Coherent confluence modulo relations and double groupoids

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Abstract – A coherent presentation of an n-category is a presentation by generators, relations and relations among relations. Confluent and terminating presentations by rewriting systems give coherent presentations, whose relations among relations are generated by confluence diagrams of critical branchings. This article presents a procedure that computes coherent presentations from relations defined modulo a set of axioms. Our coherence results are formulated using the structure of n-category enriched in double groupoids, whose horizontal cells represent rewriting sequences, vertical cells represent the congruence generated by the axioms and square cells represent coherence cells induced by diagrams of confluence modulo. We illustrate these constructions on rewriting modulo commutation relations in commutative monoids and isotopy relations in pivotal monoidal categories.

Keywords – Rewriting modulo, double categories, coherence of higher categories.

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Introduction

Algebraic rewriting aims at giving constructive methods based on rewriting theory to obtain properties of higher algebraic structures presented by generators and relations. This approach has been used to compute linear bases, coherent presentations or higher-syzygies. In many situations, the presentations have a great complexity due to the number of generators and relations. This is particularly the case for diagrammatic algebras, such as Temperley-Lieb algebras [37], Brauer algebras [2], Birman-Wenzl algebras [34], Jones' planar algebras [22], Khovanov-Lauda-Rouquier algebras and the associated categorifications of quantum groups [26, 33], and Khovanov's categorification of the Heisenberg algebra [7, 27]. For some of these algebraic structures, there is a huge number of relations, leading to a combinatorial explosion of cases in the proof of the confluence properties of their presentations by rewriting systems. However, many of these relations are inherent to the algebraic structure itself. For instance, some of the above algebras can be interpreted as linear 2-categories with the additional structure of a pivotal 2-category in which all the string diagrams representing 2-cells are drawn up by isotopy. The relations arising from the algebraic structure create obstructions to prove local rewriting properties. To deal with this problem, we develop a rewriting theoretical approach based on rewriting modulo the axioms of the algebraic structure. In this article, we tackle the problem of computing coherent presentations modulo a set of axioms.

Coherence and rewriting modulo

Coherence by confluence. A *coherent presentation* of a 1-category extends the notion of presentation of the category by globular cells that generate all the 2-syzygies of the presentation. Explicitly, a coherent presentation is defined by a directed graph X, a set R of globular relations on the free category on X, and an acyclic set Γ of 2-spheres of the free $(2,1)$-category $R^\top$ generated by the presentation (X,R). The acyclicity property means that the quotient of the $(2,1)$-category $R^\top$ by the congruence generated by Γ is aspherical. This notion of coherent presentation extends to $(n-1)$-categories, for $n>1$, presented by n-polygraphs: a coherent presentation is an n-polygraph P_n extended by an acyclic cellular extension P_{n+1} of the free $(n,n-1)$-category on P_n. When the n-polygraph is convergent, that is confluent and terminating, it can be extended into a coherent presentation by adding generating $(n+1)$-cells defined by a family of confluence diagrams of the form

$$\begin{tikzcd}
& v \arrow[dr, dotted, "f'"] \arrow[dd, phantom, "\Downarrow A_{f,g}"] & \\
u \arrow[ur, bend left, "f"] \arrow[dr, bend right, "g"'] & & w \\
& v' \arrow[ur, dotted, "g'"'] &
\end{tikzcd}$$

for every critical branching (f,g) of the n-polygraph P_n. Coherent presentations constructed in this way generalize rewriting systems by keeping track of the cells generated by confluence diagrams. This construction was initiated by Squier in [35] for monoids and generalized to n-categories in [17].

Rewriting modulo. The aim of this article is to extend these constructions to rewriting systems defined modulo a set of fixed relations. Rewriting modulo appears naturally in algebraic rewriting when studied reductions are defined modulo the axioms of an ambiant algebraic structure, eg. rewriting in commutative, groupoidal, linear, pivotal, weak structures. Furthermore, rewriting modulo facilitates the analysis of confluence. In particular, rewriting modulo a set of relations makes the property of confluence easier to prove for two reasons:

— the family of critical branchings that should be considered in the analysis of coherence by confluence is reduced,

— non-orientation of a part of the relations allows more flexibility when reaching confluence.

In Section 3 we introduce the notion of n-*polygraph modulo* as a data (R, E, S) made of an n-polygraph R, whose generating n-cells are called *primary rules*, an n-polygraph E such that $E_k = R_k$ for $k \leqslant n-2$ and $E_{n-1} \subseteq R_{n-1}$, whose generating n-cells are called *modulo rules*, and S is a cellular extension of R^*_{n-1} depending on both cellular extensions R_n and E_n. In this way, a presentation modulo is split into two parts: oriented rules in the set R_n and non-oriented equations in the set E_n.

The most naive approach of rewriting modulo is to consider the rewriting system ${}_E R_E$ consisting in rewriting on congruence classes modulo E. This approach works for some equational theories, such as associative and commutative theory. However, it appears inefficient in general for the analysis of confluence. Indeed, the reducibility of an equivalence class needs to explore all the class, hence it requires all equivalence classes to be finite. Another approach of rewriting modulo has been considered by Huet in [21], where rewriting sequences involve only oriented rules and no equivalence steps, and the confluence property is formulated modulo equivalence. However, for algebraic rewriting systems such rewriting modulo is too restrictive for computations, see [24]. Peterson and Stickel introduced in [31] an extension of Knuth-Bendix's completion procedure, [28], to reach confluence of a rewriting system modulo an equational theory, for which a finite, complete unification algorithm is known. They applied their procedure to rewriting systems modulo axioms of associativity and commutation, in order to rewrite in free commutative groups, commutative unitary rings, and distributive lattices. Jouannaud and Kirchner enlarged this approach in [23] with the definition of rewriting properties for any rewriting system modulo S such that $R \subseteq S \subseteq {}_E R_E$. They also proved a critical branching lemma and developed a completion procedure for a rewriting system modulo ${}_E R$, whose one-step reductions consist in application of a rule in R using E-matching. Their completion procedure is based on a finite E-unification algorithm. Bachmair and Dershowitz in [1] developed a generalization of Jouannaud-Kirchner's completion procedure using inference rules. Several other approaches have also been studied for term rewriting systems modulo to deal with various equational theories, see [29, 38].

In Section 3, we define termination property for polygraphs modulo and we recall from [21] Huet's principle of double induction, that we use in many proofs in this article. We define confluence properties on polygraphs modulo, as introduced by Huet in [21] and by Jouannaud and Kirchner in [23], and we present a completion procedure for the n-polygraph modulo ${}_E R$ in terms of critical branchings that implements inference rules for completion modulo given by Bachmair and Dershowitz in [1].

Confluence modulo and double categories. We formulate the notion of coherence modulo for an $(n-1)$-category using the structure of $(n-1)$-category enriched in double groupoids. The notion of double category was first introduced by Ehresmann in [15] as an internal category in the category of categories. The notion of double groupoids, that is internal groupoids in the category of groupoids, and its higher-dimensional versions have been widely used in homotopy theory, [4, 6], see [5] and [3] for a complete account on the theory. A double category gives four related categories: a vertical category, an horizontal category and two categories of squares with either vertical or horizontal cells as sources and targets. A square cell A is pictured by

$$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, "e"'] \arrow[dr, phantom, "\Downarrow A"] & v \arrow[d, "e'"] \\ u' \arrow[r, "g"'] & v' \end{tikzcd}$$

where f, g are horizontal cells, and e, e' are vertical cells. In [17], rewriting sequences with respect to an n-polygraph are interpreted by n-cells in the free category generated by the polygraph. In Section 3, we give an interpretation of confluence modulo for an n-polygraph modulo (R, E, S) in a free $(n-1)$-category enriched in double categories, where the horizontal cells are the n-cells of the free n-category S^* generated by the n-polygraph S and the vertical cells are the n-cells of the free $(n, n-1)$-category $E^\top$ generated by the n-polygraph E. In this way, horizontal cells represent rewriting sequences modulo, vertical cells represent the congruence generated by E and square cells represent *coherence cells modulo*. We define a *branching modulo* E of an n-polygraph modulo (R, E, S) as a triple (f, e, g), where f and g are n-cells of S^* and e is an n-cell of $E^\top$, that we picture as follows

$$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, "e"'] & u' \\ v \arrow[r, "g"'] & v' \end{tikzcd}$$

Such a branching is *confluent modulo* E if there exist n-cells f' and g' in S^* and an n-cell e' in $E^\top$ as in the following diagram:

$$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, "e"'] & u' \arrow[r, dotted, "f'"] & u'' \arrow[d, dotted, "e'"] \\ v \arrow[r, "g"'] & v' \arrow[r, dotted, "g'"'] & w'' \end{tikzcd}$$

Double coherence from confluence

Coherent confluence modulo. The notion of coherent presentation modulo introduced in this article is based on an adaptation of the structure of polygraph known in the globular setting, [9, 32, 36], to a cubical setting. We define a *double* $(n+1, n-1)$*-polygraph* as a data $P = (P^v, P^h, P^s)$ made of two n-polygraphs P^v and P^h with the same underlying $(n-1)$-polygraph, and a square extension P^s made of generating squares of the form

$$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, "e"'] & u' \arrow[d, "e'"] \\ v \arrow[r, "g"'] & v' \end{tikzcd}$$

where f, g are n-cells of the free $(n, n-1)$-category $(P^v)^\top$ generated by P^v and e, e' are n-cells of the free $(n, n-1)$-category $(P^h)^\top$ generated by P^h. We define a *double coherent presentation of an*

$(n-1)$-*category* $\mathcal{C}$ as a double $(n+1, n-1)$-polygraph $P = (P^v, P^h, P^s)$ such that the coproduct of the polygraphs P^v and P^h is a presentation of the category $\mathcal{C}$ and that the square extension P^s is acyclic, that is for any square S constructed on the vertical $(n, n-1)$-category $(P^v)^\top$ and the horizontal $(n, n-1)$-category $(P^h)^\top$, there exists a square $(n+1)$-cell A in the free $(n-1)$-category $P^{\top\!\top}$ enriched in double groupoids generated by P, defined in Subsection 2.2, whose boundary is S.

In Section 4, we define the notion of confluence modulo of an n-polygraph modulo (R, E, S) with respect to a square extension Γ of the pair of n-categories $(E^\top, S^*)$. Explicitly, we say that S is Γ-*confluent modulo* E if for any branching (f, e, g) of S modulo E, there exist n-cells f', g' in S^*, e' in $E^\top$ and an $(n+1)$-cell

$$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, "e"'] & u' \arrow[r, dotted, "f'"] \arrow[d, phantom, "\Downarrow A"] & w \arrow[d, dotted, "e'"] \\ v \arrow[r, "g"'] & v' \arrow[r, dotted, "g'"'] & w' \end{tikzcd}$$

in the free $(n-1)$-category enriched in double categories generated by the square extension Γ and an action of E on Γ as defined in Subsection 4.1. We deduce coherent confluence of an n-polygraph modulo from local coherent confluence properties. In particular, Theorem 4.1.4 is a formulation of the Newman lemma for confluence modulo. Explicitly, given an n-polygraph modulo (R, E, S) such that ${}_E R_E$ is terminating, and Γ a square extension of $(E^\top, S^*)$, when S is locally Γ-confluent modulo E, we prove that S is Γ-confluent modulo E. Finally, with Theorem 4.2.2 we give a coherent formulation of the critical branching lemma modulo, deducing coherent local confluence from coherent confluence of some critical branchings modulo.

Coherent completion modulo. In Section 5, we present several ways to extend a presentation of an $(n-1)$-category by a polygraph modulo into a double coherent presentation of this category. Starting with an n-polygraph modulo, we show how to construct a double coherent presentation of the $(n-1)$-category presented by this polygraph. Theorem 5.2.2 gives conditions for an n-polygraph modulo (R, E, S) to extend a square extension Γ on the vertical and horizontal $(n, n-1)$-categories $E^\top$ and $S^\top$ into an acyclic extension.

Recall from [17] that a convergent n-polygraph E can be extended into a coherent globular presentation of the category it presents by considering a family of generating confluences of E as a cellular extension of the free $(n, n-1)$-category $E^\top$ that contains exactly one globular $(n+1)$-cell

$$\begin{tikzcd} & v \arrow[dr, bend left, "e_1"] \arrow[dd, phantom, "\Downarrow E_{e,e'}"] & \\ u \arrow[ur, bend left, "e"] \arrow[dr, bend right, "e'"'] & & w \\ & v' \arrow[ur, bend right, "e'_1"'] & \end{tikzcd}$$

for every critical branching (e, e') of E, where (e_1, e'_1) is a chosen confluence. Any $(n+1, n)$-polygraph obtained from E by adjunction of a chosen family of generating confluences of E is a globular coherent presentation of the $(n-1)$-category $\overline{E}$, [17]. In Subsection 5.1, we define a *coherent completion* of an n-polygraph modulo (R, E, S) as a square extension of the pair of $(n+1, n)$-categories $(E^\top, S^\top)$ whose elements are the generating square $(n+1)$-cells

$$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, "e"'] & u' \arrow[r, "f'"] \arrow[d, phantom, "\Downarrow"] & w \arrow[d, "e'"] \\ u \arrow[r, "g"'] & v \arrow[r, "g'"'] & w' \end{tikzcd}$$

for any critical branchings (f, e, g) of S modulo E. As a consequence of Theorem 5.2.2, we show how to extend a coherent completion Γ of S modulo E and a coherent completion Γ_E of E into an acyclic extension. In particular, when E is empty, we recover Squier's coherence theorem for convergent n-polygraphs as given in [17, Theorem 5.2.], see also [18].

We prove that an acyclic extension of a pair $(E^{\top}, S^{\top})$ of $(n, n-1)$-categories coming from a polygraph modulo (R, E, S) can also be obtained from an assumption of commuting normalization strategies for the polygraphs S and E. We say that a normalization strategy σ with respect to S *weakly commute* with a normalization strategy ρ with respect to E if for any $(n-1)$-cell u in R^*_{n-1}, there exists an n-cell η_u in S^* as in the following diagram:

$$\begin{tikzcd} u \arrow[r, "\sigma_u"] \arrow[d, "\rho_u"'] & \hat{u} \arrow[d, "\rho_{\hat{u}}"] \\ \widetilde{u} \arrow[r, dotted, "\eta_u"'] & \widetilde{\hat{u}} \end{tikzcd}$$

Theorem 5.3.3 explains how to construct an acyclic extension for an n-polygraph modulo (R, E, S) when there exist commuting normalization strategies for the polygraphs S and E.

Globular coherence from double coherence. In the last section, we explain how to deduce a globular coherent presentation for an n-category from a double coherent presentation generated by a polygraph modulo. Our construction is based on the structure of *dipolygraph* as a presentation by generators and relations for ∞-categories whose underlying k-categories are not necessarily free, defined in Subsection 2.1. We define an $(n+2, n)$-*dipolygraph* as a variation of the notion of $(n+2, n)$-polygraph for which the globular extensions are defined on quotient categories. We define in Subsection 2.3 a quotient functor

$$\mathsf{V} : \mathsf{DbPol}_{(n+2,n)} \to \mathsf{DiPol}_{(n+2,n)}$$

from the category of double $(n+2, n)$-polygraph to the category of $(n+2, n)$-dipolygraphs.

The last result of this article gives conditions to quotient a double coherent presentation generated by a polygraph modulo. Explicitly, given an n-polygraph modulo (R, E, S) such that E is convergent, S is convergent modulo E, and $\mathrm{Irr}(E)$ is E-normalizing with respect to S, Theorem 6.1.2 shows how to deduce from a coherent completion Γ of S modulo E a globular coherent presentation of the $(n-1)$-category $(R^*_{n-1})_E$, whose generating n-cells are defined by quotient of n-cells of Γ by the cellular extension E. Finally, we illustrate this method by showing how to construct coherent presentations for commutative monoids in Subsection 6.2 and for pivotal monoidal categories modulo isotopy relations defined by adjunction in Subsection 6.3.

Organization of the article

In Section 1, we set up notations and terminology on higher-dimensional globular n-categories and globular n-polygraphs. We refer the reader to [17] for a deeper presentation on rewriting properties of n-polygraphs. We also recall from [15] the notions of double categories and of double groupoids. In Section 2 we define the notions of double polygraphs and dipolygraphs, giving double coherent presentations of globular n-categories. We explicit following [14] a construction of a free n-category enriched in double groupoids generated by a double n-polygraph, in which our coherence results will be formulated. Finally, we explain how to deduce a globular coherent presentation from a double coherent presentation. As examples, we explicit the notion of coherent presentation on the cases of groups, commutative monoids and pivotal categories. Section 3 is devoted to the study of rewriting properties of polygraphs defined modulo relations. We formulate the notions of termination, confluence, local confluence and confluence modulo for these polygraphs. Following [1] we give a completion procedure in terms of critical branchings for confluence modulo of the polygraph modulo ${}_E R$. In Section 4, we develop the notion of coherent confluence modulo and we prove a coherent version of Newman's lemma and critical branching lemma for polygraphs modulo to prove coherent confluence of a polygraph modulo from coherent confluence modulo of some critical branchings. In Section 5, we define the notion of coherent completion modulo, and we show how to construct a double coherent presentation of an n-category presented by a polygraph modulo from a such a coherent completion. Finally, in Section 6 we explain how to deduce a globular coherent presentation for an n-category from a double coherent presentation generated by a polygraph modulo. We apply our construction in the situation of commutative monoids and pivotal monoidal categories modulo isotopy relations.

1. Preliminaries

In this preliminary section, we give notations on higher-dimensional categories used in this article. In particular, we recall the structure of polygraph from [9, 32, 36] and we refer the reader to [17, 19, 20] for rewriting properties of polygraphs. We recall the notion of double categories from [15] and we refer the reader to [6, 13, 14] for deeper presentations on double categories and double groupoids.

1.1. Higher-dimensional categories and polygraphs

Throughout this article, n denotes either a fixed natural number or ∞.

1.1.1. Higher-dimensional categories. We will denote by Cat_n the category of (small, strict and globular) n-categories. If $\mathcal{C}$ is an n-category, we denote by $\mathcal{C}_k$ the set of k-cells of $\mathcal{C}$. If f is a k-cell of $\mathcal{C}$, then $\partial_{-,i}(f)$ and $\partial_{+,i}(f)$ respectively denote the i-source and i-target of f, while $(k-1)$-source and $(k-1)$-target will be denoted by $\partial_-(f)$ and $\partial_+(f)$ respectively. The source and target maps satisfy the *globular relations*:

$$\partial_{\alpha,i}\partial_{\alpha,i+1} \;=\; \partial_{\alpha,i}\partial_{\beta,i+1}, \tag{1.1.2}$$

for all α, β in $\{-,+\}$. Two k-cells f and g are i-*composable* when $\partial_{+,i}(f) = \partial_{-,i}(g)$. In that case, their i-composite is denoted by $f \star_i g$, or by fg when $i = 0$. The compositions satisfy the *exchange relations*:

$$(f_1 \star_i g_1) \star_j (f_2 \star_i g_2) \;=\; (f_1 \star_j f_2) \star_i (g_1 \star_j g_2). \tag{1.1.3}$$

for all $i \neq j$ and for all cells f_1, f_2, g_1, g_2 such that both sides are defined. If f is a k-cell, we denote by 1_f its identity $(k+1)$-cell. When 1_f is composed with l-cells, we simply denote it by f for $l \geqslant k+1$.

A k-cell f of an n-category $\mathcal{C}$ is i-*invertible* when there exists a (necessarily unique) k-cell g in $\mathcal{C}$, with i-source $\partial_{+,i}(f)$ and i-target $\partial_{-,i}(f)$, called the i-*inverse of* f, that satisfies

$$f \star_i g \;=\; 1_{\partial_{-,i}(f)} \qquad \text{and} \qquad g \star_i f \;=\; 1_{\partial_{+,i}(f)}.$$

When $i = k-1$, we just say that f is *invertible* and we denote by f^- its *inverse*. As in higher-dimensional groupoids, if a k-cell f is invertible and if its i-source u and i-target v are invertible, then f is $(i-1)$-invertible, with $(i-1)$-inverse given by $v^- \star_{i-1} f^- \star_{i-1} u^-$.

For a natural number $p \leqslant n$, or for $p = n = \infty$, an (n,p)-*category* is an n-category whose k-cells are invertible for every $k > p$. When $n < \infty$, this is an n-category enriched in $(n-p)$-groupoids and, when $n = \infty$, an n-category enriched in ∞-groupoids. In particular, an (n,n)-category is an n-category, and an $(n,0)$-category is an n-*groupoid*, also called a *groupoid* for $n = 1$.

A 0-*sphere of* $\mathcal{C}$ is a pair $\gamma = (f,g)$ of 0-cells of $\mathcal{C}$ and, for $1 \leqslant k \leqslant n$, a k-*sphere of* $\mathcal{C}$ is a pair $S = (f,g)$ of k-cells of $\mathcal{C}$ such that $\partial_-(f) = \partial_-(g)$ and $\partial_+(f) = \partial_+(g)$. The k-cell f (resp. g) is called the *source* (resp. target) of S denoted by $\partial_-(S)$ (resp. $\partial_+(S)$). We will denote by $\mathrm{Sph}_k(\mathcal{C})$ the set of k-spheres of $\mathcal{C}$. If f is a k-cell of $\mathcal{C}$, for $1 \leqslant k \leqslant n$, the *boundary of* f is the $(k-1)$-sphere $(\partial_-(f), \partial_+(f))$ denoted by $\partial(f)$.

1.1.4. Cellular extensions. Suppose $n < \infty$, a *cellular extension* of an n-category $\mathcal{C}$ is a set Γ equipped with a map $\gamma : \Gamma \to \mathrm{Sph}_n(\mathcal{C})$. By considering all the formal compositions of elements of Γ, seen as $(n+1)$-cells with source and target in $\mathcal{C}$, one builds the *free* $(n+1)$-*category generated by* Γ *over* $\mathcal{C}$, denoted by $\mathcal{C}[\Gamma]$. The *size* of an $(n+1)$-cell f of $\mathcal{C}[\Gamma]$ is the number denoted by $||f||_\Gamma$, of $(n+1)$-cells of Γ it contains. We denote by $\mathcal{C}^{(1)}$ the set of n-cells in $\mathcal{C}$ of size 1. We denote by $(\mathcal{C})_\Gamma$ the quotient of the n-category $\mathcal{C}$ by the congruence generated by Γ, i.e., the n-category one gets from $\mathcal{C}$ by identification of the n-cells $\partial_-(S)$ and $\partial_+(S)$, for all n-sphere S of Γ.

If $\mathcal{C}$ is an (n,p)-category and Γ is a cellular extension of $\mathcal{C}$, then the *free* $(n+1,p)$-*category generated by* Γ *over* $\mathcal{C}$ is denoted by $\mathcal{C}(\Gamma)$ and defined as follows:

$$\mathcal{C}(\Gamma) \;=\; \big(\mathcal{C}\,[\Gamma,\;\Gamma^-]\big)_{\mathrm{Inv}(\Gamma)}$$

where Γ^- contains the same $(n+1)$-cells as Γ, with source and target reversed, and $\mathrm{Inv}(\Gamma)$ is the cellular extension of $\mathcal{C}\,[\Gamma,\;\Gamma^-]$ made of two $(n+2)$-cells

$$f \star_n f^- \;\to\; 1_{\partial_-(f)} \qquad \text{and} \qquad f^- \star_n f \;\to\; 1_{\partial_+(f)}$$

for every $(n+1)$-cell f in Γ.

Let $\mathcal{C}$ be an (n,p)-category, for $p < n < \infty$. A cellular extension Γ of $\mathcal{C}$ is *acyclic* if the (n,p)-category $\mathcal{C}/\Gamma$ is aspherical, i.e., such that, for every n-sphere S of $\mathcal{C}$, there exists an $(n+1)$-cell with boundary S in the $(n+1,p)$-category $\mathcal{C}(\Gamma)$.

1.1.5. Polygraphs. Recall that an n-*polygraph* is a data $P = (P_0, P_1, \ldots, P_n)$ made of a set P_0 and, for every $0 \leqslant k < n$, a cellular extension P_{k+1} of the free k-category

$$P_k^* \;:=\; P_0[P_1]\ldots[P_k],$$

whose elements are called *generating* $(k+1)$*-cells of* P. For $0 \leqslant k \leqslant n-1$, we will denote by $P_{\leqslant k}$ the underlying k-polygraph $(P_0, P_1, \ldots, P_k)$. We will denote by P^* (resp. $P^\top$) the free n-category (resp. $(n, n-1)$-category) generated by an n-polygraph P. We will denote by $\overline{P}$ the $(n-1)$*-category presented by the polygraph* P, that is $\overline{P} := (P^*_{n-1})_{P_n}$.

Given two n-polygraphs P and Q, a *morphism* of n-polygraphs from P to Q is a pair (ξ_{n-1}, f_n) where ξ_{n-1} is a morphism of $(n-1)$-polygraphs from P_{n-1} to Q_{n-1}, and where f_n is a map from P_n to Q_n such that the following diagrams commute:

$$\begin{tikzcd} P^*_{n-1} \arrow[d, "F_{n-1}(\xi_{n-1})"'] & P_n \arrow[l, "s^P_{n-1}"'] \arrow[d, "f_n"] \\ Q^*_{n-1} & Q_n \arrow[l, "s^Q_{n-1}"] \end{tikzcd} \qquad \begin{tikzcd} P^*_{n-1} \arrow[d, "F_{n-1}(\xi_{n-1})"'] & P_n \arrow[l, "t^P_{n-1}"'] \arrow[d, "f_n"] \\ Q^*_{n-1} & Q_n \arrow[l, "t^Q_{n-1}"] \end{tikzcd}$$

Equivalently, it is a sequence of maps $(f_k : P_k \to Q_k)_k$ indexed by integers $0 \leqslant k \leqslant n-1$ such that the relations

$$f_k s^P_k = s^Q_k f_{k+1} \quad \text{and} \quad f_k t^P_k = t^Q_k f_{k+1}$$

holds for all $0 \leqslant k \leqslant n-1$.

