N-body simulation is illustrated in Fig. 7 by a projection onto the x-y plane of a 10% slice of the simulation box.

Since the simulation assumes periodic boundary conditions, some Eulerian positions situated near boundaries may have their Lagrangian antecedents at the opposite side of the simulation box. Suppressing the resulting spurious large displacements is crucial for successful reconstruction. Indeed, for a typical particle displacement of 1/20 the box size, spurious box-wide leaps of 1% of the particles will generate a contribution to the quadratic cost (34) four times larger than that of the rest. To suppress such leaps, for each Eulerian position that has its antecedent Lagrangian position at the other side of the simulation box, we add or subtract the box size from coordinates of the latter (in other words, we are considering the distance on a torus). In what follows we refer to this procedure as the *periodicity correction*.

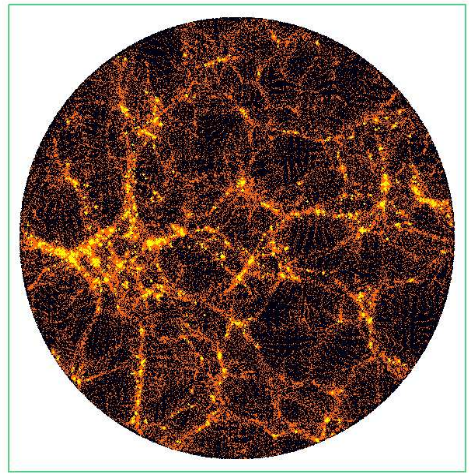


Figure 7. N-body simulation output in the Eulerian space used for testing our reconstruction method (shown is a projection onto the x-y plane of a 10% slice of the simulation box of size $200h^{-1}$ Mpc). Points are highlighted in yellow when reconstruction fails by more than $6.25\,h^{-1}$ Mpc, which happens mostly in high-density regions.

We first present reconstructions for three samples of particles initially situated on Lagrangian subgrids with meshes given by $\Delta x = 6.25\,h^{-1}$ Mpc, $\Delta x/2$ and $\Delta x/4$. To further reduce possible effects of the unphysical periodic boundary condition, we discard those points whose Eulerian positions are not within the sphere of radius $16\Delta x$ placed at the centre of the simulation box (for the largest Δx its diameter coincides with the box size). The problem is then confined to finding the pairing between the remaining Eulerian positions and the set of their periodicity-corrected Lagrangian antecedents in the N-body simulation.

The results are shown in Figs. 8–11. The main plots show the scatter of reconstructed vs. simulation Lagrangian positions for the same Eulerian positions. For these diagrams we introduce a 'quasi-periodic projection'

$$\tilde{q} \equiv (q_1 + \sqrt{2}q_2 + \sqrt{3}q_3)/(1 + \sqrt{2} + \sqrt{3}) \qquad (42)$$

of the vector $\boldsymbol{q}$, which ensures a one-to-one correspondence between $\tilde{q}$-values and points on the regular Lagrangian grid. The insets are histograms (by percentage) of distances, in reconstruction mesh units, between the reconstructed and simulation Lagrangian positions; the first darker bin, slightly less than one mesh in width, corresponds to perfect reconstruction (thereby allowing a good determination of the peculiar velocities of galaxies).

With the mesh size Δx, Lagrangian positions of 62% of the sample of 17,178 points are reconstructed perfectly and about 75% are placed within not more than one mesh. With the $\Delta x/2$ grid, we still have 35% of exact reconstruction out

[12] In a flavour of N-body codes called particle-mesh (PM) codes, Newtonian forces acting on particles are interpolated from the gravitational field computed on a uniform mesh. In very dense regions, precision is increased by adaptively refining the mesh and by direct calculation of local particle-particle (PP) interactions; codes of this type are correspondingly called *adaptive* P^3M.

[13] The use of a ΛCDM model instead of the model without a cosmological constant (Appendix A) leads to some modifications in basic equations but does not change formulas used for the MAK reconstruction.

[14] Data of the first year Wilkinson Microwave Anisotropy Probe (Spergel et al. 2003; see also Bridle et al. 2003) suggest a value $\sigma_8 = 0.84 \pm 0.04$, marginally smaller than the one used here. This may slightly extend the range of scales favourable for the MAK reconstruction.

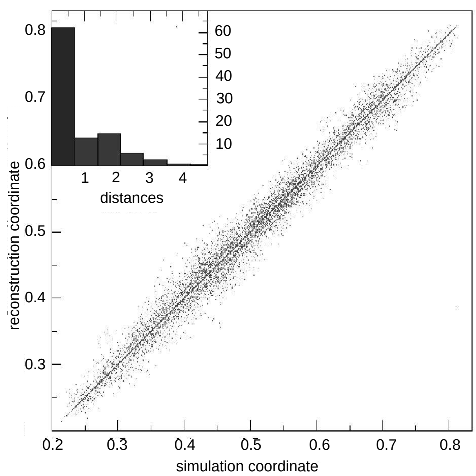


Figure 8. Test of the MAK reconstruction for a sample of $N' = 17,178$ points initially situated on a cubic grid with mesh $\Delta x = 6.25\,h^{-1}$ Mpc. The scatter diagram plots true versus reconstructed initial positions using a quasi-periodic projection which ensures one-to-one correspondence with points on the cubic grid. The histogram inset gives the distribution (in percentages) of distances between true and reconstructed initial positions; the horizontal unit is the sample mesh. The width of the first bin is less than unity to ensure that only exactly reconstructed points fall in it. Note that more than sixty percent of the points are exactly reconstructed.

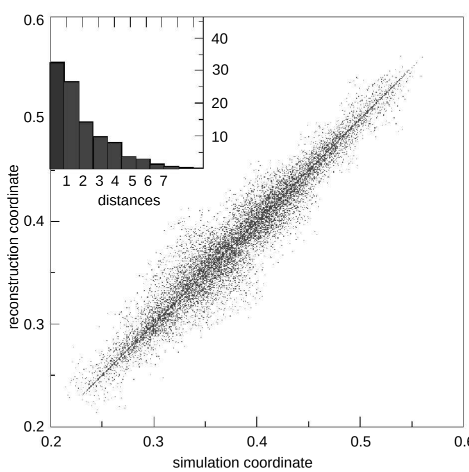


Figure 9. Same as Fig. 8 but with $N' = 19,187$ and a sample mesh of $\Delta x/2 = 3.125\,h^{-1}$ Mpc. Exact reconstruction is down to 35%.

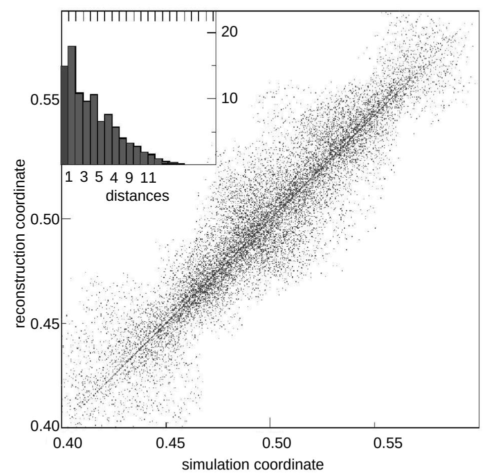


Figure 10. Same as Fig. 8 but with $N' = 23,111$ and a sample mesh of $\Delta x/4 = 1.56\,h^{-1}$ Mpc. Exact reconstruction is down to 14%.

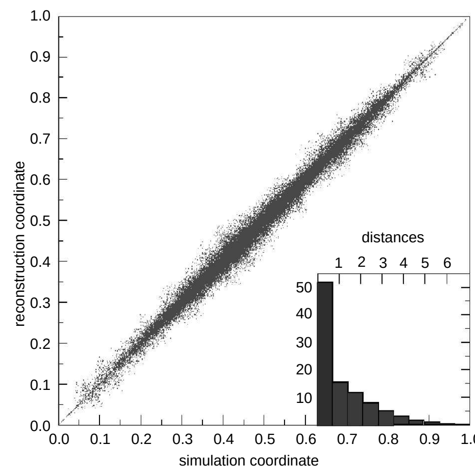


Figure 11. Same as Fig. 8 with $N' = 10^5$ points selected at random, neighbouring points being typically $4.35\,h^{-1}$ Mpc apart. Exact reconstruction is in excess of 50%.

of 19,187 points, but only 14% for the $\Delta x/4$ grid with 23,111 points.

We also performed a reconstruction on a random sample of 100,000 Eulerian positions taken with their periodicity-corrected Lagrangian antecedents out of the whole set of 128^3 particles, without any restrictions. This reconstruction, with the effective mesh size (average distance between neighbouring points) of $4.35h^{-1}$Mpc, gives 51% of perfect reconstruction (Fig. 11).

We compared these results with those of the PIZA reconstruction method (see Section 4.1 and Croft & Gaztañaga 1997), which gives a 2-monotone but not necessarily optimal pairing between Lagrangian and Eulerian positions. We applied the PIZA method on the Δx grid and obtained typically 30–40% exactly reconstructed positions, but severe non-uniqueness: for two different seeds of the random generator used to set up the initial tentative assignment, only about half of the exactly reconstructed positions were the same (see figs. 3 and 7 of Mohayaee et al. (2003) for an illustration). We also implemented a modification of the PIZA method establishing 3-monotonicity (monotonicity with respect to interchanges of 3 points instead of pairs) and checked that it does not give a significant improvement over the original PIZA.

In comoving coordinates, the typical displacement of a mass element is about 1/20 the box size, that is about $10\,h^{-1}$ Mpc. This is not much larger than the coarsest grid of $6.25\,h^{-1}$ Mpc used in testing MAK which gave 62% of exact reconstruction. Nevertheless there are 18 other grid points within $10\,h^{-1}$ Mpc of any given grid point, so that this high percentage cannot be trivially explained by the smallness of the displacement. Note that without the periodicity correction, the percentage of exact reconstruction for the coarsest grid degraded significantly (from 62% to 45%) and the resulting cost was far from the true minimum.

For real catalogues, reconstruction has to be performed for galaxies whose positions are specified in the *redshift space*, where they appear to be displaced radially (along the line of sight) by an amount proportional to the radial component of the peculiar velocity. Thus, at the present epoch, the redshift position $\boldsymbol{s}$ of a mass element situated at the point $\boldsymbol{x}$ in the physical space is given by

$$\boldsymbol{s} = \boldsymbol{x} + \hat{\boldsymbol{x}}\beta\,(\boldsymbol{v}\cdot\hat{\boldsymbol{x}})\,, \tag{43}$$

where $\boldsymbol{v}$ is the peculiar velocity in the comoving coordinates $\boldsymbol{x}$ and the linear growth factor time τ, $\hat{\boldsymbol{x}}$ denotes the unit normal in the direction of $\boldsymbol{x}$, and the parameter β equals 0.486 in our ΛCDM model.

Following Valentine et al. (2000; see also Monaco & Efstathiou 1999), we use the Zel'dovich approximation (ZA) to render our MAK quadratic cost function in the $\boldsymbol{s}$ variable. As follows from (11), in this approximation the peculiar velocity is given by

$$\boldsymbol{v} = \frac{1}{\tau}(\boldsymbol{x}-\boldsymbol{q}). \tag{44}$$

At the present time, since $\tau_0 = 1$, this together with (43) gives

$$(\boldsymbol{s}-\boldsymbol{q})\cdot\hat{\boldsymbol{x}} \;=\; (1+\beta)(\boldsymbol{x}-\boldsymbol{q})\cdot\hat{\boldsymbol{x}}, \tag{45}$$

$$|\boldsymbol{s}-\boldsymbol{q}|^2 \;=\; |\boldsymbol{x}-\boldsymbol{q}|^2 + \beta(\beta+2)\big((\boldsymbol{x}-\boldsymbol{q})\cdot\hat{\boldsymbol{x}}\big)^2. \tag{46}$$

Combining now these two equations and using the fact that, by (43), the vectors $\boldsymbol{x}$ and $\boldsymbol{s}$ are collinear and therefore $\hat{\boldsymbol{x}} = \pm\hat{\boldsymbol{s}}$, we may write the quadratic cost function as

$$\frac{1}{2}|\boldsymbol{x}-\boldsymbol{q}|^2 = \frac{1}{2}|\boldsymbol{s}-\boldsymbol{q}|^2 - \frac{\beta(\beta+2)}{2(\beta+1)^2}\big((\boldsymbol{s}-\boldsymbol{q})\cdot\hat{\boldsymbol{s}}\big)^2. \tag{47}$$

The redshift-space reconstruction is then in principle reduced to the physical-space reconstruction. Note however that the redshift transformation of Eulerian positions may fail to be one-to-one if the peculiar component of velocity field in the proper space coordinates exceeds the Hubble expansion component. This undermines the simple reduction outlined above for catalogues confined to small distances.

