COHERENCE OF STRING REWRITING SYSTEMS BY DECREASINGNESS

CLÉMENT ALLEAUME – PHILIPPE MALBOS

Abstract – Squier introduced a homotopical method in order to describe all the relations amongst rewriting reductions of a confluent and terminating string rewriting system. From a string rewriting system he constructed a 2-dimensional combinatorial complex whose 2-cells are generated by relations induced by the rewriting rules. When the rewriting system is confluent and terminating, the homotopy of this complex can be characterized in term of confluence diagrams induced by the critical branchings of the rewriting system. Such a construction is now used to solve coherence problems for monoids using confluent and terminating string rewriting systems.
In this article, we show how to weaken the termination hypothesis in the description of all the relations amongst rewriting reductions. Our construction uses the decreasingness method introduced by van Oostrom. We introduce the notion of decreasing two-dimensional polygraph and we give sufficient conditions for a decreasing polygraph to be extended in a coherent way. In particular, we show how a confluent and quasi-terminating polygraph can be extended into a coherent presentation.

Keywords – string rewriting systems, coherence, termination, decreasingness.

1. INTRODUCTION

At the end of the eighties, using a homological argument, Squier showed that there are finitely presented monoids with a decidable word problem that cannot be presented by a finite convergent (*i.e.*, confluent and terminating) string rewriting system, [15, 16]. He linked the existence of a finite convergent presentation for a finitely presented monoid to a homological property by showing that the critical branchings of a convergent string rewriting system generate the module of the 2-homological syzygies of the presentation. A purely combinatorial approach is then presented in [17] to the question of whether or not a finitely presented monoid admits a finite convergent presentation. The existence of such a presentation is linked to a finiteness condition of finitely presented monoids, called *finite derivation type*, that extends the properties of being finitely generated and finitely presented.

Beyond the questions of decidability of the word problem and of the existence of finite convergent presentations, the graph-theoretical tools associated to convergent presentations of monoids developped in [17] were applied to question of coherence problems for monoids (*e.g.*, Artin monoids [4] or plactic monoids [8]) and monoidal categories [5]. In particular, one of the problems is to compute a *coherent presentation* of a monoid presented by a string rewriting system. Such a presentation extends the generators and the rules by homotopy generators taking into account all the relations amongst the rewriting sequences. A method is given in [17] to solve this problem from a convergent string rewriting system. However, in some situations it is difficult to get both confluence and termination on a finite set of generators and a finite set of rules.

In this article, using decreasingness methods from [18], we show how to weaken the termination hypothesis in the construction of coherent presentations. As an application we show how to extend a confluent and quasi-terminating string rewriting system into a coherent presentation.

Squier's two-dimensional complex. To a string rewriting system Σ_2 on an alphabet Σ_1 Squier, Otto and Kobayashi associated in [17] a 2-dimensional cellular complex $S(\Sigma)$, defined independently by Kilibarda [10] and Pride [14]. The complex $S(\Sigma)$ has only one 0-cell, its 1-cells are the strings in the free monoid Σ_1^* generated by the alphabet Σ_1 and its 2-cells are induced by the rewriting rules $\alpha : u \Rightarrow v$ in Σ_2 and the set Σ_2^- of their inverses $\alpha^- : v \Rightarrow u$. That is, there is a 2-cell in $S(\Sigma)$ between each pair of strings with shape wuw' and wvw' such that $\Sigma_2 \sqcup \Sigma_2^-$ contains the relation $u \Rightarrow v$. This 2-dimensional complex is extended with 3-cells, called *Peiffer confluences*, filling all the 2-spheres of the following form

$$\begin{tikzcd}
 & wv_1w'u_2w'' \arrow[dr, Rightarrow, "wv_1w'\alpha_2w''"] \arrow[dd, phantom, "\Rrightarrow"] & \\
wu_1w'u_2w'' \arrow[ur, Rightarrow, "w\alpha_1w'u_2w''"] \arrow[dr, Rightarrow, "wu_1w'\alpha_2w''"'] & & wv_1w'v_2w'' \\
 & wu_1w'v_2w'' \arrow[ur, Rightarrow, "w\alpha_1w'v_2w''"'] &
\end{tikzcd}$$

where $\alpha_1 : u_1 \Rightarrow v_1$ and $\alpha_2 : u_2 \Rightarrow v_2$ are in $\Sigma_2 \sqcup \Sigma_2^-$ and w, w' and w'' are strings in Σ_1^*. The Peiffer confluences make homotopic the 2-cells corresponding to the application of rewriting steps on non-overlapping strings.

A *homotopy basis* of the complex $S(\Sigma)$ is defined as a set Σ_3 of additional 3-cells that makes $S(\Sigma)$ aspherical, that is any 2-dimensional sphere can be "filled up" by the 3-cells of Σ_3. The presentation Σ is called of finite derivation type (FDT) if it is finite and it admits a finite homotopy basis. The FDT property is an invariant property for finitely presented monoids, that is, if Σ and Υ are two finite string rewriting systems that present the same monoid, then Σ has FDT if and only if Υ has FDT, [17].

Squier's completion. Given a convergent string rewriting system Σ, the set made of one 3-cell filling a confluence diagram induced by each critical branching forms a homotopy basis of $S(\Sigma)$, [17]. Such a set of 3-cells is called a *family of generating confluences of* Σ. In others words, any diagram defined by two parallel rewriting sequences can be filled up by confluence diagrams induced by the critical branchings and by the Peiffer confluences. This result corresponds to a homotopical version of Newman's Lemma, [13]. In particular, when the presentation is finite, it has finitely many critical branchings, hence a finite family of generating confluences. This is a way to prove that finite convergent presentations have FDT, [17].

Squier's completion without termination. The above result starts from a convergent presentation and the construction of homotopy bases is made by Noetherian induction. In some situations, it is difficult to get both confluence and termination without adding new generators, as in the case of plactic monoids [8] or Artin monoids [4]. Moreover, the addition of new generators implies as much new relations and thus new potentially non confluent critical branchings. For instance, the Artin monoid on the symmetric group S_2 is the monoid of braids on three strands $\mathbf{B}_3^+$ generated by two elements s and t and one relation $sts = tst$. Kapur and Narendran proved that this monoid does not admit a finite convergent presentation with only two generators, [9]. Note that a finite convergent presentation can be obtained by Knuth-Bendix completion on the presentation with three generators s, t, a and the two rules $sts \Rightarrow tst$ and $st \Rightarrow a$, where a is a redundant generator.

Coherence for quasi-terminating polygraphs. In this article, we weaken the termination hypothesis and we give a construction of homotopy bases for decreasing and quasi-terminating string rewriting systems. The notion of quasi-termination weakens termination in the sense that if there is an infinite rewriting sequence it must contain infinitely many occurrences of the same 1-cell. In that case, Noetherian induction cannot be used to construct a coherent presentation. For this reason we proceed by using a well-founded labelling on the rewriting system, called the labelling to the quasi-normal form. For example, the monoid $\mathbf{B}_3^+$ admits the following confluent and quasi-terminating presentation

$$\big\langle s, t \mid sts \Rightarrow tst,\ tst \Rightarrow sts \big\rangle.$$

We obtain a homotopy basis of the monoid $\mathbf{B}_3^+$ containing five 3-cells. This presentation can be homotopically reduced to obtain an empty homotopy basis.

Summary of results. In this work, we use the categorical description of string rewriting systems by 2-polygraphs, that are recalled in Section 2. We introduce the notion of decreasing 2-polygraph from the corresponding one introduced by van Oostrom for abstract rewriting systems in [18]. We will use van Oostrom's decreasingness techniques to prove our main result. However, decreasingness for string rewriting systems needs to take into account the structure of rewriting on strings. In particular, we introduce the notion of Peiffer decreasingness in order to take into account the confluence diagrams induced by application of rewriting steps on non-overlapping strings and the notion of compatibility with contexts for taking into account the contexts of the rules.

In Section 3, we extend Squier's completion known on convergent 2-polygraphs to decreasing 2-polygraphs. We define a *Squier's decreasing completion* of a decreasing 2-polygraph Σ as an extension of Σ by the globular extension of loops, containing one 3-cell for each equivalence class of elementary 2-loop and the globular extension of generating decreasing confluences, containing a decreasing confluence diagram for each critical branching of Σ.

Our main result states that a strictly decreasing 2-polygraph whose labelling is compatible with contexts and Peiffer decreasing can be extended into a coherent presentation, Theorem 3.2.1. As a consequence of this result, we show how to compute a coherent presentation from a confluent and quasi-terminating 2-polygraph. Finally, we show how our construction generalizes the one given in [17] for convergent rewriting systems and we deduce some homological and homotopical consequences.

2. Decreasing polygraphs

In this section, we recall categorical notions used in this work to describe string rewriting systems and relations between rewriting sequences. We refer the reader to [7] for a deeper presentation of these notions. Then we introduce decreasing 2-polygraphs from the corresponding notion for abstract rewriting systems introduced by van Oostrom in [18].

2.1. Two-dimensional polygraphs and extended presentations

2.1.1. Two-dimensional polygraphs. A 1-*polygraph* Σ is a directed graph made of a set of 0-cells Σ_0, a set of 1-cells Σ_1 and source and target maps $s_0, t_0 : \Sigma_1 \to \Sigma_0$. We denote by Σ_1^* the free category generated by Σ_1. A *globular extension* of the free category Σ_1^* is a set Σ_2 equipped with two maps

$s_1, t_1 : \Sigma_2 \to \Sigma_1^*$ such that, for every α in Σ_2, the pair $(s_1(\alpha), t_1(\alpha))$ is a 1-*sphere* in the category Σ_1^*, that is, $s_0s_1(\alpha) = s_0t_1(\alpha)$ and $t_0s_1(\alpha) = t_0t_1(\alpha)$. A 2-*polygraph* is a triple $\Sigma = (\Sigma_0, \Sigma_1, \Sigma_2)$, where (Σ_0, Σ_1) is a 1-polygraph and Σ_2 is a globular extension of Σ_1^*, whose elements are called the 2-*cells* of the 2-polygraph. A presentation of a category **C** is a 2-polygraph such that the quotient of the free category Σ_1^* by the congruence generated by Σ_2 is isomorphic to **C**. Note that a monoid being a category with a single object, is presented in the same way by a 2-polygraph with only one 0-cell.