We will denote by Pol_n the category of n-polygraphs and their morphisms, and by $U^{\mathrm{Pol}}_n : \mathsf{Cat}_n \to \mathsf{Pol}_n$ the forgetful functor sending an n-category on its underlying n-polygraph.

For $p \leqslant n$, an (n,p)*-polygraph* is a data P made of an p-polygraph $(P_0, \ldots, P_p)$, and for every $p \leqslant k < n$, a cellular extension P_{k+1} of the free (k,p)-category

$$P^\top_k \;:=\; P^*_p(P_{n+1}) \cdots (P_k).$$

Note that an (n,n)-polygraph is an n-polygraph.

1.1.6. Contexts in n-categories. A *context* of an n-category $\mathcal{C}$ is a pair (S, C) made of an $(n-1)$-sphere S of $\mathcal{C}$ and an n-cell C in $\mathcal{C}[S]$ such that $||C||_S = 1$. We often denote simply by C, such a context. Recall from [17, Proposition 2.1.3] that every context of $\mathcal{C}$ has a decomposition

$$f_n \star_{n-1} (f_{n-1} \star_{n-2} \cdots (f_1 \star_0 S \star_0 g_1) \cdots \star_{n-2} g_{n-1}) \star_{n-1} g_n,$$

where S is an $(n-1)$-sphere and, for every k in $\{1, \ldots, n\}$, f_k and g_k are n-cells of $\mathcal{C}$. Moreover, one can choose these cells so that f_k and g_k are (the identities of) k-cells. A *whisker of* $\mathcal{C}$ is a context with a decomposition

$$f_{n-1} \star_{n-2} \cdots (f_1 \star_0 S \star_0 g_1) \cdots \star_{n-2} g_{n-1}$$

such that, for every k in $\{1, \ldots, n-1\}$, f_k and g_k are k-cells.

Given an n-polygraph P, recall from [17, Proposition 2.1.5] that every n-cell f in P^* with size $k \geqslant 1$ has a decomposition

$$f \;=\; C_1[\gamma_1] \star_{n-1} \cdots \star_{n-1} C_k[\gamma_k],$$

where $\gamma_1, \ldots, \gamma_k$ are generating n-cells of P and $C_1, \ldots, C_k$ are whiskers of P^*.

1.2. Double groupoids

In this subsection, we recall the notion of double category introduced in [15]. It can be defined as an internal category in the category Cat of all (small) categories and functors. Recall that given $\mathcal{V}$ be a category with finite limits, an *internal category* $\mathbf{C}$ in $\mathcal{V}$ is a data $(\mathbf{C}_1, \mathbf{C}_0, \partial_-^{\mathbf{C}}, \partial_+^{\mathbf{C}}, \circ_{\mathbf{C}}, i_{\mathbf{C}})$, where

$$\partial_-^{\mathbf{C}}, \partial_+^{\mathbf{C}} : \mathbf{C}_1 \longrightarrow \mathbf{C}_0, \qquad i_{\mathbf{C}} : \mathbf{C}_0 \longrightarrow \mathbf{C}_1, \qquad \circ_{\mathbf{C}} : \mathbf{C}_1 \times_{\mathbf{C}_0} \mathbf{C}_1 \longrightarrow \mathbf{C}_1$$

are morphisms of $\mathcal{V}$ satisfying the usual axioms of a category, and where $\mathbf{C}_1 \times_{\mathbf{C}_0} \mathbf{C}_1$ denotes the pullback in $\mathcal{V}$ over morphisms $\partial_-^{\mathbf{C}}$ and $\partial_+^{\mathbf{C}}$. An internal functor from $\mathbf{C}$ to $\mathbf{D}$ is a pair of morphisms $\mathbf{C}_1 \to \mathbf{D}_1$ and $\mathbf{C}_0 \to \mathbf{D}_0$ in $\mathcal{V}$ commuting in the obvious way. We denote by $\mathsf{Cat}(\mathcal{V})$ the category of internal categories in $\mathcal{V}$ and their functors.

In the same way, we define an *internal groupoid* $\mathbf{G}$ in $\mathcal{V}$ as an internal category $(\mathbf{G}_1, \mathbf{G}_0, \partial_-^{\mathbf{G}}, \partial_+^{\mathbf{G}}, \circ_{\mathbf{G}}, i_{\mathbf{G}})$ with an additional morphism

$$(\cdot)^-_{\mathbf{G}} : \mathbf{G}_1 \to \mathbf{G}_1$$

satisfying the axioms of groups, that is

$$\partial_-^{\mathbf{G}} \circ (\cdot)^-_{\mathbf{G}} = \partial_+^{\mathbf{G}}, \quad \partial_+^{\mathbf{G}} \circ (\cdot)^-_{\mathbf{G}} = \partial_-^{\mathbf{G}}, \tag{1.2.1}$$

$$i_{\mathbf{G}} \circ \partial_-^{\mathbf{G}} = \circ_{\mathbf{G}} \circ (\mathrm{id} \times (\cdot)^-_{\mathbf{G}}) \circ \Delta, \quad i_{\mathbf{G}} \circ \partial_+^{\mathbf{G}} = \circ_{\mathbf{G}} \circ ((\cdot)^-_{\mathbf{G}} \times \mathrm{id}) \circ \Delta, \tag{1.2.2}$$

where $\Delta : \mathbf{G}_1 \to \mathbf{G}_1 \times \mathbf{G}_1$ is the diagonal functor. We denote by $\mathsf{Grpd}(\mathcal{V})$ the category of internal groupoids in $\mathcal{V}$ and their functors.

1.2.3. Double categories and double groupoids. The category of *double categories* is defined as the category $\mathsf{Cat}(\mathsf{Cat})$, and the category of *double groupoids* is defined as the category $\mathsf{Grpd}(\mathsf{Grpd})$ of internal groupoids in the category Grpd of groupoids and their functors. Explicitly, a double category is an internal category $(\mathbf{C}_1, \mathbf{C}_0, \partial_-^{\mathbf{C}}, \partial_+^{\mathbf{C}}, \circ_{\mathbf{C}}, i_{\mathbf{C}})$ in Cat, that gives four related categories:

$$\begin{aligned} \mathbf{C}^{sv} &:= (\mathbf{C}^s, \mathbf{C}^v, \partial^v_{-,1}, \partial^v_{+,1}, \diamond^v, i^v_1), & \qquad \mathbf{C}^{sh} &:= (\mathbf{C}^s, \mathbf{C}^h, \partial^h_{-,1}, \partial^h_{+,1}, \diamond^h, i^h_1), \\ \mathbf{C}^{vo} &:= (\mathbf{C}^v, \mathbf{C}^o, \partial^v_{-,0}, \partial^v_{+,0}, \circ^v, i^v_0), & \qquad \mathbf{C}^{ho} &:= (\mathbf{C}^h, \mathbf{C}^o, \partial^h_{-,0}, \partial^h_{+,0}, \circ^h, i^h_0), \end{aligned}$$

where $\mathbf{C}^{sh}$ is the category $\mathbf{C}_1$ and $\mathbf{C}^{vo}$ is the category $\mathbf{C}_0$. The sources, target and identity maps pictured in the following diagram

$$\begin{tikzcd}
& \mathbf{C}^s \arrow[dl, shift right=3, "\partial^v_{+,1}"'] \arrow[dl, shift left=3, "\partial^v_{-,1}"] \arrow[dr, shift left=3, "\partial^h_{+,1}"] \arrow[dr, shift right=3, "\partial^h_{-,1}"'] & \\
\mathbf{C}^v \arrow[ur, "i^v_1"'] \arrow[dr, shift right=3, "\partial^v_{-,0}"'] \arrow[dr, shift left=3, "\partial^v_{+,0}"] & & \mathbf{C}^h \arrow[ul, "i^h_1"] \arrow[dl, shift left=3, "\partial^h_{-,0}"] \arrow[dl, shift right=3, "\partial^h_{+,0}"'] \\
& \mathbf{C}^o \arrow[ul, "i^v_0"] \arrow[ur, "i^h_0"'] &
\end{tikzcd}$$

satisfy the following relations:

i) $\partial^{h}_{\alpha,0}\partial^{h}_{\beta,1} = \partial^{v}_{\beta,0}\partial^{v}_{\alpha,1}$, for all α, β in $\{-,+\}$,

ii) $\partial^{\mu}_{\alpha,1} i^{\eta}_{1} = i^{\mu}_{0}\partial^{\eta}_{\alpha,0}$, for all α in $\{-,+\}$ and μ,η in $\{v,h\}$,

iii) $i^{v}_{1}i^{v}_{0} = i^{h}_{1}i^{h}_{0}$,

iv) $\partial^{\mu}_{\alpha,1}(A \diamond^{\mu} B) = \partial^{\mu}_{\alpha,1}(A) \circ^{\mu} \partial^{\mu}_{\alpha,1}(B)$, for all $\alpha \in \{-,+\}$, $\mu \in \{v,h\}$ and any squares A, B such that both sides are defined,

v) *middle four interchange law* :

$$(A \diamond^{v} A') \diamond^{h} (B \diamond^{h} B') = (A \diamond^{h} B) \diamond^{v} (A' \diamond^{h} B'), \tag{1.2.4}$$

for any cells A, A', B, B' in $\mathbf{C}^{s}$ such that both sides are defined.

Elements of $\mathbf{C}^{o}$ are called *point cells*, the elements of $\mathbf{C}^{h}$ and $\mathbf{C}^{v}$ are respectively called *horizontal cells* and *vertical cells* and pictured by

$$x_1 \xrightarrow{f} x_2 \qquad\qquad \begin{tikzcd} x_1 \arrow[d, "e"'] \\ x_2 \end{tikzcd}$$

Following relations **i)**, the elements of $\mathbf{C}^{s}$ are called *square cells* and can be pictured by squares as follows:

$$\begin{tikzcd} \cdot \arrow[r, "\partial^{h}_{-,1}(A)"] \arrow[d, "\partial^{v}_{-,1}(A)"'] \arrow[dr, phantom, "\Downarrow A"] & \cdot \arrow[d, "\partial^{v}_{+,1}(A)"] \\ \cdot \arrow[r, "\partial^{h}_{+,1}(A)"'] & \cdot \end{tikzcd}$$

and by the followings squares for identities

$$\begin{tikzcd} x_1 \arrow[r, "f"] \arrow[d, "i^{v}_{0}(x_1)"'] \arrow[dr, phantom, "\Downarrow i^{h}_{1}(f)"] & x_2 \arrow[d, "i^{v}_{0}(x_2)"] \\ x_1 \arrow[r, "f"'] & x_2 \end{tikzcd} \quad \begin{tikzcd} x \arrow[r, "i^{h}_{0}(x)"] \arrow[d, "e"'] \arrow[dr, phantom, "\Downarrow i^{v}_{1}(e)"] & x \arrow[d, "e"] \\ y \arrow[r, "i^{h}_{0}(y)"'] & y \end{tikzcd} \quad \text{or simply by} \quad \begin{tikzcd} x_1 \arrow[r, "f"] \arrow[d, "\shortparallel"'] \arrow[dr, phantom, "i^{h}_{1}(f)"] & x_2 \arrow[d, "\shortparallel"] \\ x_1 \arrow[r, "f"'] & x_2 \end{tikzcd} \quad \begin{tikzcd} x \arrow[r, "="] \arrow[d, "e"'] \arrow[dr, phantom, "i^{v}_{1}(e)"] & x \arrow[d, "e"] \\ y \arrow[r, "="'] & y \end{tikzcd}$$

The compositions $\diamond^{v}$ (resp. $\diamond^{h}$) are called respectively *vertical* and *horizontal compositions*, and can be pictured as follows

$$\begin{tikzcd} x_1 \arrow[r, "f_1"] \arrow[d, "e_1"'] \arrow[dr, phantom, "\Downarrow A"] & x_2 \arrow[r, "f_2"] \arrow[d, "e_2"] \arrow[dr, phantom, "\Downarrow B"] & x_3 \arrow[d, "e_3"] \\ y_1 \arrow[r, "g_1"'] & y_2 \arrow[r, "g_2"'] & y_3 \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} x_1 \arrow[r, "f_1 \circ^{h} f_2"] \arrow[d, "e_1"'] \arrow[dr, phantom, "A \Downarrow\diamond^{v} B"] & x_3 \arrow[d, "e_3"] \\ y_1 \arrow[r, "g_1 \circ^{h} g_2"'] & y_3 \end{tikzcd}$$

for all x_i, y_i in $\mathbf{C}^o$, f_i, g_i in $\mathbf{C}^h$, e_i in $\mathbf{C}^v$ and A, B in $\mathbf{C}^s$,

$$\begin{tikzcd} x_1 \arrow[r, "f"] \arrow[d, "e_1"'] \arrow[dr, phantom, "\Downarrow A"] & x_2 \arrow[d, "e_2"] \\ y_1 \arrow[r, "g"'] \arrow[d, "e_1'"'] \arrow[dr, phantom, "\Downarrow A'"] & y_2 \arrow[d, "e_2'"] \\ z_1 \arrow[r, "h"'] & z_2 \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} x_1 \arrow[r, "f"] \arrow[d, "e_1 \circ^v e_1'"'] \arrow[dr, phantom, "A \diamond^h A'"] & x_2 \arrow[d, "e_2 \circ^v e_2'"] \\ z_1 \arrow[r, "h"'] & z_2 \end{tikzcd}$$

for all x_i, y_i, z_i in $\mathbf{C}^o$, f, g, h in $\mathbf{C}^h$, e_i, e_i' in $\mathbf{C}^v$ and A, A' in $\mathbf{C}^s$.

Similarly a double groupoid is given by the same data $(\mathbf{G}_1, \mathbf{G}_0, \partial_-^{\mathbf{G}}, \partial_+^{\mathbf{G}}, \circ_{\mathbf{G}}, i_{\mathbf{G}})$, with an inverse operation $(\cdot)_{\mathbf{G}}^- : \mathbf{G}_1 \to \mathbf{G}_1$ satisfying the relations (1.2.1) and (1.2.2). As a consequence the four related categories $\mathbf{G}^{sv}$, $\mathbf{G}^{sh}$, $\mathbf{G}^{vo}$ and $\mathbf{G}^{ho}$ are groupoids. For any square cell

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "e"'] \arrow[dr, phantom, "\Downarrow A"] & \cdot \arrow[d, "e'"] \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$

in $\mathbf{G}^s$, the inverse square cell with respect to $\diamond^\mu$, for $\mu \in \{v, h\}$, is denoted by $A^{-,\mu}$ and satisfy the following relations

$$A \diamond^\mu (A^{-,\mu}) = i_1^\mu(\partial_{-,1}^\mu(A)), \qquad (A^{-,\mu}) \diamond^\mu A = i_1^\mu(\partial_{+,1}^\mu(A)). \tag{1.2.5}$$

The sources and targets of these inverse are given as follows

$$\begin{tikzcd} \cdot \arrow[r, "f^-"] \arrow[d, "e'"'] \arrow[dr, phantom, "\Downarrow A^{-,v}"] & \cdot \arrow[d, "e"] \\ \cdot \arrow[r, "g^-"'] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "g"] \arrow[d, "e^-"'] \arrow[dr, phantom, "\Downarrow A^{-,h}"] & \cdot \arrow[d, "(e')^-"] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd}$$

1.2.6. Squares. A *square* of a double category $\mathbf{C}$ is a quadruple (f, g, e, e') such that f, g are horizontal cells and e, e' are vertical cells that compose as follows:

$$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, "e"'] & v \arrow[d, "e'"] \\ u' \arrow[r, "g"'] & v' \end{tikzcd}$$

The *boundary* of a square cell A in $\mathbf{C}$ is the square $(\partial_{-,h}(A), \partial_{+,h}(A), \partial_{-,v}(A), \partial_{+,v}(A))$, denoted by $\partial(A)$. We will denote by $\mathrm{Sqr}(\mathbf{C})$ the set of square cells of $\mathbf{C}$.

1.2.7. n-categories enriched in double categories. The coherence results for rewriting systems modulo presented in this article are formulated using the notion of n-categories enriched in double categories and double groupoids. Let us expand the latter notion for $n > 0$. Consider the category $\mathsf{Cat}(\mathsf{Grpd})$ equipped with the cartesian product defined by

$$\mathbf{C} \times \mathbf{D} = (\mathbf{C}_1 \times \mathbf{D}_1, \mathbf{C}_0 \times \mathbf{D}_0, s_{\mathbf{C}} \times t_{\mathbf{C}}, c_{\mathbf{C}} \times c_{\mathbf{D}}, i_{\mathbf{C}} \times i_{\mathbf{D}}),$$

for any double groupoids $\mathbf{C}$ and $\mathbf{D}$. The terminal double groupoid $\mathbf{T}$ has only one point cell, denoted by $\bullet$, and identities $i_0^v(\bullet)$, $i_0^h(\bullet)$, $i_1^v i_0^h(\bullet) = i_1^h i_0^v(\bullet)$.

An n-category enriched in double groupoids is an n-category $\mathcal{C}$ such that for any x, y in $\mathcal{C}_{n-1}$ the homset $\mathcal{C}_n(x,y)$ has a double groupoid structure, whose point cells are the n-cells in $\mathcal{C}_n(x,y)$. We will denote by $\mathcal{C}_{n+1}^v$ (resp. $\mathcal{C}_{n+1}^h$, $\mathcal{C}_{n+2}^s$) the union of sets $\mathcal{C}_n(x,y)^v$ (resp. $\mathcal{C}_n(x,y)^h$, $\mathcal{C}_n(x,y)^s$) for all x, y in $\mathcal{C}_{n-1}$. An $(n+2)$-cell A in $\mathcal{C}_{n+2}^s$ can be represented by the following diagrams:

$$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, "e"'] \arrow[dr, phantom, "\Downarrow A"] & v \arrow[d, "e'"] \\ u' \arrow[r, "g"'] & v' \end{tikzcd}$$

with u, u', v, v' in $\mathcal{C}_n$, f, g in $\mathcal{C}_{n+1}^h$ and e, e' in $\mathcal{C}_{n+1}^v$. The compositions of the $(n+2)$-cells and the identities $(n+2)$-cells are induced by the functors of double categories

$$\star_{n-1}^{x,y,z} : \mathcal{C}_n(x,y) \times \mathcal{C}_n(y,z) \to \mathcal{C}_n(x,z), \qquad\qquad 1_x : \mathbf{T} \to \mathcal{C}_n(x,x),$$

for all $(n-1)$-cells x, y, z. The $(n-1)$-composite of an $(n+2)$-cell A in $\mathcal{C}_n(x,y)$ with an $(n+2)$-cell B in $\mathcal{C}_n(y,z)$ of the form

$$\begin{tikzcd} u_1 \arrow[r, "f_1"] \arrow[d, "e_1"'] \arrow[dr, phantom, "\Downarrow A"] & v_1 \arrow[d, "e_1'"] \\ u_1' \arrow[r, "g_1"'] & v_1' \end{tikzcd} \qquad\qquad \begin{tikzcd} u_2 \arrow[r, "f_2"] \arrow[d, "e_2"'] \arrow[dr, phantom, "\Downarrow B"] & v_2 \arrow[d, "e_2'"] \\ u_2' \arrow[r, "g_2"'] & v_2' \end{tikzcd}$$

is defined by $\star_{n-1}$ compositions of n-cells, vertical $(n+1)$-cells and horizontal $(n+1)$-cells and denoted by:

$$\begin{tikzcd} u_1 \star_{n-1} u_2 \arrow[r, "f_1 \star_{n-1} f_2"] \arrow[d, "e_1 \star_{n-1} e_2"'] \arrow[dr, phantom, "\Downarrow A \star_{n-1} B"] & v_1 \star_{n-1} v_2 \arrow[d, "e_1' \star_{n-1} e_2'"] \\ u_1' \star_{n-1} u_2' \arrow[r, "g_1 \star_{n-1} g_2"'] & v_1' \star_{n-1} v_2' \end{tikzcd}$$

By functoriality, the $(n-1)$-composition satisfies the following exchange relations:

$$(A \diamond^{\mu} A') \star_{n-1} (B \diamond^{\mu} B') \;=\; (A \star_{n-1} B) \diamond^{\mu} (A' \star_{n-1} B'), \tag{1.2.8}$$

$$(A \diamond^{\mu} A') \star_{n-1} (B \diamond^{\eta} B') \;=\; ((A \star_{n-1} B) \diamond^{\mu} (A' \star_{n-1} B)) \diamond^{\eta} ((A \star_{n-1} B') \diamond^{\mu} (A' \star_{n-1} B')). \tag{1.2.9}$$

Using middle four interchange law (1.2.4), the identity (1.2.9) is equivalent to the following identity

$$(A \diamond^{\mu} A') \star_{n-1} (B \diamond^{\eta} B') \;=\; ((A \star_{n-1} B) \diamond^{\eta} (A \star_{n-1} B')) \diamond^{\mu} ((A' \star_{n-1} B) \diamond^{\eta} (A' \star_{n-1} B'))$$

for all $\mu \neq \eta$ in $\{v, h\}$ and any $(n+2)$-cells A, A', B, B' such that both sides are defined.

We will denote by $\mathsf{Cat}_n(\mathsf{DbCat})$ (resp. $\mathsf{Cat}_n(\mathsf{DbGrpd})$) the category of n-categories enriched in double categories (resp. double groupoids) and enriched n-functors.

1.2.10. 2-categories as double categories. From a 2-category $\mathcal{C}$, one can construct two canonical double categories, by setting the vertical or horizontal cells to be only identities in $\mathcal{C}$. In this way, 2-categories can be considered as special cases of double categories. The *quintet construction* gives another way to associate a double category, called the *double category of quintets in* $\mathcal{C}$ and denoted by $\mathbf{Q}(\mathcal{C})$ to a 2-category $\mathcal{C}$. The vertical and horizontal categories of $\mathbf{Q}(\mathcal{C})$ are both equal to $\mathcal{C}$, and there is a square cell

$$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow A"] & u' \arrow[d, "k"] \\ v \arrow[r, "h"'] & v' \end{tikzcd}$$

in $\mathbf{Q}(\mathcal{C})$ whenever there is a 2-cell $A : f \star_1 k \Rightarrow g \star_1 h$ in $\mathcal{C}$. This defines a functor $\mathbf{Q} : \mathsf{Cat}_2 \to \mathsf{DbCat}$. Similarly, for $n \geqslant 2$ one can associate to an n-category an $(n-2)$-category enriched in double categories by a quintet construction.

2. Double coherent presentations

Recall from [16] that a *coherent presentation* of a 1-category $\mathcal{C}$ is a $(3,1)$-polygraph P whose underlying 2-polygraph $P_{\leqslant 2}$ is a presentation of $\mathcal{C}$ and P_3 is an acyclic extension of the free $(2,1)$-category $P_2^{\top}$. This notion extends to n-categories generated by n-polygraphs. Namely, a coherent presentation of such an n-category $\mathcal{C}$ is an $(n{+}2, n)$-polygraph P such that the underlying $(n{+}1)$-polygraph $P_{\leqslant (n+1)}$ is a presentation of $\mathcal{C}$ and P_{n+2} is an acyclic extension of the free $(n+1, n)$-category generated by P. In Subsection 2.1, we introduce dipolygraphs in order to extend the notion of coherent presentation to n-categories whose underlying $(n-1)$-category is not free. We also introduce the notion of double n-polygraph generating n-categories enriched in double groupoids. In Section 4, we will formulate coherence results modulo using the structure of double n-polygraph. Finally, we introduce in Subsection 2.2 double coherent presentations of n-categories. This notion allows us to obtain coherent presentations from polygraphs modulo as it will be explained in 6.

2.1. Double polygraphs and dipolygraphs

2.1.1. Square extensions. Let $(\mathbf{C}^v, \mathbf{C}^h)$ be a pair of n-categories with the same underlying $(n-1)$-category $\mathbf{B}$. A *square extension* of the pair $(\mathbf{C}^v, \mathbf{C}^h)$ is a set Γ equipped with four maps $\partial^{\mu}_{\alpha,n}$, with $\alpha \in \{-,+\}$, $\mu \in \{1,2\}$, as depicted by the following diagram:

$$\begin{tikzcd}
& \Gamma \arrow[dl, shift right, "\partial^{\nu}_{+,n}"'] \arrow[dl, shift left, "\partial^{\nu}_{-,n}"] \arrow[dr, shift left, "\partial^{h}_{+,n}"] \arrow[dr, shift right, "\partial^{h}_{-,n}"'] & \\
\mathbf{C}^{\nu} \arrow[dr, shift left, "\partial^{\nu}_{+,n-1}"] \arrow[dr, shift right, "\partial^{\nu}_{-,n-1}"'] & & \mathbf{C}^{h} \arrow[dl, shift right, "\partial^{h}_{+,n-1}"'] \arrow[dl, shift left, "\partial^{h}_{-,n-1}"] \\
& \mathbf{B} &
\end{tikzcd}$$

and satisfying the following relations:

$$\partial^{\nu}_{\alpha,n-1}\partial^{\nu}_{\beta,n} = \partial^{h}_{\beta,n-1}\partial^{h}_{\alpha,n},$$

for all α, β in $\{-,+\}$. The point cells of a square A in Γ are the $(n-1)$-cells of $\mathbf{B}$ of the form

$$\partial^{\mu}_{\alpha,n-1}\partial^{\eta}_{\beta,n}(A)$$

with α, β in $\{-,+\}$, and η, μ in $\{h,\nu\}$. Note that by construction these four $(n-1)$-cells have the same $(n-2)$-source and $(n-2)$-target in $\mathbf{B}$ respectively denoted by $\partial_{-,n-2}(A)$ and $\partial_{+,n-2}(A)$.