We have performed a MAK reconstruction with the redshift-modified cost function (47). The redshift positions were computed for the simulation data with peculiar velocities smoothed over a sphere with radius of 1/100 the box size ($2\,h^{-1}$ Mpc). This reconstruction led to 43% of exactly reconstructed positions and 60% which are within not more than one Δx mesh from their correct positions (see Fig. 12; a scatter diagram is omitted because it is quite similar to that in Fig. 8). A comparison of the redshift-space MAK reconstruction with the physical-space MAK reconstruction shows that almost 50% of exactly reconstructed positions correspond to the same points. This test shows that the MAK method is robust with respect to systematic errors introduced by the redshift transformation.

Our results demonstrate the essentially potential character of the Lagrangian map above $\sim 6\,h^{-1}$ Mpc (within the ΛCDM model) and perhaps at somewhat smaller scales.

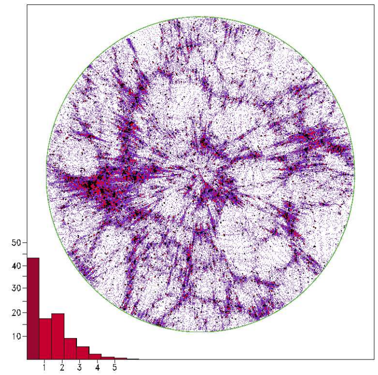


Figure 12. Test of the redshift-space variant of the MAK reconstruction based on the same data as Fig. 8. The circular redshift map (violet points) corresponds to the same physical-space slice as displayed in Fig. 7 (the observer is taken at the center of the simulation box). Points are highlighted in red when reconstruction fails by more than one mesh.

Although it is not our intention in this paper to actually implement the MAK reconstruction on real catalogues, a few remarks are in order. The effect of the catalogue selection function can be handled by standard techniques; for instance one can assign each galaxy a 'mass' inversely proportional to the catalog selection function (Nusser & Branchini 2000; Valentine et al. 2000; Branchini et al. 2002). Biasing can be taken into account in a similar manner (Nusser & Branchini 2000). Both these modifications and the natural scatter of masses in the observational catalogues require that massive objects be represented by clusters of multiple Eulerian points of unit mass (with the correspondingly increased number of points on a finer grid in the Lagrangian space), which reduces the problem to the usual assignment.

In the redshift-space modification, more accurate determination of peculiar velocities can be done using second-order Lagrangian perturbation theory. Note also that, for the observational catalogues, the motion of the local group itself should also be accounted for (Taylor & Valentine 1999).

6 RECONSTRUCTION OF THE FULL SELF-GRAVITATING DYNAMICS

The MAK reconstruction discussed in Sections 3 and 4 was performed under the assumption of a potential Lagrangian map and of the absence of multi-streaming. The tests done in Section 5 indicate that potentiality works well at scales above $6\,h^{-1}$ Mpc, whereas multi-streaming is mostly believed to be unimportant above a few megaparsecs. There could thus remain a substantial range of scales over which the quality of the reconstruction can be improved by relaxing the potentiality assumption and using the full self-gravitating dynamics. Here we show that, as long as the dynamics can be described by a solution to the Euler–Poisson equations, the prescription of the present density field still determines a unique solution to the full reconstruction problem. We give only the main ideas, technical details being left for Appendix D (a mathematically rigorous proof may be found in Loeper (2003)). In order to make the exposition self-contained, we also give in Appendix C an elementary introduction to convexity and duality which are used for the derivation (and also elsewhere in this paper).

We shall start from an Eulerian variational formulation of the Euler–Poisson equations in an Einstein–de Sitter universe, which is an adaptation of a variational principle given by Giavalisco et al. (1993). We minimize the action

$$I = \frac{1}{2}\int_0^{\tau_0} \mathrm{d}\tau \int \mathrm{d}^3\boldsymbol{x}\, \tau^{3/2} \left(\rho|\boldsymbol{v}|^2 + \frac{3}{2}|\nabla_{\boldsymbol{x}}\varphi_{\mathrm{g}}|^2\right), \tag{48}$$

under the following four constraints: the Poisson equation (3), the mass conservation equation (2) and the boundary conditions that the density field be unity at $\tau = 0$ and prescribed at the present time $\tau = \tau_0$. The constraints can be handled by the standard method of Lagrange multipliers (here functions of space and time), which allows to vary independently the fields ρ, φ_{g} and $\boldsymbol{v}$. The vanishing of the variation in $\boldsymbol{v}$ gives $\boldsymbol{v} = \tau^{-3/2}\nabla_{\boldsymbol{x}}\theta$, where $\theta(\boldsymbol{x}, \tau)$ is the Lagrange multiplier for the mass conservation constraint. Hence, the velocity is curl-free. The vanishing of the variation in ρ gives then

$$\partial_\tau\theta + \frac{1}{2\tau^{3/2}}|\nabla_{\boldsymbol{x}}\theta|^2 + \frac{3}{2\tau}\psi = 0. \tag{49}$$

By taking the gradient, this equation goes over into the momentum equation (1), repeated here for convenience:

$$\partial_\tau\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}})\boldsymbol{v} = -\frac{3}{2\tau}(\boldsymbol{v} + \nabla_{\boldsymbol{x}}\varphi_{\mathrm{g}}). \tag{50}$$

It is noteworthy that, if in the action we replace 3/2 both in the exponent of τ and in the gravitational energy term by $3\alpha/2$, we obtain (50) but also with a $3\alpha/(2\tau)$ factor in the right-hand side. The Zel'dovich approximation and the associated MAK reconstruciton amount clearly to setting $\alpha = 0$, so as to recover the 'free-streaming action'

$$I = \frac{1}{2}\int_0^{\tau_0} \mathrm{d}\tau \int \mathrm{d}^3\boldsymbol{x}\; \rho|\boldsymbol{v}|^2. \tag{51}$$

Assuming the action (48) to be finite, existence of a minimum is mostly a consequence of the action being manifestedly non-negative. Here it is interesting to observe that the Lagrangian, which is the *difference* between the kinetic energy and the potential energy, is positive whereas the Hamiltonian which is their *sum* does not have a definite sign. As a consequence, our two-point boundary problem is, as we shall see, well posed but the initial-value problem for the Euler–Poisson system is not well posed since formation of caustics after a finite time cannot be ruled out.[15]

Does the variational formulation imply uniqueness of the solution? This would be the case if the action were a strictly convex functional (see Appendix C1), which is guaranteed to have one and only one minimum. The action as written in (48) is not convex in the ρ and $\boldsymbol{v}$ variables, but can be rendered so by introducing the mass flux $\boldsymbol{J} = \rho\,\boldsymbol{v}$; the kinetic energy term becomes then $|\boldsymbol{J}|^2/(2\rho)$, which is convex in the $\boldsymbol{J}$ and ρ variables.

Strict convexity is particularly cumbersome to establish, but there is an alternative way, known as duality: by a Legendre-like transformation the variational problem is carried into a dual problem written in terms of dual variables; the minimum value for the original problem is the maximum for the dual problem. It turns out that the difference of these equal values can be rewritten as a sum of non-negative terms, each of which must thus vanish. This is then used to prove (i) that the difference between any two solutions to the variational problem vanishes and (ii) that any curl-free solution to the Euler–Poisson equations with the prescribed boundary conditions for the density also minimizes the action. All this together establishes uniqueness. For details see Appendix D.

Several of the issues raised in connection with the MAK reconstruction appear in almost the same form for the Euler–Poisson reconstruction. First, we are faced again with the problem that, when reconstructing from a finite patch of the present universe, we need either to know the shape of the initial domain or to make some hypothesis as to the present distribution of matter outside this patch. Second, just as for the MAK reconstruction, the proof of uniqueness still holds when the present density $\rho_0(\boldsymbol{x})$ has a singular part, that is, when some matter is concentrated. Again, we shall have full information on the initial shape of collapsed regions but not on the initial fluctuations inside them. The particular solution obtained from the variational formulation is the only solution which stays smooth for all times prior to τ_0.

We also note that, at this moment and probably for quite some time, 3D catalogues sufficiently dense to allow reconstruction will be limited to fairly small redshifts. Eventually, it will however become of interest to perform reconstruction 'along our past light-cone' with data not all at τ_0. The variational approach can in principle be adapted to handle such reconstruction.

In previous sections we have seen how to implement reconstruction using MAK, which is equivalent to using the simplified action (51). Implementation using the full Euler–Poisson action (48) is mostly beyond the scope of this paper, but we shall indicate some possible directions. In principle it should be possible to adapt to the Euler–Poisson reconstruction the method of the augmented Lagrangian which has been applied to the two-dimensional Monge–Ampère equation (Benamou & Brenier 2000). An alternative strategy, which allows reduction to MAK-type problems, uses the idea of 'kicked burgulence' (Bec, Frisch & Khanin 2000) in which, in order to solve the one or multi-dimensional Burgers equation

$$\partial_\tau\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}})\boldsymbol{v} = \boldsymbol{f}(\boldsymbol{x},\tau), \quad \boldsymbol{v} = -\nabla_{\boldsymbol{x}}\varphi_{\mathrm{v}}, \tag{52}$$

one approximates the force by a sum of delta-functions in time:

$$\boldsymbol{f}(\boldsymbol{x},\tau) \approx \sum_i \delta(\tau - \tau_i)\boldsymbol{g}_i(\boldsymbol{x}). \tag{53}$$

In the present case, the $\boldsymbol{g}_i(\boldsymbol{x})$ are proportional to the right-hand side of (50) evaluated at the kicking times τ_i. The action becomes then a sum of free-streaming Zel'dovich-type actions plus discrete gravitational contributions stemming from the kicking times. Between kicks one can use our MAK solution. At kicking times the velocity undergoes a discontinuous change which is related to the gravitational potential (and thus to the density) at those times. The densities

[15] If we had considered electrostatic repulsive interactions the conclusions would be reversed.

at kicking times can be determined by an iterative procedure. The kicking strategy also allows to do redshift-space reconstruction by applying the redshift-space modified cost (Section 5) at the last kick.

7 COMPARISON WITH OTHER RECONSTRUCTION METHODS

Reconstruction started with Peebles' (1989) work, in which he compared reconstructed and measured peculiar velocities for a small number of Local Group galaxies, situated within a few Mpc. The focus of reconstruction work has now moved to tackling the rapidly growing large 3D surveys (see, e.g. Frieman & Szalay 2000). It is not our intention here to review all the work on reconstruction;[16] rather we shall discuss how some of the previously used methods can be reinterpreted in the light of the optimization approach to reconstruction. For convenience we shall divide methods into perturbative (Section 7.1), probabilistic (Section 7.2), and variational (Section 7.3). Methods such as POTENT (Dekel et al. 1990), whose purpose is to obtain the full peculiar velocity field from its radial components using the (Eulerian) curl-free property, are not directly within our scope. Note that in its original Lagrangian form (Bertschinger & Dekel 1989; Dekel et al. 1990) POTENT was assuming a curl-free velocity in Lagrangian coordinates, an assumption closely related to the potential assumption made for MAK, as already pointed out in Section 3.1.