2.1.2. Free 2-categories. Recall that a 2-*category* (resp. $(2,1)$-*category*) $\mathcal{C}$ is a category enriched in category (resp. groupoid). Equivalently, a $(2,1)$-category is a 2-category in which all 2-cells are invertible for the 1-composition. We denote by $\mathcal{C}_2$ the set of 2-cells of $\mathcal{C}$ and the 0-*composition* (resp. 1-*composition*) of two 2-cells f and g in $\mathcal{C}$ is denoted by $f \star_0 g$, or by fg (resp. $f \star_1 g$) . We will denote by s_i (resp. t_i) the i-*source map* (resp. i-*target map*) defined on 1-cells and 2-cells of a 2-category. A 2-*sphere* in $\mathcal{C}$ is a pair (f, g) of 2-cells of $\mathcal{C}$ such that $s_1(f) = s_1(g)$ and $t_1(f) = t_1(g)$.

Given a 2-polygraph Σ, we will denote by Σ_2^* the free 2-category generated by Σ and by $\Sigma_2^\top$ the free $(2,1)$-category generated by Σ, that is the free 2-category generated by Σ in which all the 2-cells are invertible.

2.1.3. Rewriting sequences. A *rewriting step* with respect to a 2-polygraph Σ is a 2-cell of Σ_2^* of the form $u\varphi v$ where u and v are 1-cells in Σ_1^* and φ is a 2-cell of Σ_2. We denote Σ_{stp} the set of rewriting steps of Σ. A *rewriting sequence* with respect to Σ is a finite or infinite sequence $f_0 \cdot f_1 \cdot \ldots \cdot f_i \cdot \cdots$, where the f_i are rewriting steps such that $t_1(f_i) = s_1(f_{i+1})$ for all $i \geqslant 0$. A 1-cell u *rewrites into* a 1-cell v if there is a rewriting sequence $f_0 \cdot \ldots \cdot f_n$ such that $s_1(f_0) = u$ and $t_1(f_n) = v$.

For any rewriting sequence $f_0 \cdot f_1 \cdot \ldots \cdot f_n$ from u to v there is a corresponding 2-cell $f_0 \star_1 f_1 \star_1 \ldots \star_1 f_n$ in the 2-category Σ_2^* with source $s_1(f_0) = u$ and target $t_1(f_n) = v$. Conversely, any 2-cell f in the 2-category Σ_2^* can be decomposed as a composite $f_0 \star_1 \ldots \star_1 f_n$ of rewriting steps. Note that, this decomposition is unique up to Peiffer relations.

The *length* of a finite rewriting sequence f is the number, denoted by $\ell(f)$, of rewriting steps occurring in the sequence. Given two 1-cells u and v such that u can be reduced to v, the *distance* from u to v, denoted by $d(u, v)$, is the length of the shortest rewriting sequence from u to v.

2.1.4. Support of a 2-cell. Let Σ be a 2-polygraph. Any 2-cell f in Σ_2^* can be written as a 1-composite of finitely many rewriting steps $u_1\varphi_1v_1, \ldots, u_k\varphi_kv_k$, where the u_i and v_i are 1-cells in Σ_1^* and φ_i is a 2-cell in Σ_2. We define the *support* of the 2-cell f as the multiset, denoted by $\mathrm{Supp}(f)$, consisting of the 2-cells φ_i occurring in this decomposition. The support is well-defined because any decomposition of f in Σ_2^* into a 1-composite of rewriting steps involves the same rewriting steps. Note also that any such a decomposition is finite and thus the support of a 2-cell is a finite multiset. As a consequence, the multiset inclusion is a well-founded order on supports, allowing us to prove some properties by induction on the support of 2-cells.

2.1.5. Branchings. A *(finite) branching* of a 2-polygraph Σ is a pair (f, g) of (finite) rewriting sequences of Σ with a common source $u = s_1(f) = s_1(g)$. Such a branching will be denoted by $(f, g) : u \Rightarrow (t_1(f), t_1(g))$. A *confluence* of a 2-polygraph Σ is a pair (f', g') of rewriting sequences of Σ with a common target $v = t_1(f') = t_1(g')$. Such a confluence will be denoted by $(f', g') : (t_1(f), t_1(g)) \Rightarrow v$.

A branching (f, g) is *local* (resp. *aspherical*) if f and g are in Σ_{stp} (resp. $f = g$). A *Peiffer branching* of Σ is a local branching (fv, ug) with source uv where u, v are composable 1-cells and f, g are in Σ_{stp}. An *overlapping branching* of Σ is a local branching that is not aspherical or Peiffer. An overlapping branching is called a *critical branching* if it is minimal for the order $\sqsubseteq$ on local branchings generated by $(f, g) \sqsubseteq (wfw', wgw')$, for any local branching (f, g) composable with 1-cells w and w' in Σ_1^*.

2.1.6. Termination and quasi-termination. A 2-polygraph Σ is *terminating* if it has no infinite rewriting sequence, that is there is no sequence $(u_n)_{n\in\mathbb{N}}$ of 1-cells such that for each n in $\mathbb{N}$, there is a rewriting step from u_n to u_{n+1}. In that case, every 1-cell u of Σ_1^* has at least one normal form $\widehat{u}$, that is, there is no rewriting step with source $\widehat{u}$.

Following [2], we say that a 2-polygraph Σ is *quasi-terminating* if for each sequence $(u_n)_{n\in\mathbb{N}}$ of 1-cells such that for each n in $\mathbb{N}$ there is a rewriting step from u_n to u_{n+1}, the sequence $(u_n)_{n\in\mathbb{N}}$ contains an infinite number of occurrences of the same 1-cell.

Let Σ be a 2-polygraph. A 1-cell u of Σ_1^* is called a *quasi-normal form* if for any rewriting step with source u leading to a 1-cell v, there exists a rewriting sequence from v to u. A quasi-normal form of a 1-cell u is a quasi-normal form $\widetilde{u}$ such that there exists a rewriting sequence from u to $\widetilde{u}$. If Σ is quasi-terminating, any 1-cell u of Σ_1^* admits a quasi-normal form. Note that, this quasi-normal form is neither irreducible nor unique in general.

2.1.7. Example. Let us consider the 2-polygraph defined by the following 2-graph

$$\begin{tikzcd} c \arrow[r, Rightarrow, bend right] \arrow[rrr, Rightarrow, bend left=40] & d \arrow[l, Rightarrow, bend right] \arrow[r, Rightarrow] & b \arrow[r, Rightarrow, bend right] & a \arrow[l, Rightarrow, bend right] \end{tikzcd}$$

The 1-cell d has two quasi-normal forms which are a and b. The 1-cell c is not a quasi-normal form because there is a rewriting step from c to a and a cannot be rewritten into c.

2.1.8. Confluence and convergence. A 2-polygraph Σ is *confluent* (resp. *locally confluent*) if every branching (resp. local branching) (f, g) of Σ can be completed by a confluence $(f', g') : (t_1(f), t_1(g)) \Rightarrow v$. We say that Σ is *convergent* (resp. *quasi-convergent*) if it is confluent and it terminates (resp. quasi-terminates).

2.1.9. Example. The 2-polygraph $\Sigma(\mathbf{B}_3^+) = \langle s, t \mid \alpha : sts \Rightarrow tst,\ \beta : tst \Rightarrow sts \rangle$ presents the monoid $\mathbf{B}_3^+$. This polygraph is not terminating but it is quasi-terminating. It has four critical branchings $(\alpha t, s\beta)$, $(\beta s, t\alpha)$, $(\alpha ts, st\alpha)$ and $(\beta st, ts\beta)$. These four branchings are confluent as follows

$$\begin{tikzcd}[column sep=small] & tst^2 \arrow[dr, Rightarrow, "\beta t"] & \\ stst \arrow[ur, Rightarrow, "\alpha t"] \arrow[dr, Rightarrow, "s\beta"'] & & stst \\ & s^2ts \arrow[ur, Rightarrow, "s\alpha"'] & \end{tikzcd} \quad \begin{tikzcd}[column sep=small] & sts^2 & \\ tsts \arrow[ur, Rightarrow, "\beta s"] \arrow[dr, Rightarrow, "t\alpha"'] & & tsts \arrow[ul, Rightarrow, "\beta s"'] \\ & t^2st \arrow[ur, Rightarrow, "t\beta"'] & \end{tikzcd} \quad \begin{tikzcd}[column sep=small] & tst^2s \arrow[dr, Rightarrow, "\beta ts"] & \\ ststs \arrow[ur, Rightarrow, "\alpha ts"] \arrow[dr, Rightarrow, "st\alpha"'] & & ststs \\ & st^2st \arrow[ur, Rightarrow, "st\beta"'] & \end{tikzcd} \quad \begin{tikzcd}[column sep=small] & sts^2t & \\ tstst \arrow[ur, Rightarrow, "\beta st"] \arrow[dr, Rightarrow, "ts\beta"'] & & tstst \arrow[ul, Rightarrow, "\beta st"'] \\ & ts^2ts \arrow[ur, Rightarrow, "ts\alpha"'] & \end{tikzcd}$$

2.1.10. Extended presentations. Let Σ be a 2-polygraph. A *globular extension* of the $(2,1)$-category $\Sigma_2^\top$ is a set Γ together with two maps $s_2, t_2 : \Gamma \to \Sigma_2^\top$ satisfying the *globular relations* $s_1 s_2 = s_1 t_2$ and $t_1 s_2 = t_1 t_2$. Two 2-cells f and g in $\Sigma_2^\top$ are *equal with respect to* Γ, and we denote $f \equiv_\Gamma g$, if f and g are equal in the quotient 2-category $\Sigma_2^\top/\Gamma$ of the 2-category $\Sigma_2^\top$ by the congruence on 2-cells generated by Γ.

Relations between rewriting sequences can be described using the notion of extended presentation. Recall from [6] that a $(3, 1)$-*polygraph* is a pair (Σ, Σ_3) made of a 2-polygraph Σ and a globular extension Σ_3 of the free $(2, 1)$-category $\Sigma_2^{\top}$, that is a set together with two maps $s_2, t_2 : \Sigma_3 \to \Sigma_2^{\top}$ satisfying the *globular relations* $s_1 s_2 = s_1 t_2$ and $t_1 s_2 = t_1 t_2$. We will denote by $\Sigma_3^{\top}$ the free $(3, 1)$-category generated by the $(3, 1)$-polygraph (Σ, Σ_3). An *extended presentation* of a category **C** is a $(3, 1)$-polygraph whose underlying 2-polygraph is a presentation of **C**.

2.1.11. Coherent presentations. A *coherent presentation* of a category **C** is an extended presentation (Σ, Σ_3), such that the globular extension Σ_3 is a homotopy basis of the $(2, 1)$-category $\Sigma_2^{\top}$. That is, for every 2-sphere (f, g) of $\Sigma_2^{\top}$, there exists a 3-cell from f to g in the free $(3, 1)$-category generated by the $(3, 1)$-polygraph (Σ_2, Σ_3).

2.2. Rewriting loops

In this part, Σ denotes a 2-polygraph.