A pair of n-categories $(\mathbf{C}^{\nu},\mathbf{C}^{h})$ has two canonical square extensions, the empty one, and the full one that contains all squares on $(\mathbf{C}^{\nu},\mathbf{C}^{h})$, denoted by $\mathrm{Sqr}(\mathbf{C}^{\nu},\mathbf{C}^{h})$. We will write $\mathrm{Sph}(\mathbf{C}^{\nu},1)$ (resp. $\mathrm{Sph}(1,\mathbf{C}^{h})$) the square extension of $(\mathbf{C}^{\nu},\mathbf{C}^{h})$ made of all squares of the form

$$\begin{tikzcd}
u \arrow[r, "="] \arrow[d, "e"'] & u \arrow[d, "e'"] \\
v \arrow[r, "="'] & v
\end{tikzcd}
\qquad \Big(\text{resp.} \quad
\begin{tikzcd}
u \arrow[r, "f"] \arrow[d, "\shortparallel"'] & u' \arrow[d, "\shortparallel"] \\
u \arrow[r, "g"'] & u'
\end{tikzcd}
\quad \Big)$$

for all n-cells e, e' in $\mathbf{C}^{\nu}$ (resp. n-cells in f, g in $\mathbf{C}^{h}$).

The *Peiffer square extension* of the pair $(\mathbf{C}^{\nu},\mathbf{C}^{h})$ is the square extension of $(\mathbf{C}^{\nu},\mathbf{C}^{h})$, denoted by $\mathrm{Peiff}(\mathbf{C}^{\nu},\mathbf{C}^{h})$, containing the squares of the form

$$\begin{tikzcd}
u \star_i v \arrow[r, "f \star_i v"] \arrow[d, "u \star_i e"'] & u' \star_i v \arrow[d, "u' \star_i e"] \\
u \star_i v' \arrow[r, "f \star_i v'"'] & u' \star_i v'
\end{tikzcd}
\qquad
\begin{tikzcd}
w \star_i u \arrow[r, "w \star_i f"] \arrow[d, "e' \star_i u"'] & w \star_i u' \arrow[d, "e' \star_i u'"] \\
w' \star_i u \arrow[r, "w' \star_i f"'] & w' \star_i u'
\end{tikzcd}$$

for all n-cells e, e' in $\mathbf{C}^{\nu}$ and n-cell f in $\mathbf{C}^{h}$.

2.1.2. Double polygraphs. We define a *double n-polygraph* as a data $P = (P^{\nu}, P^{h}, P^{s})$ made of

i) two $(n+1)$-polygraphs P^{ν} and P^{h} such that $P^{\nu}_{\leqslant n} = P^{h}_{\leqslant n}$,

ii) a square extension P^{s} of the pair of free $(n+1)$-categories $((P^{\nu})^{*}, (P^{h})^{*})$.

Such a data can be pictured by the following diagram

$$
\begin{tikzcd}
 & & P^s \arrow[dll, shift right, Rightarrow, "\partial^v_{+,n+1}"'] \arrow[dll, shift left, Rightarrow, "\partial^v_{-,n+1}"] \arrow[drr, shift left, Rightarrow, "\partial^h_{+,n+1}"] \arrow[drr, shift right, Rightarrow, "\partial^h_{-,n+1}"'] & & \\
(P^v)^* \arrow[drr, shift left, Rightarrow, "\partial^v_{-,n}"] \arrow[drr, shift right, Rightarrow, "\partial^v_{+,n}"'] & & & & (P^h)^* \arrow[dll, shift right, Rightarrow, "\partial^h_{-,n}"'] \arrow[dll, shift left, Rightarrow, "\partial^h_{+,n}"] \\
P^v \arrow[u, "\iota^v_{n+1}"] \arrow[rr, shift left, Rightarrow, "\partial^v_{+,n}"] \arrow[rr, shift right, Rightarrow, "\partial^v_{-,n}"'] & & P^*_{n+1} \arrow[d, shift right, Rightarrow, "\partial_{+,n-1}"'] \arrow[d, shift left, Rightarrow, "\partial_{-,n-1}"] & & P^h \arrow[u, "\iota^h_{n+1}"'] \arrow[ll, shift right, Rightarrow, "\partial^h_{+,n}"'] \arrow[ll, shift left, Rightarrow, "\partial^h_{-,n}"] \\
 & & P^*_n & &
\end{tikzcd}
$$

For $0 \leqslant k \leqslant n$, the k-cells of the $(n+1)$-polygraphs P^v and P^h are called *generating* k*-cells of* P. The $(n+1)$-cells of P^v (resp. P^h) are called *generating vertical* $(n+1)$*-cells of* P (resp. *generating horizontal* $(n+1)$*-cells of* P), and the elements of P^s are called *generating square* $(n+2)$*-cells* of P.

2.1.3. The category of double n-polygraphs. Given two double n-polygraphs $P = (P^v, P^h, P^s)$ and $Q = (Q^v, Q^h, Q^s)$, a *morphism of double* n*-polygraphs* from P to Q is a triple (f^v, f^h, f^s) made of two morphisms of $(n+1)$-polygraphs

$$
f^v : P^v \to Q^v, \qquad f^h : P^h \to Q^h,
$$

and a map $f^s : P^s \to Q^s$ such that the following diagrams commute:

$$
\begin{tikzcd}
P^\mu_{n+1} \arrow[d, "f^\mu_{n+1}"'] & P^s \arrow[l, "\partial^{\mu,P}_{-,n-1}"'] \arrow[d, "f^s"] \\
Q^\mu_{n+1} & Q^s \arrow[l, "\partial^{\mu,Q}_{-,n-1}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
P^\mu_{n+1} \arrow[d, "f^\mu_{n+1}"'] & P^s \arrow[l, "\partial^{\mu,P}_{+,n-1}"'] \arrow[d, "f^s"] \\
Q^\mu_{n+1} & Q^s \arrow[l, "\partial^{\mu,Q}_{+,n-1}"]
\end{tikzcd}
$$

for μ in $\{v, h\}$. We will denote by DbPol_n the category of double n-polygraphs and their morphisms.

Let us explicit two full subcategories of DbPol_n used in the sequel to formulate coherence and confluence results for polygraphs modulo. We define a *double* $(n+2, n)$*-polygraph* as a double n-polygraph whose square extension P^s is defined on the pair of $(n+1, n)$-categories $((P^v)^\top, (P^h)^\top)$. We denote by $\mathsf{DbPol}_{(n+2,n)}$ the category of double $(n+2,n)$-polygraphs. In some situations, we will also consider double n-polygraphs whose square extension is defined on the pair of $(n+1)$-categories $((P^v)^\top, (P^h)^*)$ (resp. $((P^v)^*, (P^h)^\top)$). We will respectively denote by DbPol^v_n (resp. DbPol^h_n) the full subcategories of DbPol_n they form.

2.1.4. Dipolygraphs. We define the structure of dipolygraph as presentation by generators and relations for ∞-categories whose underlying k-categories are not necessarily free. Note that a similar notion was introduced by Burroni in [8]. Let us define the notion of n-dipolygraph by induction on $n \geqslant 0$. A 0-*dipolygraph* is a set. A 1-*dipolygraph* is a triple $((P_0, P_1), Q_1)$, where (P_0, Q_1) is a 1-polygraph and P_1 is a cellular extension of the quotient category $(P_0^*)_{Q_1}$. For $n \geqslant 2$, an n-*dipolygraph* is a data $(P, Q) = ((P_i)_{0\leqslant i\leqslant n}, (Q_i)_{1\leqslant i\leqslant n})$ made of

i) a 1-dipolygraph $((P_0, P_1), Q_1)$,

ii) for every $2 \leqslant k \leqslant n$, a cellular extension Q_k of the $(k-1)$-category

$$[P_{k-2}]_{Q_{k-1}}[P_{k-1}],$$

where $[P_{k-2}]_{Q_{k-1}}$ denotes the $(k-2)$-category

$$((((P_0^*)_{Q_1}[P_1])_{Q_2}[P_2])_{Q_3}\dots[P_{k-2}])_{Q_{k-1}},$$

iii) for every $2 \leqslant k \leqslant n$, a cellular extension P_k of the $(k-1)$-category

$$[P_{k-1}]_{Q_k}.$$

For $0 \leqslant k \leqslant n-1$, we will denote by $(P, Q)_{\leqslant k}$ the underlying k-dipolygraph $((P_i)_{0\leqslant i\leqslant k}, (Q_i)_{1\leqslant i\leqslant k})$.

2.1.5. For $0 \leqslant p \leqslant n$, an (n, p)-*dipolygraph* is a data $((P_i)_{0\leqslant i\leqslant n}, (Q_i)_{1\leqslant i\leqslant n})$ such that:

i) $((P_i)_{0\leqslant i\leqslant p+1}, (Q_i)_{1\leqslant i\leqslant p+1})$ is a $(p+1)$-dipolygraph,

ii) for every $p+2 \leqslant k \leqslant n$, Q_k is a cellular extension of the $(k-1, p)$-category

$$([P_p]_{Q_{p+1}})(P_{p+1})_{Q_{p+2}}\dots(P_{k-1}),$$

iii) for every $p+2 \leqslant k \leqslant n$, P_k is a cellular extension of the $(k-1, p)$-category

$$((([P_p]_{Q_{p+1}})(P_{p+1}))_{Q_{p+2}}\dots(P_{k-1}))_{Q_k}.$$

2.1.6. We define a *morphism of* (n, p)-*dipolygraphs*

$$((P_i)_{0\leqslant i\leqslant n}, (Q_i)_{1\leqslant i\leqslant n}) \to ((P'_i)_{0\leqslant i\leqslant n}, (Q'_i)_{1\leqslant i\leqslant n})$$

as a family of pairs $((f_k, g_k))_{1\leqslant k\leqslant n}$, where $f_k : P_k \to P'_k$ and $g_k : Q_k \to Q'_k$ are maps such that the following diagram commute

$$\begin{tikzcd}
Q_k \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "g_k"'] & {[P_{k-2}]_{Q_{k-1}}[P_{k-1}]} \arrow[d, "\widetilde{f}_{k-1}"] \\
Q'_k \arrow[r, shift left] \arrow[r, shift right] & {[P'_{k-2}]_{Q'_{k-1}}[P'_{k-1}]}
\end{tikzcd}
\qquad
\begin{tikzcd}
P_k \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "f_k"'] & {[P_{k-1}]_{Q_k}} \arrow[d, "{[f_{k-1}]_{g_k}}"] \\
P'_k \arrow[r, shift left] \arrow[r, shift right] & {[P'_{k-1}]_{Q'_k}}
\end{tikzcd}$$

for any $1 \leqslant k \leqslant p+1$, and such that the following diagrams commute

$$
\begin{tikzcd}
Q_k \arrow[r, Rrightarrow] \arrow[d, "g_k"'] & ([P_p]_{Q_p})(P_{p+1})_{Q_{p+2}} \dots (P_{k-1}) \arrow[d, "\widetilde{f}_{k-1}"] \\
Q'_k \arrow[r, Rrightarrow] & ([P'_p]_{Q'_p})(P'_{p+1})_{Q'_{p+2}} \dots (P'_{k-1})
\end{tikzcd}
\qquad
\begin{tikzcd}
P_k \arrow[r, Rrightarrow] \arrow[d, "f_k"'] & ((([P_p]_{Q_{p+1}})(P_{p+1}))_{Q_{p+2}} \dots (P_{k-1}))_{Q_k} \arrow[d, "{[f_{k-1}]_{g_k}}"] \\
P'_k \arrow[r, Rrightarrow] & ((([P'_p]_{Q'_{p+1}})(P'_{p+1}))_{Q'_{p+2}} \dots (P'_{k-1}))_{Q'_k}
\end{tikzcd}
$$

for any $p+2 \leqslant k \leqslant n$, where the map $\widetilde{f}_{k-1}$ is induced by the map f_{k-1} and the map $[f_{k-1}]_{g_k}$ is defined by the following commutative diagram:

$$
\begin{tikzcd}
(([P_p]_{Q_{p+1}})(P_{p+1}))_{Q_{p+2}} \dots (P_{k-1}) \arrow[r, "\pi"] \arrow[d, "\widetilde{f}_{k-1}"'] & ((([P_p]_{Q_{p+1}})(P_{p+1}))_{Q_{p+2}} \dots (P_{k-1}))_{Q_k} \arrow[d, "{[f_{k-1}]_{g_k}}"] \\
(([P'_p]_{Q'_{p+1}})(P'_{p+1}))_{Q'_{p+2}} \dots (P'_{k-1}) \arrow[r, "\pi'"'] & ((([P'_p]_{Q'_{p+1}})(P'_{p+1}))_{Q'_{p+2}} \dots (P'_{k-1}))_{Q'_k}
\end{tikzcd}
$$

We will denote by $\mathsf{DiPol}_{(n,p)}$ the category of (n,p)-dipolygraphs and their morphisms.

2.1.7. Presentations by dipolygraphs. The $(n-1)$-category presented by an n-dipolygraph (P,Q) is defined by

$$\overline{(P,Q)} := ([P_{n-1}]_{Q_n})_{P_n}.$$

Let $\mathcal{C}$ be an $(n-1)$-category. A *presentation of* $\mathcal{C}$ is an n-dipolygraph (P,Q) whose presented category $\overline{(P,Q)}$ is isomorphic to $\mathcal{C}$. A *coherent presentation of* $\mathcal{C}$ is an $(n+1,n-1)$-dipolygraph (P,Q) satisfying the following conditions

i) the underlying n-dipolygraph $(P,Q)_{\leqslant n}$ is a presentation of $\mathcal{C}$,

ii) the cellular extension P_{n+1} is acyclic,

iii) the cellular extension Q_{n+1} is empty.

2.2. Double coherent presentations

In this subsection, we introduce the notion of double coherent presentation of an n-category, defined using the structure of double n-polygraph. Let us first explicit the construction of a free n-category enriched in double categories generated by a double n-polygraph.

2.2.1. What is a free double category like ? The question of the construction of free double categories was considered in several works, [11–14]. In particular, Dawson and Pare gave in [14] constructions of free double categories generated by double graphs and double reflexive graphs. Such free double categories always exist, and they show how to describe their cells explicitly in geometrical terms. However, they show that free double categories generated by double graphs cannot describe many of the possible compositions in free double categories. They fixed this problem by considering double reflexive graphs as generators.

2.2.2. The coherence results that we will state in Section 5 are formulated in free n-categories enriched in double categories generated by double n-polygraphs. For every $n \geqslant 0$, let us consider the forgetful functor

$$W_n : \mathsf{Cat}_n(\mathsf{DbCat}) \to \mathsf{DbPol}_n \tag{2.2.3}$$

that sends an n-category enriched in double categories $\mathcal{C}$ on the double n-polygraph, denoted by

$$W_n(\mathcal{C}) = (W^v_{n+1}(\mathcal{C}), W^h_{n+1}(\mathcal{C}), W^s_{n+2}(\mathcal{C})),$$

where $W^v_{n+1}(\mathcal{C})$ (resp. $W^h_{n+1}(\mathcal{C})$) is the underlying $(n+1)$-polygraph of the $(n+1)$-category obtained as the extension of the underlying n-category of $\mathcal{C}$ by the vertical (resp. horizontal) $(n+1)$-cells and $W^s_{n+2}(\mathcal{C})$ is the square extension generated by all squares of $\mathcal{C}$. Explicitly, for $\mu \in \{v, h\}$, consider $\mathcal{C}^\mu_{n+1}$ the $(n+1)$-category whose

i) underlying $(n-1)$-category coincides with the underlying $(n-1)$-category of $\mathcal{C}$,

ii) set of n-cells is given by

$$(\mathcal{C}^\mu_{n+1})_n := \coprod_{x,y \in \mathcal{C}_{n-1}} (\mathcal{C}_n(x,y))^o,$$

iii) set of $(n+1)$-cells is given by

$$(\mathcal{C}^\mu_{n+1})_{n+1} := \coprod_{x,y \in \mathcal{C}_{n-1}} (\mathcal{C}_n(x,y))^\mu.$$

The $(n-1)$-composition of n-cells and $(n+1)$-cells of $\mathcal{C}^\mu_{n+1}$ are defined by enrichment. The n-composition of $(n+1)$-cells of $\mathcal{C}^\mu_{n+1}$ are induced by the composition $\circ^\mu$. We define $W^\mu_{n+1}(\mathcal{C})$ as the underlying $(n+1)$-polygraph of the $(n+1)$-category $\mathcal{C}^\mu_{n+1}$:

$$W^\mu_{n+1}(\mathcal{C}) := U^{\mathrm{Pol}}_{n+1}(\mathcal{C}^\mu_{n+1}).$$

Finally, the square extension $W^s_{n+2}(\mathcal{C})$ is defined on the pair of $(n+1)$-categories $(\mathcal{C}^v_{n+1}, \mathcal{C}^h_{n+1})$ by

$$W^s_{n+2}(\mathcal{C}) := \coprod_{x,y \in \mathcal{C}_{n-1}} \mathcal{C}_n(x,y)^s.$$

2.2.4. Proposition. *For every $n \geqslant 0$, the forgetful functor W_n defined in (2.2.3) admits a left adjoint functor F_n.*

The proof of this result consists in constructing explicitly in 2.2.5 the free n-category enriched in double categories generated by a double n-polygraph and the proof in 2.2.6 of universal property of free object.

2.2.5. Free n-category enriched in double categories. Consider a double n-polygraph $P = (P^v, P^h, P^s)$. We construct the *free n-category enriched in double categories on* P, denoted by $P^\square$, as follows:

i) the underlying n-category of $P^\square$ is the free n-category P_n^*,

ii) for all $(n-1)$-cells x and y of P_{n-1}^*, the hom-double category $P^\square(x,y)$ is constructed as follows

a) the point cells of $P^\square(x,y)$ are the n-cells in $P_n^*(x,y)$,

b) the vertical cells of $P^\square(x,y)$ are the $(n+1)$-cells of the free $(n+1)$-category $(P^v)^*$ with $(n-1)$-source x and $(n-1)$-target y,

c) the horizontal cells of $P^\square(x,y)$ are the $(n+1)$-cells of the free $(n+1)$-category $(P^h)^*$ with $(n-1)$-source x and $(n-1)$-target y,

d) the set of square cells of $P^\square(x,y)$ is defined recursively and contains

– the square cells A of P^s such that $\partial_{-,n-1}(A) = x$ and $\partial_{+,n-1}(A) = y$,

– the square cells $C[A]$ for any context C of the n-category P_n^* and A in P^s, such that $\partial_{-,n-1}(C[A]) = x$ and $\partial_{+,n-1}(C[A]) = y$,

– identities square cells $i_1^h(f)$ and $i_1^v(e)$, for any $(n+1)$-cells f in $(P^h)^*$ and $(n+1)$-cell e in $(P^v)^*$ whose $(n-1)$-source (resp. $(n-1)$-target) in P_{n-1}^* is x (resp. y),

– all formal pastings of these elements with respect to $\diamond^h$-composition and $\diamond^v$-composition.

e) two square cells constructed as such formal pastings are identified by the associativity, and identity axioms of compositions $\diamond^v$ and $\diamond^h$ and middle four interchange law given in (1.2.4),

iii) for all $(n-1)$-cells x, y, z of P_{n-1}^*, the composition functor

$$\star_{n-1} : P^\square(x,y) \times P^\square(y,z) \longrightarrow P^\square(x,z)$$

is defined for any

$$\begin{tikzcd} u_1 \arrow[r, "f_1"] \arrow[d, "e_1"'] \arrow[dr, phantom, "\Downarrow A_1"] & v_1 \arrow[d, "e_1'"] \\ u_1' \arrow[r, "g_1"'] & v_1' \end{tikzcd} \quad \text{in } P^\square(x,y), \text{ and } \quad \begin{tikzcd} u_2 \arrow[r, "f_2"] \arrow[d, "e_2"'] \arrow[dr, phantom, "\Downarrow A_2"] & v_2 \arrow[d, "e_2'"] \\ u_2' \arrow[r, "g_2"'] & v_2' \end{tikzcd} \quad \text{in } P^\square(y,z),$$

by

$$\begin{tikzcd} u_1 \star_{n-1} u_2 \arrow[r, "f_1 \star_{n-1} f_2"] \arrow[d, "e_1 \star_{n-1} e_2"'] \arrow[dr, phantom, "\Downarrow A_1 \star_{n-1} A_2"] & v_1 \star_{n-1} v_2 \arrow[d, "e_1' \star_{n-1} e_2'"] \\ u_1' \star_{n-1} u_2' \arrow[r, "g_1 \star_{n-1} g_2"'] & v_1' \star_{n-1} v_2' \end{tikzcd}$$

where the square cell $A_1 \star_{n-1} A_2$ is defined recursively using exchanges relations (1.2.8-1.2.9) from functoriality of the composition $\star_{n-1}$, and the middle four identities (1.2.4),

iv) for all $(n-1)$-cell x of P_{n-1}^*, the identity map $\mathbf{T} \longrightarrow P^\square(x,x)$, where $\mathbf{T}$ is the terminal double groupoid, sends the one point cell $\bullet$ on x and the identity $i_\alpha^\mu(\bullet)$ on $i_\alpha^\mu(x)$ for all $\mu \in \{v,h\}$ and $\alpha \in \{0,1\}$.

2.2.6. The functor $F_n : \mathsf{DbPol}_n \to \mathsf{Cat}_n(\mathsf{DbCat})$ defined by $F_n(P) = P^{\square}$ for any double n-polygraph P satisfies the universal property of a free object in $\mathsf{Cat}_n(\mathsf{DbCat})$. Indeed, given a double n-polygraph $P = (P^v, P^h, P^s)$, a morphism $\eta_P : P \to W_n(F_n(P))$ of double n-polygraphs, an n-category enriched in double categories $\mathcal{C}$, and a morphism $\varphi : P \to W_n(\mathcal{C})$ of double n-polygraphs, there exists a unique enriched morphism $\widetilde{\varphi} : F_n(P) \to \mathcal{C}$ such that the following diagram commutes

$$\begin{tikzcd} P \arrow[r, "\eta_P"] \arrow[dr, "\varphi"'] & W_n(F_n(P)) \arrow[d, "W_n(\widetilde{\varphi})"] \\ & W_n(\mathcal{C}) \end{tikzcd}$$

The functor $\widetilde{\varphi} = (\widetilde{\varphi}_k)_{0 \leqslant k \leqslant n+2}$ is defined as follows.

i) By construction, the morphism φ induces morphisms of $(n+1)$-polygraphs $\varphi^{\mu} : P^{\mu} \to W^{\mu}_{n+1}(\mathcal{C})$, for $\mu \in \{v, h\}$. The morphism φ^{μ} extends by universal property of free $(n+1)$-categories into a functor $\widetilde{\varphi}^{\mu} : (P^{\mu})^* \to \mathcal{C}^{\mu}_{n+1}$. We set $\widetilde{\varphi}_k = \varphi^v_k = \varphi^h_k$ for $0 \leqslant k \leqslant n$, and

$$\widetilde{\varphi}_{n+1}(f) = \varphi^h(f), \qquad \widetilde{\varphi}_{n+1}(e) = \varphi^v(e),$$

for every horizontal $(n+1)$-cell f and every vertical $(n+1)$-cell e.

ii) By construction, any square $(n+2)$-cell A in $F_n(P)$ is a composite of generating square $(n+2)$-cells in P^s with respect to the compositions $\diamond^v$, $\diamond^h$ and $\star_{n-1}$. Moreover, following [13, Theorem 1.2], if a compatible arrangement of square cells in a double category is composable in two different ways, the results are equal modulo the associativity, identity axioms of compositions $\diamond^v$ and $\diamond^h$, and middle four interchange law (1.2.4). We extend the functor φ to the functor $\widetilde{\varphi}$ by setting

$$\widetilde{\varphi}(A \diamond^{\mu} B) = \varphi(A) \diamond^{\mu} \varphi(B), \qquad \widetilde{\varphi}(A \star_{n-1} B) = \varphi(A) \star_{n-1} \varphi(B),$$

for every $\mu \in \{v, h\}$ and all square generating $(n+2)$-cells A, B in P^s whenever the composites are defined.

2.2.7. Free n-categories enriched in double groupoids. By a similar construction to the free n-category enriched in double categories on a double n-polygraph $P = (P^v, P^h, P^s)$ given in 2.2.5, we construct the free n-category enriched in double groupoids generated by a double $(n+2, n)$-polygraph $P = (P^v, P^h, P^s)$, that we denote by $P^{\top\!\top}$. It is obtained as the free n-category enriched in double categories $P^{\square}$ having in addition

- inverse vertical $(n+1)$-cells e^- for any generating vertical $(n+1)$-cell e,
- inverse horizontal $(n+1)$-cells f^- for any generating vertical $(n+1)$-cell f,
- inverse square $(n+2)$-cells $A^{-,\mu}$ for any generating square $(n+2)$-cell A in P^s,

that satisfy the inverses axioms of groupoids for vertical and horizontal cells and the relations (1.2.5) for square cells.

Finally, we will also consider the free n-category enriched in double categories, whose vertical category is a groupoid, generated by a double n-polygraph $P = (P^v, P^h, P^s)$ in DbPol^v, that we denote by $P^{\top\!\top,v}$. In that case, we only require the invertibility of vertical $(n+1)$-cells and the invertibility of square $(n+2)$-cells with respect to $\diamond^h$-composition.