7.1 Perturbative methods

Nusser & Dekel (1992) have proposed using the Zel'dovich approximation backwards in time to obtain the initial velocity fluctuations and thus (by slaving) the density fluctuations. Schematically, their procedure involves two steps: (i) obtaining the present potential velocity field and (ii) integrating the Zel'dovich–Bernouilli equation back in time. Using the equality (in our notation) of the velocity and gravitational potentials, they point out that the velocity potential can be computed from the present density fluctuation field by solving the Poisson equation. This is a perturbative approximation to reconstruction in so far as it replaces the Monge–Ampère equation (19) by a linearized form. Indeed, when using the Zel'dovich approximation we have $\boldsymbol{q} = \boldsymbol{x} - \tau\boldsymbol{v} = \boldsymbol{x} + \tau\nabla_{\boldsymbol{x}}\varphi_{\rm v}(\boldsymbol{x})$. We know that $\boldsymbol{q} = \nabla_{\boldsymbol{x}}\Theta(\boldsymbol{x})$ with Θ satisfying the Monge–Ampère equation. The latter can thus be rewritten as

$$\det\left(\delta_{ij} + \tau\nabla_{x_i}\nabla_{x_j}\varphi_{\rm v}(\boldsymbol{x})\right) = \rho(\boldsymbol{x}), \qquad (54)$$

where δ_{ij} denotes the identity matrix. If we now use the relation $\det(\delta_{ij} + \epsilon A_{ij}) = 1 + \epsilon\sum_i A_{ii} + O(\epsilon^2)$ and truncate the expansion at order ϵ, we obtain the Poisson equation

$$\tau\nabla^2_{\boldsymbol{x}}\varphi_{\rm v}(\boldsymbol{x}) = \rho(\boldsymbol{x}) - 1 = \delta(\boldsymbol{x}). \qquad (55)$$

Of course, in one dimension no approximation is needed. From a physical point of view, equating the velocity and gravitational potentials at the present epoch amounts to using the Zel'dovich approximation in reverse and is actually inconsistent with the forward Zel'dovich approximation: the slaving which makes the two potentials equal initially does not hold in this approximation at later epochs. Replacing the Monge–Ampère equation by the Poisson equation is not consistent with a uniform initial distribution of matter and will in general lead to spurious multi-streaming in the initial distribution. Of course, if the present-epoch velocity field happens to be known one can try applying the Zel'dovich approximation in reverse. Nusser and Dekel observe that calculating the inverse Lagrangian map by $\boldsymbol{q} = \boldsymbol{x} - \tau\boldsymbol{v}$ does not work well (spurious multi-streaming appears) and instead integrate back in time the Zel'dovich–Bernouilli equation[17]

$$\partial_t\varphi_{\rm v} = \frac{1}{2}\left(\nabla_{\boldsymbol{x}}\varphi_{\rm v}\right)^2, \qquad (56)$$

which is obviously equivalent to the Burgers equation (13) with the viscosity $\nu = 0$. One way of performing this reverse integration, which guarantees the absence of multi-streaming, is to use the Legendre transformation (18) to calculate $\Phi(\boldsymbol{q})$ from $\Theta(\boldsymbol{x}) = |\boldsymbol{x}|^2/2 - \tau\varphi_{\rm v}(\boldsymbol{x})$ and then obtain the reconstructed initial velocity field as

$$\boldsymbol{v}_{\rm in}(\boldsymbol{q}) = \boldsymbol{v}_0\left(\nabla_{\boldsymbol{q}}\Phi(\boldsymbol{q})\right). \qquad (57)$$

This procedure can however lead to spurious shocks in the reconstructed initial conditions, due to inaccuracies in the present-epoch velocity data, unless the data are suitably smoothed. Finally, the improved reconstruction method of Gramann (1993) can be viewed as an approximation to the Monge–Ampère equation beyond the Poisson equation which captures part of the nonlinearity.

7.2 Probabilistic methods

Weinberg (1992) presents an original approach to reconstruction, which turns out to have hidden connections to optimal mass transportation. The key observations in his 'Gaussianization' technique are the following: (i) the initial density fluctuations are assumed to be Gaussian, (ii) the rank order of density values is hardly changed between initial and present states, (iii) the bulk displacement of large-scale features during dynamical evolution can be neglected. Assumption (i) is part of the standard cosmological paradigm. Assumption (iii) can of course be tested in N-body simulations. As we have seen in Section 5, a displacement of $10\,h^{-1}$Mpc is typical and can indeed be considered small compared to the size of the simulation boxes ($64\,h^{-1}$Mpc in Weinberg's simulations and $200\,h^{-1}$Mpc in ours). Assumption (ii) means that the correspondence between initial and present values of the density ρ (or of the contrast $\delta = \rho - 1$) is monotone. This map, which can be determined from the empirical present data, can then be applied to all the data to produce a reconstructed initial density field. Finally, by running an N-body simulation initialized on the reconstructed field one can test the validity of the procedure, which turns

[16] For a comparison of six different techniques, see Narayanan & Croft (1999).

[17] In the non-cosmological literature this equation is usually called Hamilton–Jacobi in the context of analytical mechanics (Landau & Lifshitz 1960) and Kardar–Parisi–Zhang (1986) in condensed matter physics.

out to be quite good and can be improved further by hybrid methods (Narayanan & Weinberg 1998; Kolatt et al. 1996) combining Gaussianization with the perturbative approaches of Nusser & Dekel (1992) or Gramann (1993).

This technique is actually connected with mass transportation: starting with the work of Fréchet (1957a; 1957b; see also Rachev 1984), probabilists have been asking the following question: given two random variables m_1 and m_2 with two laws, say PDFs p_1 and p_2, can one find a joint distribution of (m_1, m_2) with PDF $p_{12}(m_1, m_2)$ having the following properties: (i) p_1 and p_2 are the marginals, i.e. when p_{12} is integrated over m_2 (respectively, m_1) one recovers p_1 (respectively, p_2), (ii) the correlation $\langle m_1 m_2\rangle$ is maximum? Since $\langle m_1^2\rangle$ and $\langle m_2^2\rangle$ are obviously prescribed by the constraint that we know p_1 and p_2, maximizing the correlation is the same as minimizing the quadratic distance $\langle (m_1 - m_2)^2\rangle$. This is precisely an instance of the mass transportation problem with quadratic cost, as we defined it in Section 3.3. As we know, the optimal solution is obtained by a map from the space of m_1 values to that of m_2 values which is the gradient of a convex function. If m_1 and m_2 are scalar variables, the map is just monotone, as in the Gaussianization method (in the discrete setting this was already observed in Section 4.1). Hence Weinberg's method may be viewed as requiring maximum correlation (or minimum quadratic distance in the above sense) between initial and present distributions of density fluctuations.

In principle the Gaussianization method can be extended to multipoint distributions, leading to a difficult multidimensional mass transportation problem which can be discretized into an assignment problem just as in Section 4.1. The contact of the maximum correlation assumption to the true dynamics is probably too flimsy to justify using such heavy machinery.

7.3 Variational methods

All variational approaches to reconstruction, starting with that of Peebles (1989), have common features: one uses a suitable Lagrangian and poses a two-point variational problem with boundary conditions prescribed at the present epoch by the observed density field, and at early times by requiring a quasi-uniform distribution of matter (more precisely, as we have seen in Section 2.1, by requiring that the solutions not be singular as $\tau \to 0$).

The Path Interchange Zel'dovich Approximation (PIZA) method of Croft & Gaztañaga (1997) and our MAK reconstruction techniques use a free-streaming Lagrangian in linear growth rate time. As we have seen in Section 3.1, this amounts to assuming adhesion dynamics. Once discretized for numerical purposes, the variational problem becomes an instance of the assignment problem. Croft & Gaztañaga (1997) have proposed a restricted procedure for solving it, which yields non-unique approximate solutions. As we have seen in Sections 4 and 5, the exact and unique solution can be found with reasonable CPU resources.

Turning now to the Peebles least action method, let us first describe it schematically, using our notation. In its original formulation it is applied to a discrete set of galaxies (assumed of course to trace mass) in an Einstein–de Sitter universe. The action, in our notation, can be written as

$$I = \int_0^{\tau_0} \mathrm{d}\tau \, \frac{3}{2\tau^{1/2}} \left(\sum_i \frac{m_i \tau^2}{3} \left| \frac{d\boldsymbol{x}_i}{d\tau} \right|^2 + \frac{3G}{2} \sum_{i \neq j} \frac{m_i m_j}{|\boldsymbol{x}_i - \boldsymbol{x}_j|} + \pi G \bar{\varrho}_0 \sum_i m_i |\boldsymbol{x}_i|^2 \right), \qquad (58)$$

where m_i is the mass and $\boldsymbol{x}_i$ the comoving coordinate of ith galaxy (see also Nusser & Branchini 2000). This is supplemented by the boundary condition that the present positions of the galaxies are known and that the early-time velocities satisfy[18]

$$\tau^{3/2} \frac{d\boldsymbol{x}_i}{d\tau} \to 0 \qquad \text{for} \quad \tau \to 0. \qquad (59)$$

This particle approach was extended by Giavalisco et al. (1993) to a continuous distribution in Eulerian coordinates and leads then to the action analogous to (48) which we have used in Section 6. The procedure also involves a 'Galerkin truncation' of the particle trajectories to finite sums of trial functions of the form

$$x_i^\mu(\tau) = x_i^\mu(\tau_0) + \sum_{n=0}^{N-1} C_{i,n}^\mu f_n(\tau), \qquad (60)$$

$$f_n(\tau) = \tau^n(\tau_0 - \tau), \quad n = 0, 1, \ldots, N-1. \qquad (61)$$

The reconstructed peculiar velocities for the Local Group were used by Peebles to calibrate the Hubble and density parameters, which turned out to differ from the previously assumed values. However the peculiar velocity of one dwarf galaxy, N6822, failed to match the observed value (see Fig. 13). This led Peebles (1990) to partially relax the assumption of *minimum* action, allowing also for saddle points in the action. Somewhat better agreement with observations is then obtained, but at the expense of lack of uniqueness.

In the context of the present approach, various remarks can be made. The boundary condition (59) is trivially satisfied if the velocities $d\boldsymbol{x}/d\tau$ remain bounded. Actually, we have seen in Section 2.1 that, as a consequence of slaving, the velocity has a regular expansion in powers of τ, which implies its boundedness as $\tau \to 0$. The important point is that the function $f_n(\tau)$ appearing in (60) should be expandable in powers of τ, as is the case with the ansatz (61).

In Section 6 we have established uniqueness of the reconstruction with a prescribed present density and under the assumption of absence of multi-streaming (but we allow for mass concentrations). This restriction is meaningful only in the continuous case: in the discrete case, unless the particles are rather closely packed, the concept of multi-streaming is not clear but there have been attempts to relate uniqueness to absence of 'orbit crossing' (see, e.g., Giavalisco et al. 1993; Whiting 2000). Of course, at the level of the underlying dark matter, multi-streaming is certainly not ruled out at sufficiently small scales; at such scales unique reconstruction is not possible.

In the truly discrete case, e.g. when considering a dwarf

[18] This condition, which is written $a^2 d\boldsymbol{x}_i/dt \to 0$ in Peebles' notation, ensures the vanishing of the corresponding boundary term after an integration by parts in the time variable.