2.2.1. Equivalent loops. A 2*-loop* in the 2-category Σ_2^* is a 2-cell f of Σ_2^* such that $s_1(f) = t_1(f)$. Two 2-loops f and g in Σ_2^* are *equivalent* if there exist a decomposition $f = f_1 \star_1 \ldots \star_1 f_p$, where f_i is a rewriting step of Σ for any $1 \leqslant i \leqslant p$, and a circular permutation σ such that $g = f_{\sigma(1)} \star_1 \ldots \star_1 f_{\sigma(p)}$. This defines an equivalence relation on 2-cells of Σ_2^*. We will denote by $\mathcal{L}(f)$ the equivalence class of a 2-loop f in Σ_2^* for this relation.

2.2.2. Lemma. *For any equivalent* 2*-loops* f *and* g *in* Σ_2^**, there exist* 2*-cells* h *and* k *of* $\Sigma_2^{\top}$ *such that* $f = h \star_1 g \star_1 k$.

Proof. Let us decompose f into a sequence $f = f_1 \star_1 \ldots \star_1 f_p$ of rewriting steps and let σ be a circular permutation such that $g = f_{\sigma(1)} \star_1 \ldots \star_1 f_{\sigma(p)}$. Let i be the integer such that $\sigma(i) = 1$. Let k be the 2-cell $f_{\sigma(1)} \star_1 \ldots \star_1 f_{\sigma(i-1)}$. Let $h = k^-$ be the inverse of k for the 1-composition. Then, we have $f = h \star_1 g \star_1 k$. □

2.2.3. Minimal and elementary loops. We say that a 2-loop f in Σ_2^* is

- **i)** *minimal with respect to* 1*-composition*, if any decomposition $f = g \star_1 h \star_1 k$ in Σ_2^* with h a 2-loop implies that h is either an identity or equal to f,
- **ii)** *minimal by context*, if there is no decomposition $f = ugv$, where u and v are nonidentity 1-cells in Σ_1^* and g is a loop in Σ_2^*.

A 2-loop f in Σ_2^* is *elementary* if it is minimal both with respect to 1-composition and by context. As an immediate consequence of these definitions, any 2-loop f minimal for 1-composition can be written $f = ugv$, where g is an elementary loop and u, v are 1-cells in Σ_1^*.

2.2.4. Lemma. *Let* f *be a nonidentity* 2*-loop in* Σ_2^**. Then, there exists a decomposition* $f = f_1 \star_1 f' \star_1 f_2$ *in* Σ_2^**, where* f' *is a* 2*-loop minimal with respect to* 1*-composition and* f_1, f_2 *are* 2*-cells such that* $f_1 \star_1 f_2$ *is a* 2*-loop.*

Proof. Let f be a nonidentity 2-loop in Σ_2^*. The proof is by induction on the support $\mathrm{Supp}(f)$. If the 2-loop f is minimal for 1-composition, we can write $f = 1_{s_1(f)} \star_1 f \star_1 1_{s_1(f)}$. If f is not minimal for 1-composition, there exists a decomposition $f = g \star_1 h \star_1 k$, where h is a 2-loop that is neither an identity nor equal to f. Hence, $\mathrm{Supp}(h)$ is strictly included in $\mathrm{Supp}(f)$ that proves the decomposition. □

2.2.5. Globular extensions of loops. We will denote by $\mathcal{E}(\Sigma)$ the set of equivalence classes of elementary 2-loops of Σ_2^*. A *loop extension* of Σ is a globular extension of the $(2,1)$-category $\Sigma_2^\top$ made of a family of 3-cells $A_\alpha : \alpha \Rrightarrow 1_{s_1(\alpha)}$ indexed by exactly one α for each equivalence class in $\mathcal{E}(\Sigma)$.

2.2.6. Lemma. *Let $\mathcal{L}(\Sigma)$ be a loop extension of Σ. For any 2-loop f in Σ_2^*, there exists a 3-cell from f to $1_{s_1(f)}$ in the free $(3,1)$-category $\mathcal{L}(\Sigma)^\top$ generated by the $(3,1)$-polygraph $(\Sigma, \mathcal{L}(\Sigma))$.*

Proof. Let us fix a loop extension $\mathcal{L}(\Sigma)$. Let f be 2-loop in Σ_2^*. We proceed by induction on the support $\mathrm{Supp}(f)$.
Step 1. Suppose that f is elementary. By definition of $\mathcal{L}(\Sigma)$, the equivalence class $\mathcal{L}(f)$ contains an elementary 2-loop e such that $\mathcal{L}(\Sigma)$ contains a 3-cell A_e from e to $1_{s_1(e)}$. The 2-loop e being equivalent to f, by Lemma 2.2.2 there exist two 2-cells h and k of $\Sigma_2^\top$ such that $f = h \star_1 e \star_1 k$. Thus, the 3-cell $h \star_1 A_e \star_1 k$ in $\mathcal{L}(\Sigma)^\top$ goes from f to $h \star_1 k$. By construction the 2-cell $h \star_1 k$ is equal $1_{s_1(f)}$. In this way we construct a 3-cell in $\mathcal{L}(\Sigma)^\top$ from f to $1_{s_1(f)}$.
Step 2. Suppose that f is minimal with respect to 1-composition. Then, there is a decomposition $f = ugv$, where u and v are 1-cells in Σ_1^* and g is an elementary 2-loop in Σ_2^*. By Step 1, there exists a 3-cell A_g from g to $1_{s_1(g)}$ in $\mathcal{L}(\Sigma)^\top$. Thus uA_gv is a 3-cell in $\mathcal{L}(\Sigma)^\top$ from f to $1_{s_1(f)}$.
Step 3. Suppose that f is a nonidentity 2-loop. By Lemma 2.2.4, the 2-loop f can be written as $f_1 \star_1 f' \star_1 f_2$ where f' is a 2-loop minimal for 1-composition and f_1 and f_2 are 2-cells such that $f_1 \star_1 f_2$ is a 2-loop. By Step 2, there exists a 3-cell $A_{f'}$ in $\mathcal{L}(\Sigma)^\top$ from f' to $1_{s_1(f')}$. Hence, the 1-composite $f_1 \star_1 A_{f'} \star_1 f_2$ is a 3-cell from f to $f_1 \star_1 f_2$ in $\mathcal{L}(\Sigma)^\top$. The support of $f_1 \star_1 f_2$ being strictly included in the support of f, this proves the lemma by induction on the support of f. □

2.3. Labelled polygraphs

2.3.1. Labelled 2-polygraphs. A *well-founded labelled 2-polygraph* is a data $(\Sigma, W, \prec, \psi)$ made of a 2-polygraph Σ, a set W, a well-founded order $\prec$ on W and a map $\psi : \Sigma_{\mathrm{stp}} \longrightarrow W$. The map ψ is called a *well-founded labelling* of Σ and associates to a rewriting step f a *label* $\psi(f)$.

Given a rewriting sequence $f = f_1 \cdot \ldots \cdot f_k$, we denote by $L^W(f) = \{\psi(f_1), \ldots, \psi(f_k)\}$ the set of labels of rewriting steps in f. Note that two distinct rewriting sequences f and g can correspond to a same 2-cell in the free 2-category Σ_2^* despite $L^W(f)$ and $L^W(g)$ being distinct.

2.3.2. Labelling to the quasi-normal form. Consider a quasi-convergent 2-polygraph Σ. By quasi-termination, any 1-cell u admits a quasi-normal form, not unique in general. For every 1-cell u in Σ_1^*, let us fix a quasi-normal form $\widetilde{u}$. Note that by confluence hypothesis, any two congruent 1-cells of Σ_1^* have the same quasi-normal form. This defines a *quasi-normal form map* $s : \Sigma_1^* \to \Sigma_1^*$ sending a 1-cell u on $\widetilde{u}$. The *labelling to the quasi-normal form*, labelling QNF for short, associates to the map s the labelling $\psi^{\mathrm{QNF}} : \Sigma_{\mathrm{stp}} \longrightarrow \mathbb{N}$ defined by

$$\psi^{\mathrm{QNF}}(f) = d(t_1(f), \widetilde{t_1(f)}),$$

for any rewriting step f of Σ.

2.3.3. Lexicographic maximum measure, [18, Definition 3.1]. Let $(\Sigma, W, \prec, \psi)$ be a well-founded labelled 2-polygraph. Let $w = w_1 \ldots w_n$ and $w' = w'_1 \ldots w'_m$ be 1-cells in the free monoid W^* with w_i and w'_j in W. We denote by $w^{(w')}$ the 1-cell $\overline{w}_1 \ldots \overline{w}_n$ such that for every $0 \leqslant k \leqslant n$, the 1-cell $\overline{w}_k$ is defined by

$$\overline{w}_k = \begin{cases} 1 & \text{if } w_k \prec w'_j \text{ for some } 1 \leqslant j \leqslant m, \\ w_k & \text{otherwise.} \end{cases}$$

Following [18, Definition 3.1], we consider the measure $|\cdot|$ from the free monoid W^* to the set of multisets over W and defined as follows:

i) for every i in W, the multiset $|i|$ is the singleton $\{i\}$,

ii) for every i in W and every 1-cell w in W^*, we have $|iw| = |i| \cup |w^{(i)}|$.

The measure $|\cdot|$ is extended to the set of finite rewriting sequences of Σ by setting, for every rewriting sequence $f_1 \cdot \ldots \cdot f_n$, with f_i labelled by k_i for all i,

$$|f_1 \cdot \ldots \cdot f_n| = |k_1 \ldots k_n|,$$

were $k_1 \ldots k_n$ is a product in the monoid W^*. Finally, the measure $|\cdot|$ is extended to the set of finite branchings (f, g) of Σ, by setting

$$|(f, g)| = |f| \cup |g|.$$

Recall from [18, Lemma 3.2], that for every 1-cells w_1, w_2 in W^*, we have $|w_1 w_2| = |w_1| \cup |w_2^{(w_1)}|$. As a consequence, for any rewriting sequences f and g of Σ the following relation holds

$$|f \cdot g| = |f| \cup |g^{(f)}|,$$

where $|g^{(f)}|$ is defined by

$$|g^{(f)}| = |k_1 \ldots k_m^{(l_1 \ldots l_n)}|,$$

with $f = f_1 \cdot \ldots \cdot f_n$ and $g = g_1 \cdot \ldots \cdot g_m$ and f_i labelled by l_i and g_j labelled by k_j.

2.4. Decreasing two-dimensional polygraphs

Let us recall in the context of 2-polygraph the notion of decreasingness from [18, Definition 3.3].