2.2.8. Acyclicity. Let $P = (P^v, P^h, P^s)$ be a double $(n+2,n)$-polygraph. The square extension P^s of the pair of $(n+1,n)$-categories $((P^v)^\top, (P^h)^\top)$ is *acyclic* if for any square S over $((P^v)^\top, (P^h)^\top)$ there exists a square $(n+2)$-cell A in the free n-category enriched in double groupoids $P^{\top\!\!\top}$ such that $\partial(A) = S$. For example, the set of squares over $((P^v)^\top, (P^h)^\top)$ forms an acyclic extension.

2.2.9. Double coherent presentations of n-categories. Recall that a *presentation of an n-category* $\mathcal{C}$ is an $(n+1)$-polygraph P whose presented category $\overline{P}$ is isomorphic to $\mathcal{C}$. We define a *double coherent presentation of* $\mathcal{C}$ as a double $(n+2,n)$-polygraph (P^v, P^h, P^s) satisfying the two following conditions:

i) the $(n+1)$-polygraph $(P_n, P^v_{n+1} \cup P^h_{n+1})$ is a presentation of $\mathcal{C}$, where P_n is the underlying n-polygraph of P^v and P^h,

ii) the square extension P^s is acyclic.

2.3. Globular coherent presentations from double coherent presentations

2.3.1. We define a quotient functor

$$V : \mathsf{DbPol}_{(n+2,n)} \to \mathsf{DiPol}_{(n+2,n)} \tag{2.3.2}$$

that sends a double $(n+2,n)$-polygraph $P = (P^v, P^h, P^s)$ to the $(n+2,n)$-dipolygraph

$$V(P) = ((P_0, \ldots, P_{n+2}), (Q_1, \ldots, Q_{n+2})) \tag{2.3.3}$$

defined as follows:

i) $(P_0, \ldots, P_n)$ is the underlying n-polygraph $P^v_{\leqslant n} = P^h_{\leqslant n} := P_n$,

ii) for every $1 \leqslant i \leqslant n$, the cellular extension Q_i is empty,

iii) Q_{n+1} is the cellular extension $P^v_{n+1} \overset{\partial^v_{-,n}}{\underset{\partial^v_{+,n}}{\rightrightarrows}} P^*_n$,

iv) P_{n+1} is the cellular extension $P^h_{n+1} \overset{\widetilde{\partial}^h_{-,n}}{\underset{\widetilde{\partial}^h_{+,n}}{\rightrightarrows}} (P^*_n)_{P^v_{n+1}}$, where the maps $\widetilde{\partial}^h_{-,n}$ and $\widetilde{\partial}^h_{+,n}$ are defined by

$$\widetilde{\partial}^h_{\mu,n} = \partial^h_{\mu,n} \circ \pi,$$

for any μ in $\{-,+\}$, where $\pi : P^*_n \twoheadrightarrow (P^*_n)_{P^v_{n+1}}$ denotes the canonical projection sending an n-cell u in P^*_n on its class, denoted by $[u]^v$, modulo P^v_{n+1}. Moreover, for any $f : u \to v$ in P^h_{n+1}, we will denote by $[f]^v : [u]^v \to [v]^v$ the corresponding element in P_{n+1},

v) the cellular extension Q_{n+2} is empty,

vi) P_{n+2} is defined as the cellular extension $P^s \overset{\check{s}}{\underset{\check{t}}{\rightrightarrows}} (P^*_n)_{P^v_{n+1}}(P^h_{n+1})$, where the maps $\check{s}$ and $\check{t}$ are defined by the following commutative diagrams:

$$\begin{tikzcd}
P_s \arrow[d, shift left, "\partial^h_{-,n+1}"] \arrow[d, shift right, "\partial^h_{+,n+1}"'] \arrow[dr, shift left, "\check{s}"] \arrow[dr, shift right, "\check{t}"'] & \\
(P^h_{n+1})^\top \arrow[r, "F"'] \arrow[d, shift left, "\partial^h_{-,n}"] \arrow[d, shift right, "\partial^h_{+,n}"'] & (P^*_n)_{P^v_{n+1}}(P^h_{n+1}) \arrow[d, shift left, "\overline{\overline{\partial}}^h_{-,n}"] \arrow[d, shift right, "\overline{\overline{\partial}}^h_{+,n}"'] \\
P^*_n \arrow[r, "\pi"'] & (P^*_n)_{P^v_{n+1}}
\end{tikzcd}$$

where the maps $\overline{\overline{\partial}}^h_{-,n}$ and $\overline{\overline{\partial}}^h_{+,n}$ are induced from $\widetilde{\partial}^h_{-,n}$ and $\widetilde{\partial}^h_{+,n}$, and the $(n+1)$-functor F is defined by:

a) F is the identity functor on the underlying $(n-1)$-category P^*_{n-1},

b) F sends an n-cell u in P^*_n to its equivalence class $[u]^v$ modulo P^v_{n+1},

c) F sends an $(n+1)$-cell $f : u \to v$ in $(P^h_{n+1})^\top$ to the $(n+1)$-cell $[f]^v : [u]^v \to [v]^v$ in $(P^*_n)_{P^v_{n+1}}(P^h_{n+1})$ defined as follows

- for any f in P^h_{n+1}, $[f]^v$ is defined by **iv)**,
- F is extended to the $(n+1)$-cells of $(P^h_{n+1})^\top$ by functoriality by setting

$$[x_n \star_n \ldots (x_1 \star_0 g \star_0 y_1) \ldots \star_n y_n]^v = [x_n]^v \star_n x_{n-1} \star_n \ldots (x_1 \star_0 [g]^v \star_0 y_1) \ldots \star_n y_{n-1} \star_n [y_n]^v$$

for all whisker $x_n \star_n \ldots (x_1 \star_0 - \star_0 y_1) \ldots \star_n y_n$ of $(P^h_{n+1})^\top$ and $(n+1)$-cell g in $(P^h_{n+1})^\top$, and

$$[f_1 \star_n f_2]^v = [f_1]^v \star_n [f_2]^v,$$

for all $(n+1)$-cells f_1, f_2 in $(P^h_{n+1})^\top$.

2.3.4. Given a generating square $(n+2)$-cell

$$\begin{tikzcd}
u \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow A"] & u' \arrow[d, "k"] \\
v \arrow[r, "h"'] & v'
\end{tikzcd}$$

of P^s, we denote by $[A]^v$ the generating $(n+2)$-cell of the globular cellular extension P_{n+2} on $(P_n^*)_{P_{n+1}^v}(P_{n+1}^h)$ defined in (2.3.3) as follows:

$$\begin{tikzcd}[u]^v = [u']^v \arrow[rr, bend left, "{[f]^v}"] \arrow[rr, bend right, "{[g]^v}"'] \arrow[rr, phantom, "\Downarrow [A]^v"] & & {[v]^v = [v']^v}\end{tikzcd}$$

Note that by construction in the $(n+2,n)$-category $((P_n^*)_{P_{n+1}^v}(P_{n+1}^h))(P_{n+2})$ the following relations hold

$$[A]^v \star_n [A']^v = [A \diamond^v A']^v, \qquad [A]^v \star_{n+1} [A']^v = [A \diamond^h A']^v,$$

for all generating square $(n+2)$-cells A and A' in P^s such that these compositions make sense.

2.3.5. Proposition. *Let* $P = (P^v, P^h, P^s)$ *be a double* $(n+2,n)$*-polygraph. If the square extension* P^s *is acyclic then the cellular extension* P_{n+2} *of the* $(n+1)$*-category* $(P_n^*)_{P_{n+1}^v}(P_{n+1}^h)$ *defined in (2.3.3) is acyclic.*

In particular, if P *is a double coherent presentation of an* n*-category* $\mathcal{C}$*. Then, the* $(n+2,n)$*-dipolygraph* $V(P)$ *is a globular coherent presentation of the quotient* n*-category* $(P_n^*)_{P_{n+1}^v}$*, that is the* n*-category is isomorphic to* $\overline{V(P)}_{\leqslant(n+1)}$ *and* P_{n+2} *is an acyclic extension of* $(P_n^*)_{P_{n+1}^v}(P_{n+1}^h)$.

Proof. Given an $(n+1)$-sphere $\gamma := ([f]^v, [g]^v)$ in $(P_n^*)_{P_{n+1}^v}(P_{n+1}^h)$, by definition of the functor V defined in (2.3.2), there exists an $(n+1)$-square

$$S := \begin{tikzcd} u \arrow[r, "f"] \arrow[d, "e"'] & u' \arrow[d, "e'"] \\ v \arrow[r, "g"'] & v' \end{tikzcd}$$

in $((P_{n+1}^v)^\top, (P_{n+1}^h)^\top)$, such that $F(f) = [f]^v$ and $F(g) = [g]^v$ and $V(S) = \gamma$. By acyclicity assumption, there exists a square $(n+2)$-cell A in the free n-category enriched in double groupoids $(P^v, P^h, P^s)^{\top\!\!\top}$ such that $\partial(A) = S$. Then $[A]^v$ is an $(n+2)$-cell in $(P_n^*)_{P_{n+1}^v}(P_{n+1}^h))(P_{n+2})$ such that $\partial([A]^v) = \gamma$. Finally, the fact that $V(P)_{\leqslant(n+1)}$ is a presentation of the quotient n-category $(P_n^*)_{P_{n+1}^v}$ follows from the definition of the functor V and the fact that the $(n+1)$-polygraph $(P_n, P_{n+1}^v \cup P_{n+1}^h)$ is a presentation of $\mathcal{C}$. $\square$

2.4. Examples

We illustrate how to define coherent presentations of algebraic structures in terms of dipolygraphs on the cases of groups, commutative monoids and pivotal categories.

2.4.1. Coherent presentations of groups. A presentation of a group G is defined by a set X of generators and a set R of relations equipped with a map from R to the free group $F(X)$ on X such that G is isomorphic to the quotient of $F(X)$ by the normal subgroup generated by R. The free group $F(X)$ can be presented by the 2-polygraph, denoted by $\mathsf{Gp}_2(X)$, with only one 0-cell, its set of generating 1-cells is $X \cup X^-$, where $X^- := \{x^- \mid x \in X\}$ and its generating 2-cells are

$$xx^- \Rightarrow 1, \qquad x^-x \Rightarrow 1,$$

for any x in X. A coherent presentation of the group G is a $(3,1)$-dipolygraph (P, Q) such that:

i) (P_0, P_1, Q_2) is the 2-polygraph $\mathsf{Gp}_2(X)$, and the cellular extension Q_1 is empty,

ii) the cellular extension P_2 of $F(X)$ has for generating set R, its source map is the identity and its target is constant equal to 1,

iii) the cellular extension Q_3 is empty, and P_3 is an acyclic extension of the 2-group $(F(X))(R)$.

2.4.2. Coherent presentation of commutative monoids. A presentation of a commutative monoid M is defined by a set X of generators and a cellular extension R of relations on the free commutative monoid $\langle X \rangle$ on X such that M is isomorphic to the quotient of $\langle X \rangle$ by the congruence generated by R. The free commutative monoid $\langle X \rangle$ on X can be defined by the 2-polygraph, denoted by $\mathsf{Com}_2(X)$, with only one 0-cell, its set of generating 1-cells is X, and the generating 2-cells are

$$x_i x_j \Rightarrow x_j x_i$$

for any x_i, x_j in X, such that $x_i > x_j$ for a given total order $>$ on X. A coherent presentation of the commutative monoid M is a $(3,1)$-dipolygraph (P, Q) such that:

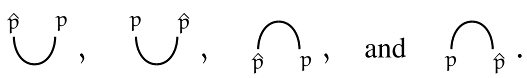
i) (P_0, P_1, Q_2) is the 2-polygraph $\mathsf{Com}_2(X)$, and the cellular extension Q_1 is empty,

ii) $P_2 = R$, Q_3 is empty, and P_3 is an acyclic extension of the 2-category $\langle X \rangle(R)$.

2.4.3. Coherent presentation of monoidal pivotal categories. Recall that a (strict monoidal) pivotal category $\mathcal{C}$ is a monoidal category, seen as 2-category with only one 0-cell, in which every 1-cell p has a right dual 1-cell $\hat{p}$, which is also a left-dual, that is there are 2-cells

$$\eta_p^- : 1 \Rightarrow \hat{p} \star_0 p, \qquad \eta_p^+ : 1 \Rightarrow p \star_0 \hat{p}, \qquad \varepsilon_p^- : \hat{p} \star_0 p \Rightarrow 1, \qquad \text{and} \qquad \varepsilon_p^+ : p \star_0 \hat{p} \Rightarrow 1, \tag{2.4.4}$$

respectively represented by the following diagrams:

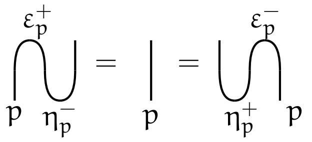


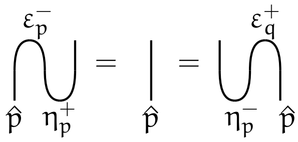
(2.4.5)

These 2-cells satisfy the relations

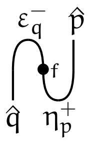
$$(\varepsilon_p^+ \star_0 1_p) \star_1 (1_p \star_0 \eta_p^-) = 1_p = (1_p \star_0 \varepsilon_p^-) \star_1 (\eta_p^+ \star_0 1_p)$$
$$(\varepsilon_p^- \star_0 1_{\hat{p}}) \star_1 (1_{\hat{p}} \star_0 \eta_p^+), = 1_{\hat{p}} = (1_{\hat{p}} \star_0 \eta_p^+) \star_1 (\eta_p^- \star_0 1_{\hat{p}}),$$

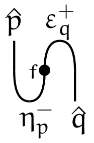
that can be diagrammatically depicted as follows





Any 2-cell $f : p \Rightarrow q$ in $\mathcal{C}$ is cyclic with respect to the biadjunctions $\hat{p} \vdash p \vdash \hat{p}$ and $\hat{q} \vdash q \vdash \hat{q}$ defined respectively by the family of 2-cells $(\eta_p^-, \eta_p^+, \varepsilon_p^-, \varepsilon_p^+)$ and $(\eta_q^-, \eta_q^+, \varepsilon_q^-, \varepsilon_q^+)$, that is $f^* = {}^*f$, where f^* and ${}^*f$ are respectively the right and left duals of f, defined using the right and left adjunction as follows:

${}^*f :=$ 

$f^* :=$ 

We refer the reader to [10, 25] for more details about the notion of pivotal monoidal category.

A presentation of a pivotal category $\mathcal{C}$ is defined by a set X_1 of generating 1-cells, a set X_2 of generating cyclic 2-cells, and a cellular extension R on the free pivotal category $\mathcal{P}(X_1, X_2)$ on the data (X_1, X_2), such that $\mathcal{C}$ is isomorphic to the quotient of $\mathcal{P}(X_1, X_2)$ by the congruence generated by R. The free pivotal category $\mathcal{P}(X_1, X_2)$ can be presented by the 3-polygraph $\mathsf{Piv}_3(X_1, X_2)$ defined as follows

i) it has only one 0-cell,

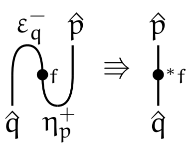
ii) its set of generating 1-cells is $X_1 \cup \widehat{X}_1$, where $\widehat{X}_1 := \{\hat{p} \mid p \in X_1\}$,

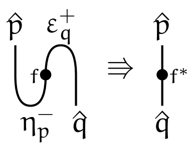
iii) its set of generating 2-cells is

$$X_2 \ \cup \ \{\eta_p^-, \eta_p^+, \varepsilon_p^-, \varepsilon_p^+ \ \mid \ p \in X_1\},$$

where the 2-cells η_p^-, η_p^+, ε_p^-, ε_p^+ are defined by (2.4.4),

iv) its generating 3-cells are





for any generating 2-cell f in X_2 or f is an identity cell.

A coherent presentation of the pivotal category $\mathcal{C}$ is a $(4, 2)$-dipolygrah (P, Q) such that:

i) (P_0, P_1, P_2, Q_3) is the 3-polygraph $\mathsf{Piv}_3(X_1, X_2)$ and the cellular extensions Q_1 and Q_2 are empty,

ii) $P_3 = R$, Q_4 is empty and P_4 is an acyclic extension of the 2-category $\mathcal{P}(X_1, X_2)(R)$.

3. Polygraphs modulo

In this section, we introduce the notion of polygraph modulo and we define the rewriting properties of termination, confluence and local confluence for these polygraphs.

3.1. Polygraphs modulo

3.1.1. Cellular extensions modulo. Consider two n-polygraphs E and R such that $E_{\leqslant n-2} = R_{\leqslant n-2}$ and $E_{n-1} \subseteq R_{n-1}$. One defines the cellular extension

$$\gamma^{\,_E R} : {}_E R \to \mathrm{Sph}_{n-1}(R^*_{n-1}),$$

where the set ${}_E R$ is defined by the following pullback in $\mathbf{Set}$:

$$\begin{tikzcd} E_n^\top \times_{R^*_{n-1}} R_n^{*(1)} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & R_n^{*(1)} \arrow[d, "\partial_{-,n-1}"] \\ E_n^\top \arrow[r, "\partial_{+,n-1}"'] & R^*_{n-1} \end{tikzcd}$$

and the map $\gamma^{\,_E R}$ is defined by $\gamma^{\,_E R}(e,f) = (\partial_{-,n-1}(e), \partial_{+,n-1}(f))$ for all e in $E^\top$ and f in $R_n^{*(1)}$. Similarly, one defines the cellular extension

$$\gamma^{R_E} : R_E \to \mathrm{Sph}_{n-1}(R^*_{n-1}),$$

where the set R_E is defined by the following pullback in $\mathbf{Set}$:

$$\begin{tikzcd} R_n^{*(1)} \times_{R^*_{n-1}} E_n^\top \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & E_n^\top \arrow[d, "\partial_{-,n-1}"] \\ R_n^{*(1)} \arrow[r, "\partial_{+,n-1}"'] & R^*_{n-1} \end{tikzcd}$$

and the map γ^{R_E} is defined by $\gamma^{R_E}(f,e) = (\partial_{-,n-1}(f), \partial_{+,n-1}(e))$ for all e in $E^\top$ and f in $R_n^{*(1)}$. Finally, one defines the cellular extension

$$\gamma^{\,_E R_E} : {}_E R_E \to \mathrm{Sph}_{n-1}(R^*_{n-1}),$$

where the set ${}_E R_E$ is defined by the following composition of pullbacks in Set:

$$
\begin{tikzcd}
E_n^{\top} \times_{R_{n-1}^*} R_n^{*(1)} \times_{R_{n-1}^*} E_n^{\top} \arrow[r, "(\pi_2, \pi_3)"] \arrow[d, "(\pi_1, \pi_2)"'] & R_n^{*(1)} \times_{R_{n-1}^*} E_n^{\top} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & E_n^{\top} \arrow[d, "\partial_{-,n-1}"] \\
E_n^{\top} \times_{R_{n-1}^*} R_n^{*(1)} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & R_n^{*(1)} \arrow[r, "\partial_{+,n-1}"'] \arrow[d, "\partial_{-,n-1}"] & R_{n-1}^* \\
E_n^{\top} \arrow[r, "\partial_{+,n-1}"'] & R_{n-1}^* &
\end{tikzcd}
$$

and the map $\gamma^{{}_E R_E}$ is defined by $\gamma^{{}_E R_E}(e, f, e') = (\partial_{-,n-1}(e), \partial_{+,n-1}(e'))$.

3.1.2. Polygraphs modulo. A n-*polygraph modulo* is a data (R, E, S) made of

i) an n-polygraph R, whose generating n-cells are called *primary rules*,

ii) an n-polygraph E such that $E_{\leqslant(n-2)} = R_{\leqslant(n-2)}$ and $E_{n-1} \subseteq R_{n-1}$, whose generating n-cells are called *modulo rules*,

iii) S is a cellular extension of R_{n-1}^* such that the inclusions of cellular extensions

$$R \subseteq S \subseteq {}_E R_E$$

holds.

If no confusion may occur, an n-polygraph modulo (R, E, S) will be simply denoted by S. For simplicity of notation, the n-polygraphs modulo $(R, E, {}_E R)$, (R, E, R_E) and $(R, E, {}_E R_E)$ will be denoted by ${}_E R$, R_E and ${}_E R_E$ respectively.

3.1.3. Given an n-polygraph modulo (R, E, S), we will consider in the sequel the following categories:

- the free n-category $R_{n-1}^*[R_n, E_n \coprod E_n^{-1}]/\mathrm{Inv}(E_n, E_n^{-1})$, denoted by $R^*(E)$.
- the free n-category generated by S, denoted by S^*,
- the free $(n, n-1)$-category generated by S, denoted by $S^{\top}$.

For any n-cell f in S^* (resp. $S^{\top}$), the *size* of f is defined as the positive integer $||f||_{R_n}$ corresponding to the number of n-cells of R_n contained in A, and denoted by $\ell(f)$.

3.1.4. Reductions. One says that an $(n-1)$-cell u *reduces* into some $(n-1)$-cell v with respect to an n-polygraph modulo (R, E, S) when there exists a non-identity n-cell from u to v in the free n-category S^*. A *reduction sequence* is a family $(u_k)_k$ of $(n-1)$-cells in R_{n-1}^* such that each u_k reduces to u_{k+1}. A *rewriting step* is an non-identity n-cell f of S^* such that $\ell(f) = 1$, that is with shape $C[\gamma]$, where γ is a generating n-cell of S and C is a whisker of R_{n-1}^*.

3.2. Termination and normal forms

In this subsection, we introduce the property of termination and the notion of normal form for polygraphs modulo. We explain how to prove termination of polygraphs modulo using a termination order compatible modulo rules. Finally, we recall the double induction principle introduced by Huet in [21] that we will use in many proofs in the sequel.

3.2.1. Termination. Recall that an n-polygraph *terminates* if it has no infinite rewriting sequence. An n-polygraph modulo (R, E, S) is *terminating* when the n-polygraph (R_{n-1}, S) is terminating. Note that, when $S \neq R$, ${}_E R$ is terminating iff R_E is terminating iff ${}_E R_E$ is terminating iff S is terminating.

An order relation $\prec$ on R^*_{n-1} is *compatible with* S *modulo* E if it satisfies the two following conditions:

i) $v \prec u$, for any $(n-1)$-cells u, v in R^*_{n-1} such that there exists an n-cell $u \to v$ in S^*,

ii) if $v \prec u$ for $(n-1)$-cells u, v in R^*_{n-1}, then $v' \prec u'$ holds for any $(n-1)$-cells u', v' in R^*_{n-1} such that there exist n-cells $e : u \to u'$ and $e' : v \to v'$ in $E^\top$.

A *termination order for* S *modulo* E is a well-founded order relation compatible with S modulo E.

In this work, many constructions will be based on the termination of the n-polygraph modulo ${}_E R_E$, which can be proved by constructing a termination order for one of the n-polygraphs modulo ${}_E R$, R_E and ${}_E R_E$. It can be also proved by constructing a termination order for R compatible with E.

3.2.2. Normal forms. A $(n-1)$-cell u in R^*_{n-1} is S-*reduced* if it cannot be reduced by n-cells of S. A S-*normal form* for an $(n-1)$-cell u in R^*_{n-1} is a S-reduced $(n-1)$-cell v such that u can be reduced to v with respect to S. We will denote by $\mathrm{Irr}(S)$ the set of S-reduced $(n-1)$-cells of R^*_{n-1}, and by $\mathrm{NF}(S, u)$ the set of S-normal forms of an $(n-1)$-cell u of R^*_{n-1}. If S is terminating, every $(n-1)$-cell has at least one S-normal form.

3.2.3. Noetherian induction. If S is terminating one can prove properties on $(n-1)$-cells of R^*_{n-1} using *Noetherian induction*. For that, one proves the property on normal forms; then one fixes an $(n-1)$-cell u, one assumes that the result holds for every $(n-1)$-cell v such that u reduces into v and one proves, under those hypotheses, that the $(n-1)$-cell u satisfies the property.

Let us recall the double Noetherian induction principle introduced by Huet in [21] to prove the equivalence between confluence modulo and local confluence modulo under a termination hypothesis. We construct an auxiliary n-polygraph S^{II} as follows. One defines

$$S^{\mathrm{II}}_k = S_k \times S_k \text{ for } 0 \leqslant k \leqslant n-1,$$

and S^{II}_n contains an n-cell $(u, v) \to (u', v')$, for all $(n-1)$-cells u, u', v, v' in any of the following situation:

i) there exists an n-cell $u \to u'$ in S^* and $v = v'$;

ii) there exists an n-cell $v \to v'$ in S^* and $u = u'$;

iii) there exist n-cells $u \to u'$ and $u \to v'$ in S^*;

iv) there exist n-cells $v \to u'$ and $v \to v'$ in S^*;

v) there exist n-cells e_1, e_2 and e_3 in $E^\top$ as in the following diagram

$$u \xrightarrow{e_1} v \xrightarrow{e_2} u' \xrightarrow{e_3} v'$$

such that $\ell(e_1) > \ell(e_3)$.

Note that this definition implies that, if there exist n-cells $u \to u'$ and $v \to v'$ in S^*, then there is an n-cell $(u,v) \to (u',v')$ in $S^{\amalg}$ given by the following composition:

$$(u,v) \to (u',v) \to (u',v')$$

Following [21, Proposition 2.2], if S_E is terminating, then so is $S^{\amalg}$. In the sequel, we will apply this Noetherian induction on $S^{\amalg}$ with the following property:

for any n-cells $f : u \to u'$, $g : v \to v'$ in S^* and $e : u \to v$ in $E^\top$, there exist n-cells $f' : u' \to u''$, $g' : v' \to w''$ in S^* and $e' : u'' \to w''$ in $E^\top$, and a square $(n+1)$-cell A in a given $(n-1)$-category enriched in groupoids, as depicted in the following diagram:

$$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, "e"'] & u' \arrow[r, dotted, "f'"] \arrow[d, phantom, "\Downarrow A"] & u'' \arrow[d, dotted, "e'"] \\ v \arrow[r, "g"'] & v' \arrow[r, dotted, "g'"'] & w'' \end{tikzcd}$$

In Section 4, we will formulate this property for such a branching (f,e,g) of S modulo E in terms of coherent confluence modulo E.