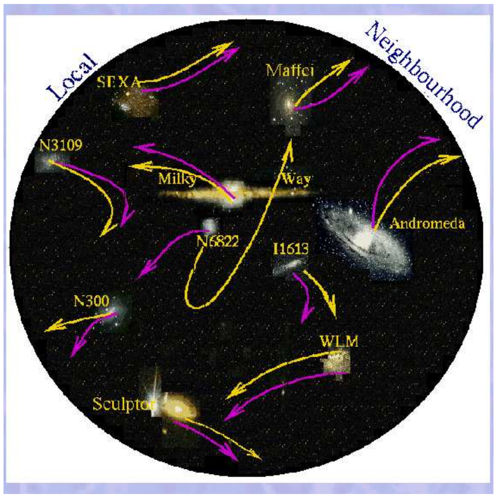


Figure 13. A schematic demonstration of Peebles' reconstruction of the trajectories of the members of the local neighbourhood using a variational approach based on the minimization of Euler–Lagrange action. The arrows go back in time, starting from the present and pointing towards the initial positions of the sources. In most cases there is more than one allowed trajectory due to orbit crossing (closely related to the multi-streaming of the underlying dark matter fluid). The pink (darker) orbits correspond to taking the minimum of the action whereas the yellow (brighter) orbits were obtained by taking the saddle-point solution. Of particular interest is the orbit of N6822 which in the former solution is on its first approach towards us and in the second solution is in its passing orbit. A better agreement between the evaluated and observed velocities was shown to correspond to the saddle-point solution.

galaxy, there is no reason to prefer the true minimum action solution over any other stationary action solution.

8 CONCLUSION

The main theoretical result of this paper is that reconstruction of the past dynamical history of the Universe, knowing only the present spatial distribution of mass, is a well-posed problem with a unique solution. More precisely, reconstruction is uniquely defined down to those scales, a few megaparsecs, where multi-streaming becomes important. The presence of concentrated mass in the form of clusters, filaments, etc is not an obstacle to a unique displacement reconstruction; the mass within each such structure originates from a collapsed region of known shape but with unknown initial density and velocity fluctuations inside. There are of course practical limitations to reconstruction stemming from the knowledge of the present mass distribution over only a limited patch of the Universe; these were discussed in Section 3.4.

In this paper we have also presented in detail and tested a reconstruction method called MAK which reduces reconstruction to an assignment problem with quadratic cost, for which effective algorithms are available. MAK, which is exact for dynamics governed by the adhesion model, works very well above $6\,h^{-1}$ Mpc and can in principle be adapted to full Euler–Poisson reconstruction.

We note that a very common method for testing ideas about the early Universe is to take some model of early density fluctuations and then run an N-body simulations with assumed cosmological parameters until the present epoch. Confrontation with the observed *statistical properties* of the present Universe helps then in selecting plausible models and in narrowing the choice of cosmological parameters. This *forward method* is conceptually very different from reconstruction; the latter not only works backward but, more importantly, it is a *deterministic* method which gives us a detailed map of the early Universe and how it relates to the present one. Reconstruction thus allows us to obtain the peculiar velocities of galaxies and is probably the only method which can hope to do this for a large number of galaxies. In those instances were we have partial information on peculiar velocities (from independent distance measurements), e.g. for the NearBy Galaxies (NBG) catalogue of Tully (1988), such information can be used to calibrate cosmological parameters or to provide additional constraints, which are in principle redundant but can improve the quality. The detailed reconstruction of early density fluctuations, which will become possible using large 3D surveys such as 2dF and SDSS (see, e.g., Frieman & Szalay 2000), will allow us to test such assumptions as the Gaussianity of density fluctuations at decoupling.

Finally we have no reason to hide the pleasure we experience in seeing this heavenly problem bring together and indeed depend crucially on so many different areas of mathematics and physics, from fluid dynamics to Monge–Ampère equations, mass transportation, convex geometry and combinatorial optimization. Probably this is the first time that one tackles the three-dimensional Monge–Ampère equation numerically for practical purposes. As usual, we can expect that the techniques, here applied to cosmic reconstruction, will find many applications, for example to the optimal matching of two holographic or tomographic images or to the correction of images in multi-dimensional colour space.

ACKNOWLEDGMENTS

Special thanks are due to E. Branchini (observational and conceptual aspects) and to D. Bertsekas (algorithmic aspects). We also thank J. S. Bagla, J. Bec, E. Bertschinger, T. Buchert, A. Domingues, H. Frisch, J. Gaite, L. Moscardini, A. Noullez, M. Rees, V. Sahni, S. Shandarin, A. Shnirelman, E. Slezak, E. Spiegel, A. Starobinsky, P. Thomas and B. Villone for comments and useful information.

This work was supported by the European Union under contract HPRN-CT-2000-00162, by the BQR program of Observatoire de la Côte d'Azur, by the TMR program of the European Union (UF), by MIUR (SM), by the French Ministry of Education, the McDonnel Foundation and the Russian RFBR (AS). RM was supported by the TMR network 'fractal structures and self-organization' under contract FM-RXCT980183 and by a Marie Curie Fellowship HPMF-CT-2002-01532.

APPENDIX A: EQUATIONS OF MOTION IN AN EXPANDING UNIVERSE

On distances covered by present and forthcoming redshift galaxy catalogues, the Newtonian description constitutes a realistic approximation to the dynamics of self-gravitating cold dark matter filling the Universe (Peebles 1980; Coles & Lucchin 2002). This description gives, in proper space coordinates denoted here by $\boldsymbol{r}$ and cosmic time t, the familiar Euler–Poisson system for the density $\varrho(\boldsymbol{r},t)$, velocity $\boldsymbol{U}(\boldsymbol{r},t)$ and the gravitational potential $\phi(\boldsymbol{r},t)$:

$$\begin{aligned} \partial_t \boldsymbol{U} + (\boldsymbol{U}\cdot\nabla_{\boldsymbol{r}})\boldsymbol{U} &= -\nabla_r \phi_{\rm g}, & \text{(A.1)}\\ \partial_t \varrho + \nabla_{\boldsymbol{r}}\cdot(\varrho \boldsymbol{U}) &= 0, & \text{(A.2)}\\ \nabla^2_{\boldsymbol{r}}\phi_{\rm g} &= 4\pi G\varrho, & \text{(A.3)} \end{aligned}$$

where G is the gravitation constant.

In a homogeneous isotropic universe, the density and velocity fields take the form

$$\varrho(\boldsymbol{r},t) = \bar\varrho(t), \quad \boldsymbol{U}(\boldsymbol{r},t) = H(t)\boldsymbol{r} = \frac{\dot a(t)}{a(t)}\boldsymbol{r}. \qquad \text{(A.4)}$$

Here the coefficient $H(t)$ is the *Hubble parameter*, and $a(t)$ is the *expansion scale factor* defined so that integration of the velocity field $\dot{\boldsymbol{r}} = \boldsymbol{U}(\boldsymbol{r},t) = H(t)\boldsymbol{r}$ yields $\boldsymbol{r} = a(t)\boldsymbol{x}$, where $\boldsymbol{x}$ is called the *comoving coordinate*.

The *background density* $\bar\varrho(t)$ gives rise to the background gravitational potential $\bar\phi_{\rm g}$, which by (A.1) and (A.4) satisfies

$$-\nabla_{\boldsymbol{r}}\bar\phi_{\rm g} = \frac{\ddot a}{a}\boldsymbol{r}. \qquad \text{(A.5)}$$

For the background density, mass conservation (A.2) gives then

$$\bar\varrho a^3 = \bar\varrho_0, \qquad \text{(A.6)}$$

where $\bar\varrho_0 = \bar\varrho(t_0)$ with t_0 the present epoch and $a(t_0)$ normalized to unity. Eqs. (A.5), (A.6), and (A.3) imply the *Friedmann equation* for $a(t)$:

$$\ddot a = -\frac{4}{3}\pi G\bar\varrho_0\frac{1}{a^2} \qquad \text{(A.7)}$$

with conditions posed at $t = t_0$:

$$a(t_0) = 1, \quad \dot a(t_0) = H_0 > 0, \qquad \text{(A.8)}$$

where H_0 is the present value of the Hubble parameter, positive for an expanding universe.

For simplicity we restrict ourselves to the case of the *critical density*, corresponding to the flat, matter-dominated Einstein-de Sitter universe (without a cosmological constant):

$$\bar\varrho_0 = \frac{3H_0^2}{8\pi G} \qquad \text{(A.9)}$$

and adjust the origin of the time axis such that the solution takes the form of a power law

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3} \qquad \text{(A.10)}$$

with $H_0 = 2/(3t_0)$ and $\bar\varrho_0 = 1/(6\pi G t_0^2)$.

The observed Hubble expansion of the Universe suggests that the density, velocity and gravitational fields may be decomposed into a sum of terms describing the uniform expansion and fluctuations against the background:

$$\varrho = \bar\varrho(t)\,\rho, \quad \boldsymbol{U} = \frac{\dot a(t)}{a(t)}\boldsymbol{r} + a(t)\boldsymbol{u}, \quad \phi_{\rm g} = \bar\phi_{\rm g} + \widetilde\varphi_{\rm g}. \qquad \text{(A.11)}$$

The term $a(t)\boldsymbol{u}$ is called the *peculiar velocity*. In cosmology, one also often employs the *density contrast* defined as $\delta = \rho - 1$, which gives the fluctuation against the normalized background density. Taking ρ, $\boldsymbol{u}$, and $\widetilde\varphi_{\rm g}$ as functions of the comoving coordinate $\boldsymbol{x} = \boldsymbol{r}/a(t)$ and using (A.5), (A.6) and (A.7), we rewrite the Euler–Poisson system in the form

$$\begin{aligned} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla_{\boldsymbol{x}})\boldsymbol{u} &= -2\frac{\dot a}{a}\boldsymbol{u} - \frac{1}{a}\nabla_{\boldsymbol{x}}\widetilde\varphi_{\rm g}, & \text{(A.12)}\\ \partial_t \rho + \nabla_{\boldsymbol{x}}\cdot(\rho\boldsymbol{u}) &= 0, & \text{(A.13)}\\ \nabla^2_{\boldsymbol{x}}\widetilde\varphi_{\rm g} &= \frac{4\pi G\bar\varrho_0}{a}(\rho - 1). & \text{(A.14)} \end{aligned}$$

Note the *Hubble drag* term $-2(\dot a/a)\boldsymbol{u}$ in the right-hand side of (A.12) representing the relative slowdown of peculiar velocities due to the uniform expansion.