2.4.1. Decreasing 2-polygraph. Let (Σ, ψ) be a well-founded labelled 2-polygraph. A local branching (f, g) of Σ is *decreasing* (resp. *strictly decreasing*) if there is a confluence diagram of the following form

$$
\begin{tikzcd}
\cdot \arrow[rrr, Rightarrow, "f"] \arrow[ddd, Rightarrow, "g"'] & & & \cdot \arrow[d, Rightarrow, "f'"] \\
& & & \cdot \arrow[d, Rightarrow, "g''"] \\
& & & \cdot \arrow[d, Rightarrow, "h_1"] \\
\cdot \arrow[r, Rightarrow, "g'"'] & \cdot \arrow[r, Rightarrow, "f''"'] & \cdot \arrow[r, Rightarrow, "h_2"'] & \cdot
\end{tikzcd}
\qquad \text{(resp.} \quad
\begin{tikzcd}
\cdot \arrow[r, Rightarrow, "f"] \arrow[d, Rightarrow, "g"'] & \cdot \arrow[d, Rightarrow, "f'"] \\
\cdot \arrow[r, Rightarrow, "g'"'] & \cdot
\end{tikzcd}
\quad \text{).}
$$

and such that the following properties hold

i) $k \prec \psi(f)$, for all k in $L^W(f')$,

ii) $k \prec \psi(g)$, for all k in $L^W(g')$,

iii) f'' is an identity or a rewriting step labelled by $\psi(f)$,

iv) g'' is an identity or a rewriting step labelled by $\psi(g)$,

v) $k \prec \psi(f)$ or $k \prec \psi(g)$, for all k in $L^W(h_1) \cup L^W(h_2)$.

Such a diagram is then called a *decreasing confluence diagram* (resp. *strictly decreasing confluence diagram*) of the branching (f, g).

A 2-polygraph Σ is *decreasing* (resp. *strictly decreasing*) if there exists a well-founded labelling $(W, \prec, \psi)$ of Σ making all its local branching decreasing (resp. strictly decreasing).

As in the case of abstract rewriting systems, [18, Corollary 3.9.], we prove that any decreasing 2-polygraph is confluent.

2.4.2. Strictly decreasing branching. We extend the notion of strict decreasingness on local branchings to branchings as follows. A branching (f, g) is *strictly decreasing* is there is a confluence diagram $(f \cdot f', g \cdot g')$ such that the two following properties hold

i) for each k' in $L^W(f')$, we have $k' \prec k$ for any k in $L^W(f)$,

ii) for each l' in $L^W(g')$, we have $l' \prec l$ for any l in $L^W(g)$.

2.4.3. Decreasingness from quasi-termination. Any quasi-convergent 2-polygraph Σ is strictly decreasing with respect to any quasi-normal form labelling ψ^{QNF}. Indeed, for any local branching $u \Rightarrow (v, w)$ there exists a quasi-normal form $\widetilde{u}$ and a confluence $(f', g') : (v, w) \Rightarrow \widetilde{u}$. The rewriting sequences f' and g' can be chosen of minimal length, thus making this confluence diagram strictly decreasing with respect the labelling ψ^{QNF}.

2.4.4. Decreasingness of Peiffer branchings. For a Peiffer branching $(fv, ug) : uv \Rightarrow (u'v, uv')$ of a 2-polygraph Σ, the confluence $(u'g, fv') : (u'v, uv') \Rightarrow u'v'$ is called the *Peiffer confluence* of the branching (fv, ug). In a decreasing 2-polygraph (Σ, ψ) every Peiffer branching can be completed into a decreasing confluence diagram. However, the confluence diagram obtained with the Peiffer confluence is not always decreasing as in the case of the following example.

2.4.5. Example. As shown in 2.4.3, a labelling QNF makes every Peiffer branching decreasing. But, it does not necessarily makes the Peiffer confluences decreasing. In particular, it is not the case when the source uv of the Peiffer confluence is already the chosen quasi-normal form. For instance, consider the quasi-convergent 2-polygraph $\Sigma = \langle a, b \mid \alpha : a \Rightarrow b,\ \beta : b \Rightarrow a \rangle$. For each 1-cel u of Σ_1^*, we set $\widetilde{u} = a^{\ell(u)}$ as a quasi-normal form. Let us now consider the following Peiffer diagram:

$$\begin{tikzcd}[row sep=small]
 & ab \arrow[dr, Rightarrow, "\alpha b"] & \\
a^2 \arrow[ur, Rightarrow, "a\alpha"] \arrow[dr, Rightarrow, "\alpha a"'] & & b^2 \\
 & ba \arrow[ur, Rightarrow, "b\alpha"'] &
\end{tikzcd}$$

This Peiffer diagram is not decreasing with respect to ψ^{QNF}. Indeed, we have $\psi^{\mathrm{QNF}}(\alpha a) = \psi^{\mathrm{QNF}}(a\alpha) = 1$ and $\psi^{\mathrm{QNF}}(\alpha b) = \psi^{\mathrm{QNF}}(b\alpha) = 2$. However, this Peiffer branching is decreasing by using the following confluence $(a\beta, \beta a) : (ab, ba) \Rightarrow a^2$, since $\psi^{\mathrm{QNF}}(a\beta) = \psi^{\mathrm{QNF}}(\beta a) = 0$.

2.4.6. Peiffer decreasingness. A decreasing (resp. strictly decreasing) 2-polygraph (Σ, ψ) is *Peiffer decreasing* with respect to a globular extension Γ of the $(2,1)$-category $\Sigma_2^{\top}$ if, for any Peiffer branching $(fv, ug) : uv \Rightarrow (u'v, uv')$, there exists a decreasing (resp. strictly decreasing) confluence diagram $(fv \cdot f', ug \cdot g')$:

$$\begin{tikzcd} & u'v \arrow[dr, Rightarrow, "u'g"] \arrow[drr, Rightarrow, bend left, "f'"] & & \\ uv \arrow[ur, Rightarrow, "fv"] \arrow[dr, Rightarrow, "ug"'] & & u'v' & u'' \\ & uv' \arrow[ur, Rightarrow, "fv'"'] \arrow[urr, Rightarrow, bend right, "g'"'] & & \end{tikzcd}$$

such that $u'g \star_1 (fv')^- \equiv_\Gamma f' \star_1 (g')^-$.

2.4.7. Example. Any 2-polygraph Σ such that any non trivial local branching (f, g) is confluent using two rewriting steps $f' : t_1(f) \Rightarrow v$ and $g' : t_1(g) \Rightarrow v$ is Peiffer decreasing. Indeed, a labelling such that all rewriting steps have the same label makes any local branching (f, g) decreasing. Moreover, with such a labelling, any Peiffer confluence is decreasing. In particular, the 2-polygraph $\Sigma(\mathbf{B}_3^+)$ is decreasing for a singleton labelling.

2.4.8. Compatibility with contexts. Let (Σ, ψ) be a well-founded labelled 2-polygraph. The labelling ψ is *compatible with contexts* if for any decreasing (resp. strictly decreasing) confluence diagram $(f \cdot f', g \cdot g')$, where (f, g) is a local branching, and for any composable 1-cells u_1 and u_2 in Σ_1^*, the following confluence diagram is decreasing (resp. strictly decreasing):

$$\begin{tikzcd} & u_1vu_2 \arrow[dr, Rightarrow, "u_1f'u_2"] & \\ u_1uu_2 \arrow[ur, Rightarrow, "u_1fu_2"] \arrow[dr, Rightarrow, "u_1gu_2"'] & & u_1u'u_2 \\ & u_1wu_2 \arrow[ur, Rightarrow, "u_1g'u_2"'] & \end{tikzcd}$$

Note that a labelling QNF is not compatible with contexts in general.

2.4.9. $\star_0$-compatibility. A well-founded labelling $(W, \psi, \prec)$ is $\star_0$-*compatible* if for any rewriting steps f and g such that $\psi(f) \prec \psi(g)$, we have $\psi(u_1fu_2) \prec \psi(u_1gu_2)$ for any composable 1-cells u_1 and u_2 in Σ_1^*. Note that the $\star_0$-compatibility does not implies the compatibility with contexts. Indeed, if (f, g) is a local branching that can be completed into a diagram

$$\begin{tikzcd} & b \arrow[dr, Rightarrow, "f'"] & \\ a \arrow[ur, Rightarrow, "f"] \arrow[dr, Rightarrow, "g"'] & & d \\ & c \arrow[ur, Rightarrow, "g'"'] & \end{tikzcd}$$

where f' and g' are rewriting steps such that $\psi(f) = \psi(f')$ and $\psi(g) = \psi(g')$, then the confluence diagram is decreasing. Even, if the labelling $(W, \psi, \prec)$ is $\star_0$-compatible, we do not necessarily have $\psi(ufv) = \psi(uf'v)$ and $\psi(ugv) = \psi(ug'v)$ for any 1-cells u and v. Thus, the following diagram is not decreasing in general:

$$\begin{tikzcd}[row sep=tiny] & ubv \arrow[dr, Rightarrow, bend left, "uf'v"] & \\ uav \arrow[ur, Rightarrow, bend left, "ufv"] \arrow[dr, Rightarrow, bend right, "ugv"'] & & udv \\ & ucv \arrow[ur, Rightarrow, bend right, "ug'v"'] & \end{tikzcd}$$

If ψ is a $\star_0$-compatible labelling, for any strictly decreasing diagram $(f \cdot f', g \cdot g')$, where (f, g) is a local branching, we have $\psi^{\mathrm{QNF}}(u_1 f' u_2) < \psi^{\mathrm{QNF}}(u_1 f u_2)$ and $\psi^{\mathrm{QNF}}(u_1 g' u_2) < \psi^{\mathrm{QNF}}(u_1 g u_2)$ for every composable 1-cells u_1 and u_2. As a consequence, any $\star_0$-compatible labelling on a strictly decreasing 2-polygraph is compatible with contexts.

2.4.10. Example. Consider the 2-polygraph Σ defined in 2.4.5. The labelling QNF defined using the quasi-normal forms of the form $\widetilde{u} = a^{\ell(u)}$ is compatible with contexts. This is a consequence of the following equality

$$\psi^{\mathrm{QNF}}(u_1 f u_2) = d(u_1, a^{\ell(u_1)}) + \psi^{\mathrm{QNF}}(f) + d(u_2, a^{\ell(u_2)})$$

for any rewriting step f and 1-cells u_1 and u_2.

If we consider an other labelling QNF of the 2-polygraph Σ associated to quasi-normal forms of the form $\widetilde{u} = a^{\ell(u)}$ for any 1-cell u such that $\ell(u) \neq 3$ and $\widetilde{u} = b^3$ for any 1-cell u such that $\ell(u) = 3$. Then the confluence diagram $(a\alpha \cdot a\beta, \alpha a \cdot \beta a)$ is decreasing with $\psi^{\mathrm{QNF}}(a\alpha) = \psi^{\mathrm{QNF}}(\alpha a) = 1$ and $\psi^{\mathrm{QNF}}(a\beta) = \psi^{\mathrm{QNF}}(\beta a) = 0$. However, the confluence diagram $(ba\alpha \cdot ba\beta, b\alpha a \cdot b\beta a)$ is not decreasing with $\psi^{\mathrm{QNF}}(ba\alpha) = \psi^{\mathrm{QNF}}(b\alpha a) = 1$ and $\psi^{\mathrm{QNF}}(ba\beta) = \psi^{\mathrm{QNF}}(b\beta a) = 2$. As a consequence this labelling QNF is not compatible with contexts.