3.3. Confluence modulo

In this subsection, we define properties of confluence and local confluence modulo for an n-polygraph modulo (R,E,S), and we explicit a classification of branchings of S modulo E.

3.3.1. Branchings. A *branching* of the n-polygraph modulo S is a pair (f,g), where f and g are n-cells of $S^*$and such that $\partial^h_{-,n-1}(f) = \partial^h_{-,n-1}(g)$. Such a branching is depicted by

$$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, equal] & u' \\ u \arrow[r, "g"'] & v' \end{tikzcd} \tag{3.3.2}$$

and will be denoted by $(f,g) : u \Rightarrow (u',v')$. The $(n-1)$-cell u is called the *source* of this branching. We do not distinguish the branchings (f,g) and (g,f). A *branching modulo* E of the n-polygraph modulo S is a triple (f,e,g) where f and g are n-cells of S^* with f non trivial and e is an n-cell of $E^\top$. Such a branching is depicted by

$$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, "e"'] & u' \\ v \arrow[r, "g"'] & v' \end{tikzcd} \qquad \Bigg(\text{resp.} \quad \begin{tikzcd} u \arrow[r, "f"] \arrow[d, "e"'] & u' \\ v & \end{tikzcd} \quad \Bigg) \tag{3.3.3}$$

when g is non trivial (resp. trivial) and denoted by

$$(f, e, g) : (u, v) \Rightarrow (u', v') \qquad \text{(resp. } (f, e) : u \Rightarrow (u', v)\text{).}$$

The pair of $(n-1)$-cells (u, v) (resp. (u, u)) is called the *source* of this branching modulo E. Note that any branching (f, g) is a branching modulo E (f, e, g) where $e = i_1^v(\partial^h_{-,(n-1)}(f)) = i_1^v(\partial^h_{-,(n-1)}(g))$.

3.3.4. Confluence and confluence modulo. A *confluence* of the n-polygraph modulo S is a pair (f', g') of n-cells of S^* such that $\partial^h_{+,(n-1)}(f') = \partial^h_{+,(n-1)}(g')$. Such a confluence is depicted by

$$\begin{tikzcd} u' \arrow[r, "f'"] & w \arrow[d, equal] \\ v' \arrow[r, "g'"'] & w \end{tikzcd}$$

and denoted by $(f', g') : (u', v') \Rightarrow w$. A *confluence modulo* E of the n-polygraph modulo S is a triple (f', e', g'), where f', g' are n-cells of S^* and e' is an n-cell of $E^\top$ such that $\partial^h_{+,(n-1)}(f') = \partial^v_{-,(n-1)}(e')$ and $\partial^h_{+,(n-1)}(g') = \partial^v_{+,(n-1)}(e')$. Such a confluence is depicted by

$$\begin{tikzcd} u' \arrow[r, "f'"] & w \arrow[d, "e'"] \\ v' \arrow[r, "g'"'] & w' \end{tikzcd}$$

and denoted by $(f', e', g') : (u', v') \Rightarrow (w, w')$.

A branching as in (3.3.2) is *confluent* (resp. *confluent modulo* E) if there exist n-cells f', g' in S^* and e' in $E^\top$ as in the following diagrams:

$$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, equal] & u' \arrow[r, dotted, "f'"] & w \arrow[d, equal] \\ u \arrow[r, "g"'] & v' \arrow[r, dotted, "g'"'] & w' \end{tikzcd} \qquad \Bigg(\text{resp.} \quad \begin{tikzcd} u \arrow[r, "f"] \arrow[d, equal] & u' \arrow[r, dotted, "f'"] & w \arrow[d, dotted, "e'"] \\ u \arrow[r, "g"'] & v' \arrow[r, dotted, "g'"'] & w' \end{tikzcd} \Bigg).$$

A branching modulo E as in (3.3.3) is *confluent modulo* E if there exist n-cells f', g' in S^* and e' in $E^\top$ as in the following diagram:

$$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, "e"'] & u' \arrow[r, dotted, "f'"] & w \arrow[d, dotted, "e'"] \\ v \arrow[r, "g"'] & v' \arrow[r, dotted, "g'"'] & w' \end{tikzcd}.$$

We say that the n-polygraph modulo S is *confluent* (resp. *confluent modulo* E) if all of its branchings (resp. branchings modulo E) are confluent (resp. confluent modulo E). Note that when S is confluent, every $(n-1)$-cell of S^* has at most one normal form with respect to S. Under the confluence modulo hypothesis, an $(n-1)$-cell may admit several S-normal forms, which are all equivalent modulo E.

3.3.5. Diconvergence. The n-polygraph modulo S is called *convergent* if it is both terminating and confluent. It is called *convergent modulo* E when it is confluent modulo E and ${}_E R_E$ is terminating. We say that S is *diconvergent* when E is convergent and S is convergent modulo E.

3.3.6. JK confluence and JK coherence. Finally, let us recall the notion of confluence modulo introduced by Jouannaud and Kirchner in [23]. We say that the n-polygraph modulo S is

i) *JK confluent modulo* E, if any branching is confluent modulo E,

ii) *JK coherent modulo* E, if for any branching modulo E $(f, e) : u \Rightarrow (u', v)$ is confluent modulo E:

$$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, "e"'] & v \arrow[r, dotted, "f'"] & v' \arrow[d, dotted, "e'"] \\ u' \arrow[rr, dotted, "g'"'] & & w \end{tikzcd}$$

in such a way that g' is a non-trivial n-cell in S^*.

3.3.7. Local branchings. A branching (f, g) of the n-polygraph modulo S is *local* if f, g are n-cells of $S^{*(1)}$. A branching (f, e, g) modulo E is *local* if f is an n-cell of $S^{*(1)}$, g is an n-cell of S^* and e an n-cell of $E^\top$ such that $\ell(g) + \ell(e) = 1$. Local branchings belong to one of the following families:

i) *local aspherical* branchings of the form:

$$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, "="'] & v \arrow[d, "="] \\ u \arrow[r, "f"'] & v \end{tikzcd}$$

where f is an n-cell of $S^{*(1)}$;

ii) *local Peiffer* branchings of the form:

$$\begin{tikzcd} u \star_i v \arrow[r, "f \star_i v"] \arrow[d, "="'] & u' \star_i v \\ u \star_i v \arrow[r, "u \star_i g"'] & u \star_i v' \end{tikzcd}$$

where $0 \leqslant i \leqslant n-2$, f and g are n-cells of $S^{*(1)}$,

iii) *local Peiffer modulo* of the forms:

$$\begin{tikzcd} u \star_i v \arrow[r, "f \star_i v"] \arrow[d, "u \star_i e"'] & u' \star_i v \\ u \star_i v' & \end{tikzcd} \qquad \begin{tikzcd} w \star_i u \arrow[r, "w \star_i f"] \arrow[d, "e' \star_i u"'] & w \star_i u' \\ w' \star_i u & \end{tikzcd} \tag{3.3.8}$$

where $0 \leqslant i \leqslant n-2$, where f is an n-cell of $S^{*(1)}$ and e, e' are n-cells of $E^{\top(1)}$;

iv) *overlapping branchings* are the remaining local branchings:

$$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, equal] & v \\ u \arrow[r, "g"'] & v' \end{tikzcd}$$

where f and g are n-cells of $S^{*(1)}$,

v) *overlapping branchings modulo* are the remaining local branchings modulo:

$$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, "e"'] & v \\ v' & \end{tikzcd} \tag{3.3.9}$$

where f is an n-cell of $S^{*(1)}$ and e is an n-cell of $E^{\top(1)}$.

Let (f, g) (resp. (f, e, g)) be a branching (resp. branching modulo E) of the n-polygraph modulo S with source u (resp. (u, v)) and a whisker $C[\partial u]$ of R^*_{n-1} composable with u and v, the pair $(C[f], C[g])$ (resp. triple $(C[f], C[e], C[g])$) is a branching (resp. branching modulo E) of the n-polygraph modulo S. If the branching (f, e, g) is local, then the branching $(C[f], C[e], C[g])$ is local. We denote by $\sqsubseteq$ the order relation on branchings modulo E of S defined by $(f, e, g) \sqsubseteq (f', e', g')$ when there exists a whisker C of R^*_{n-1} such that $(C[f], C[e], C[g]) = (f', e', g')$ hold. A branching (resp. branching modulo E) is *minimal* if it is minimal for the order relation $\sqsubseteq$. A branching (resp. branching modulo E) is *critical* if it is an overlapping branching that is minimal for the relation $\sqsubseteq$.

3.3.10. Local confluence modulo. The n-polygraph modulo S is *locally confluent modulo* E if any of its local branchings modulo E is confluent modulo E. Note that following [23], there exists a local version of JK-confluence modulo E and JK coherence modulo E, given by properties **a)** and **b)** of Proposition 4.2.1, and we will prove in the next section that all these notions are equivalent.

3.4. Completion procedure for ${}_E R$

In this subsection, we give a completion procedure for an n-polygraph modulo $(R, E, {}_E R)$, when ${}_E R$ is not confluent modulo E. The procedure computes an n-polygraph $\check{R}$ such that ${}_E \check{R}$ is confluent modulo E.

3.4.1. Completion of ${}_E R$ modulo E. Note that the property of JK coherence is trivially satisfied for ${}_E R$. Indeed, any branching (f, e) of ${}_E R$ modulo E is trivially confluent modulo E as follows:

$$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, "e"'] & v \arrow[d, equal] \\ v' \arrow[r, "e^- \cdot f"'] & v \end{tikzcd} \tag{3.4.2}$$

where $e^{-} \cdot f$ is a rewriting step of ${}_{E}R$. Following the critical branching lemma modulo, Theorem 4.2.2 given in the next section, we describe a completion procedure for confluence of ${}_{E}R$ modulo E in terms of critical branchings, similar to the Knuth-Bendix completion. From (3.4.2) and Theorem 4.2.2, when ${}_{E}R$ is terminating, ${}_{E}R$ is confluent modulo E if and only if all critical branchings (f, g) of ${}_{E}R$ modulo E with f in $({}_{E}R)^{*(1)}$ and g in $R^{*(1)}$ are confluent modulo E, as depicted by:

$$\begin{tikzcd}
u \arrow[r, "f \in ({}_{E}R)^{*(1)}"] \arrow[d, equal] & v \arrow[r, dotted, "f' \in ({}_{E}R)^{*}"] & v' \arrow[d, dotted, "e'"] \\
u \arrow[r, "g \in R^{*(1)}"'] & w \arrow[r, dotted, "g' \in ({}_{E}R)^{*}"'] & w'
\end{tikzcd}$$

We denote by $\mathrm{CP}({}_{E}R, R)$ the set of such critical branchings.

3.4.3. Completion procedure for ${}_{E}R$. Let us consider R and E two n-polygraphs such that $E_{\leqslant n-2} = R_{\leqslant n-2}$ and $E_{n-1} \subseteq R_{n-1}$, and $\prec$ a termination order compatible with R modulo E. In this paragraph, we describe a procedure to compute a completion $\check{R}$ of the n-polygraph R such that ${}_{E}\check{R}$ is confluent modulo E. We denote by $\hat{u}^{{}_{E}R}$ a normal form of an element u in R^{*}_{n-1} with respect to ${}_{E}R$. To simplify the notations, for any $(n-1)$-cells u and v in R^{*}_{n-1}, we denote $u \approx_{E} v$ if there exists an n-cell $e : u \to v$ in $E^{\top}$.

Input:
- R and E 2-polygraphs over a 1-polygraph X.
- $\prec$ a termination order for R compatible with E,
  which is total on the set of ${}_{E}R$-irreducible elements.

begin
  $C \leftarrow \mathrm{CP}({}_{E}R, R)$;
  while $C \neq \emptyset$ **do**
    Pick any branching $c = (f : u \Rightarrow v, g : u \Rightarrow w)$ in $\mathcal{C}$, with f in ${}_{E}R^{*}$ and g in R^{*};
    Reduce v to $\hat{v}^{{}_{E}R}$ a ${}_{E}R$-normal form;
    Reduce w to $\hat{w}^{{}_{E}R}$ a ${}_{E}R$-normal form;
    $C \leftarrow C \backslash \{c\}$;
    if $\hat{v}^{{}_{E}R} \not\approx_{E} \hat{w}^{{}_{E}R}$ **then**
      if $\hat{w}^{{}_{E}R} \prec \hat{v}^{{}_{E}R}$ **then**
        $R \leftarrow R \cup \{\hat{v}^{{}_{E}R} \overset{\alpha}{\Rrightarrow} \hat{w}^{{}_{E}R}\}$;
      end
      if $\hat{v}^{{}_{E}R} \prec \hat{w}^{{}_{E}R}$ **then**
        $R \leftarrow R \cup \{\hat{w}^{{}_{E}R} \overset{\alpha}{\Rrightarrow} \hat{v}^{{}_{E}R}\}$;
      end
    end
    $C \leftarrow C \cup \{({}_{E}R, R)\text{-critical branchings created by } \alpha\}$;
  end
end

This procedure may not be terminating. However, it does not fail because of the hypothesis that $\prec$ is total on the set of ${}_E R$-irreducible elements.

3.4.4. Proposition. *When it terminates, the completion procedure for* ${}_E R$ *returns an* n*-polygraph* $\check{R}$ *such that* ${}_E\check{R}$ *is confluent modulo* E.

Proof. The proof of soundness of the completion procedure for ${}_E R$ is a consequence of the inference system given by Bachmair and Dershowitz in [1] in order to get a set of rules $\check{R}$ such that ${}_E\check{R}$ is confluent modulo E. Given two n-polygraphs R and E and a termination order $>$ compatible with R modulo E, their inference system is given by the following six elementary rules:

1) Orienting an equation:

$$(A \cup \{s = t\}, R) \quad \rightsquigarrow \quad (A, R \cup \{s \to t\}) \text{ if } s > t.$$

2) Adding an equational consequence:

$$(A, R) \quad \rightsquigarrow \quad (A \cup \{s = t\}, R) \text{ if } s \overset{*}{\longleftarrow}_{R \cup E} u \overset{*}{\longrightarrow}_{R \cup E} t.$$

3) Simplifying an equation:

$$(A \cup \{s = t\}, R) \quad \rightsquigarrow \quad (A \cup \{u = t\}, R) \text{ if } s \overset{{}_E R}{\to} u.$$

4) Deleting an equation:

$$(A \cup \{s = t\}, R) \quad \rightsquigarrow \quad (A, R) \text{ if } s \approx_E t.$$

5) Simplifying the right-hand side of a rule:

$$(A, R \cup \{s \to t\}) \quad \rightsquigarrow \quad (A, R \cup \{s \to u\}) \text{ if } t \overset{{}_E R}{\to} u.$$

6) Simplifying the left-hand side of a rule:

$$(A, R \cup \{s \to t\}) \quad \rightsquigarrow \quad (A \cup \{u = t\}, R) \text{ if } s \overset{{}_E R}{\to} u.$$

The soundness of Procedure 3.4.3 is a consequence of the following arguments:

i) For any critical branching $(f : u \to v, g : u \to w)$ in $\mathrm{CP}({}_E R, R)$, we can add an equation $v = w$ using the rule *Adding an equational consequence*, and simplify it to $\hat{v}^{{}_E R} = \hat{w}^{{}_E R}$ using the rule *Symplifying an equation*.

ii) If $\hat{v}^{{}_E R} \approx_E \hat{w}^{{}_E R}$, we can delete the equation using the rule *Deleting an equation*.

iii) Otherwise, we can always orient it using the rule *Orienting an equation*.

Thus, each step of this completion procedure comes from one of the inference rules given by Bachmair and Dershowitz. Following [1], it returns a set R of rules so that ${}_E R$ is confluent modulo E. $\square$

3.4.5. Completion procedure for ${}_E R_E$. By definition, the polygraph ${}_E R$ is confluent modulo E if and only if the polygraph ${}_E R_E$ is confluent modulo E. We can extend the above completion procedure in the case of the polygraph modulo ${}_E R_E$. In that case, the critical branchings of the form (f, e) with f in ${}_E R_E^{*(1)}$ and e in $E^{\top(1)}$ are still trivially confluent. Let us denote by $\mathrm{CP}({}_E R_E, R)$ the set of critical branchings of ${}_E R_E$ modulo R. All these critical branchings can be written as a pair $(f \cdot e, g)$, where (f, g) is a critical branching in $\mathrm{CP}({}_E R, R)$ and e is an n-cell in $E^\top$.

As a consequence, the completion procedure for ${}_E R$ given in 3.4.3 can be adapted for the polygraph modulo ${}_E R_E$. In that case, the procedure differs from 3.4.3 by the fact that when adding a rule $\alpha : u \Rightarrow v$ in R, one can choose as target of α any element of the equivalence class of v with respect to E. We prove in the same way than when it terminates, this completion procedure returns an n-polygraph $\check{R}$ such that ${}_E R_E$ is confluent modulo E.

4. Coherent confluence modulo

In this section, we introduce the property of coherent confluence modulo defined by the adjunction of a square cell for each confluence diagram modulo. Under a termination hypothesis, Theorem 4.1.4 shows how to deduce coherent confluence modulo for a polygraph modulo from coherent local confluence modulo. This result is a coherent version of Newman's lemma that states the equivalence between local confluence and confluence under a termination hypothesis, [30]. Theorem 4.2.2 formulates a coherent version of the critical branching lemma, it shows how to deduce local coherent confluence modulo from the coherent confluence modulo of critical branchings.

4.1. Coherent Newman's lemma modulo

4.1.1. Biaction of $E^\top$ on $\mathrm{Sqr}(E^\top, S^*)$. Let (R, E, S) be an n-polygraph modulo. Let Γ be a square extension of the pair of n-categories $(E^\top, S^*)$. As the inclusions $R \subseteq S \subseteq {}_E R_E$ of cellular extensions hold, any n-cell f in S^* can be decomposed in $f = e_1 \star_{n-1} f_1 \star_{n-1} e_2 \star_{n-1} f_2$ with f_1 in $R^{*(1)}$, f_2 in S^* such that $\ell(f_2) = \ell(f) - 1$, e_1 and e_2 are n-cells in $E^\top$ possibly identities, and $\star_{n-1}$ denoting for the composition along $(n-1)$-cells in the free n-category generated by $R \cup E$.

Thus, a branching (f, e, g) of S modulo E with a choice of a generating confluence (f', e', g') may correspond to different squares in $\mathrm{Sqr}(E^\top, S^*)$. For instance, if g can be decomposed $g = e_1 \star_{n-1} g_1 \star_{n-1} e_2$, the following squares in $\mathrm{Sqr}(E^\top, S^*)$ correspond to the same branching of S modulo E:

$$
\begin{tikzcd}
u \arrow[r, "f"] \arrow[d, "e"'] & v \arrow[r, dotted, "f'"] & v' \arrow[d, dotted, "e'"] \\
u \arrow[r, "g"'] & w \arrow[r, dotted, "g'"'] & w'
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
u \arrow[r, "f"] \arrow[d, "e \star_{n-1} e_1"'] & v \arrow[r, dotted, "f'"] & v' \arrow[d, dotted, "e'"] \\
u_1 \arrow[r, "g_1 e_2"'] & w \arrow[r, dotted, "g'"'] & w'
\end{tikzcd}
$$

When computing a coherent presentation of S modulo E, one does not want to consider two different elements in a coherent completion of S modulo E, as defined in 5.1, to tile these squares which are not equal in the free n-category enriched in double category generated by the double $(n-1)$-polygraph $(E, S, \Gamma \cup \mathrm{Peiff}(E^\top, S^*))$.

In order to avoid these redundant squares, we define a *biaction* of $E^{\top}$ on $\mathrm{Sqr}(E^{\top}, S^*)$. For any n-cells e_1 and e_2 in $E^{\top}$ and any $(n+1)$-cell

$$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, "e"'] \arrow[dr, phantom, "\Downarrow A"] & u' \arrow[d, "e'"] \\ u \arrow[r, "g"'] & v' \end{tikzcd}$$

in $\mathrm{Sqr}(E^{\top}, S^*)$ satisfying the following conditions

i) $\partial_{+,n-1}(e_1) = \partial^h_{-,n-1}\partial^v_{-,n}(A)$,

ii) $\partial_{-,n-1}(e_2) = \partial^h_{+,n-1}\partial^v_{-,n}(A)$,

iii) $e_1\partial^h_{-,n}(A) \in S$,

iv) $e_2^- \partial^h_{+,n}(A) \in S$,

we define the square $(n+1)$-cell ${}^{e_1}_{e_2}A$ as follows:

$$\begin{tikzcd} u_1 \arrow[r, "e_1 f"] \arrow[d, "e_1 e e_2"'] \arrow[dr, phantom, "{\parallel\ {}^{e_1}_{e_2}A\ \Downarrow}"] & u' \arrow[d, "e'"] \\ u_2 \arrow[r, "e_2^- g"'] & v' \end{tikzcd}$$

where $u_1 = \partial_{-,n-1}(e_1)$ and $u_2 = \partial_{+,n-1}(e_2)$. For a square extension Γ of $(E^{\top}, S^*)$, we denote by $E \rtimes \Gamma$ the set containing all elements ${}^{e_1}_{e_2}A$ with A in Γ and e_1, e_2 in $E^{\top}$, whenever it is well defined. For any e_1,e_2 in $E^{\top}$ and A,A' in Γ, the following equalities hold whenever both sides are defined:

i) ${}^{e_1'}_{e_2'}({}^{e_1}_{e_2}A) = {}^{e_1' e_1}_{e_2 e_2'}A$;

ii) ${}^{e_1}_{e_2}(A \diamond^v A') = ({}^{e_1}_{e_2}A) \diamond^v A'$;

iii) ${}^{e_1}_{e_2}(A \diamond^h A') = ({}^{e_1}_{1}A) \diamond^h ({}^{1}_{e_2}A')$.

4.1.2. Coherent confluence modulo. Let (R, E, S) be an n-polygraph modulo. Let Γ be a square extension of the pair of n-categories $(E^{\top}, S^*)$. Let us denote

$$\Gamma^{\curlyvee} \;:=\; (E, S, E \rtimes \Gamma \cup \mathrm{Peiff}(E^{\top}, S^*))^{\top\!\!\top, v}$$

the free $(n-1)$-category enriched in double categories, whose vertical n-cells are invertible, generated by the double $(n-1)$-polygraph $(E, S, E \rtimes \Gamma \cup \mathrm{Peiff}(E^{\top}, S^*))$ in DbPol^v_{n-1}.

A branching modulo E as in (3.3.3) is Γ-*confluent modulo* E if there exist n-cells f', g' in S^*, e' in $E^{\top}$ and an $(n+1)$-cell A in $\Gamma^{\curlyvee}$ as in the following diagram:

$$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, "e"'] & u' \arrow[r, dotted, "f'"] \arrow[d, phantom, "\Downarrow A"] & w \arrow[d, dotted, "e'"] \\ v \arrow[r, "g"'] & v' \arrow[r, dotted, "g'"'] & w' \end{tikzcd} .$$

We say that S is Γ-*confluent* (resp. *locally* Γ-confluent, resp. *critically* Γ-*confluent*) *modulo* E if every branching (resp. local branching, resp. critical branching) modulo E is Γ-confluent modulo E, and that S is Γ-*convergent* if it is Γ-confluent modulo E and ${}_E R_E$ is terminating. The polygraph modulo S is called Γ-*diconvergent*, when it is Γ-convergent and E is convergent. Note that when $\Gamma = \mathrm{Sqr}(E^\top, S^*)$ (resp. $\Gamma = \mathrm{Sph}(S^*)$), the property of Γ-confluence modulo E corresponds to the property of confluence modulo E (resp. confluence) given in 3.3.

In the sequel, proofs of confluence modulo results will be based on Huet's double Noetherian induction principle on the n-polygraph $S^{\amalg}$ defined in 3.2.3 and the property $\mathcal{P}$ on $R^*_{n-1} \times R^*_{n-1}$ defined, for any u, v in R^*_{n-1}, by

$$\mathcal{P}(u, v) : \text{any branching } (f, e, g) \text{ of } S \text{ modulo } E \text{ with source } (u, v) \text{ is } \Gamma\text{-confluent modulo } E.$$

4.1.3. Proposition (Coherent half Newman's modulo lemma). *Let* (R, E, S) *be an* n*-polygraph modulo such that* ${}_E R_E$ *is terminating, and* Γ *be a square extension of* $(E^\top, S^*)$. *If* S *is locally* Γ*-confluent modulo* E *then the two following conditions hold*

i) *any branching* (f, e) *of* S *modulo* E *with* f *in* $S^{*(1)}$ *and* e *in* $E^\top$ *is* Γ*-confluent modulo* E*,*

ii) *any branching* (f, e) *of* S *modulo* E *with* f *in* S^* *and* e *in* $E^{\top(1)}$ *is* Γ*-confluent modulo* E*,*

Proof. We prove condition **i)**, the proof of condition **ii)** is similar. Let us assume that S is locally Γ-confluent modulo E, we proceed by double induction.

We denote by u the source of the branching (f, e). If u is irreducible with respect to S, then f is an identity n-cell, and the branching is trivially Γ-confluent.