Formally linearizing (A.12)–(A.14) around the trivial zero solution, one obtains the following ODE for the *linear growth factor* $\tau(t)$ of density fluctuations:

$$\frac{d}{dt}(a^2\dot\tau) = 4\pi G\bar\varrho_0\frac{\tau}{a}. \qquad \text{(A.15)}$$

The only solution of this equation that stays bounded (indeed, vanishes) at small times is usually referred to as the *growing mode*. As we shall shortly see, it is convenient to choose the amplitude factor τ of the growing mode to be a new 'time variable,' which in an Einstein–de Sitter universe is proportional to $t^{2/3}$. It is normalized such that $\tau_0 = \tau(t_0) = 1$. Rescaling the peculiar velocity and the gravitational potential according to

$$\boldsymbol{u} = \dot\tau\boldsymbol{v}, \quad \widetilde\varphi_{\rm g} = \frac{4\pi G\bar\varrho_0\tau}{a}\varphi_{\rm g} \qquad \text{(A.16)}$$

and using the fact that in an Einstein–de Sitter universe $d\ln(a^2\dot\tau)/d\tau = 3/(2\tau)$, we arrive at the following form of the *Euler–Poisson system*, which we use throughout this paper:

$$\begin{aligned} \partial_\tau \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}})\boldsymbol{v} &= -\frac{3}{2\tau}(\boldsymbol{v} + \nabla_{\boldsymbol{x}}\varphi_{\rm g}), & \text{(A.17)}\\ \partial_\tau \rho + \nabla_{\boldsymbol{x}}\cdot(\rho\boldsymbol{v}) &= 0, & \text{(A.18)}\\ \nabla^2_{\boldsymbol{x}}\varphi_{\rm g} &= \frac{\rho - 1}{\tau}. & \text{(A.19)} \end{aligned}$$

Suppose initially, i.e. at $\tau = 0$, a mass element is located at a point with the comoving coordinate $\boldsymbol{q}$. Transported by the peculiar velocity field in the comoving coordinates, this element describes a trajectory $\boldsymbol{x}(\boldsymbol{q},\tau)$. Using the *Lagrangian coordinate* $\boldsymbol{q}$ to parametrize the whole continuum of mass elements, we recast (A.17) and (A.19) in the form

$$\begin{aligned} D_\tau^2\boldsymbol{x} &= -\frac{3}{2\tau}\left(D_\tau\boldsymbol{x} + \nabla_{\boldsymbol{x}}\varphi_{\rm g}\right), & \text{(A.20)}\\ \nabla^2_{\boldsymbol{x}}\varphi_{\rm g} &= \frac{1}{\tau}\left[(\det\nabla_{\boldsymbol{q}}\boldsymbol{x})^{-1} - 1\right]. & \text{(A.21)} \end{aligned}$$

The density and peculiar velocity in Lagrangian variables are given by

$$\begin{aligned} &\rho(\boldsymbol{x}(\boldsymbol{q},\tau),\tau) = (\det\nabla_{\boldsymbol{q}}\boldsymbol{x})^{-1}, \\ &\boldsymbol{v}(\boldsymbol{x}(\boldsymbol{q},\tau),\tau) = D_\tau\boldsymbol{x}(\boldsymbol{q},\tau), \end{aligned} \qquad \text{(A.22)}$$

which automatically satisfy the mass conservation law (A.18). Here D_τ is the operator of Lagrangian time derivative, which in Lagrangian variables is the usual partial time derivative at constant $\boldsymbol{q}$ and in Eulerian variables coincides with the material derivative $\partial_\tau + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}$. The notation $\nabla_{\boldsymbol{x}}$ in Lagrangian variables stands for the $\boldsymbol{x}(\boldsymbol{q},\tau)$-dependent differential operator with components $\nabla_{x_i} \equiv (\partial q_j/\partial x_i)\nabla_{q_j}$, which expresses the Eulerian gradient rewritten in Lagrangian coordinates, using the inverse Jacobian matrix. Note that $\nabla_{\boldsymbol{x}}$ and D_τ do not commute and that terms with $\nabla_{\boldsymbol{x}}$ in the Lagrangian equations are implicitly non-linear.

In one dimension, (A.21) has an interesting consequence:

$$\nabla_x \varphi_{\rm g} = -\frac{x-q}{\tau}. \tag{A.23}$$

Indeed, in one dimension (A.21) takes the form

$$\nabla_x^2 \varphi_{\rm g} = \frac{1}{\tau}\left[(\nabla_q x)^{-1} - 1\right]. \tag{A.24}$$

Multiplying this equation by $\nabla_q x$ and expressing the first of the two x-derivatives acting on $\varphi_{\rm g}$ as a q-derivative, we obtain

$$\nabla_q\left(\nabla_x\varphi_{\rm g}\right) = \nabla_q \frac{q-x}{\tau}. \tag{A.25}$$

Eq. (A.23) is obtained from (A.25) by integrating in q. The absence of an arbitrary τ-dependent constant is established either by assuming vanishing at large distances of both $\varphi_{\rm g}$ and of the displacement $x-q$ or, in the space-periodic case, by assuming the vanishing of period averages.

Using (A.23) to eliminate the $\varphi_{\rm g}$ term in (A.20) and introducing the notation ξ for the displacement $x-q$, we obtain

$$D_\tau^2 \xi = -\frac{3}{2\tau}\left(D_\tau \xi - \frac{\xi}{\tau}\right). \tag{A.26}$$

The only solution to this equation that remains well-behaved for $\tau \to 0$ is the linear one $\xi \propto \tau$. This solution has the two terms on the right-hand side of the one-dimensional version of (A.20) cancelling each other and hence gives a vanishing 'acceleration' $D_\tau^2 x$.

An approximate vanishing of acceleration takes place in higher dimensions as well. For early times, the *Lagrangian map* $\boldsymbol{x}(\boldsymbol{q},\tau)$ stays close to the identity, with displacements $\boldsymbol{\xi}(\boldsymbol{q},\tau) = \boldsymbol{x}(\boldsymbol{q},\tau) - \boldsymbol{q}$ small. Linearizing (A.20) and (A.21) around zero displacement, we get the system

$$D_\tau^2 \boldsymbol{\xi} = -\frac{3}{2\tau}(D_\tau \boldsymbol{\xi} + \nabla_{\boldsymbol{q}}\varphi_{\rm g}), \tag{A.27}$$

$$\nabla_{\boldsymbol{q}}^2 \varphi_{\rm g} = -\frac{1}{\tau}\nabla_{\boldsymbol{q}}\cdot\boldsymbol{\xi}. \tag{A.28}$$

Here we use the fact that $\nabla_{\boldsymbol{x}} \simeq \nabla_{\boldsymbol{q}}$ and $\det \nabla_{\boldsymbol{q}}\boldsymbol{x} \simeq 1 + \nabla_{\boldsymbol{q}}\cdot\boldsymbol{\xi}$. Using (A.28) to eliminate $\varphi_{\rm g}$ in (A.27), we get for $\theta \equiv \nabla_{\boldsymbol{q}}\cdot\boldsymbol{\xi}$ an equation that coincides with (A.26) up to the change of variable $\xi \mapsto \theta$. Choosing the well-behaved linear solution for θ, solving for $\boldsymbol{\xi}$ and using the above argument to eliminate a τ-dependent constant, we see that, in the linearized equations, terms in the right-hand side of (A.27) cancel each other and the acceleration vanishes. This simplification justifies using the linear growth factor τ as a time variable.

APPENDIX B: HISTORY OF MASS TRANSPORTATION

The subject of mass transportation was started by Gaspard Monge (1781) in a paper[19] entitled *Théorie des déblais et des remblais* (Theory of cuts and fills) whose preamble is worth quoting entirely (our translation):

> When earth is to be moved from one place to another, the usage is to call *cuts* the volumes of earth to be transported and *fills* the space to be occupied after transportation.
>
> The cost of transporting one molecule being, all things otherwise equal, proportional to its weight and to the distance [*espace*] travelled and consequently the total cost being proportional to the sum of products of molecules each multiplied by the distance travelled, it follows that for given shapes and positions of the cuts and fills, it is not indifferent that any given molecule of the cuts be transported to this or that place in the fills, but there ought to be a certain distribution of molecules of the former into the latter, according to which the sum of these products will be the least possible, and the cost of transportation will be a *minimum.*

Although clearly posed, the 'mass transportation problem' was not solved, in more than one dimension, until Leonid Kantorovich (1942) formulated a 'relaxed' version, now called the Monge–Kantorovich problem: instead of a 'distribution of molecules of the former *into* the latter,' he allowed a distribution in the product space where more than one position in the fills could be associated with a position in the cuts and where the initial and final distributions are prescribed marginals (see Section 3.3). In *cosmospeak*, he allowed multi-streaming with given initial and final mass distributions. Using the techniques of duality and of linear programming that he had invented (see Appendix C2), Kantorovich was then able to solve the mass transportation problem in this relaxed formulation. The techniques developed by Kantorovich found many applications, notably in economics, which in fact was his original motivation (he was awarded, together with T.C. Koopmans, the 1975 Nobel prize in this field).

Before turning to more recent developments we must say a few words about the history of the Monge–Ampère equation. To the best of our knowledge the equation, in its two-dimensional form, appears for the first time in Ampère (1820), a huge (188 pages) mathematical memoir in which the equation is to be found on p. 65. Ampère also pointed out the way the equation changes under Legendre transformations but there is no physical interpretation in terms of Lagrangian coordinates.[20] Again, to the best of our knowledge, Monge had nothing to do with the particular equation (19). In his 1820 paper Ampère mentions a number of other non-linear partial differential equations which he attributes to Monge and, certainly, Monge was the first to develop general geometrical techniques for partial differential equations. The name Monge–Ampère for (19) appears already in the

[19] The author's name appears in this paper as 'M. Monge,' where the 'M.' stands for 'Monsieur.'

[20] According to the biography of Ampère by L. Pearce Williams in the *Dictionary of Scientific Biography*, Ampère's paper was written – after he had switched from mathematics to chemistry and physics – with the purpose of facilitating his election to the Paris Academy of Science; one can then speculate that his mention of the Legendre transformation was influenced by Legendre's presence in this academy.

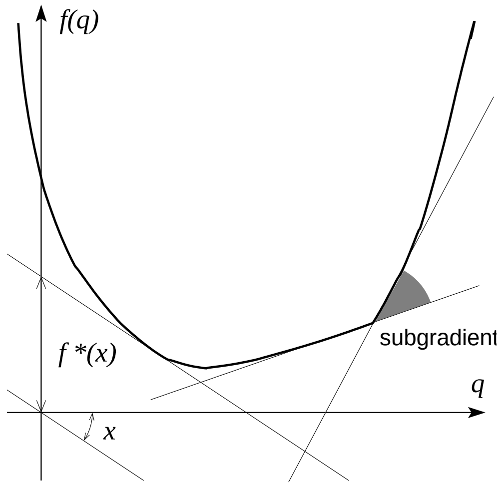


Figure C1. A convex function $f(q)$ and the geometrical construction of its Legendre transform $f^*(x)$. Also illustrated is the subgradient of $f(q)$ at a non-smooth point.

early twentieth century on pp. 581–582 of Goursat (1905). It may be that in the 19th century it has designated a larger class of non-linear second-order partial differential equations in two independent variables, but more historical research would be needed to ascertain this.

The subjects of mass transportation and of the Monge–Ampère equation came together when one of us (YB) showed the equivalence of the elliptic Monge–Ampère equation and of the mass transportation problem with quadratic cost: when initial and final distributions are non-singular, the optimal solution is actually one-to-one, so that nothing is lost by the Kantorovich relaxation trick (Brenier 1987, 1991). For an extension of this result to general convex costs and a review of the many recent papers on the subject, see Gangbo & McCann (1996).

APPENDIX C: BASICS OF CONVEXITY AND DUALITY

C1 Convexity and the Legendre transformation

A convex body may be defined by the condition that it coincides with the intersection of all half-spaces containing it. Obviously, it is sufficient to take only those half-spaces limited by planes that touch the body; such planes are called *supporting.*

Take now a convex function $f(\boldsymbol{q})$, so that the set of points in the $(3+1)$-dimensional $(\boldsymbol{q}, f)$ space lying above its graph is convex. It follows that we can write

$$f(\boldsymbol{q}) = \max_{\boldsymbol{x}}\, \boldsymbol{x}\cdot\boldsymbol{q} - f^*(\boldsymbol{x}), \tag{C.1}$$

where the expression $\boldsymbol{x}\cdot\boldsymbol{q} - f^*(\boldsymbol{x})$ specifies a supporting plane with the slope $\boldsymbol{x}$ for the set of points lying above the graph of f (see Fig. C1 for the one-dimensional case). The function $f^*(\boldsymbol{x})$, which specifies how high one should place a supporting plane to touch the graph, is called the *Legendre transform* of $f(\boldsymbol{q})$.[21]

From Eq. (C.1) follows the inequality (known as the *Young inequality*)

$$f(\boldsymbol{q}) + f^*(\boldsymbol{x}) \geqslant \boldsymbol{x}\cdot\boldsymbol{q} \text{ for all } \boldsymbol{x},\, \boldsymbol{q}, \tag{C.2}$$

where both sides coincide if and only if the supporting plane with the slope $\boldsymbol{x}$ touches the graph of f at $\boldsymbol{q}$. This fact, together with the obvious symmetry of this inequality, implies that

$$f^*(\boldsymbol{x}) = \max_{\boldsymbol{q}}\, \boldsymbol{x}\cdot\boldsymbol{q} - f(\boldsymbol{q}). \tag{C.3}$$

Thus, the Legendre transform of a convex function is itself convex and the Legendre transform of the Legendre transform recovers the initial convex function.