2.4.11. Example. Consider the 2-polygraph $\Sigma(\mathbf{B}_3^+)$ given in 2.1.9. We define a QNF labelling ψ^{QNF} on $\Sigma(\mathbf{B}_3^+)$ by associating to each 1-cell u of $\Sigma(\mathbf{B}_3^+)_1^*$ the quasi-normal form $\widetilde{u}$ defined as follows. Setting $N_u = \max\{n \mid u = (sts)^n v \text{ holds in } \mathbf{B}_3^+\}$, we define $\widetilde{u} = (sts)^{N_u} v$. The maximality of N_u ensures the unicity of such a quasi-normal form. Indeed, let us consider the following convergent presentation of the monoid $\mathbf{B}_3^+$:

$$\Upsilon = \big\langle s, t, a \;\big|\; sts \Rightarrow a,\ tst \Rightarrow a, sa \Rightarrow at, ta \Rightarrow as \big\rangle.$$

Suppose that a 1-cell u of $\Sigma(\mathbf{B}_3^+)_1^*$ has two distinct quasi-normal forms $(sts)^{N_u} v$ and $(sts)^{N_u} w$. Those two 1-cells have respectively $a^{N_u} v$ and $a^{N_u} w$ as normal forms with respect to Υ_2. Indeed, there is no occurrence of a in v and w, and the 1-cells sts and tst cannot divide v and w by maximality of N_u. By unicity of the normal forms in a convergent 2-polygraph, the 1-cells $a^{N_u} v$ and $a^{N_u} w$ are not equal in the monoid $\mathbf{B}_3^+$, hence they are not the normal forms of a same 1-cell in Υ_1^*. Thus, the 1-cells $(sts)^{N_u} v$ and $(sts)^{N_u} w$ are not the quasi-normal forms of a same 1-cell in $\Sigma(\mathbf{B}_3^+)_1^*$, which contradicts our assumption.

The labelling defined in this way is $\star_0$-compatible. Indeed, for any rewriting steps f and g of $\Sigma(\mathbf{B}_3^+)$ such that $\psi^{\mathrm{QNF}}(g) < \psi^{\mathrm{QNF}}(f)$ and for any composable 1-cells u_1 and u_2, we have $\psi^{\mathrm{QNF}}(u_1 f u_2) < \psi^{\mathrm{QNF}}(u_1 g u_2)$. Hence, the labelling ψ^{QNF} is compatible with contexts.

2.4.12. Multiset order. Given a well-founded set of labels $(W, \prec)$, we consider the partial order $\prec_{\text{mul}}$ on the multisets over W defined in [3, 18] as follows. For any multisets M and N over W, we set $M \prec_{\text{mul}} N$ if there exist multisets X, Y and Z such that:

i) $M = Z \cup X$, $N = Z \cup Y$ and Y is not empty,

ii) for every i in W such that $X(i) \neq 0$, there exists j in W such that $Y(j) \neq 0$ and $i \prec j$.

The order $\prec_{\text{mul}}$ is well-founded because $\prec$ is. We call $\preccurlyeq_{\text{mul}}$ the symmetric closure of $\prec_{\text{mul}}$.

Let us mention a particular case of [18, Lemma 3.6.], that will be used in the proof of our main result.

2.4.13. Lemma. *Let Σ be a decreasing 2-polygraph. For every diagram in Σ_2^* of the following form*

$$\begin{tikzcd} & {} \arrow[rr, Rightarrow, bend left, "f_2"] \arrow[dr, Rightarrow, "f_1'"] & & {} \\ {} \arrow[ur, Rightarrow, bend left, "f_1"] \arrow[dr, Rightarrow, bend right, "g_1"'] & & {} & \\ & {} \arrow[ur, Rightarrow, bend right, "g_1'"'] & & \end{tikzcd}$$

where f_1 is a non empty rewriting sequences, f_2 and g_1 are rewriting sequence and the confluence diagram $(f_1 \cdot f_1', g_1 \cdot g_1')$ is strictly decreasing, the inequality $|(f_1', f_2)| \prec_{\text{mul}} |(g_1, f_1 \cdot f_2)|$ holds.

2.4.14. Proposition. *Let (Σ, ψ) be a well-founded labelled 2-polygraph. Then Σ is strictly decreasing if and only if any branching of Σ is strictly decreasing.*

Proof. One implication is obvious. Let us assume that Σ is strictly decreasing and let (f, g) be a branching of Σ. We prove by induction on $|(f, g)|$ that (f, g) is strictly decreasing. If f or g is an empty rewriting sequence, the strict decreasingness of (f, g) is trivial. Else, we can write

$$\begin{tikzcd} & {} \arrow[rr, Rightarrow, bend left, "f'"] \arrow[dr, Rightarrow, "f''"] & & {} \\ {} \arrow[ur, Rightarrow, bend left, "f_1"] \arrow[dr, Rightarrow, bend right, "g_1"'] & & {} & \\ & {} \arrow[ur, Rightarrow, "g''"] \arrow[dr, Rightarrow, "g'"'] & & \\ & & {} & \end{tikzcd}$$

such that the confluence diagram $(f_1 \cdot f'', g_1 \cdot g'')$ is strictly decreasing. By Lemma 2.4.13, we have $|(f', f'')| \prec_{\text{mul}} |(f_1, g)|$. Thus, we have $|(f', f'')| \prec_{\text{mul}} |(f, g)|$ and we can use the induction hypothesis to construct a strictly decreasing confluence diagram $(f' \cdot k_1, f'' \cdot k_2)$. By using again Lemma 2.4.13, we have $|(g'' \cdot k_2, g')| \prec_{\mathrm{mul}} |(f, g)|$. Thus, by applying again the induction hypothesis, we have a diagram

$$\begin{tikzcd}
 & \cdot \arrow[rr, Rightarrow, "f'"] \arrow[dr, Rightarrow, "f''"] & & \cdot \arrow[dr, Rightarrow, "k_1"] & \\
\cdot \arrow[ur, Rightarrow, "f_1"] \arrow[dr, Rightarrow, "g_1"'] & & \cdot \arrow[rr, Rightarrow, "k_2"] & & \cdot \arrow[dd, Rightarrow, "l_1"] \\
 & \cdot \arrow[ur, Rightarrow, "g''"] \arrow[dr, Rightarrow, "g'"'] & & & \\
 & & \cdot \arrow[rr, Rightarrow, "l_2"'] & & \cdot
\end{tikzcd}$$

where the diagram $(f \cdot k_1 \cdot l_1, g \cdot l_2)$ is strictly decreasing. □

3. Coherence by decreasingness

In this section, we extend to decreasing 2-polygraphs the notion of Squier's completion known for convergent 2-polygraphs. We give sufficient conditions on the labelling of a decreasing 2-polygraph making the Squier's decreasing completion a coherent presentation. In particular, we show how to extend a quasi-convergent 2-polygraph into a coherent presentation.

3.1. Squier's decreasing completion

Squier's completion provides a way to extend a convergent 2-polygraph into a coherent presentation, see [7, 17].

3.1.1. Squier's completion. A *family of generating confluences* of a 2-polygraph Σ is a globular extension of the $(2,1)$-category $\Sigma_2^\top$ that contains exactly one 3-cell of the following form

$$\begin{tikzcd}
 & v \arrow[dr, Rightarrow, "f'"] \arrow[dd, Rrightarrow] & \\
u \arrow[ur, Rightarrow, "f"] \arrow[dr, Rightarrow, "g"'] & & u' \\
 & w \arrow[ur, Rightarrow, "g'"'] &
\end{tikzcd}$$

for each critical branching (f, g) of Σ. If Σ is confluent, it always admits such a family A *Squier's completion* of a convergent 2-polygraph Σ is a $(3,1)$-polygraph that extends Σ by a chosen family of generating confluences. Any Squier's completion of a convergent 2-polygraph Σ is a coherent presentation of the category presented by Σ, [17], see also [7].

3.1.2. Generating decreasing confluences. Let (Σ, ψ) be a decreasing 2-polygraph. A *family of generating decreasing confluences of* Σ *with respect to* ψ is a globular extension of the $(2,1)$-category $\Sigma_2^\top$ that contains, for every critical branching $(f, g) : u \Rightarrow (v, w)$ of Σ, exactly one 3-cell $D^\psi_{f,g}$ of the following form

$$
\begin{tikzcd}
 & v \arrow[dr, Rightarrow, bend left, "f'"] \arrow[d, phantom, "\Downarrow D^{\psi}_{f,g}"] & \\
u \arrow[ur, Rightarrow, bend left, "f"] \arrow[dr, Rightarrow, bend right, "g"'] & {} & u' \\
 & w \arrow[ur, Rightarrow, bend right, "g'"'] &
\end{tikzcd}
$$

and where the confluence diagram $(f \cdot f', g \cdot g')$ is decreasing with respect to ψ. Any decreasing 2-polygraph admits such a family of generating decreasing confluences. Indeed, any critical branching is local and thus confluent by decreasingness hypothesis. However, note that such a family is not unique in general.

For a strictly decreasing 2-polygraph Σ, we define in the same way a *family of generating strictly decreasing confluences* of Σ, but where the confluence diagrams are strictly decreasing with respect to ψ.

3.1.3. Squier's decreasing completion. Let (Σ, ψ) be a decreasing 2-polygraph. A *Squier's decreasing completion of* Σ *with respect to* ψ is a $(3,1)$-polygraph that extends the 2-polygraph Σ by a globular extension

$$\mathcal{O}(\Sigma, \psi) \cup \mathcal{L}(\Sigma)$$

where $\mathcal{O}(\Sigma, \psi)$ is a chosen family of generating decreasing confluences with respect to ψ and $\mathcal{L}(\Sigma)$ is a loop extension of Σ defined in 2.2.5. If (Σ, ψ) is a strictly decreasing 2-polygraph, a *strictly decreasing Squier's completion* is a Squier's decreasing completion, whose the generating decreasing confluences are required strict.