Suppose that f is not an identity and assume that for any pair (u', v') of $(n-1)$-cells in R^*_{n-1} such that there is an n-cell $(u, u) \to (u', v')$ in $S^{\amalg}$, any branching (f', e', g') of source (u', v') is Γ-confluent modulo E. Prove that the branching (f, e) is Γ-confluent modulo E.

We proceed by induction on $\ell(e) \geqslant 1$. If $\ell(e) = 1$, (f, e) is a local branching of S modulo E and it is Γ-confluent modulo E by local Γ-confluence of S modulo E. Now, let us assume that for $k \geqslant 1$, any branching (f'', e'') of S modulo E such that $\ell(e'') = k$ is Γ-confluent modulo E, and let us consider a branching (f, e) of S modulo E such that $\ell(e) = k + 1$, with source u. We choose a decomposition $e = e_1 \star_{n-1} e_2$ with e_1 in $E^{\top(1)}$ and e_2 in $E^\top$. Using local Γ-confluence on the branching (f, e_1) of source u, there exist n-cells f' and f_1 in S^*, an n-cell $e'_1 : t_{n-1}(f') \to t_{n-1}(f_1)$ in $E^\top$ and an $(n+1)$-cell A in $\Gamma^\curlyvee$ such that $\partial^h_{-,n}(A) = f \star_{n-1} f'$ and $\partial^h_{+,n}(A) = f_1$. Then, we choose a decomposition $f_1 = f_1^1 \star_{n-1} f_1^2$ with f_1^1 in $S^{*(1)}$ and f_1^2 in S^*. Using the induction hypothesis on the branching (f_1^1, e_2) of S modulo E of source $u_1 := t_{n-1}(e_1) = s_{n-1}(e_2)$, there exist n-cells f'_1 and g in S^*, an n-cell $e_2 : t_{n-1}(f'_1) \to t_{n-1}(g)$ in $E^\top$ and an $(n+1)$-cell B in $\Gamma^\curlyvee$ such that $\partial^h_{-,n}(B) = f_1^1 \star_{n-1} f'_1$ and $\partial^h_{+,n}(B) = g$. This can be represented by the following diagram:

$$
\begin{tikzcd}
u \arrow[r, "f"] \arrow[d, "e_1"'] \arrow[dr, phantom, "\text{Local }\Gamma\text{-conf mod }E"] & u' \arrow[r, "f'"] & u'' \arrow[d, "e_1'"] \\
u_1 \arrow[r, "f_1^1"] \arrow[d, equal] \arrow[r, phantom, shift right=4ex, "i_1^h(f_1^1)"] & u_1' \arrow[r, "f_1^2"] \arrow[d, equal] & u_1'' \\
u_1 \arrow[r, "f_1^1"] \arrow[d, "e_2"'] \arrow[dr, phantom, "\text{Induction on }\ell(e)"] & u_1' \arrow[r, "f_1'"] & u_2' \arrow[d, "e_2'"] \\
v \arrow[rr, "g"'] & & v'
\end{tikzcd}
$$

Now, there is an n-cell $(u, u) \to (u_1', u_1')$ in $S^{\amalg}$ given by the composition

$$(u, u) \to (u_1, u_1) \to (u_1, u_1') \to (u_1', u_1')$$

where the first step exists because $\ell(e_1) > 0$ and the remaining composition is as in 3.2.3. Then, we apply double induction on the branching (f_1^2, f_1') of S modulo E of source (u_1', u_1'): there exist n-cells f_2 and f_2' in S^* and an n-cell $e_3 : t_{n-1}(f_2) \to t_{n-1}(f_2')$ in $E^{\top}$. By a similar argument, we can apply double induction on the branchings $(f_2, (e_1')^-)$ and (f_2', e_2') of S modulo E, so that there exist n-cells f'',f_3, f_3' and g' in S^* and n-cells $e_1'' : t_{n-1}(f'') \to t_{n-1}(f_3)$ and $e_2'' : t_{n-1}(f_3') \to t_{n-1}(g')$ as in the following diagram:

$$
\begin{tikzcd}
u \arrow[r, "f"] \arrow[d, "e_1"'] \arrow[dr, phantom, "\text{Local }\Gamma\text{-conf mod }E"] & u' \arrow[r, "f'"] & u'' \arrow[rr, "f''"] \arrow[d, "e_1'"] \arrow[drr, phantom, "\text{Db Ind.}"] & & u''' \arrow[d, "e_1''"] \\
u_1 \arrow[r, "f_1^1"] \arrow[d, equal] \arrow[r, phantom, shift right=4ex, "i_1^h(f_1^1)"] & u_1' \arrow[r, "f_1^2"] \arrow[d, equal] \arrow[drr, phantom, "\text{Db Ind.}"] & u_1'' \arrow[r, "f_2"] & w_1 \arrow[r, "f_3"] \arrow[d, "e_3"] & w_1' \\
u_1 \arrow[r, "f_1^1"] \arrow[d, "e_2"'] \arrow[dr, phantom, "\text{Induction on }\ell(e)"] & u_1' \arrow[r, "f_1'"] & u_2' \arrow[r, "f_2'"] \arrow[d, "e_2'"] \arrow[drr, phantom, "\text{Db Ind.}"] & w_2 \arrow[r, "f_3'"] & w_2' \arrow[d, "e_2''"] \\
v \arrow[rr, "g"'] & & v' \arrow[rr, "g'"'] & & v''
\end{tikzcd}
$$

We can then repeat the same process using double induction on the branching (f_3, e_3, f_3') of S modulo E of source (w_1, w_2) and so on, and this process terminates in finitely many steps, otherwise it leads to an infinite rewriting sequence wrt S starting from u_1, which is not possible since ${}_E R_E$, and thus S, is terminating. This yields the Γ-confluence of the branching (f, e). □

4.1.4. Theorem (Coherent Newman's lemma modulo). *Let* (R, E, S) *be an* n*-polygraph modulo such that* ${}_E R_E$ *is terminating, and* Γ *be a square extension of* $(E^\top, S^*)$*. If* S *is locally* Γ*-confluent modulo* E *then it is* Γ*-confluent modulo* E*.*

Proof. Prove that any branching (f, e, g) of S modulo E is Γ-confluent modulo E. Let us choose such a branching and denote by (u, v) its source. We assume that any branching (f', e', g') of S modulo E of source (u', v') such that there is an n-cell $(u, v) \to (u', v')$ in $S^{\amalg}$ is Γ-confluent modulo E. We follow the proof scheme used by Huet in [21, Lemma 2.7]. Let us denote by $n := \ell(f)$ and $m := \ell(g)$. We assume without loss of generality that $n > 0$ and we fix a decomposition $f = f_1 \star_{n-1} f_2$ with f_1 in $S^{*(1)}$ and f_2 in S^*.

If $m = 0$, by Proposition 4.1.3 on the branching (f_1, e) of S modulo E, there exist n-cells f_1' and g' in S^*, an n-cell $e' : t_{n-1}(f_1') \to t_{n-1}(g')$ and an $(n+1)$-cell A in $\Gamma^{\curlyvee}$ such that $\partial^h_{-,n}(A) = f_1 \star_{n-1} f_1'$ and $\partial^h_{+,n}(A) = g'$. Then, since there is an n-cell $(u, u) \to (u_1, u_1)$ in $S^{\amalg}$ with $u_1 := t_{n-1}(f_1)$, we can apply double induction on the branching (f_2, f_1') of S modulo E as in the following diagram:

$$
\begin{tikzcd}
u \arrow[r, "f_1"] \arrow[d, equal] \arrow[dr, phantom, "i_1^h(f_1)"] & u_1 \arrow[r, "f_2"] \arrow[d, equal] \arrow[drr, phantom, "\text{Db Ind.}"] & u_2 \arrow[r, "f_2'"] & u_2' \arrow[d] \\
u \arrow[r, "f_1"] \arrow[d, "e"'] \arrow[drr, phantom, "\text{Prop. 4.1.3}"] & u_1 \arrow[r, "f_1'"] & u_2 \arrow[r, "f_1''"] \arrow[d, "e'"] & u_2' \\
v \arrow[rr, "g'"'] & & v' &
\end{tikzcd}
$$

We finish the proof of this case with a similar argument than in 4.1.3, using repeated double inductions that can not occur infinitely many times since S is terminating.

Now, assume that $m > 0$ and fix a decomposition $g = g_1 \star_{n-1} g_2$ of g with g_1 in $S^{*(1)}$ and g_2 in S^*. By Step 1 on the branching (f_1, e) of S modulo E, there exist n-cells f_1' and h_1 in S^*, an n-cell $e_1 : t_{n-1}(f_1') \to t_{n-1}(h_1)$ in $E^\top$ and an $(n+1)$-cell A in $\Gamma^{\curlyvee}$ such that $\partial^h_{-,n}(A) = f_1 \star_{n-1} f_1'$ and $\partial^h_{+,n}(A) = h_1$. We distinguish two cases whether h_1 is trivial or not.

If h_1 is trivial, the Γ-confluence of the branching (f, e, g) of S modulo E is given by the following diagram

$$
\begin{tikzcd}
u \arrow[r, "f_1"] \arrow[d, equal] \arrow[dr, phantom, "i_1^h(f_1)"] & u_1 \arrow[r, "f_2"] \arrow[d, equal] & u_2 \arrow[rrr, "f_2'"] \arrow[drrr, phantom, "\text{Db Ind.}"] & & & u_2' \arrow[d] & \\
u \arrow[r, "f_1"] \arrow[d, "e"'] \arrow[drr, phantom, "\text{Prop. 4.1.3}"] & u_1 \arrow[r, "f_1'"] & u_1' \arrow[rr, "f_3"] \arrow[d, "e'"] \arrow[drr, phantom, "\text{Prop. 4.1.3}"] & & u_3 \arrow[r, "f_4"] \arrow[d, "e_1"] \arrow[drr, phantom, "\text{Db Ind.}"] & u_4 \arrow[r, "f_5"] & u_5 \arrow[d] \\
v \arrow[rr, "1_v"] \arrow[d, equal] \arrow[drr, phantom, "i_1^h(1_v)"] & & v \arrow[r, "g_1"] \arrow[d, equal] \arrow[dr, phantom, "i_1^h(g_1)"] & v_1' \arrow[r, "g_1'"] \arrow[d, equal] \arrow[drr, phantom, "\text{Db Ind.}"] & v_1'' \arrow[r, "g_1''"] & w_1 \arrow[r, "g_3"] \arrow[d] & w_3 \\
v \arrow[rr, "1_v"'] & & v \arrow[r, "g_1"] & v_1' \arrow[r, "g_2"'] & v_2 \arrow[r, "g_2'"'] & w_2 &
\end{tikzcd}
$$

where the branchings (f_1, e) and (g_1, e') of S modulo E are Γ-confluent by Proposition 4.1.3, double induction applies on the branchings $(f_2, f_1' \star_{n-1} f_3)$, (g_1', g_2) and (f_4, e_1, g_1'') since there are n-cells

$$(u, v) \to (u, u) \to (u_1, u_1)\ ,\ (u, v) \to (v, v) \to (v, v_1') \to (v_1', v_1') \text{ and } (u, v) \to (u_3, v) \to (u_3, v_1'')$$

in S^{II} and one can check that this process of double induction can be repeated, terminating in a finite number of steps since S is terminating and yields a Γ-confluence of the branching (f, e, g) modulo E.

If h_1 is not trivial, let us fix a decomposition $h_1 = h_1^1 \star_{n-1} h_1^2$ with h_1^1 in $S^{*(1)}$ and h_1^2 in S^*. The Γ-confluence of the branching (f, e, g) of S modulo E is given by the following diagram:

$$
\begin{tikzcd}
u \arrow[r, "f_1"] \arrow[d, equal] \arrow[dr, phantom, "i_1^h(f_1)"] & u_1 \arrow[r, "f_2"] \arrow[d, equal] \arrow[drr, phantom, "\text{Db Ind.}"] & u_2 \arrow[r, "f_2'"] & u_2' \arrow[d] & \\
u \arrow[r, "f_1"] \arrow[d, "e"'] \arrow[drr, phantom, "\text{Prop. 4.1.3}"] & u_1 \arrow[r, "f_1'"] & u_1' \arrow[r, "f_3"] \arrow[d] \arrow[drr, phantom, "\text{Db Ind.}"] & u_3 \arrow[r, "f_4"] & u_4 \arrow[d] \\
v \arrow[r, "h_1^1"] \arrow[d, equal] \arrow[dr, phantom, "i_1^h(h_1^1)"] & v_1 \arrow[r, "h_1^2"] \arrow[d, equal] \arrow[drr, phantom, "\text{Db Ind.}"] & w_1 \arrow[r, "h_2"] & w_2 \arrow[r, "h_2'"] \arrow[d] & w_2' \\
v \arrow[r, "h_1^1"] \arrow[d, equal] \arrow[drr, phantom, "\text{Local }\Gamma\text{-conf mod }E"] & v_1 \arrow[r, "h_1'"] & w_1' \arrow[r, "h_3"] \arrow[d] \arrow[drr, phantom, "\text{Db Ind.}"] & w_3 \arrow[r, "h_3'"] & w_3' \arrow[d] \\
v \arrow[r, "g_1"] \arrow[d, equal] \arrow[dr, phantom, "i_1^h(g_1)"] & v' \arrow[r, "g_1'"] \arrow[d, equal] \arrow[drr, phantom, "\text{Db Ind.}"] & v_1' \arrow[r, "g_2'"] & v_2' \arrow[r, "g_3'"] \arrow[d] & v_3' \\
v \arrow[r, "g_1"'] & v' \arrow[r, "g_2"'] & v_2 \arrow[r, "g_3"'] & v_3 &
\end{tikzcd}
$$

where the branching (f_1, e) modulo E is Γ-confluent by Proposition 4.1.3, the branching (h_1^1, g_1) is Γ-confluent by assumption of local Γ-confluence of S, and one can check that double induction applies on the branchings (f_2, f_1'), (h_1^2, h_1'), (g_1', g_2), (f_3, h_2) and (h_3, g_2'). This process of double induction can be repeated, terminating in a finite number of steps since S is terminating and yields a Γ-confluence of the branching (f, e, g) modulo E.

□

4.2. Coherent critical branching lemma modulo

In this subsection, we show how to prove coherent local confluence of an n-polygraph modulo from coherent confluence of some critical branchings. In particular, we show that we do not need to consider all the local branchings.

4.2.1. Proposition. *Let (R, E, S) be an n-polygraph modulo such that ${}_E R_E$ is terminating, and Γ be a square extension of $(E^\top, S^*)$. Then S is Γ-locally confluent modulo E, if and only if the two following conditions hold:*

a) *any local branching $(f, g) : u \Rightarrow (v, w)$ with f in $S^{*(1)}$ and g in $R^{*(1)}$ is Γ-confluent modulo E:*

$$
\begin{tikzcd}
u \arrow[r, "f"] \arrow[d, equal] & v \arrow[r, dotted, "f'"] \arrow[d, phantom, "\Downarrow A"] & v' \arrow[d, dotted, "e'"] \\
u \arrow[r, "g"'] & w \arrow[r, dotted] & w'
\end{tikzcd}
$$

b) *any local branching* $(f, e) : u \Rightarrow (v, u')$ *modulo* E *with* f *in* $S^{*(1)}$ *and* e *in* $E^{\top(1)}$ *is* Γ*-confluent modulo* E*:*

$$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, "e"'] & v \arrow[r, dotted, "f'"] \arrow[d, phantom, "\Downarrow B"] & v' \arrow[d, dotted, "e'"] \\ u' \arrow[rr, dotted, "g'"'] & & w \end{tikzcd}$$

Proof. We prove this result using Huet's double Noetherian induction principle on the n-polygraph $S^{\amalg}$ defined in 3.2.3 and the property $\mathcal{P}$ on $R^*_{n-1} \times R^*_{n-1}$ defined by: for any u, v in R^*_{n-1},

$\mathcal{P}(u, v)$: any branching (f, e, g) of S modulo E of source (u, v) is Γ-confluent modulo E.

The only part is trivial because properties **a)** and **b)** correspond to Γ-confluence of some local branchings of S modulo E. Conversely, assume that S satisfy properties **a)** and **b)** and let us prove that any local branching is Γ-confluent modulo E. We consider a local branching (f, e, g) of S modulo E, and assume without loss of generality that f is a non-trivial n-cell in $S^{*(1)}$. There are two cases: either g is trivial, and the local branching (f, e) of S modulo E is Γ-confluent by **b)**, or e is trivial. In that case, if g is in $R^{*(1)}$, then Γ-confluence of the branching (f, g) is given by **a)**. Otherwise, let us choose a decomposition $g = e_1 \star_{n-1} g' \star_{n-1} e_2$ with e_1,e_2 in $E^{\top}$ and g' in $R^{*(1)}$. Now, let us prove the confluence of the branching

$$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, "e_1"'] & v \\ u' \arrow[r, "g'e_2"'] & v' \end{tikzcd}$$

of S modulo E, where $g'e_2$ is an n-cell in $S^{*(1)}$. We will then prove the Γ-confluence of the branching (f, g) using the biaction of $E^{\top}$ on $\mathrm{Sqr}(E^{\top}, S^*)$. Using Proposition 4.1.3 on the branching (f, e_1) of S modulo E, there exist n-cells f' and f_1 in S^*, an n-cell $e' : t_{n-1}(f') \to t_{n-1}(f_1)$ and an $(n+1)$-cell A in $\Gamma^{\curlyvee}$ such that $\partial^h_{-,n}(A) = f \star_{n-1} f'$ and $\partial^h_{+,n}(A) = f_1$. Using property **a)** on the local branching $(g', g'e_2) \in R^{*(1)} \times S^{*(1)}$ and the trivial confluence given by the right vertical cell e_2, there exists an $(n+1)$-cell B in $\Gamma^{\curlyvee}$ such that $\partial^h_{-,n}(B) = g'$ and $\partial^h_{+,n}(B) = g'e_2$. Let us choose a decomposition $f_1 = f_1^1 \star_{n-1} f_1^2$ with f_1^1 in $S^{*(1)}$ and f_1^2. By property **a)** on the local branching (f_1^1, g'), there exist n-cells f_1' and g_1' in S^*, an n-cell $e'' : t_{n-1}(f_1') \to t_{n-1}(g_1')$ and an $(n+1)$-cell C in $\Gamma^{\curlyvee}$ such that $\partial^h_{-,n}(C) = f_1^1 \star_{n-1} f_1'$ and $\partial^h_{+,n}(C) = g' \star_{n-1} g_1'$ as depicted on the following diagram:

$$\begin{tikzcd}
u \arrow[r, "f"] \arrow[d, "e_1"'] & u' \arrow[r, "f'"] \arrow[d, phantom, "\Downarrow A"] & u'' \arrow[d, "e_1'"] \\
u_1 \arrow[r, "f_1^1"] \arrow[d, equal] \arrow[r, phantom, shift right=4ex, "i_1^h(f_1^1)"] & u_1' \arrow[r, "f_1^2"] \arrow[d, equal] & u_1'' \\
u_1 \arrow[r, "f_1^1"] \arrow[d, equal] & u_1' \arrow[r, "f_1'"] \arrow[d, phantom, "\Downarrow C"] & u_2' \arrow[d, "e_2'"] \\
v \arrow[r, "g'"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow B"] & v_1 \arrow[r, "g_2'"'] \arrow[d, "e_2"] & v_2 \\
v \arrow[r, "g'e_2"'] & v' &
\end{tikzcd}$$

There are n-cells $(u, u) \to (u_1', u_1')$ and $(u, u) \to (v_1, v_1)$ in $S^{\amalg}$ given by the following compositions

$$(u, u) \to (u_1, u_1) \to (u_1, u_1') \to (u_1', u_1')$$

$$(u, u) \to (u_1, u_1) \to (u_1, v) \to (v, v) \to (v, v_1) \to (v_1, v_1)$$

so that we can apply double induction on the branchings (f_1^2, f_1') and (g_2', e_2) of S modulo E, and we finish the proof of Γ-confluence of the branching $(f, e_1, g'e_2)$ using repeated double inductions, terminating in a finite number of steps since S is terminating.

Now, we get the Γ-confluence of the branching (f, g) of S by the following diagram:

$$\begin{tikzcd}
u \arrow[r, "f"] \arrow[d, equal] & u' \arrow[r, "f'"] \arrow[d, phantom, "{}^{1}_{e_1}A\Downarrow"] & u'' \arrow[d, "e_1'"] \\
u_1 \arrow[r, "e_1 f_1^1"] \arrow[d, equal] \arrow[r, phantom, shift right=4ex, "i_1^h(e_1 f_1^1)"] & u_1' \arrow[r, "f_1^2"] \arrow[d, equal] & u_1'' \\
u_1 \arrow[r, "e_1 f_1^1"] \arrow[d, equal] & u_1' \arrow[r, "f_1'"] \arrow[d, phantom, "{}^{e_1}_{e_1^-}C\Downarrow"] & u_2' \arrow[d, "e_2'"] \\
v \arrow[r, "e_1 g'"] \arrow[d, equal] \arrow[dr, phantom, "{}^{e_1}_{e_1^-}B\Downarrow"] & v_1 \arrow[r, "g_2'"'] \arrow[d, "e_2"] & v_2 \\
v \arrow[r, "e_1 g' e_2"'] & v' &
\end{tikzcd}$$

since the top rectangle is by definition tiled by the $(n+1)$-cell ${}^{1}_{e_1}A$, the bottom rectangle is tiled by the $(n+1)$-cell ${}^{e_1}_{e_1^-}B$ and the remaining rectangle is tiled by the $(n+1)$-cell ${}^{e_1}_{e_1^-}C$. The rest of the diagram is tiled in the same way than above. $\square$

4.2.2. Theorem (Coherent critical branching lemma modulo). *Let (R, E, S) be an n-polygraph modulo such that ${}_E R_E$ is terminating, and Γ be a square extension of $(E^\top, S^*)$. Then S is Γ-locally confluent modulo E, if and only if the two following conditions hold*

a_0) *any critical branching $(f, g) : u \Rightarrow (v, w)$ with f in $S^{*(1)}$ and g in $R^{*(1)}$ is Γ-confluent modulo E:*

$$\begin{tikzcd}
u \arrow[r, "f"] \arrow[d, equal] & v \arrow[r, dotted, "f'"] \arrow[d, phantom, "\Downarrow A"] & v' \arrow[d, dotted, "e'"] \\
u \arrow[r, "g"'] & w \arrow[r, dotted] & w'
\end{tikzcd}$$

b_0) *any critical branching $(f, e) : u \Rightarrow (v, u')$ modulo E with f in $S^{*(1)}$ and e in $E^{\top(1)}$ is Γ-confluent modulo E:*

$$\begin{tikzcd}
u \arrow[r, "f"] \arrow[d, "e"'] & v \arrow[r, dotted, "f'"] \arrow[d, phantom, "\Downarrow B"] & v' \arrow[d, dotted, "e'"] \\
u' \arrow[rr, dotted, "g'"'] & & w
\end{tikzcd}$$

Proof. By Proposition 4.2.1, the local Γ-confluence is equivalent to both conditions **a)** and **b)**. Let us prove that the condition **a)** (resp. **b)**) holds if and only if the condition **a_0**) (resp. **b_0**)) holds. One implication is trivial. Suppose that condition **b_0**) holds and prove condition **b)**. The proof of the other implication is similar. We examine all the possible forms of local branchings modulo given in 3.3.7. Local aspherical branchings modulo and local Peiffer branchings modulo of the forms (3.3.8) are trivially confluent modulo:

$$\begin{tikzcd}
u \star_i v \arrow[r, "f \star_i v"] \arrow[d, "u \star_i e"'] & u' \star_i v \arrow[d, "u' \star_i e"] \\
u \star_i v' \arrow[r, "f \star_i v'"'] & u' \star_i v'
\end{tikzcd}
\qquad
\begin{tikzcd}
w \star_i u \arrow[r, "w \star_i f"] \arrow[d, "e' \star_i u"'] & w \star_i u' \arrow[d, "e' \star_i u'"] \\
w' \star_i u \arrow[r, "w' \star_i f"'] & w' \star_i u'
\end{tikzcd}$$

and Γ-confluent modulo by definition of Γ-confluence. The other local branchings modulo are overlapping branchings modulo $(f, e) : u \Rightarrow (u', v)$ of the form (3.3.9), where f is an n-cell of $S^{*(1)}$ and e is an n-cell of $E^{\top(1)}$. By definition, there exists a whisker C on R^*_{n-1} and a critical branching $(f', e') : u_0 \Rightarrow (u'_0, v_0)$ such that $f = C[f']$ and $e = C[e']$. Following condition **b_0**) the branching (f', e') is Γ-confluent, that is there exists a Γ-confluence modulo E:

$$\begin{tikzcd}
u \arrow[r, "f'"] \arrow[d, "e'"'] & v \arrow[r, dotted, "f''"] \arrow[d, phantom, "\Downarrow A"] & v' \arrow[d, dotted, "e''"] \\
u' \arrow[rr, dotted, "g'"'] & & w
\end{tikzcd}$$

inducing a Γ-confluence for (f, e):

$$\begin{tikzcd} C[u] \arrow[r, "C[f']"] \arrow[d, "C[e']"'] & C[v] \arrow[r, dotted, "C[f'']"] \arrow[d, phantom, "\Downarrow C[A]"] & v' \arrow[d, dotted, "C[e'']"] \\ C[u'] \arrow[rr, dotted, "C[g']"'] & & w \end{tikzcd}$$

This proves the condition **b)**. $\square$

5. Coherent completion modulo

In this section, we show how to construct a double coherent presentation of an $(n-1)$-category $\mathcal{C}$ starting with a presentation of this $(n-1)$-category by an n-polygraph modulo. We explain how the results presented in this section generalize to n-polygraphs modulo the coherence results from n-polygraphs as given in [17, 18].