If however we apply (C.1) to a *nonconvex* function f, we obtain a convex function f^*, whose own Legendre transform will give the *convex hull* of f, the largest convex function whose graph lies below that of f.

When f is both convex and differentiable, (C.2) becomes an equality for $\boldsymbol{x} = \nabla_{\boldsymbol{q}} f(\boldsymbol{q})$. If f^* is also differentiable, then one also has $\boldsymbol{q} = \nabla_{\boldsymbol{x}} f^*(\boldsymbol{x})$. This is actually Legendre's original definition of the transformation, which is thus limited to smooth functions. Furthermore, if the original function is not convex and thus has the same gradient at separated locations, Legendre's purely local definition will give a multivalued Legendre transform. (In the context of the present paper this corresponds to multi-streaming.)

Not all convex functions are differentiable (e.g. $f(\boldsymbol{q}) = |\boldsymbol{q}|$). But the Young inequality can be employed to define a useful generalization of the gradient: the *subgradient* of f at $\boldsymbol{q}$ is the set of all $\boldsymbol{x}$ for which the equality in (C.2) holds (see Fig. C1). If f is smooth at $\boldsymbol{q}$, then $\nabla_{\boldsymbol{q}} f(\boldsymbol{q})$ will be the only such point; otherwise, there will be a (convex) set of them.

If a convex function has the same subgradient at more than one point, the function is said to lack *strict convexity.* In fact, strict convexity and smoothness are complementary: lack of one in a convex function implies lack of the other in the Legendre transform.

For further background on convex analysis and geometry, see Rockafellar (1970).

C2 Duality in optimization

Suppose we want to minimize a convex function $\Phi(\boldsymbol{q})$ subject to a set of linear constraints that may be written in matrix notation as $A\boldsymbol{q} = \boldsymbol{b}$ (vectors $\boldsymbol{q}$ satisfying this constraint are called *admissible* in optimization parlance). We now observe that

$$\inf_{A\boldsymbol{q}=\boldsymbol{b}} \Phi(\boldsymbol{q}) = \inf_{\boldsymbol{q}} \sup_{\boldsymbol{x}}\, \Phi(\boldsymbol{q}) - \boldsymbol{x}\cdot(A\boldsymbol{q} - \boldsymbol{b}). \tag{C.4}$$

Indeed, should $A\boldsymbol{q}$ not equal $\boldsymbol{b}$, the sup operation in $\boldsymbol{x}$ will give infinity, so such $\boldsymbol{q}$ will not contribute to minimization. Here we use the inf/sup notation instead of min/max because the extremal values may not be reached, e.g., when they are infinite.

[21] It was introduced in the one-dimensional case by Mandelbrojt (1939) and then generalized by Fenchel (1949).

Using (C.1), we rewrite this in the form

$$\begin{aligned}&\inf_{\boldsymbol{q}}\sup_{\boldsymbol{x},\boldsymbol{y}} \boldsymbol{y}\cdot\boldsymbol{q}-\Phi^*(\boldsymbol{y})-\boldsymbol{x}\cdot(A\boldsymbol{q}-\boldsymbol{b})\\ &=\inf_{\boldsymbol{q}}\sup_{\boldsymbol{x},\boldsymbol{y}}(\boldsymbol{y}-A^T\boldsymbol{x})\cdot\boldsymbol{q}-\Phi^*(\boldsymbol{y})+\boldsymbol{x}\cdot\boldsymbol{b},\end{aligned} \tag{C.5}$$

where $\Phi^*(\boldsymbol{y})$ is the Legendre transform of $\Phi(\boldsymbol{q})$ and A^T is the transpose of A. Taking inf in $\boldsymbol{q}$ first, we see that the expression in the right-hand side will be infinite unless $\boldsymbol{y}=A^T\boldsymbol{x}$. We then obtain the optimization problem of finding

$$\sup_{\boldsymbol{x}} \boldsymbol{x}\cdot\boldsymbol{b}-\Phi^*(A^T\boldsymbol{x}), \tag{C.6}$$

which is called *dual* to the original one. Note that there are no constraints on the dual variable $\boldsymbol{x}$: any value is admissible.

Denoting solutions of problems (C.4) and (C.6) by $\boldsymbol{q}^*$ and $\boldsymbol{x}^*$, we see that

$$\Phi(\boldsymbol{q}^*)+\Phi^*(A^T\boldsymbol{x}^*)-\boldsymbol{x}^*\cdot\boldsymbol{b}=0, \tag{C.7}$$

because the optimal values of both problems are given by (C.5) and thus coincide. Furthermore, for any admissible $\boldsymbol{q}$ and $\boldsymbol{x}$

$$\Phi(\boldsymbol{q})+\Phi^*(A^T\boldsymbol{x})-\boldsymbol{x}\cdot\boldsymbol{b}\geqslant 0, \tag{C.8}$$

because the right-hand sides of (C.4) and (C.6) cannot pass beyond their optimal values.

Moreover, let equality (C.7) be satisfied for some admissible $\boldsymbol{q}^*$ and $\boldsymbol{x}^*$; then such $\boldsymbol{q}^*$ and $\boldsymbol{x}^*$ must solve the problems (C.4) and (C.6). Indeed, taking e.g. $\boldsymbol{x}^*$ for $\boldsymbol{x}$ in (C.8) and using (C.7), we see that for any other admissible $\boldsymbol{q}$

$$\Phi(\boldsymbol{q}^*)\leqslant\Phi(\boldsymbol{q}), \tag{C.9}$$

i.e., that $\boldsymbol{q}^*$ solves the original optimization problem (C.4).

Convex optimization problems with linear constraints considered in this section are called *convex programs*. Their close relatives are *linear programs*, namely optimization problems of the form

$$\inf_{A\boldsymbol{q}=\boldsymbol{b},\,\boldsymbol{q}\geqslant 0}\boldsymbol{c}\cdot\boldsymbol{q}=\inf_{\boldsymbol{q}\geqslant 0}\sup_{\boldsymbol{x}}\boldsymbol{c}\cdot\boldsymbol{q}-\boldsymbol{x}\cdot(A\boldsymbol{q}-\boldsymbol{b}), \tag{C.10}$$

where notation $\boldsymbol{q}\geqslant 0$ means that all components of the vector $\boldsymbol{q}$ are nonnegative. Proceeding essentially as above with $\boldsymbol{c}\cdot\boldsymbol{q}$ instead of $\Phi(\boldsymbol{q})$, we observe that in order not to obtain infinity when minimizing in $\boldsymbol{q}$ in (C.5), we have now to require that $A^T\boldsymbol{x}\leqslant\boldsymbol{c}$ (i.e. $\boldsymbol{c}-A^T\boldsymbol{x}\geqslant 0$). The dual problem thus takes the form

$$\sup_{A^T\boldsymbol{x}\leqslant\boldsymbol{c}}\boldsymbol{x}\cdot\boldsymbol{b} \tag{C.11}$$

with an admissibility constraint on $\boldsymbol{x}$. Instead of (C.7) and (C.8) we obtain

$$\boldsymbol{x}^*\cdot\boldsymbol{b}=\boldsymbol{c}\cdot\boldsymbol{q}^*\quad\text{or}\quad(A^T\boldsymbol{x}^*-\boldsymbol{c})\cdot\boldsymbol{q}^*=0 \tag{C.12}$$

and

$$\boldsymbol{x}\cdot\boldsymbol{b}\leqslant\boldsymbol{c}\cdot\boldsymbol{q}\quad\text{or}\quad(A^T\boldsymbol{x}-\boldsymbol{c})\cdot\boldsymbol{q}\leqslant 0, \tag{C.13}$$

the latter inequality being automatically satisfied for any admissible $\boldsymbol{x},\boldsymbol{q}$. Note that for linear programs, the fact that (C.12) holds for some admissible $\boldsymbol{q}^*,\boldsymbol{x}^*$ also implies that $\boldsymbol{q}^*$ and $\boldsymbol{x}^*$ solve their respective optimization problems.

For further background on optimization and duality, see, e.g., Papadimitriou & Steiglitz (1982).

C3 Why the analogue computer of Section 4.2 solves the assignment problem

We suppose that the analogue computer described in Section 4.2 has settled into equilibrium, which minimizes its potential energy

$$U=\sum_{i=1}^N\alpha_i-\sum_{j=1}^N\beta_j, \tag{C.14}$$

under the set of constraints

$$\alpha_i-\beta_j\geqslant C-c_{ij}, \tag{C.15}$$

for all i,j. Our goal is here is to show that the set of equilibrium forces f_{ij}, acting on studs between row and column rods, solves the original linear programming problem of minimizing

$$\tilde{I}=\sum_{i,j=1}^N c_{ij}f_{ij} \tag{C.16}$$

under constraints

$$f_{ij}\geqslant 0,\qquad \sum_{k=1}^N f_{kj}=\sum_{k=1}^N f_{ik}=1, \tag{C.17}$$

for all i,j and that in fact forces f_{ij} take only zero and unit values, thus providing the solution to the assignment problem.

Note first that if a row rod A_i and a column rod B_j are not in contact at equilibrium, then the corresponding force vanishes ($f_{ij}=0$); if they are, then $f_{ij}\geqslant 0$. Take now a particular pair of rods A_i and B_j that are in contact. At equilibrium, the force f_{ij} must equal forces exerted on the corresponding stud by A_i and B_j. We claim that both these forces must be integer. To see this, let us compute the force exerted by A_i. This rod contributes its weight, $+1$, possibly decreased by the force that it feels from other column rods that are in contact with A_i. Each of these takes -1 (its 'buoyancy') out of the total force, but we may have to add the force it feels in turn from other row rods with which it might be in contact. Proceeding this way from one rod to another, we see that all contributions, positive or negative, are unity, so their sum f_{ij} must be integer. The same argument applies to rod B_j.

Does this process indeed finish or, at some stage, do we come back at an already visited stud and thus end up in an infinite cycle? In fact, for general set of stud lengths $C-c_{ij}$, the latter cannot happen, because otherwise an alternating sum of some subset of stud lengths would give exactly zero – a zero probability event for a set of arbitrary real numbers.

Consider now a row rod A_i. It is in contact with one or more column rods, whose combined upward push must equilibrate the unit weight of A_i. Since any of the latter rods exerts a nonnegative integer force, it follows that exactly one of these forces is unity, and all the other ones are zero. A similar argument holds for any column rod B_j.

We have thus shown that all f_{ij} in the equilibrium equal 1 or 0. One can of course ignore the vanishing forces. Then each row rod A_i is supported by exactly one column rod B_j, and each B_j supports exactly one A_i. This defines a one-to-one pairing, and we are only left with a check that this pairing minimizes (C.16).

Observe that pushing a column rod down by some distance Δ and simultaneously increasing by Δ the length of all studs attached to this rod will have no effect on positions and constraints of all other rods, hence on the equilibrium network of contacts. Moreover, due to constraints (C.17), the corresponding change in coefficients c_{ij} will not change the cost function (C.16) in any essential way, except of just subtracting Δ.

We can use this observation to put all column rods at the same level, say at $z = 0$, adjusting c_{ij} to some new values c'_{ij}. Thus, for every i, the row rod A_i rests on the stud with the largest height $C - c'_{ij}$, so the equilibrium pairing maximizes the sum

$$\sum_{i,j=1}^{N} (C - c'_{ij}) f_{ij} \tag{C.18}$$

and thus minimizes (C.16).[22]

APPENDIX D: DETAILS OF THE VARIATIONAL TECHNIQUE FOR THE EULER–POISSON SYSTEM

In this appendix, we explain details of the variational procedure outlined in Section 6, which proves that prescription of the density fields at terminal epochs $\tau = 0$ and $\tau = \tau_0$ uniquely determines a regular and thus curl-free solution to the Euler–Poisson system (A.17)–(A.19).