3.1.4. Lemma. *Let* (Σ, ψ) *be a strictly decreasing 2-polygraph. Let* $\mathcal{S}^{sd}(\Sigma, \psi)$ *be a strictly decreasing Squier's completion of* Σ*. Suppose that* ψ *is compatible with contexts and that* (Σ, ψ) *is Peiffer decreasing with respect to the extension* $\mathcal{S}^{sd}(\Sigma, \psi)$*. Then, for any 2-sphere* (f, g) *in* Σ_2^**, there exists a 3-cell from* f *to* g *in the* $(3,1)$*-category* $\mathcal{S}^{sd}(\Sigma, \psi)^\top$*.*

Proof. We proceed in two steps.
Step 1. We prove that, for every local branching $(f, g) : u \Rightarrow (v, w)$ of Σ, there exists a confluence $(f', g') : (v, w) \Rightarrow u'$ of Σ and a 3-cell $A : f \star_1 f' \Rrightarrow g \star_1 g'$ in $\mathcal{S}^{sd}(\Sigma, \psi)^\top$ such that the confluence diagram $(f \cdot f', g \cdot g')$ is strictly decreasing.

In the case of an aspherical branching, we can choose f' and g' to be identity 2-cells, A to be an identity 3-cell and the confluence diagram (f, f) is trivially strictly decreasing.

Suppose that (f, g) is a Peiffer branching $(f_1 v_1, u_1 g_1) : u_1 v_1 \Rightarrow (u_1' v_1, u_1 v_1')$. By hypothesis, the Peiffer confluence $(f_1 v_1 \cdot u_1' g_1, u_1 g_1 \cdot f_1 v_1')$ is equivalent to a strictly decreasing confluence diagram $(f_1 v_1 \cdot f_1', u_1 g_1 \cdot g_1')$. Hence, there exists a 3-cell $A : f_1 v_1 \star_1 f_1' \Rrightarrow u_1 g_1 \star_1 g_1'$ in the $(3,1)$-category $\mathcal{S}^{sd}(\Sigma, \psi)^\top$.

If (f, g) is an overlapping branching, we have $(f, g) = (whw', wkw')$ with (h, k) a critical branching. We consider the 3-cell $D^{\psi}_{h,k} : h \star_1 h' \Rrightarrow k \star_1 k'$ of $\mathcal{O}(\Sigma, \psi)$ corresponding to the strict generating decreasing confluence of the critical branching (h, k) with respect to the labelling ψ, or its inverse. Let us define the 2-cells $f' = wh'w'$ and $g' = wk'w'$ and the 3-cell $A = wD^{\psi}_{h,k}w'$. The labelling ψ being compatible with contexts, the confluence diagram corresponding to the 3-cell A is strictly decreasing.

Step 2. Let (f, g) be a 2-sphere in Σ_2^*. This 2-sphere defines a branching with source $s_1(f) = s_1(g)$. The 2-polygraph Σ being strictly decreasing, we prove the lemma by well-founded induction on the measure

$|(f, g)|$ of the branching (f, g). If f or g is an identity 2-cell, say $g = 1$, the 2-cell f is a 2-loop. By Lemma 2.2.6, there exists a 3-cell $E : f \Rrightarrow 1_{s_1(f)}$ in the $(3, 1)$-category $\mathcal{L}(\Sigma)^{\top}$. Else, we have decompositions $f = f_1 \star_1 f_2$ and $g = g_1 \star_1 g_2$ in Σ_2^* where (f_1, g_1) is a local branching. Note that f_2 or g_2 can be equal to an identity 2-cell. The local branching (f_1, g_1) is confluent by decreasingness. Moreover, by Step 1, there exists a 3-cell $A : f_1 \star_1 f_1' \Rrightarrow g_1 \star_1 g_1'$ in the $(3, 1)$-category $\mathcal{S}^{sd}(\Sigma, \psi)^{\top}$, where the confluence diagram $(f_1 \cdot f_1', g_1 \cdot g_1')$ is strictly decreasing.

The branchings (f_1', f_2) is confluent by decreasingness. Moreover, the 2-polygraph Σ being strictly decreasing, by Lemma 2.4.14, there exist rewriting sequences h and k as indicated in the following diagram:

$$
\begin{tikzcd}
& u_1 \arrow[dr, Rightarrow, "f_1'"] \arrow[drrr, Rightarrow, bend left, "f_2"] \arrow[dd, phantom, "\Downarrow A"] \arrow[drr, phantom, "B"] & & & \\
u \arrow[ur, Rightarrow, bend left, "f_1"] \arrow[dr, Rightarrow, bend right, "g_1"'] & & u' \arrow[r, Rightarrow, "h"] & {} & \hat{u} \arrow[l, Rightarrow, "k"'] \\
& v_1 \arrow[ur, Rightarrow, "g_1'"] \arrow[urrr, Rightarrow, bend right, "g_2"'] \arrow[urr, phantom, "C"] & & &
\end{tikzcd}
$$

such that the confluence diagrams $(f_1' \cdot h, f_2 \cdot k)$ is strictly decreasing.

Consider the multiset order $\prec_{mul}$ associated to the order $\prec$. The confluence diagram $(f_1 \cdot f_1', g_1 \cdot g_1')$ being strictly decreasing, for any k in $L^W(f_1')$ and any l in $L^W(g_1')$, we have $k \prec \psi(f_1)$ and $l \prec \psi(g_1)$. Thus $|f_1 \cdot f_1'| = |f_1|$ and $|g_1 \cdot g_1'| = |g_1|$. This implies the following equality

$$|(f_1, g_1)| = |(f, g)|.$$

The confluence diagram $(f_1' \cdot h, f_2 \cdot k)$ being strictly decreasing, by the same argument, we have

$$|(f_1' \cdot h, f_2 \cdot k)| = |(f_1', f_2)|.$$

Moreover, by Lemma 2.4.13, we have $|(f_1', f_2)| \prec_{mul} |(f, g_1)|$. It follows that

$$|(f_1' \cdot h, f_2 \cdot k)| \prec_{mul} |(f, g)|.$$

By induction hypothesis, we deduce that there exists a 3-cell $B : f_2 \star_1 k \Rrightarrow f_1' \star_1 h$ in $\mathcal{S}^{sd}(\Sigma, \psi)^{\top}$.

Finally, let us prove that there exists a 3-cell $C : g_1' \star_1 h \Rrightarrow g_2 \star_1 k$ in $\mathcal{S}^{sd}(\Sigma, \psi)^{\top}$. We have

$$|(g_1' \cdot h, g_2 \cdot k)| = |g_1'| \cup |h^{(g_1')}| \cup |g_2| \cup |k^{(g_2)}|.$$

On the other hand, we have

$$|(f, g)| = |f| \cup |g| = |f| \cup |g_1| \cup |g_2^{(g_1)}|.$$

Furthermore, there exists a multiset R, possibly empty, such that $|g_2| = |g_2^{(g_1)}| \cup R$. Hence

$$|(g_1' \cdot h, g_2 \cdot k)| = |g_2^{(g_1)}| \cup X \quad \text{and} \quad |(f, g)| = |g_2^{(g_1)}| \cup Y.$$

where $X = |g_1'| \cup |h^{(g_1')}| \cup R \cup |k^{(g_2)}|$ and $Y = |f| \cup |g_1|$. Moreover, we check that for every i in W such that $X(i) \neq 0$, there exists j in W such that $Y(j) \neq 0$ and $i \prec j$. Hence, we have

$$|(g_1' \cdot h, g_2 \cdot k)| \prec_{mul} |(f, g)|.$$

The existence of the 3-cell C follows by induction hypothesis. In this way, we have constructed a 3-cell in $\mathcal{S}^{sd}(\Sigma, \psi)^\top$ from f to g obtained by composition of the 3-cells A, B and C. $\square$

3.1.5. Example. The 2-polygraph $\Sigma(\mathbf{B}_3^+)$ given in 2.1.9 is strictly decreasing for the labelling QNF ψ^{QNF} defined in 2.4.11. It has four confluent critical branchings. Thus, a strictly decreasing Squier's completion of the 2-polygraph $\Sigma(\mathbf{B}_3^+)$ is given by the following 3-cells:

$$\begin{tikzcd}
 & tst^2 \arrow[dr, Rightarrow, "\beta t"] \arrow[dd, Rightarrow, "D^{\psi^{QNF}}_{t\alpha,\beta s}"] & \\
stst \arrow[ur, Rightarrow, "\alpha t"] \arrow[dr, Rightarrow, "s\beta"'] & & stst \\
 & s^2ts \arrow[ur, Rightarrow, "s\alpha"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & sts^2 \arrow[dd, Rightarrow, "D^{\psi^{QNF}}_{\beta s,t\alpha}"] & \\
tsts \arrow[ur, Rightarrow, "\beta s"] \arrow[dr, Rightarrow, "t\alpha"'] & & tsts \arrow[ul, Rightarrow, "\beta s"'] \\
 & t^2st \arrow[ur, Rightarrow, "t\beta"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & tst^2s \arrow[dr, Rightarrow, "\beta ts"] \arrow[dd, Rightarrow, "D^{\psi^{QNF}}_{\alpha ts,st\alpha}"] & \\
ststs \arrow[ur, Rightarrow, "\alpha ts"] \arrow[dr, Rightarrow, "st\alpha"'] & & ststs \\
 & st^2st \arrow[ur, Rightarrow, "st\beta"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & sts^2t \arrow[dd, Rightarrow, "D^{\psi^{QNF}}_{\beta st,ts\beta}"] & \\
tstst \arrow[ur, Rightarrow, "\beta st"] \arrow[dr, Rightarrow, "ts\beta"'] & & tstst \arrow[ul, Rightarrow, "\beta st"'] \\
 & ts^2ts \arrow[ur, Rightarrow, "ts\alpha"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & tst \arrow[dr, Rightarrow, "\beta"] \arrow[d, Rightarrow, "E_{\alpha,\beta}"] & \\
sts \arrow[ur, Rightarrow, "\alpha"] \arrow[rr, Rightarrow, "1_{sts}"'] & {} & sts
\end{tikzcd}$$

where $D^{\psi^{QNF}}_{t\alpha,\beta s}$, $D^{\psi^{QNF}}_{\beta s,t\alpha}$, $D^{\psi^{QNF}}_{\alpha ts,st\alpha}$ and $D^{\psi^{QNF}}_{\beta st,ts\alpha}$ are the generating decreasing confluences and $E_{\alpha\star_1\beta}$ is an elementary 2-loop of Σ. Each of these confluences is decreasing because:

$$\psi^{QNF}(\alpha t) = \psi^{QNF}(s\beta) = 1 \quad \text{and} \quad \psi^{QNF}(\beta t) = \psi^{QNF}(s\alpha) = 0,$$
$$\psi^{QNF}(\beta s) = 0,\ \psi^{QNF}(t\alpha) = 2 \quad \text{and} \quad \psi^{QNF}(t\beta) = 1, \psi^{QNF}(\beta s) = 0,$$
$$\psi^{QNF}(\alpha ts) = \psi^{QNF}(st\alpha) = 1 \quad \text{and} \quad \psi^{QNF}(\beta ts) = \psi^{QNF}(st\beta) = 0,$$
$$\psi^{QNF}(\beta st) = 0,\ \psi^{QNF}(ts\beta) = 2 \quad \text{and} \quad \psi^{QNF}(ts\alpha) = 1, \psi^{QNF}(\beta st) = 0.$$

3.2. Coherence by decreasingness

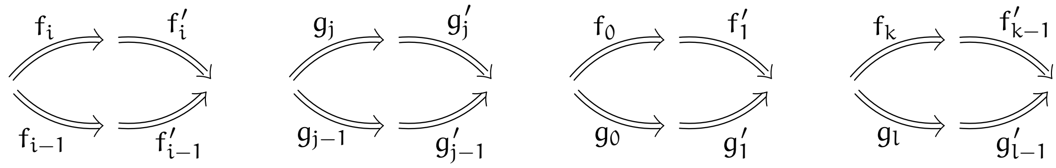
The following theorem is the main result of this article.