5.1. Coherent completion modulo

We recall the notion of coherent completion of a convergent n-polygraph and introduce the notion of coherent completion modulo for polygraphs modulo, given by adjunction of a square cell for any confluence diagram of critical branching modulo.

5.1.1. Coherent completion. Recall from [17] that a convergent n-polygraph can be extended into a coherent globular presentation of the category it presents. Explicitly, given a convergent n-polygraph E, we consider a family of generating confluences of E as a cellular extension of the free $(n, n-1)$-category $E^{\top}$ that contains exactly one globular $(n+1)$-cell

$$\begin{tikzcd} & v \arrow[dr, bend left, "e_1"] \arrow[d, phantom, "\Downarrow E_{e,e'}"] & \\ u \arrow[ur, bend left, "e"] \arrow[dr, bend right, "e'"'] & {} & w \\ & v' \arrow[ur, bend right, "e'_1"'] & \end{tikzcd}$$

for every critical branching (e, e') of E, where (e_1, e'_1) is a chosen confluence. Any $(n+1, n)$-polygraph obtained from E by adjunction of a chosen family of generating confluences of E is a globular coherent presentation of the $(n-1)$-category $\overline{E}$, [17]. This result was originally proved by Squier in [35] for $n = 2$. From such an $(n+1, n)$-polygraph we will consider a double $(n+1, n-1)$-polygraph $(E, \emptyset, \Gamma_E)$, where Γ_E is a square extension of the $(n, n-1)$-categories $(E^{\top}, 1)$ seen as an n-category enriched in double groupoids that contains exactly one square $(n+1)$-cell

$$\begin{tikzcd} u \arrow[r, "="] \arrow[d, "e"'] \arrow[dr, phantom, "\Vert"] & u \arrow[d, "e'"] \\ v \arrow[d, dotted, "e_1"'] \arrow[dr, phantom, "E_{e,e'} \Downarrow"] & v' \arrow[d, dotted, "e'_1"] \\ w \arrow[r, dotted, "="'] & w \end{tikzcd}$$

for every critical branching (e, e') of E, where (e_1, e'_1) is a chosen confluence.

5.1.2. Coherent completion modulo. Let (R, E, S) be an n-polygraph modulo. A *coherent completion modulo* E of S is a square extension of the pair of $(n+1, n)$-categories $(E^{\top}, S^{\top})$ whose elements are the square $(n+1)$-cells $A_{f,g}$ and $B_{f,e}$ of the following form:

$$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, "="'] & u' \arrow[r, "f'"] \arrow[d, phantom, "\Downarrow A_{f,g}"] & w \arrow[d, "e'"] \\ u \arrow[r, "g"'] & v \arrow[r, "g'"'] & w' \end{tikzcd} \qquad \begin{tikzcd} u \arrow[r, "f"] \arrow[d, "e"'] & u' \arrow[r, "f'"] \arrow[d, phantom, "\Downarrow B_{f,e}"] & w \arrow[d, "e'"] \\ v \arrow[rr, "g'"'] & & w' \end{tikzcd} \qquad (5.1.3)$$

for any critical branchings (f, g) and (f, e) of S modulo E, where f, g and e are n-cells of $S^{*(1)}$, $R^{*(1)}$ and $E^{\top(1)}$ respectively. Note that such completion is not unique in general and depends on the n-cells f', g', e' chosen to obtain the confluence of the critical branchings.

5.2. Coherence by E-normalization

In this subsection, we show how to obtain an acyclic square extension of a pair of categories $(E^{\top}, S^{\top})$ coming from a polygraph modulo (R, E, S), under an assumption of confluence modulo E and of normalization of S with respect to E.

5.2.1. Normalization in polygraphs modulo. Let us recall the notion of normalization strategy in an n-polygraph P. Denote by $\mathcal{C}$ the $(n-1)$-category presented by P. Consider a section $s : \mathcal{C} \to P_n^*$ of the canonical projection $\pi : P_n^* \to \mathcal{C}$, that sends any $(n-1)$-cell u in $\mathcal{C}$ on an $(n-1)$-cell in P_{n-1}^* denoted by $\hat{u}$ such that $\pi(\hat{u}) = u$. A *normalization strategy for* P *with respect to* s is a map

$$\sigma : P_{n-1}^* \to P_n^*$$

that sends every $(n-1)$-cell u of P_{n-1}^* to an $(n+1)$-cell

$$\sigma_u : u \to \hat{u}.$$

Let (R, E, S) be an n-polygraph modulo. The n-polygraph modulo S is *normalizing* if any $(n-1)$-cell u admits at least one normal with respect to S, that is $\mathrm{NF}(S, u)$ is not empty.

A set X of $(n-1)$-cells in R_{n-1}^* is E-*normalizing with respect to* S if for any u in X, the set $\mathrm{NF}(S, u) \cap \mathrm{Irr}(E)$ is not empty. The n-polygraph modulo S is E-*normalizing* if it normalizing and R_{n-1}^*

is E-normalizing. When S is E-normalizing, a E-*normalization strategy* σ for S, associates to each $(n-1)$-cell u in R_{n-1}^* an n-cell $\sigma_u : u \to \hat{u}$ in S^*, where $\hat{u}$ belongs to $\mathrm{NF}(S, u) \cap \mathrm{Irr}(E)$. Note that a normalizing cellular extension modulo ${}_E R_E$ is E-normalizing.

5.2.2. Theorem. *Let* (R, E, S) *be an* n*-polygraph modulo, and* Γ *be a square extension of the pair of* $(n+1, n)$*-categories* $(E^\top, S^\top)$ *such that* S *is* Γ*-diconvergent. If* $\mathrm{Irr}(E)$ *is* E*-normalizing with respect to* S*, then the square extension* $E \rtimes \Gamma \cup \mathit{Peiff}(E^\top, S^*) \cup \Gamma_E$ *is acyclic.*

Proof. Let Γ be a square extension of $(E^\top, S^\top)$. We will denote by $\mathcal{C}$ the free n-category enriched in double groupoid $(E, S, E \rtimes \Gamma \cup \mathrm{Peiff}(E^\top, S^*) \cup \Gamma_E)^{\top\!\!\top}$ generated by the double $(n+1, n-1)$-polygraph $(E, S, E \rtimes \Gamma \cup \mathrm{Peiff}(E^\top, S^*) \cup \Gamma_E)$. We will denote by $\widetilde{u}$ the unique normal form of an $(n-1)$-cell u in R_{n-1}^* with respect to E and we fix a normalization strategy $\rho_u : u \to \widetilde{u}$ for E.

By termination of ${}_E R_E$, the n-polygraph modulo S is normalizing. Let us fix a E-normalization strategy $\sigma_u : u \to \hat{u}$ for S. Let us consider a square

$$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, "e"'] & v \arrow[d, "e'"] \\ u' \arrow[r, "g"'] & v' \end{tikzcd} \tag{5.2.3}$$

in $\mathcal{C}$. By definition the n-cell f in $S^\top$ can be decomposed (in general in a non unique way) into a zigzag sequence $f_0 \star_{n-1} f_1^- \star_{n-1} \cdots \star_{n-1} f_{2n} \star_{n-1} f_{2n+1}^-$ with source u and target v where the $f_{2k} : u_{2k} \to u_{2k+1}$ and $f_{2k+1} : u_{2k+2} \to u_{2k+1}$, for all $0 \leqslant k \leqslant n$ are n-cell of S^*, with $u_0 = u$ and $u_{2n+2} = v$.

By Γ-confluence modulo E there exist n-cells e_{f_i} in $E^\top$ and $(n+1)$-cells σ_{f_i} in $\mathcal{C}$ as in the following diagrams:

$$\begin{tikzcd} u_{2k} \arrow[r, "f_{2k}"] \arrow[d, "\rho_{u_{2k}}"'] & u_{2k+1} \arrow[r, "\sigma_{u_{2k+1}}"] \arrow[d, phantom, "\Downarrow \sigma_{f_{2k}}"] & \widehat{u_{2k+1}} \arrow[d, dotted, "e_{f_{2k}}"] \\ \widetilde{u_{2k}} \arrow[rr, "\sigma_{\widetilde{u_{2k}}}"'] & & \widehat{\widetilde{u_{2k}}} \end{tikzcd} \qquad \begin{tikzcd} u_{2k+2} \arrow[r, "f_{2k+1}"] \arrow[d, "\rho_u"'] & u_{2k+1} \arrow[r, "\sigma_{u_{2k+1}}"] \arrow[d, phantom, "\Downarrow \sigma_{f_{2k+1}}"] & \widehat{u_{2k+1}} \arrow[d, dotted, "e_{f_{2k+1}}"] \\ \widetilde{u_{2k+2}} \arrow[rr, "\sigma_{\widetilde{u_{2k+2}}}"'] & & \widehat{\widetilde{u_{2k+2}}} \end{tikzcd}$$

for all $0 \leqslant k \leqslant n$. By definition of the normalization strategy σ, for any $0 \leqslant i \leqslant 2n+1$, the $(n-1)$-cell $\widehat{\widetilde{u}}$ is a normal form with respect to E, and by convergence of the n-polygraph E it follows that $\widehat{\widetilde{u_i}} = \widehat{\widetilde{u_{i+1}}}$.

Moreover, for any $1 \leqslant i \leqslant 2n+1$, there exists a square $(n+1)$-cell in $\mathcal{C}$ as in the following diagram:

$$\begin{tikzcd} \widehat{u_{i+1}} \arrow[r, "="] \arrow[d, "e_{f_i}"'] \arrow[dr, phantom, "\Downarrow E_{i+1}"] & \widehat{u_{i+1}} \arrow[d, "e_{f_{i+1}}"] \\ \widehat{\widetilde{u_i}} \arrow[r, "="'] & \widehat{\widetilde{u_{i+2}}} \end{tikzcd}$$

We define a square $(n+1)$-cell σ_f in $\mathcal{C}$ as the following $\diamond^v$-composition:

$$\sigma_{f_0} \diamond^v E_1 \diamond^v \sigma_{f_1} \diamond^v \sigma_{f_2} \diamond^v \ldots \diamond^v \sigma_{f_{2n}} \diamond^v E_{2n+1} \diamond^v \sigma_{f_{2n+1}}$$

For an even integer $i \geqslant 0$

$$\begin{tikzcd}
u_i \arrow[r, "f_i"] \arrow[d, "\rho_{u_i}"'] & u_{i+1} \arrow[r, "\sigma_{u_{i+1}}"] \arrow[d, phantom, "\Downarrow \sigma_{f_i}"] & \widehat{u_{i+1}} \arrow[r, "="] \arrow[d, "e_{f_i}"'] & \widehat{u_{i+1}} \arrow[d, "e_{f_{i+1}}"] \arrow[l, phantom, "\Downarrow E_{i+1}"] & u_{i+1} \arrow[l, "\sigma_{u_{i+1}}"'] \arrow[d, phantom, "\Downarrow \sigma_{f_{i+1}}"] & u_{i+2} \arrow[l, "f_{i+1}"'] \arrow[r, "f_{i+2}"] \arrow[d, "\rho_{u_2}"] & u_{i+3} \arrow[r, "\sigma_{u_{i+3}}"] \arrow[d, phantom, "\Downarrow \sigma_{f_{i+2}}"] & \widehat{u_{i+3}} \arrow[r, "="] \arrow[d, "e_{f_{i+2}}"'] & \widehat{u_{i+3}} \arrow[d, "e_{f_{i+3}}"] \arrow[l, phantom, "\Downarrow E_{i+3}"] \\
\widetilde{u_i} \arrow[rr, "\sigma_{\widetilde{u_i}}"'] & & \widehat{\widetilde{u}}_i \arrow[r, "="'] & \widehat{\widetilde{u_{i+2}}} & & \widetilde{u_{i+2}} \arrow[ll, "\sigma_{\widetilde{u_{i+2}}}"] \arrow[rr, "\sigma_{\widetilde{u_{i+2}}}"'] & & \widehat{\widetilde{u_{i+2}}} \arrow[r, "="'] & \widehat{\widetilde{u_{i+4}}}
\end{tikzcd}.$$

In this way, we have constructed a square $(n+1)$-cell

$$\begin{tikzcd}
u \arrow[r, "f"] \arrow[d, "\rho_u"'] \arrow[dr, phantom, "\Downarrow \sigma_f"] & v \arrow[d, "\rho_v"] \\
\widetilde{u} \arrow[r, "\sigma_{\widetilde{u}} \sigma_{\widetilde{v}}^-"'] & \widetilde{v}
\end{tikzcd}$$

Similarly, we construct a square $(n+1)$-cell σ_g as follows:

$$\begin{tikzcd}
\widetilde{u} \arrow[r, "\sigma_{\widetilde{u}} \sigma_{\widetilde{v}}^-"] & \widetilde{v} \\
u' \arrow[u, "\rho_{u'}"] \arrow[r, "g"'] \arrow[ur, phantom, "\Uparrow \sigma_g"] & v' \arrow[u, "\rho_{v'}"']
\end{tikzcd}$$

using that $\widetilde{u} = \widetilde{u'}$ and $\widetilde{v} = \widetilde{v'}$ by convergence of E. We obtain a square $(n+1)$-cell $E_e \diamond^v (\sigma_f \diamond^h \sigma_g^-) \diamond^v E_{e'}$ filling the square (5.2.3), as in the following diagram:

$$\begin{tikzcd}
u \arrow[r, "="] \arrow[dd, "e"'] \arrow[ddr, phantom, "\Downarrow E_e"] & u \arrow[rr, "f"] \arrow[d, "\rho_u"] \arrow[drr, phantom, "\Downarrow \sigma_f"] & & v \arrow[r, "="] \arrow[d, "\rho_v"'] \arrow[ddr, phantom, "\Downarrow E_{e'}"] & v \arrow[dd, "e'"] \\
 & \widetilde{u} \arrow[r, "\sigma_{\widetilde{u}}"] & \widehat{\widetilde{u}} = \widehat{\widetilde{v}} & \widetilde{v} \arrow[l, "\sigma_{\widetilde{v}}"'] & \\
u' \arrow[r, "="'] & u' \arrow[rr, "g"'] \arrow[u, "\rho_{u'}"] \arrow[urr, phantom, "\Uparrow \sigma_g"] & & v' \arrow[r, "="'] \arrow[u, "\rho_{v'}"'] & v'
\end{tikzcd}$$

□

5.2.4. Corollary. *Let (R, E, S) be a diconvergent n-polygraph modulo. If $\mathrm{Irr}(E)$ is E-normalizing with respect to S, then for any coherent completion Γ of S modulo E and any coherent completion Γ_E of E, the square extension $E \rtimes \Gamma \cup \mathrm{Peiff}(E^\top, S^*) \cup \Gamma_E$ is acyclic.*

Note that, when E is empty in Corollary 5.2.4, we recover Squier's coherence theorem [35, Theorem 5.2] for convergent n-polygraphs, [17, Proposition 4.3.4].

5.2.5. Decreasing orders for E-normalization. Let (R, E, S) be an n-polygraph modulo. We describe a way to prove that the set $\mathrm{Irr}(E)$ is E-normalizing, laying on the definition of a termination order for R.

Given an n-polygraph P, one defines a *decreasing order operator* for P as a family of functions

$$\Phi_{p,q} : P^*_{n-1}(p, q) \to \mathbb{N}^{m(p,q)}$$

indexed by pairs of $(n-2)$-cells p and q in P^*_{n-2} satisfying the following conditions:

i) For any $(n-1)$-cells u and v in $P^*_{n-1}(p, q)$ such that there exists an n-cell $f : u \to v$ in P^*, the function $\Phi_{p,q}$ satisfy $\Phi_{p,q}(u) > \Phi_{p,q}(v)$, where $>$ is the lexicographic order on $\mathbb{N}^{m(p,q)}$. We denote by $>_{\mathrm{lex}}$ the partial order on P^*_{n-1} defined by $u >_{\mathrm{lex}} v$ if and only if u and v have same source p and target q and $\Phi_{p,q}(u) > \Phi_{p,q}(v)$.

ii) For any u and v in P^*_{n-1} and any whisker C on P^*_{n-1}, $u >_{\mathrm{lex}} v$ implies that $C[u] >_{\mathrm{lex}} C[v]$.

iii) The normal forms in $P^*_{n-1}(p, q)$ with respect to P are sent to the tuple $(0, \ldots, 0)$ in $\mathbb{N}^{m(p,q)}$.

Note that if an n-polygraph P admits a decreasing order operator, it is terminating. Actually, such a decreasing order is a terminating order for P which is similar to a monomial order, but that we do not require to be total.

5.2.6. Proving coherence modulo using a decreasing order. Consider an n-polygraph modulo (R, E, S) such that E is terminating. A decreasing order operator Φ for E is *compatible with* R if for any n-cell $f : u \to v$ in R^*, then $\Phi_{p,q}(u) \geqslant \Phi_{p,q}(v)$.

In that case, the set $\mathrm{Irr}(E)$ is E-normalizing with respect to R, since if u in R^*_{n-1} is a normal form with respect to E, $\Phi_{p,q}(u) = (0, \ldots, 0)$ in $\mathbb{N}^{m(p,q)}$ and by compatibility with R, for any n-cell $f : u \to v$ in R^*, we get $\Phi_{p,q}(v) = (0, \ldots, 0)$ so v is still a normal form with respect to E. We can also prove that $\mathrm{Irr}(E)$ is E-normalizing with respect to ${}_E R$ using this method, provided for any $(n-1)$-cell u in $\mathrm{Irr}(E)$ irreducible by R, any $(n-1)$-cell u' such that there is an n-cell $u \to u'$ in $E^\top$ is also irreducible by R. This is for instance the case if R is *left-disjoint from* E, that is for any $(n-1)$-cell u in $s(R)$, we have $G_R(u) \cap E_{n-1} = \emptyset$ where:

- $s(R)$ is the set of $(n-1)$-sources in R^*_{n-1} of generating n-cells in R_n,
- for any u in R^*_{n-1}, $G_R(u)$ is the set of generating $(n-1)$-cells in R_{n-1} contained in u.

With these conditions, we can apply Theorem 5.2.2 to obtain acyclic extensions of R or ${}_E R$.

5.3. Coherence by commutation

In this subsection, we prove that an acyclic extension of a pair $(E^\top, S^\top)$ coming from a polygraph modulo (R, E, S) can be obtained from an assumption of commuting normalization strategies for the polygraphs S and E. In particular, with further assumptions about this commutation we show how to prove E-normalization.

5.3.1. Commuting normalization strategies. Let (R, E, S) be an n-polygraph modulo. Let σ (resp. ρ) a normalization strategy with respect to S (resp. with respect to E). The normalization strategies σ and ρ are *weakly commuting* if for any u in R^*_{n-1}, there exists an n-cell η_u in S^* as in the following diagram:

$$
\begin{tikzcd}
u \arrow[r, "\sigma_u"] \arrow[d, "\rho_u"'] & \widehat{u} \arrow[d, "\rho_{\widehat{u}}"] \\
\widetilde{u} \arrow[r, dotted, "\eta_u"'] & \widetilde{\widehat{u}}
\end{tikzcd}
\tag{5.3.2}
$$

Given weakly commuting normalization strategies σ and ρ, we will denote by $N(\sigma, \rho)$ the square extension of the pair $(E^\top, S^\top)$ made of squares of the form (5.3.2), for every $(n-1)$-cell u in R^*_{n-1}.

The normalization strategies σ and ρ are said to be *commuting* if $\eta_u = \sigma_{\widetilde{u}}$ holds for all $(n-1)$-cell u in R^*_{n-1}. Note that, by definition σ and ρ commute if and only if the equality $\widehat{\widetilde{u}} = \widetilde{\widehat{u}}$ hold for all $(n-1)$-cells of R^*_{n-1}.

5.3.3. Theorem. *Let (R, E, S) be an n-polygraph modulo, and Γ be a square extension of the pair of $(n+1, n)$-categories $(E^\top, S^\top)$ such that S is Γ-diconvergent. If σ and ρ are weakly commuting normalization strategies for S and E respectively, then the square extension $E \rtimes \Gamma \cup$ Peiff$(E^\top, S^*) \cup \Gamma_E \cup N(\sigma, \rho)$ is acyclic.*

Proof. Denote by $\mathcal{C}$ the free n-category enriched in double groupoids $(E, S, E \rtimes \Gamma \cup \mathrm{Peiff}(E^\top, S^*) \cup \Gamma_E \cup N(\sigma, \rho))^{\top\!\!\top}$. For u in R^*_{n-1}, we denote by N_u the square $(n+1)$-cell in $\mathcal{C}$ corresponding to the square (5.3.2).

We prove that for any n-cell $f : u \to v$ in S^*, there exists a square $(n+1)$-cell $\widetilde{\sigma_f}$ in $\mathcal{C}$ of the following form

$$
\begin{tikzcd}
\widehat{u} \arrow[d, "\rho_{\widehat{u}}"'] & u \arrow[l, "\sigma_u"'] \arrow[r, "f"] \arrow[d, phantom, "\Downarrow \widetilde{\sigma_f}"] & v \arrow[r, "\sigma_v"] & \widehat{v} \arrow[d, "\rho_{\widehat{v}}"] \\
\widetilde{\widehat{u}} \arrow[rrr, "="'] & & & \widetilde{\widehat{v}}
\end{tikzcd}
$$

The square $(n+1)$-cell $\widetilde{\sigma_f}$ is obtained as the following composition:

$$
\begin{tikzcd}
\widehat{u} \arrow[d, "\rho_{\widehat{u}}"'] \arrow[r, phantom, "\Downarrow N_u"] & u \arrow[l, "\sigma_u"'] \arrow[r, "f"] \arrow[d, "\rho_u"] & v \arrow[r, "\sigma_v"] \arrow[d, phantom, "\Downarrow \eta_f"] & \widehat{v} \arrow[r, "="] & \widehat{v} \arrow[r, "="] \arrow[d, dotted, "e_{\eta_u}"'] \arrow[r, phantom, "\overset{\|}{\Downarrow} E_{e_{\eta_u}, e_{\widehat{v}}}", shift right=8] & \widehat{v} \arrow[r, "="] \arrow[d, dotted, "e_{\widehat{v}}"] \arrow[r, phantom, "\Downarrow \gamma_v", shift right=8] & \widehat{v} \arrow[d, "\rho_{\widehat{v}}"] \\
\widetilde{\widehat{u}} & \widetilde{u} \arrow[l, "\eta_u"] \arrow[rr, "\eta_u"'] & & \widetilde{\widehat{v}} \arrow[r, "\sigma_{\widetilde{\widehat{v}}}"'] & \widehat{\widetilde{\widehat{v}}} \arrow[r, "="'] & \widehat{\widetilde{\widehat{v}}} & \widetilde{\widehat{v}} \arrow[l, "\sigma_{\widetilde{\widehat{v}}}"]
\end{tikzcd}
$$

where the n-cell e_{η_u} and the square $(n+1)$-cell η_f (resp. the n-cell $e_{\widehat{v}}$ and the square $(n+1)$-cell γ_v) belong to $\mathcal{C}$ by Γ-confluence modulo E of S, and the square $(n+1)$-cell $E_{e_{\eta_u}, e_{\widehat{v}}}$ belongs to Γ_E.

Now, let consider a square

$$
\begin{tikzcd}
u \arrow[r, "f"] \arrow[d, "e"'] & v \arrow[d, "e'"] \\
u' \arrow[r, "g"'] & v'
\end{tikzcd}
\tag{5.3.4}
$$

in $\mathcal{C}$. By definition the n-cell f in $S^\top$ can be decomposed (in general in a non unique way) into a zigzag sequence

$$f_0 \star_{n-1} f_1^- \star_{n-1} \cdots \star_{n-1} f_{2n} \star_{n-1} f_{2n+1}^-$$

with source u and target v where the $f_{2k} : u_{2k} \to u_{2k+1}$ and $f_{2k+1} : u_{2k+2} \to u_{2k+1}$, for all $0 \leqslant k \leqslant n$ are n-cell of S^*, with $u_0 = u$ and $u_{2n+2} = v$. We define a square $(n+1)$-cell σ_f as the following vertical composition:

$$N_u \diamond^v \widetilde{\sigma_{f_0}} \diamond^v \widetilde{\sigma_{f_1}} \diamond^v \ldots \diamond^v \widetilde{\sigma_{f_{2n+1}}} \diamond^v N_v$$

as depicted on the following diagram

$$\begin{tikzcd}
u_0 \arrow[r, "\sigma_{u_0}"] \arrow[d, "\rho_{u_0}"'] \arrow[dr, phantom, "\Downarrow N_{u_0}"] & \widehat{u_0} \arrow[d, "\rho_{\widehat{u_0}}"] & u_0 \arrow[l, "\sigma_{u_0}"'] \arrow[r, "f_0"] \arrow[d, phantom, "\Downarrow \widetilde{\sigma_{f_0}}"] & u_1 \arrow[r, "\sigma_{u_1}"] & \widehat{u_1} \arrow[d, "\rho_{\widehat{u_1}}"'] & u_1 \arrow[l, "\sigma_{u_1}"'] \arrow[d, phantom, "\Downarrow \widetilde{\sigma_{f_1}}"] & u_2 \arrow[l, "f_1"'] \arrow[r, "\sigma_{u_2}"] & \widehat{u_2} \arrow[r, "\sigma_{u_2}"] \arrow[d, "\rho_{\widehat{u_2}}"] & u_2 \arrow[r, "f_2"] \arrow[d, phantom, "\Downarrow \widetilde{\sigma_{f_2}}"] & u_3 \arrow[r, "\sigma_{u_3}"] & \widehat{u_3} \arrow[d, "\rho_{\widehat{u_3}}"'] & \cdots \\
\widetilde{u_0} \arrow[r, "\eta_{u_0}"'] & \widehat{\widetilde{u_0}} \arrow[rrr, "="'] & & & \widehat{\widetilde{u_1}} \arrow[rrr, "="'] & & & \widehat{\widetilde{u_2}} \arrow[rrr, "="'] & & & \widehat{\widetilde{u_3}} & \cdots
\end{tikzcd}$$

In this way, we have constructed a square $(n+1)$-cell

$$\begin{tikzcd}
u \arrow[r, "f"] \arrow[d, "\rho_u"'] \arrow[dr, phantom, "\Downarrow \sigma_f"] & v \arrow[d, "\rho_v"] \\
\widetilde{u} \arrow[r, "\eta_u \eta_v^-"'] & \widetilde{v}
\end{tikzcd}$$

Similarly, we construct a square $(n+1)$-cell σ_g as follows:

$$\begin{tikzcd}
\widetilde{u} \arrow[r, "\eta_u \eta_v^-"] \arrow[dr, phantom, "\Uparrow \sigma_g"] & \widetilde{v} \\
u' \arrow[u, "\rho_{u'}"] \arrow[r, "g"'] & v' \arrow[u, "\rho_{v'}"']
\end{tikzcd}$$

using that $\widetilde{u} = \widetilde{u'}$ and $\widetilde{v} = \widetilde{v'}$ by convergence of E. We obtain a square $(n+1)$-cell filling the square (5.3.4), as in the proof of Theorem 5.2.2. □

5.3.5. Remarks. Note that when σ and ρ are commuting, $\mathrm{Irr}(E)$ is E-normalizing with respect to S since $\widehat{\widetilde{u}} = \widetilde{\widehat{u}}$ implies that the normal form $\widehat{\widetilde{u}}$ with respect to S also is a normal form with respect to E. Then Theorem 5.2.2 applies, to prove that $E \rtimes \Gamma \cup \mathrm{Peiff}(E^\top, S^*) \cup \Gamma_E$ is acyclic.