The variational problem is posed for the functional

$$I = \frac{1}{2}\int_0^{\tau_0} \mathrm{d}\tau \int \mathrm{d}^3\boldsymbol{x}\,\tau^{3/2}\left(\rho|\boldsymbol{v}|^2 + \frac{3}{2}|\nabla_{\boldsymbol{x}}\varphi_{\mathrm{g}}|^2\right) \tag{D.1}$$

with four constraints: the Poisson equation (A.19), which we repeat here for convenience,

$$\nabla^2_{\boldsymbol{x}}\varphi_{\mathrm{g}} = \frac{\rho - 1}{\tau}, \tag{D.2}$$

the mass conservation (A.18), also repeated here,

$$\partial_\tau \rho + \nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{v}) = 0, \tag{D.3}$$

and the two boundary conditions

$$\rho(\boldsymbol{x}, 0) = 1 \quad \text{and} \quad \rho(\boldsymbol{x}, \tau_0) = \rho_0(\boldsymbol{x}). \tag{D.4}$$

In the sequel, we shall always denote by $\iint$ the double integration over $0 \leqslant \tau \leqslant \tau_0$ and over the whole space domain in $\boldsymbol{x}$ provided that the integrand vanishes at infinity sufficiently fast, or over the periodicity box in the case of periodic boundary conditions. A single integral sign $\int$ will always denote the integration over the relevant space domain in $\boldsymbol{x}$.

First, we make this problem convex by rewriting the functional and constraints in a new set of variables with the mass flux $\boldsymbol{J}(\boldsymbol{x}, t) = \rho(\boldsymbol{x}, t)\,\boldsymbol{v}(\boldsymbol{x}, t)$ instead of the velocity $\boldsymbol{v}$. The mass conservation constraint, which was the only nonlinear one in the old variables, becomes now linear:

$$\partial_\tau \rho + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{J} = 0, \tag{D.5}$$

and one can check that the density of kinetic energy takes the form

$$\frac{1}{2}\rho|\boldsymbol{v}|^2 = \frac{1}{2\rho}|\boldsymbol{J}|^2 = \max_{c,\,\boldsymbol{m}:\; c + |\boldsymbol{m}|^2/2 \leqslant 0} (\rho c + \boldsymbol{J}\cdot\boldsymbol{m})$$

or

$$\frac{|\boldsymbol{J}|^2}{2\rho} = \max_{c,\,\boldsymbol{m}} (\rho c + \boldsymbol{J}\cdot\boldsymbol{m} - F(c, \boldsymbol{m})), \tag{D.6}$$

where

$$F(c, \boldsymbol{m}) = \begin{cases} 0 & \text{if } c + |\boldsymbol{m}|^2/2 \leqslant 0 \\ +\infty & \text{otherwise.} \end{cases} \tag{D.7}$$

Note that in (D.6) the variables $c, \boldsymbol{m}$, as well as $\rho, \boldsymbol{J}$, are functions of $(\boldsymbol{x}, \tau)$. The action functional may now be written as

$$I = \frac{1}{2}\iint \left(\frac{1}{\rho}|\boldsymbol{J}|^2 + \frac{3}{2}|\nabla_{\boldsymbol{x}}\phi|^2\right)\tau^{3/2}\,\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}\tau, \tag{D.8}$$

and turns out to be convex.

To see this, first note that the operation of integration is linear and thus preserves convexity of the integrand. The integrand is a positive quadratic function of $\nabla_{\boldsymbol{x}}\phi$ and therefore is convex in ϕ; furthermore, (D.6) implies that it is also convex in $(\rho, \boldsymbol{J})$, since the kinetic energy density $|\boldsymbol{J}|^2/2\rho$ is the Legendre transform of the function $F(c, \boldsymbol{m})$, which itself is convex.

Note also that by representing the kinetic energy density in the form (D.6), we may safely allow ρ to take negative values: the right-hand side being in that case $+\infty$, it will not contribute to minimizing (D.1).

We now derive the dual optimization problem. We introduce the scalar Lagrange multipliers $\psi(\boldsymbol{x}, t)$, $\vartheta_{\mathrm{in}}(\boldsymbol{x})$, $\vartheta_0(\boldsymbol{x})$ and $\theta(\boldsymbol{x}, t)$ for the Poisson equation (D.2), the boundary conditions (D.4), and the constraints of mass conservation (D.5), respectively, and observe that the variational problem may now be written in the form

$$\begin{aligned}
\inf_{\rho,\boldsymbol{J},\phi}\ \sup_{\substack{c,\boldsymbol{m},\theta,\psi,\vartheta_0,\vartheta_T:\\ c+|\boldsymbol{m}|^2/2\leqslant 0}} & \iint \mathrm{d}^3\boldsymbol{x}\,\mathrm{d}\tau \left[\frac{3}{2}\psi\left(\nabla^2_{\boldsymbol{x}}\phi - \frac{\rho - 1}{\tau}\right)\right.\\
& \left. + \theta(\partial_\tau\rho + \nabla_{\boldsymbol{x}}\cdot\boldsymbol{J}) + \tau^{3/2}\left(\rho c + \boldsymbol{J}\cdot\boldsymbol{m} + \frac{3}{4}|\nabla_{\boldsymbol{x}}\phi|^2\right)\right] \\
& + \int \vartheta_{\mathrm{in}}(\boldsymbol{x})(\rho(\boldsymbol{x}, 0) - 1)\,\mathrm{d}^3\boldsymbol{x} \\
& - \int \vartheta_0(\boldsymbol{x})(\rho(\boldsymbol{x}, \tau_0) - \rho_0(\boldsymbol{x}))\,\mathrm{d}^3\boldsymbol{x}.
\end{aligned} \tag{D.9}$$

To see that (D.9) is indeed equivalent to minimizing (D.1) under the constraints (D.3) or (D.5), (D.2), and (D.4), observe that for those $\rho, \boldsymbol{J}, \phi$ that do not satisfy the constraints, the sup operation over $\theta, \psi, \vartheta_{\mathrm{in}}, \vartheta_0$ will give positive infinity; the sup will be finite (and thus contribute to the subsequent minimization) only if all constraints are satisfied. (This argument is the functional version of what is explained in Appendix C2 for the finite-dimensional case.)

Performing an integration by parts in the τ variable in (D.9) and using the boundary conditions on the mass density (D.4), we find that $\vartheta_{\mathrm{in}}(\boldsymbol{x}) = \theta(\boldsymbol{x}, 0)$ and $\vartheta_0(\boldsymbol{x}) = \theta(\boldsymbol{x}, \tau_0)$. Integrating further by parts in the $\boldsymbol{x}$ variable, assuming that boundary terms at infinity vanish (or that we have periodic boundary conditions in space) and rearranging terms, we get

[22] Those readers familiar with linear programming will recognize that the proof just presented is based on two ideas: (i) the total unimodularity of the matrix of constraints in terms of which the equalities in (C.17) can be written and (ii) the complementary slackness (see, e.g., Papadimitriou & Steiglitz 1982, sections 3.2 and 13.2).

$$\inf_{\rho,\boldsymbol{J},\phi} \sup_{\substack{c,\boldsymbol{m},\theta,\psi:\\ c+|\boldsymbol{m}|^2/2\leqslant 0}} \iint \mathrm{d}^3\boldsymbol{x}\,\mathrm{d}\tau \Big(\rho\,(c\tau^{3/2} - \partial_\tau\theta - \frac{3}{2\tau}\psi) + \boldsymbol{J}\cdot(\boldsymbol{m}\tau^{3/2} - \nabla_{\boldsymbol{x}}\theta) + \frac{3}{4\tau^{3/2}}\left|\nabla_{\boldsymbol{x}}\psi - \tau^{3/2}\nabla_{\boldsymbol{x}}\varphi_{\rm g}\right|^2 - \frac{3}{4\tau^{3/2}}\left|\nabla_{\boldsymbol{x}}\psi\right|^2 + \frac{3}{2\tau}\psi\Big) - \int \theta(\boldsymbol{x},0)\,\mathrm{d}^3\boldsymbol{x} + \int \theta(\boldsymbol{x},\tau_0)\,\rho_0(\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{x}. \tag{D.10}$$

Performing minimization with respect to $\rho, \boldsymbol{J}, \phi$ first, as in (C.5) of Appendix C2, we see that the following two equalities must hold (remember that ρ need not be positive at this stage):

$$c = \frac{1}{\tau^{3/2}}\left(\partial_\tau\theta + \frac{3\psi}{2\tau}\right), \qquad \boldsymbol{m} = \frac{1}{\tau^{3/2}}\nabla_{\boldsymbol{x}}\theta, \tag{D.11}$$

so that terms linear in ρ and $\boldsymbol{J}$ vanish in (D.10). It follows that c and $\boldsymbol{m}$ are determined by θ and ψ and that the constraint $c + |\boldsymbol{m}|^2/2 \leqslant 0$ can be written

$$\partial_\tau\theta + \frac{1}{2\tau^{3/2}}|\nabla_{\boldsymbol{x}}\theta|^2 + \frac{3}{2\tau}\psi \leqslant 0. \tag{D.12}$$

Also, the inf with respect to ϕ is straightforward and gives

$$\tau^{3/2}\nabla_{\boldsymbol{x}}\varphi_{\rm g} = \nabla_{\boldsymbol{x}}\psi. \tag{D.13}$$

Using (D.11) and (D.13) in (D.10), we arrive at the optimization problem of maximizing

$$J = \iint \left(\frac{3}{2\tau}\psi - \frac{3}{4\tau^{3/2}}|\nabla_{\boldsymbol{x}}\psi|^2\right)\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}\tau + \int \theta(\boldsymbol{x},\tau_0)\,\rho_0(\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{x} - \int \theta(\boldsymbol{x},0)\,\mathrm{d}^3\boldsymbol{x} \tag{D.14}$$

under constraint (D.12). Eqs. (D.14) and (D.12) constitute a variational problem *dual* to the original one.

As both the original and the dual variational problems have the same saddle-point formulation (D.9) or (D.10), the optimal values of the two functionals (D.1) and (D.14) are equal. Let $(\rho, \boldsymbol{J}, \varphi_{\rm g})$ be a solution to the original variational problem and θ, ψ be a solution to the dual one. Subtracting the (equal) optimal values from each other, we may now write, similarly to (C.7),

$$\iint \left(\frac{\tau^{3/2}}{2\rho}|\boldsymbol{J}|^2 + \frac{3\tau^{3/2}}{4}|\nabla_{\boldsymbol{x}}\varphi_{\rm g}|^2 + \frac{3}{4\tau^{3/2}}|\nabla_{\boldsymbol{x}}\psi|^2 - \frac{3}{2\tau}\psi\right)\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}\tau + \int \theta(\boldsymbol{x},0)\,\mathrm{d}^3\boldsymbol{x} - \int \theta(\boldsymbol{x},\tau_0)\,\rho_0(\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{x} = 0. \tag{D.15}$$

We are going to show that the left-hand side of (D.15) may be given the form of a sum of three nonnegative terms, each of which will therefore have to vanish. First, we rewrite the last two integrals, using the mass conservation constraint (D.5) and integrations by parts, in the form

$$-\iint \partial_\tau(\theta\rho)\,\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}\tau = -\iint (\partial_\tau\theta\,\rho + \nabla_{\boldsymbol{x}}\theta\cdot\boldsymbol{J})\,\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}\tau.$$