3.2.1. Theorem. *Let (Σ, ψ) be a strictly decreasing 2-polygraph. Let $\mathcal{S}^{sd}(\Sigma, \psi)$ be a strictly decreasing Squier's completion of Σ. If ψ is compatible with contexts and (Σ, ψ) is Peiffer decreasing with respect to the extension $\mathcal{S}^{sd}(\Sigma, \psi)$, then $\mathcal{S}^{sd}(\Sigma, \psi)$ is a coherent presentation of the category presented by Σ.*

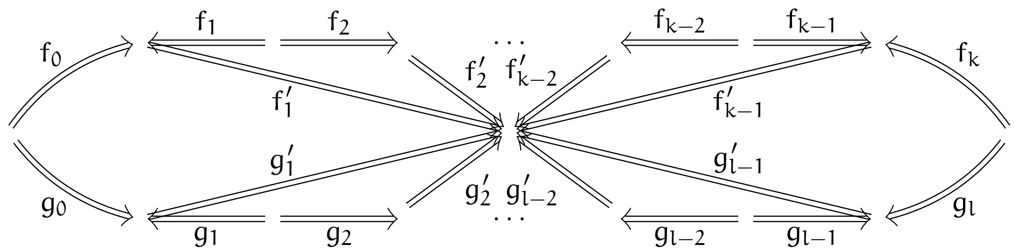
Proof. Let (f, g) be a 2-sphere of the $(2,1)$-category $\Sigma_2^\top$. By definition of $\Sigma_2^\top$, the 2-cell $f \star_1 g^-$ can be decomposed into a zigzag

$$\begin{tikzcd}
 & {} & {} \arrow[l, Rightarrow, "f_1"'] \arrow[r, Rightarrow, "f_2"] & {} & \cdots & {} & {} \arrow[l, Rightarrow, "f_{k-2}"'] \arrow[r, Rightarrow, "f_{k-1}"] & {} & \\
{} \arrow[ur, Rightarrow, "f_0"] \arrow[dr, Rightarrow, "g_0"'] & & & & & & & & {} \arrow[ul, Rightarrow, "f_k"'] \arrow[dl, Rightarrow, "g_l"] \\
 & {} & {} \arrow[l, Rightarrow, "g_1"] \arrow[r, Rightarrow, "g_2"'] & {} & \cdots & {} & {} \arrow[l, Rightarrow, "g_{l-2}"] \arrow[r, Rightarrow, "g_{l-1}"'] & {} &
\end{tikzcd}$$

where the 2-cells $f_0, \ldots, f_k$ and $g_0, \ldots, g_l$ are 2-cells of the 2-category Σ_2^*. Note that some of those 2-cells can be identities. By confluence of the 2-polygraph Σ, there exist families of 2-spheres of Σ_2^*



with same 1-target, for all $2 \leqslant i \leqslant k-1$ and $2 \leqslant j \leqslant l-1$. Note that some of these 2-spheres can be trivial. Then the 2-sphere (f, g) can be filled up by these 2-spheres as follows:



By Lemma 3.1.4, these 2-spheres can be filled up by 3-cells of the $(3,1)$-category $\mathcal{S}^{sd}(\Sigma, \psi)^\top$. Finally, the composition of these 3-cells gives a 3-cell of $\mathcal{S}^{sd}(\Sigma, \psi)^\top$ from f to g. $\square$

Strict decreasingness is a required condition in Theorem 3.2.1 as shown by the following example.

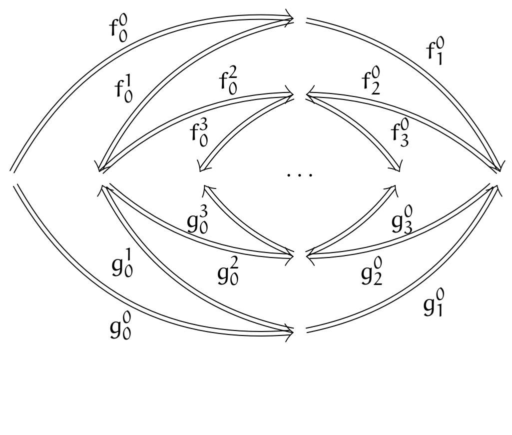
3.2.2. Example. Consider the 2-polygraph Σ without 2-loop and containing two families $(f_j^i)_{i,j \in \mathbb{N}, ij=0}$ and $(g_j^i)_{i,j \in \mathbb{N}, ij=0}$ of 2-cells satisfying the following conditions:

i) the sequences $(f_n^0)_{n \in \mathbb{N}}$, $(f_0^n)_{n \in \mathbb{N}}$, $(g_n^0)_{n \in \mathbb{N}}$ and $(g_0^n)_{n \in \mathbb{N}}$ are infinite rewriting paths,

ii) for any odd integer n, we have $t_1(f_0^n) = t_1(g_0^n)$ and $t_1(f_n^0) = t_1(g_n^0)$,

iii) for any even integer n, we have $t_1(f_0^n) = t_1(f_n^0)$ and $t_1(g_0^n) = t_1(g_n^0)$,

as indicated in the following diagram



and such that the only critical branchings of Σ are of one of the following forms:

$$(f_0^n, g_0^n),\ (f_n^0, g_n^0),\ \text{for } n \text{ even, and } (f_n^0, f_0^n),\ (g_n^0, g_0^n),\ \text{for } n \text{ odd.}$$

Let us consider the globular extension Γ of the free $(2,1)$-polygraph $\Sigma_2^\top$, defined by the following infinite family of 2-spheres:

$$(f_0^n \star_1 f_0^{n+1}, f_n^0 \star_1 f_{n+1}^0) \quad \text{and} \quad (g_0^n \star_1 g_0^{n+1}, g_n^0 \star_1 g_{n+1}^0) \quad \text{for } n \text{ odd,}$$

and

$$(f_0^n \star_1 f_0^{n+1}, g_0^n \star_1 g_0^{n+1}) \quad \text{and} \quad (f_n^0 \star_1 f_{n+1}^0, g_n^0 \star_1 g_{n+1}^0) \quad \text{for } n \text{ even.}$$

The globular extension Γ contains one generating confluence for each critical branching of Σ. However, we cannot define a 3-cell in the free $(3,1)$-category generated by (Σ, Γ) with 2-source $f_0^0 \star_1 f_1^0$ and 2-target $g_0^0 \star_1 g_1^0$. As a consequence, Γ does not form a homotopy basis of the $(2,1)$-category $\Sigma_2^\top$. In fact, we note that the 2-polygraph Σ is not strictly decreasing, because no labelling of Σ is well-founded, but decreasing with the singleton labelling.

Following 2.4.3, any quasi-convergent 2-polygraph Σ is strictly decreasing with respect to any quasi-normal form labelling ψ^{QNF}. The following result is a consequence of Theorem 3.2.1.

3.2.3. Corollary. *Let Σ be a quasi-convergent 2-polygraph and let ψ^{QNF} be a quasi-normal form labelling of Σ. Let $\mathcal{S}^{sd}(\Sigma, \psi^{\mathrm{QNF}})$ be a strictly decreasing Squier's completion of Σ. If the labelling ψ^{QNF} is compatible with contexts and $(\Sigma, \psi^{\mathrm{QNF}})$ is Peiffer decreasing with respect to the extension $\mathcal{S}^{sd}(\Sigma, \psi^{\mathrm{QNF}})$, then $\mathcal{S}^{sd}(\Sigma, \psi^{\mathrm{QNF}})$ is a coherent presentation of the category presented by Σ.*

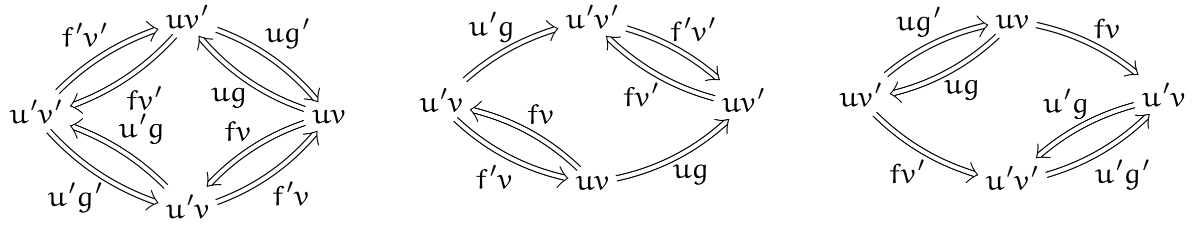
3.2.4. Example. By Theorem 3.2.1, the five 3-cells given in 3.1.5 form a homotopy basis of the 2-polygraph $\Sigma(\mathbf{B}_3^+)$. Indeed, the 2-polygraph $\Sigma(\mathbf{B}_3^+)$ is strictly decreasing for the labelling QNF defined in Example 2.4.11. This labelling being compatible with contexts, the only remaining point concerns the Peiffer confluences. Let us show that any Peiffer confluence is equivalent to a decreasing confluence diagram. Consider a Peiffer branching $(fv, ug) : uv \Rightarrow (u'v, uv')$ of $\Sigma(\mathbf{B}_3^+)$ and its Peiffer confluence $(u'g, fv') : (u'v, uv') \Rightarrow u'v'$:

$$\begin{tikzcd}[row sep=small] & u'v \arrow[dr, Rightarrow, "u'g"] & \\ uv \arrow[ur, Rightarrow, "fv"] \arrow[dr, Rightarrow, "ug"'] & & u'v' \\ & uv' \arrow[ur, Rightarrow, "fv'"'] & \end{tikzcd}$$

By definition of $\Sigma(\mathbf{B}_3^+)$, there exist rewriting steps $f' : u' \Rightarrow u$ and $g' : v' \Rightarrow v$. It follows that this Peiffer confluence is equivalent with respect to $\mathcal{L}(\Sigma(\mathbf{B}_3^+))$ to each of the following Peiffer confluence:

$$(f'v' \cdot ug', u'g' \cdot f'v), \qquad (u'g \cdot f'v', f'v \cdot ug), \qquad (ug' \cdot fv, fv' \cdot u'g').$$

The equivalences are proved by the following diagrams:



Finally, in each family of such four Peiffer confluences, one of them is decreasing with respect to the labelling ψ^{QNF}.