Second, we note that

$$\iint \left(\frac{3\tau^{3/2}}{4}|\nabla_{\boldsymbol{x}}\varphi_{\rm g}|^2 + \frac{3}{4\tau^{3/2}}|\nabla_{\boldsymbol{x}}\psi|^2\right)\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}\tau = \iint \left(\frac{3}{4\tau^{3/2}}|\tau^{3/2}\nabla_{\boldsymbol{x}}\varphi_{\rm g} - \nabla_{\boldsymbol{x}}\psi)|^2 - \frac{3}{2\tau}\psi(\rho-1)\right)\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}\tau,$$

which follows from the Poisson constraint (D.2). Taking all this into account in (D.15), we get, after a rearrangement of terms,

$$\iint \frac{\rho}{2\tau^{3/2}}\left|\frac{\tau^{3/2}}{\rho}\boldsymbol{J} - \nabla_{\boldsymbol{x}}\theta\right|^2 \mathrm{d}^3\boldsymbol{x}\,\mathrm{d}\tau + \iint -\rho\left(\partial_\tau\theta + \frac{1}{2\tau^{3/2}}|\nabla_{\boldsymbol{x}}\theta|^2 + \frac{3}{2\tau}\psi\right)\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}\tau + \iint \frac{3}{4\tau^{3/2}}|\tau^{3/2}\nabla_{\boldsymbol{x}}\varphi_{\rm g} - \nabla_{\boldsymbol{x}}\psi)|^2\,\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}\tau = 0. \tag{D.16}$$

The left-hand side is a sum of three nonnegative terms (the second is so by (D.12)), all of which must thus vanish. This gives

$$\boldsymbol{v} = \frac{1}{\rho}\boldsymbol{J} = \frac{1}{\tau^{3/2}}\nabla_{\boldsymbol{x}}\theta, \qquad \nabla_{\boldsymbol{x}}\varphi_{\rm g} = \frac{1}{\tau^{3/2}}\nabla_{\boldsymbol{x}}\psi \tag{D.17}$$

and

$$\partial_\tau\theta + \frac{1}{2\tau^{3/2}}|\nabla_{\boldsymbol{x}}\theta|^2 + \frac{3}{2\tau}\psi = 0, \tag{D.18}$$

wherever ρ is non-vanishing (otherwise the left-hand-side is non-positive by (D.12)). The last equality turns into the Euler equation

$$\partial_\tau\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}})\boldsymbol{v} = -\frac{3}{2\tau}(\boldsymbol{v} + \nabla_{\boldsymbol{x}}\varphi_{\rm g}) \tag{D.19}$$

by taking the gradient and using (D.17).

By (D.17) and (D.18), any two hypothetically different minimizing solutions for either variational problem give rise to the same velocity potential and to the same gravitational potential (up to insignificant constants) and thus define the same solution $(\rho, \boldsymbol{v}, \varphi_{\rm g})$ to the Euler–Poisson equations with the boundary conditions (D.4) and the condition of curl-free velocity.

Moreover, for any such solution $(\rho, \boldsymbol{v}, \varphi_{\rm g})$, one can use (D.17) to define θ and ψ that satisfy (D.18) and thus (D.12). By (D.16), the values of functionals I and $\bar{I}$ evaluated at these functions will coincide; together with convexity this implies, by an argument similar to that given in Appendix C2 concerning (C.9), that such $(\rho, \boldsymbol{v}, \varphi_{\rm g})$ and (θ, ψ) in fact minimize both functionals under the corresponding constraints.

This means that a (curl-free) velocity field, a gravitational field and a density fields $(\boldsymbol{v}, \varphi_{\rm g}, \rho)$ will satisfy the Euler–Poisson equations (A.17)–(A.19) (repeated as (D.19), (D.3), and (D.2) in this Appendix) and the boundary conditions (D.4) if and only if they minimize (D.1) under the corresponding constraints. This establishes uniqueness.

REFERENCES

Ampère A.-M., 1820, J. de L'École Royale Polytechnique, 11, 1

Arnol'd V. I., Shandarin S. F., Zel'dovich Y. B., 1982, Geophys. Astrophys. Fluid Dynamics, 20, 111

Balinski M. L., 1986, Math. Programming, 34, 125
Bec J., Frisch U., Khanin K., 2000, J. Fluid Mech., 416, 239
Benamou J.-D., Brenier Y., 2000, Numer. Math., 84, 375
Bernardeau F., Colombi S., Gaztañaga E., Scoccimarro R., 2002, Phys. Rep., 367, 1
Bertschinger E., Dekel A., 1989, ApJ, 336, L5
Bertsekas D. P., 1981, Math. Programming, 21, 152
Bertsekas D. P., 1992, Comput. Optim. Appl., 1, 7
Bertsekas D., 2001, in Encyclopedia of optimization, Vol. I. Kluwer Academic Publishers, Dordrecht
Branchini E., Eldar A., Nusser A., 2002, MNRAS, 335, 53
Brenier Y., 1987, C. R. Acad. Sci. Paris Sér. I Math., 305, 805
Brenier Y., 1991, Comm. Pure Appl. Math., 44, 375
Bridle S. L., Lahav O., Ostriker J. P., Steinhardt P. J., 2003, Sci, 299, 1532
Buchert T., 1992, MNRAS, 254, 729
Buchert T., Dominguez A., 1998, A&A, 335, 395
Buchert T., Ehlers J., 1993, MNRAS, 264, 375
Burkard R. E., Derigs U., 1980, Assignment and matching problems: solution methods with FORTRAN programs. Vol. 184 of Lecture Notes in Economics and Mathematical Systems, Springer-Verlag, Berlin
Caffarelli L., Li Y. Y., 2001, An extension to a theorem of Jörgens, Calabi, and Pogorelov, preprint (www.math.utexas.edu/users/combs/Caffarelli/extension.pdf)
Caffarelli L. A., 1999, in Christ M., Kenig C. E., Sadosky C., eds, Chicago Lectures in Math., Harmonic analysis and partial differential equations. Univ. Chicago Press, Chicago, IL, pp 117–126
Caffarelli L. A., Milman M., eds, 1999, Monge Ampère equation: applications to geometry and optimization. Vol. 226 of Contemporary Mathematics, American Mathematical Society, Providence, RI
Catelan P., 1995, MNRAS, 276, 115
Catelan P., Lucchin F., Matarrese S., Moscardini L., 1995, MNRAS, 276, 39
Coles P., Lucchin F., 2002, Cosmology: The Origin and Evolution of Cosmic Structure. John Wiley & Sons, Chichester et al.
Couchman H. M. P., Thomas P. A., Pearce F. R., 1995, ApJ, 452, 797
Croft R. A. C., Gaztañaga E., 1997, MNRAS, 285, 793
Dekel A., Bertschinger E., Faber S. M., 1990, ApJ, 364, 349
Fenchel W., 1949, Canadian J. Math., 1, 73
Fréchet M., 1957a, C. R. Acad. Sci. Paris Sér. I Math., 244, 689
Fréchet M., 1957b, Publ. Inst. Statist. Univ. Paris, 6, 183
Frieman J. A., Szalay A. S., 2000, Phys. Rep., 333, 215
Frisch U., Bec J., 2002, in Lesieur M., Yaglom A., David F., eds, École de physique des Houches, session LXXIV, New trends in turbulence. EDP Sciences and Springer, pp 341–384
Frisch U., Matarrese S., Mohayaee R., Sobolevski A., 2002, Nat, 417, 260
Gangbo W., McCann R. J., 1996, Acta Math., 177, 113
Giavalisco M., Mancinelli B., Mancinelli P. J., Yahil A., 1993, ApJ, 411, 9
Goursat É., 1905, Cours d'analyse mathématique. Vol. II, Gauthier–Villars, Paris
Gramann M., 1993, ApJ, 405, 449
Gurbatov S. N., Saichev A. I., 1984, Izv. Vysš. Učebn. Zaved. Radiofizika, 27, 456
Gurbatov S. N., Saichev A. I., Shandarin S. F., 1989, MNRAS, 236, 385
Hénon M., 1995, C. R. Acad. Sci. Paris Sér. I Math., 321, 741
Hénon M., 2002, A mechanical model for the transportation problem, preprint (arXiv:math.OC/0209047)
Jeans J. H., 1919, Problems of Cosmogony and Stellar Dynamics. Cambridge University Press, Cambridge
Kantorovich L. V., 1942, C. R. (Doklady) Acad. Sci. USSR, 321, 199
Kardar M., Parisi G., Zhang Y., 1986, Phys. Rev. Lett., 56, 889
Kolatt T., Dekel A., Ganon G., Willick J. A., 1996, ApJ, 458, 419
Kuhn H. W., 1955, Naval Res. Logist. Quart., 2, 83
Landau L. D., Lifshitz E. M., 1960, Mechanics. Vol. 1 of Course of Theoretical Physics, Pergamon Press, Oxford
Loeper G., 2003, The inverse problem for the Euler–Poisson system in cosmology, preprint, Laboratoire J.A. Dieudonné, Université de Nice–Sophia–Antipolis
Mandelbrojt S., 1939, C. R. Acad. Sci. Paris, 209, 977
Minkowski H., 1897, Nachr. Ges. Wiss. Göttingen (Math. Phys. Klasse), pp 198–219
Mohayaee R., Frisch U., Matarrese S., Sobolevskiĭ A., 2003, Reconstruction of the primordial Universe by a Monge–Ampère–Kantorovich optimisation scheme, preprint (astro-ph/0301641)
Monaco P., Efstathiou G., 1999, MNRAS, 308, 763
Monge G., 1781, Histoire de l'Académie Royale des Sciences, pp 666–704
Moutarde F., Alimi J.-M., Bouchet F. R., Pellat R., Ramani A., 1991, ApJ, 382, 377
Munshi D., Sahni V., Starobinsky A. A., 1994, ApJ, 436, 517
Narayanan V. K., Croft R. A. C., 1999, ApJ, 515, 471
Narayanan V. K., Weinberg D. H., 1998, ApJ, 508, 440
Nusser A., Branchini E., 2000, MNRAS, 313, 587
Nusser A., Dekel A., 1992, ApJ, 391, 443
Papadimitriou C. H., Steiglitz K., 1982, Combinatorial optimization: algorithms and complexity. Prentice-Hall Inc., Englewood Cliffs, NJ
Peebles P. J. E., 1980, The large-scale structure of the universe. Princeton University Press, Princeton, NJ
Peebles P. J. E., 1989, ApJ, 344, L53
Peebles P. J. E., 1990, ApJ, 362, 1
Pogorelov A. V., 1978, The Minkowski multidimensional problem. V. H. Winston & Sons, Washington, D.C.
Rachev S. T., 1984, Teor. Veroyatnost. i Primenen., 29, 625
Rockafellar R. T., 1970, Convex Analysis. Princeton Univ. Press, Princeton, NJ
Sathyaprakash B. S., Sahni V., Shandarin S., Fisher K. B., 1998, ApJ, 507, L109
Shandarin S. F., Zel'dovich Y. B., 1989, Rev. Modern Phys., 61, 185
Spergel D. N., Verde L., Peiris H. V., Komatsu E., Nolta M. R., Bennett C. L., Halpern M., Hinshaw G., Jarosik N., Kogut A., Limon M., Meyer S. S., Page L., Tucker G. S., Weiland J. L., Wollack E., Wright E. L., 2003, First year Wilkinson microwave anisotropy probe (WMAP) observations: determination of cosmological parameters, preprint

(astro-ph/0302209)
Susperregi M., Binney J., 1994, MNRAS, 271, 719
Taylor A., Valentine H., 1999, MNRAS, 306, 491
Tomizawa N., 1971, Networks, 1, 173
Tully R. B., 1988, Nearby galaxies catalog. Cambridge University Press, Cambridge and New York
Valentine H., Saunders W., Taylor A., 2000, MNRAS, 319, L13
Vergassola M., Dubrulle B., Frisch U., Noullez A., 1994, A&A, 289, 325
Weinberg D. H., 1992, MNRAS, 254, 315
Weinberg D. H., Gunn J. E., 1990, MNRAS, 247, 260
Whiting A. B., 2000, ApJ, 533, 50
Zel'dovich Y. B., 1970, A&A, 5, 84