3.2.5. Decreasingness from termination. Given a confluent and terminating 2-polygraph Σ, any 1-cell u of Σ_1^* has a unique normal form denoted by $\widehat{u}$. We define the *labelling to the normal form* $\psi^{NF} : \Sigma_{stp} \to \Sigma_1^*$ by setting for each rewriting step f, $\psi^{NF}(f) = t_1(f)$. We choose on Σ_1^* the order induced by the rewrite relation defined by Σ_2. This labelling is compatible with contexts and makes the 2-polygraph Σ strictly decreasing and Peiffer decreasing. Moreover, Σ being terminating it does not have loop and in particular the decreasing Squier completion coincides with the Squier completion. In this way, the Squier coherence theorem obtained for convergent string rewriting systems in [17] is a consequence of Theorem 3.2.1:

3.2.6. Corollary ([17, Theorem 5.2]). *Let Σ be a convergent 2-polygraph. Any Squier's completion $\mathcal{S}(\Sigma)$ of Σ is a coherent presentation of the category presented by Σ.*

3.3. Finiteness homotopical and homological conditions by decreasingness

3.3.1. Finite derivation type. A 2-polygraph Σ has *finite derivation type*, FDT for short, if the free $(2,1)$-category $\Sigma_2^\top$ has a finite homotopy basis, see [7, Section 4]. Squier proved that this property is invariant for finite string rewriting systems: if Σ and Υ are two finite 2-polygraphs, then Σ has FDT if and only if Υ has FDT. As a consequence, the property can be defined on finitely presented monoids: a finitely presented monoid has FDT if it has a presentation by a 2-polygraph that has FDT.

For a convergent 2-polygraph Σ, its is well known that a family of generating confluences forms a homotopy basis of $\Sigma_2^\top$. A finite convergent 2-polygraph having a finite number of critical branchings, then it has FDT. However, a finite decreasing 2-polygraph can have an infinite decreasing Squier's completion. Indeed, the set of decreasing confluences is always finite for a finite 2-polygraph but the set of elementary 2-loops may be infinite. As a consequence of Theorem 3.2.1 we can formulate the following result.

3.3.2. Proposition. *Let (Σ, ψ) be a strictly decreasing and quasi-convergent 2-polygraph such that the labelling ψ is compatible with contexts and Peiffer decreasing. If Σ has a finite set of 2-cells and a finite set of elementary 2-loops, then it has finite derivation type.*

3.3.3. Example. Let us consider the 2-polygraph Σ with only one 0-cell, $\Sigma_1 = \{a, b, c, d, d'\}$ and $\Sigma_2 = \{ab \Rightarrow a, ac \Rightarrow da, da \Rightarrow d'a, d'a \Rightarrow ac\}$. This 2-polygraph presents a monoid which has not FDT, see [12, Section 5]. Moreover, it has only one elementary 2-loop up to equivalence and a finite number of critical branchings. As a consequence, there is no well-founded labelling compatible with contexts making the 2-polygraph Σ strictly decreasing and Peiffer decreasing.

3.3.4. Finite homological type FP_3 by decreasingness. As a final remark, let us mention another application to computation of low-dimensional homological properties of monoids. Let $\mathbf{M}$ be a monoid and Σ be a coherent presentation of $\mathbf{M}$. Following [7, Proposition 5.3.2.], there is a partial resolution

$$\mathbb{Z}\mathbf{M}[\Sigma_3] \xrightarrow{d_3} \mathbb{Z}\mathbf{M}[\Sigma_2] \xrightarrow{d_2} \mathbb{Z}\mathbf{M}[\Sigma_1] \xrightarrow{d_1} \mathbb{Z}\mathbf{M} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

of left-modules over the free ring $\mathbb{Z}\mathbf{M}$ over $\mathbf{M}$, where $\mathbb{Z}$ denotes the trivial $\mathbb{Z}\mathbf{M}$-module and $\mathbb{Z}\mathbf{M}[\Sigma_i]$ denotes the free $\mathbb{Z}\mathbf{M}$-module generated by Σ_i. The morphisms of $\mathbb{Z}\mathbf{M}$-modules are defined by $\varepsilon(u) = 1$, for any u in $\mathbf{M}$, and d_1, d_2 and d_3 are defined on the generators by

$$d_1(x) = x - 1, \qquad d_2(\alpha) = [s_1(\alpha)] - [t_1(\alpha)], \qquad d_3(A) = [s_2(A)] - [t_2(A)],$$

for any x in Σ_1, α in Σ_2 and A in Σ_3 , and with the bracket notations of [7, Section 5].

In particular, by Theorem 3.2.1, if (Σ, ψ) is a strictly decreasing 2-polygraph such that ψ is compatible with contexts and Peiffer decreasing, the coherent presentation given by the strictly decreasing Squier completion $\mathcal{S}^{sd}(\Sigma, \psi)$ induces such a partial resolution. If moreover Σ has a finite set of 2-cells and a finite set of elementary 2-loops, then it has finite homological type FP_3. We expect that our construction can be extended in higher-dimension of homology producing infinite lenght resolutions for monoids presented by quasi-convergent presentations, and thus weakening the termination hypothesis required in construction of such resolutions as in [1, 11].

3.3.5. Example. Following Example 3.2.4, the monoid $\mathbf{B}_3^+$, admits a coherent presentation with two 1-cells s and t, two 2-cells $\alpha : sts \Rightarrow tst$ and $\beta : tst \Rightarrow sts$ and the five 3-cells $D^{\psi^{QNF}}_{t\alpha,\beta s}$, $D^{\psi^{QNF}}_{\beta s,t\alpha}$, $D^{\psi^{QNF}}_{\alpha ts,st\alpha}$, $D^{\psi^{QNF}}_{\beta st,ts\beta}$, $E_{\alpha,\beta}$. Using the homotopical reduction procedure introduced in [4, 2.3.1.] with a collapsible part made of the 3-cell $E_{\alpha,\beta}$, we can reduce this coherent presentation to a coherent presentation of the monoid $\mathbf{B}_3^+$ with the same k-cells for $k \leqslant 2$ and with no 3-cells. Hence, we obtain the following resolution

$$0 \longrightarrow \mathbb{Z}\mathbf{M}[\alpha, \beta] \xrightarrow{d_2} \mathbb{Z}\mathbf{M}[s, t] \xrightarrow{d_1} \mathbb{Z}\mathbf{M} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

We deduce the homology of the monoid $\mathbf{B}_3^+$ with integral coefficients: $H_n(\mathbf{M}, \mathbb{Z}) = \mathbb{Z}$ for $n \leqslant 2$ and $H_n(\mathbf{M}, \mathbb{Z}) = 0$, for $n \geqslant 3$.

References

[1] David J. Anick. On the homology of associative algebras. *Trans. Amer. Math. Soc.*, 296(2):641–659, 1986.

[2] Nachum Dershowitz. Termination of rewriting. *J. Symbolic Comput.*, 3(1-2):69–115, 1987. Rewriting techniques and applications (Dijon, 1985).

[3] Nachum Dershowitz and Zohar Manna. Proving termination with multiset orderings. *Comm. ACM*, 22(8):465–476, 1979.

[4] Stéphane Gaussent, Yves Guiraud, and Philippe Malbos. Coherent presentations of Artin monoids. *Compos. Math.*, 151(5):957–998, 2015.

[5] Yves Guiraud and Philippe Malbos. Coherence in monoidal track categories. *Math. Structures Comput. Sci.*, 22(6):931–969, 2012.

[6] Yves Guiraud and Philippe Malbos. Higher-dimensional normalisation strategies for acyclicity. *Adv. Math.*, 231(3-4):2294–2351, 2012.

[7] Yves Guiraud and Philippe Malbos. Polygraphs of finite derivation type. *Mathematical Structures in Computer Science*, pages 1–47, 009 2016.

[8] Nohra Hage and Philippe Malbos. Knuth's Coherent Presentations of Plactic Monoids of Type A. ArXiv e-prints 1609.01460, 2016.

[9] Deepak Kapur and Paliath Narendran. A finite Thue system with decidable word problem and without equivalent finite canonical system. *Theoret. Comput. Sci.*, 35(2-3):337–344, 1985.

[10] Vesna Kilibarda. On the algebra of semigroup diagrams. *Internat. J. Algebra Comput.*, 7(3):313–338, 1997.

[11] Yuji Kobayashi. Complete rewriting systems and homology of monoid algebras. *J. Pure Appl. Algebra*, 65(3):263–275, 1990.

[12] Yves Lafont. A new finiteness condition for monoids presented by complete rewriting systems (after Craig C. Squier). *J. Pure Appl. Algebra*, 98(3):229–244, 1995.

[13] Maxwell Newman. On theories with a combinatorial definition of "equivalence". *Ann. of Math. (2)*, 43(2):223–243, 1942.

[14] Stephen J. Pride. Low-dimensional homotopy theory for monoids. *Internat. J. Algebra Comput.*, 5(6):631–649, 1995.

[15] Craig Squier and Friedrich Otto. The word problem for finitely presented monoids and finite canonical rewriting systems. In *Rewriting techniques and applications (Bordeaux, 1987)*, volume 256 of *Lecture Notes in Comput. Sci.*, pages 74–82. Springer, Berlin, 1987.

[16] Craig C. Squier. Word problems and a homological finiteness condition for monoids. *J. Pure Appl. Algebra*, 49(1-2):201–217, 1987.

[17] Craig C. Squier, Friedrich Otto, and Yuji Kobayashi. A finiteness condition for rewriting systems. *Theoret. Comput. Sci.*, 131(2):271–294, 1994.

[18] Vincent van Oostrom. Confluence by decreasing diagrams. *Theoret. Comput. Sci.*, 126(2):259–280, 1994.

CLÉMENT ALLEAUME
`clement.alleaume@univ-st-etienne.fr`
Univ Lyon, Université Claude Bernard Lyon 1
CNRS UMR 5208, Institut Camille Jordan
43 blvd. du 11 novembre 1918
F-69622 Villeurbanne cedex, France

PHILIPPE MALBOS
`malbos@math.univ-lyon1.fr`
Univ Lyon, Université Claude Bernard Lyon 1
CNRS UMR 5208, Institut Camille Jordan
43 blvd. du 11 novembre 1918
F-69622 Villeurbanne cedex, France

— January 29, 2017 - 10:42 —