One can recover the fact that with the hypothesis of Theorem 5.3.3 and the assumption that the equality $\eta_u = \sigma_{\widetilde{u}}$ holds for any u in R_{n-1}^*, we do not need the square $(n+1)$-cells N_u in the coherent extension, using the following lemma on the square (5.3.2).

5.3.6. Lemma. *Let S be an n-polygraph modulo such that ${}_E R_E$ is terminating, and Γ be a square extension of the pair of $(n+1, n)$-categories $(E^\top, S^\top)$ such that S is Γ-confluent modulo E. Then any* *square in $\Gamma^{\curlyvee}$ of the form*

$$\begin{tikzcd}
u \arrow[r, "f"] \arrow[d, "e"'] & v \arrow[r, "f'"] & w \arrow[d, "e'"] \\
u' \arrow[r, "g"'] & v' \arrow[r, "g'"'] & w'
\end{tikzcd} \tag{5.3.7}$$

such that w and w' are normal forms with respect to S is the boundary of a square $(n+1)$-cell in $\Gamma^{\curlyvee}$.

Proof. Let us consider a square as in (5.3.7). By Γ-confluence of S modulo E on the branching (f, e, g), there exists a Γ-confluence as in the following diagram:

$$\begin{tikzcd}
u \arrow[r, "f"] \arrow[d, "e"'] & v \arrow[r, "f_1"] \arrow[d, phantom, "\Downarrow A"] & v_1 \arrow[d, "e''"] \\
u' \arrow[r, "g"'] & v' \arrow[r, "g_1"'] & v_1'
\end{tikzcd}$$

By Γ-confluence on the branchings (f', f_1) and (g_1, g') of S, there exist square $(n+1)$-cells B and B' as follows:

$$\begin{tikzcd}
u \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "i_1^h(f)"] & v \arrow[rr, "f'"] \arrow[d, equal] & & w \arrow[d, "e_1"] \\
u \arrow[r, "f"] \arrow[d, "e"'] & v \arrow[r, "f_1"] \arrow[d, phantom, "\Downarrow A"] \arrow[ur, phantom, "\Downarrow B"] & v_1 \arrow[r, "f_2"] \arrow[d, "e_2"] & v_2 \\
u' \arrow[r, "g"] \arrow[d, equal] \arrow[dr, phantom, "i_1^h(g)"] & v' \arrow[r, "g_1"] \arrow[d, equal] & v_1' \arrow[r, "g_2"] \arrow[d, phantom, "\Downarrow B'"] & v_2' \arrow[d, "e_3"] \\
u' \arrow[r, "g"] & v' \arrow[rr, "g'"] & & w'
\end{tikzcd}$$

Then, we use Huet's double induction as in Section 4 to prove that the square

$$\begin{tikzcd}
v_1 \arrow[r, "f_2"] \arrow[d, "e_2"'] & v_2 \arrow[d, "e_1^- e' e_2"] \\
v_1' \arrow[r, "g_2"'] & v_2'
\end{tikzcd}$$

is the boundary of a square $(n+1)$-cell in $\Gamma^{\curlyvee}$. $\square$

6. Globular coherence from double coherence

In this section we explain how to deduce a globular coherent presentation for an n-category from a double coherent presentation generated by a polygraph modulo. We apply this construction in the situation of commutative monoids in Subsection 6.2 and to pivotal monoidal categories in Subsection 6.3.

6.1. Globular coherence by convergence modulo

Let (R, E, S) be an n-polygraph modulo and Γ be a square extension on $(E^\top, S^\top)$. Consider the double $(n+1, n-1)$-polygraph given by $(E, S, E \rtimes \Gamma \cup \mathrm{Peiff}(E^\top, S^*) \cup \Gamma_E)$, where Γ_E is the square extension defined in 5.1.1. Let us denote by $((P_i)_{0\leqslant i\leqslant n+1}, (Q_i)_{1\leqslant i\leqslant n+1})$ the associated $(n+1, n-1)$-dipolygraph $V(E, S, E \rtimes \Gamma \cup \mathrm{Peiff}(E^\top, S^*) \cup \Gamma_E)$ given by the functor V defined in 2.3.2. The cellular extension S being defined modulo the cellular extension E in the sense of 3.1.1, we adapt the construction of the n-functor F in the quotient functor V defined in 2.3.1-**vi)** as follows.

a) F is the identity functor on the underlying $(n-2)$-category R^*_{n-2}, that coincides with E^*_{n-2},

b) F sends an $(n-1)$-cell u in R^*_{n-1} to its equivalence class $[u]^v$ modulo E_n,

c) F sends an n-cell $f : u \to v$ in $S^\top$ to the n-cell $[f]^v : [u]^v \to [v]^v$ in $(R^*_{n-1})_{E_n}(P_n)$ defined as in 2.3.1, **iv)-c)**, but by setting

$$[f]^v = [f_1]^v \star_{n-1} [f_2]^v \star_{n-1} \ldots \star_{n-1} [f_k]^v,$$

for any decomposition of $f = e_1 \star_{n-1} f_1 \star_{n-1} e_2 \star_{n-1} f_2 \star_{n-1} \ldots \star_{n-1} e_k \star_{n-1} f_k$ in $S^\top$, where the n-cells e_i and f_i are in $E^\top$ and $R^\top$ respectively and may be identity cells.

As a consequence of Proposition 2.3.5 and Corollary 5.2.4, we get the following result:

6.1.1. Proposition. *Let* (R, E, S) *be a diconvergent* n*-polygraph modulo. If* $\mathrm{Irr}(E)$ *is* E*-normalizing with respect to* S*, then for any coherent completion* Γ *of* S *modulo* E*, the* $(n+1, n-1)$*-dipolygraph* $V(E, S, E \rtimes \Gamma \cup \mathit{Peiff}(E^\top, S^*) \cup \Gamma_E)$ *is a globular coherent presentation of the* $(n-1)$*-category* $(R^*_{n-1})_E$.

6.1.2. Theorem. *Let* (R, E, S) *be a diconvergent* n*-polygraph modulo such that* $\mathrm{Irr}(E)$ *is* E*-normalizing with respect to* S*. Let* Γ *be a coherent completion of* S *modulo* E*, then the cellular extension*

$$[\Gamma]^v := \{[A]^v \mid A \in \Gamma\}$$

extends the n*-category* $(R^*_{n-1})_{E_n}(R_n)$ *into a globular coherent presentation of the* $(n-1)$*-category* $(R^*_{n-1})_E$.

Proof. The quotient functor V sends the cellular extension $E \rtimes \Gamma \cup \mathrm{Peiff}(E^\top, S^*) \cup \Gamma_E$ to $[\Gamma]^v$. Indeed, any square $(n+1)$-cell $E_{e,e'}$ in Γ_E yields an identity $(n+1)$-cell in the $(n+1)$-category $(R^*_{n-1})_{E_n}(S_n)(P_{n+1})$:

$$\begin{tikzcd} u \arrow[r, "="] \arrow[d, "e"'] \arrow[dr, phantom, "\Vert"] & u \arrow[d, "e'"] \\ v \arrow[d, dotted, "e_1"'] \arrow[r, phantom, "E_{e,e'}"] \arrow[dr, phantom, "\Downarrow"] & v' \arrow[d, dotted, "e'_1"] \\ w \arrow[r, dotted, "="'] & w \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd}[column sep=huge] {[u]^v = [w]^v} \arrow[r, bend left, "{[i_0^h(u)]^v}"] \arrow[r, bend right, "{[i_0^h(w)]^v}"'] \arrow[r, phantom, "\Downarrow"] & {[u]^v = [w]^v} \end{tikzcd}$$

Similarly, any $(n+1)$-cell in $\mathrm{Peiff}(E^\top, S^*)$ yields an identity $(n+1)$-cell in the $(n+1)$-category $(R^*_{n-1})_{E_n}(S_n)(P_{n+1})$. Finally, two square $(n+1)$-cells in the same orbit for the biaction of the $(n, n-1)$-category $E^\top$ on $\mathrm{Sqr}(E^\top, S^*)$ are sent on the same globular $(n+1)$-cell in $(R^*_{n-1})_{E_n}(S_n)(P_{n+1})$. $\square$

6.1.3. Gobular coherent completion procedure for $_ER$. Given a diconvergent n-polygraph modulo (R, E, S), Corollary 5.2.4 gives a method to construct an acyclic square extension of the pair of $(n, n-1)$-categories $(E^\top, S^\top)$. In many applications, this result is applied with $S = {}_ER$ and in situations where $_ER$ is not confluent modulo E. When $_ER$ is equipped with a termination order compatible with R modulo E, one can apply the completion procedure of Subsection 3.4 to obtain an n-polygraph $\check{R}$ such that $_E\check{R}$ is confluent modulo E. Moreover, following Corollary 6.1.2 the only square cells that we have to consider in the construction of the globular coherent presentation through the quotient functor V are the square cells $A_{f,g}$ and $B_{f,e}$ of (5.1.3) of a coherent completion of S modulo E. In the particular case of $_ER$, we do not have to consider square cells of the form $B_{f,e}$. Indeed, the critical branchings (f, e) where f is an n-cell in $S^{*(1)}$ and e is an n-cell in $E^{\top(1)}$ are trivially confluent from 3.4.1, and the square $(n+1)$-cell $B_{f,e}$ obtained by the following choice of a confluence modulo E:

$$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, "e"'] \arrow[dr, phantom, "\Downarrow B_{f,e}"] & v \arrow[d, "\shortparallel"] \\ u' \arrow[r, "e^- \cdot f"'] & v \end{tikzcd}$$

yields an identity $(n+1)$-cell

$$\begin{tikzcd} {[u]^\nu = [u']^\nu} \arrow[r, bend left, "{[f]^\nu}"] \arrow[r, bend right, "{[e^- \cdot f]^\nu = [f]^\nu}"'] \arrow[r, phantom, "\Downarrow i_{[f]^\nu}"] & {[v]^\nu} \end{tikzcd}$$

in the $(n+1)$-category $((R^*_{n-1})_{E_n}(P_n))(P_{n+1})$. As a consequence, one only needs to choose a family of square $(n+1)$-cells

$$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, "\shortparallel"'] & u' \arrow[r, "f'"] \arrow[d, phantom, "\Downarrow A_{f,g}"] & w \arrow[d, "e'"] \\ u \arrow[r, "g"'] & v \arrow[r, "g'"'] & w' \end{tikzcd}$$

for a choice of confluence modulo E of any critical branching (f, g) of S modulo E, where f is an n-cell of $_ER^{*(1)}$ and g is an n-cell of $R^{*(1)}$. Applying the quotient functor V of 2.3.2 on the set of square $(n+1)$-cells $A_{f,g}$, following Theorem 6.1.1, we obtain an acyclic extension of the n-category $(R^*_{n-1})_{E_n}(P_n)$ given by

$$\{\ [A_{f,g}]^\nu \mid (f, g) \text{ is a critical branching of } S \text{ modulo } E\ \},$$

where bracket notation $[-]^\nu$ is defined in 2.3.4.

6.2. Commutative monoids

We illustrate the completion procedure 6.1.3 to show how to compute a coherent presentation of a commutative monoid presented by a 2-polygraph modulo $(R, E, {}_ER_E)$, where E is the 2-polygraph $\mathsf{Com}_2(X)$ for a finite set X defined in 2.4.2. The 2-cell of the 2-polygraph $\mathsf{Com}_2(X)$ are oriented with respect to a deglex order induced by a total order on X, hence $\mathsf{Com}_2(X)$ is terminating. It is also confluent by confluence of any critical branching depicted as follows:

$$
\begin{tikzcd}[column sep=small]
 & x_i x_k x_j \arrow[r, Rightarrow, "\alpha_{i,k} x_j"] & x_k x_i x_j \arrow[dr, Rightarrow, "x_k \alpha_{i,j}"] & \\
x_i x_j x_k \arrow[ur, Rightarrow, "x_i \alpha_{j,k}"] \arrow[dr, Rightarrow, "\alpha_{i,j} x_k"'] & & & x_k x_j x_i \\
 & x_j x_i x_k \arrow[r, Rightarrow, "x_j \alpha_{i,k}"'] & x_j x_k x_i \arrow[ur, Rightarrow, "\alpha_{j,k} x_i"'] &
\end{tikzcd}
$$

for any x_i, x_j, x_k in X such that $x_i > x_j > x_k$, and the 2-cells $\alpha_{-,-}$ are the generating 2-cell of $\mathsf{Com}_2(X)$.

6.2.1. Example. Consider such a 2-polygraph modulo with $X = \{x_1, x_2, x_3, x_4\}$, and

$$
R_2 = \{x_1 x_3 \overset{\beta}{\Rightarrow} x_2 x_4,\ x_1 x_2 \overset{\gamma}{\Rightarrow} x_1\}.
$$

There is a critical branching of ${}_ER_E$ modulo E given by

$$
\begin{tikzcd}
x_1 x_2 x_3 \arrow[rr, Rightarrow, "\alpha^-_{2,3} \cdot \beta"] \arrow[d, Rightarrow, "\shortparallel"'] & & x_2 x_4 x_2 \\
x_1 x_2 x_3 \arrow[r, Rightarrow, "\gamma"'] & x_1 x_3 \arrow[r, Rightarrow, "\beta"] & x_2 x_4
\end{tikzcd}
\tag{6.2.2}
$$

where $\alpha^-_{2,3} \cdot \beta$ is the rewriting step of ${}_ER_E$ defined by $x_1 x_2 x_3 \overset{\alpha^-_{2,3}}{\Longrightarrow} x_1 x_3 x_2 \overset{\beta x_2}{\Longrightarrow} x_2 x_4 x_2$. As any permutation of the x_i in $x_2 x_4 x_2$ and $x_2 x_4$ are irreducible with respect to R_2, the 1-cells $x_2 x_4 x_2$ and $x_2 x_4$ are normal forms with respect to ${}_ER_E$, so the branching (6.2.2) is not confluent modulo E. Following the completion procedure 3.4.5, we define the following 2-cell

$$
\delta : x_2 x_2 x_4 \Rightarrow x_2 x_4,
$$

and we set $R := R \cup \{\gamma\}$. The degree lexicographic order induced by $x_1 > x_2 > x_3 > x_4$ is a termination order compatible with R_2 modulo E, so that ${}_ER_E$ is terminating and $\mathrm{Irr}(E)$ is trivially E-normalizing with respect to ${}_ER_E$. Moreover, the 2-polygraph modulo ${}_ER_E$ is confluent modulo E. Indeed, all its critical branchings modulo, depicted in (6.2.3) and (6.2.4), are confluent modulo.

$$
\begin{tikzcd}
x_1 x_2 x_3 \arrow[r, Rightarrow, "\alpha^-_{2,3} \cdot \beta"] \arrow[d, Rightarrow, "\shortparallel"'] & x_2 x_4 x_2 \arrow[r, Rightarrow, "\alpha^-_{2,4} \cdot \delta"] \arrow[d, Rightarrow, "A"] & x_2 x_4 \arrow[d, Rightarrow, "\shortparallel"] \\
x_1 x_2 x_3 \arrow[r, Rightarrow, "\gamma"'] & x_1 x_3 \arrow[r, Rightarrow, "\beta"'] & x_2 x_4
\end{tikzcd}
\qquad
\begin{tikzcd}
x_2 x_2 x_4 x_1 \arrow[r, Rightarrow, "\alpha_{2,4} \cdot \gamma"] \arrow[d, Rightarrow, "\shortparallel"'] & x_2 x_4 x_1 \arrow[r, Rightarrow, "\alpha^-_{1,4} \alpha^-_{1,2} \cdot \gamma"] \arrow[d, Rightarrow, "B"] & x_2 x_4 \arrow[d, Rightarrow, "\shortparallel"] \\
x_2 x_2 x_4 x_1 \arrow[r, Rightarrow, "\delta x_1"'] & x_2 x_4 x_1 \arrow[r, Rightarrow, "\alpha^-_{1,4} \alpha^-_{1,2} \cdot \gamma"'] & x_2 x_4
\end{tikzcd}
\tag{6.2.3}
$$

$$
\begin{tikzcd}
x_2x_4x_2x_4x_2 \arrow[r, Rightarrow, "\alpha^-_{2,4}\cdot\delta"] \arrow[d, Rightarrow, "="'] & x_2x_4x_4x_2 \arrow[r, Rightarrow, "(\alpha^-_{2,4})^2\cdot\delta"] \arrow[d, Rightarrow, "C"] & x_2x_4x_4 \arrow[d, Rightarrow, "="] \\
x_2x_4x_2x_4x_2 \arrow[r, Rightarrow, "\alpha^-_{2,4}\cdot\delta"'] & x_2x_4x_2x_4 \arrow[r, Rightarrow, "\alpha^-_{2,4}\cdot\delta"'] & x_2x_4x_4
\end{tikzcd}
\tag{6.2.4}
$$

Following procedure 6.1.3, one shows that an acyclic extension of the commutative monoid generated by X and submitted to relations in R_2 can be computed from the the square extension $\{A, B, C\}$ of $(E^\top, {}_E R_E^\top)$. This acyclic extension is made of the following 3-cells.

$$
\begin{tikzcd}
{[x_1x_2x_3]} \arrow[r, bend left, Rightarrow, "{[\beta]\star_1[\delta]}"] \arrow[r, bend right, Rightarrow, "{[\gamma]\star_1[\beta]}"'] \arrow[r, phantom, "\Downarrow{[A]}"] & {[x_2x_4]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[x_1x_2x_2x_4]} \arrow[r, bend left, Rightarrow, "{[\delta]\star_1[\gamma]}"] \arrow[r, bend right, Rightarrow, "{[\delta]\star_1[\gamma]}"'] \arrow[r, phantom, "\Downarrow{[B]}"] & {[x_2x_4]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[x_2x_2x_2x_4x_4]} \arrow[r, bend left, Rightarrow, "{[\delta]\star_1[\delta]}"] \arrow[r, bend right, Rightarrow, "{[\delta]\star_1[\delta]}"'] \arrow[r, phantom, "\Downarrow{[C]}"] & {[x_2x_4x_4]}
\end{tikzcd}
$$

Note that if we take the commutation 2-cells as rewriting rules, the Knuth-Bendix completion is infinite, requiring to add a 2-cell $\varepsilon_n : x_4x_3^n x_2x_2 \Rightarrow x_4x_3^n x_2$ for any $n \geqslant 0$. This yields acyclic extension made of an infinite set of 3-cells

$$
\begin{tikzcd}
& x_4x_3^{n+1}x_2x_2 \arrow[dr, Rightarrow, "\varepsilon_{n+1}"] & \\
x_4x_3^n x_2x_2x_3 \arrow[ur, Rightarrow, "\alpha^2_{2,3}"] \arrow[dr, Rightarrow, "\varepsilon_n x_3"'] \arrow[rr, phantom, "\Downarrow D_n"] & & x_4x_3^{n+1}x_2 \\
& x_4x_3^n x_2x_3 \arrow[ur, Rightarrow, "\alpha_{2,3}"'] &
\end{tikzcd}
$$

6.3. Pivotal monoidal categories

We present an application of the coherence Theorem 5.2.2 on a toy example in the context of diagrammatic rewriting. We consider a presentation of a pivotal monoidal category, seen as a pivotal 2-category with only one 0-cell presented by a 3-polygraph. The pivotal structure implies that two isotopic string diagrams represent the same 2-cell of the 2-category. Such relations produce many critical branching with primary rules of the presentation. In this example, using the structure of polygraph modulo, we show how to manage such isotopy rules with the primary rules in the computation of a coherent presentation of the given monoidal category. In particular, we illustrate this method on an example of relation arising in many presentations of monoidal categories, relation (6.3.7), see for instance Khovanov-Lauda's 2-category introduced in [26] which categorifies quantum groups associated with symmetrizable Kac-Moody algebras, or in the definition of Heisenberg category as given by Khovanov in [27], extended by Brundan in [7].

6.3.1. Example. We consider the 3-polygraph P defined by the following data:

i) only one generating 0-cell,

ii) two generating 1-cells $\curlywedge$ and $\curlyvee$,

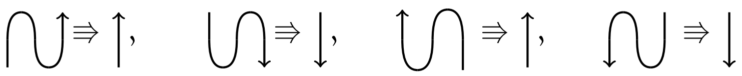
iii) eight generating 2-cells pictured by

(6.3.2)

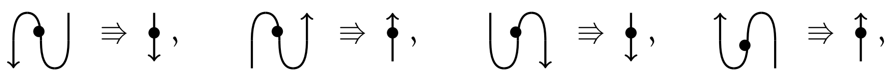
(6.3.3)

iv) the generating 3-cells of P are given by:

a) the three families of generating 3-cells of the 3-polygraph of pearls from [17]:



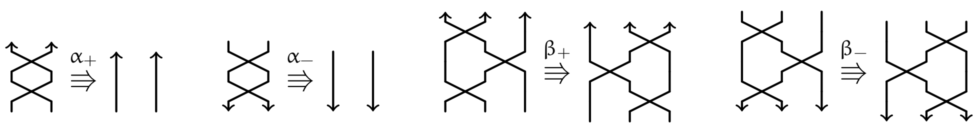
(6.3.4)



(6.3.5)

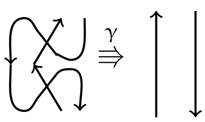
(6.3.6)

b) the generating 3-cells of the 3-polygraph of permutations for both upward and downward orientations of strands:



(6.3.7)

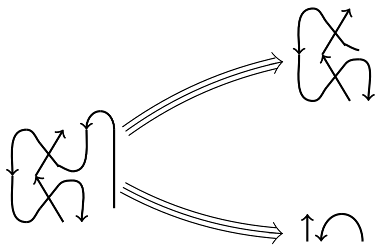
c) a generating 3-cell



(6.3.8)

Note that the relations (6.3.4 – 6.3.6) correspond to the fact that the generating 1-cells $\curlyvee$ and $\curlywedge$ are biadjoints in the 2-category $\overline{\mathrm{P}}$ presented by P, and cups and caps 2-cells are units and counits for these adjunctions. Relations implying dots also ensure that the dot 2-cell is a cyclic 2-morphism in the sense of [10] for the biadjunction $\curlyvee \vdash \curlywedge \vdash \curlyvee$, making $\overline{\mathrm{P}}$ into a *pivotal 2-category*.

6.3.9. Confluence of polygraph P. The 3-polygraph P is not confluent since the branching



(6.3.10)

is not confluent. Moreover, solving this obstruction to confluence using a Knuth-Bendix completion may create a great number of relations, making analysis of confluence from critical branchings inefficient. To tackle this issue, we use rewriting modulo isotopy.

6.3.11. Confluence modulo isotopy. We consider the 3-polygraph E defined by the following data

- **i)** $E_{\leqslant 1} = P_{\leqslant 1}$,
- **ii)** it has six 2-cells given in (6.3.3) and the dot 2-cells in 6.3.2,
- **iii)** the isotopy 3-cells (6.3.4 – 6.3.6) of the 3-polygraph of pearls.

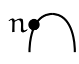
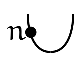
Let R be a 3-polygraph such that $R_{\leqslant 2} = P_{\leqslant 2}$ where P is the 3-polygraph of 6.3.1, and whose 3-cells are given by $(\alpha_\pm, \beta_\pm, \gamma)$ of (6.3.7 – 6.3.8), and let us consider the 3-polygraph modulo ${}_E R$. Following 3.4.1, the only critical branchings we have to consider are those of the form (f, g) with f in ${}_E R^{*(1)}$ and g in $R^{*(1)}$. The branching (6.3.10) is not such a branching because the top 3-cell belongs to $E^\top$, and the top-right 2-cell is not reducible by R. Moreover, one can check that the only critical branchings we have to consider are given by pairs (f, g) of 3-cells both in $R^{*(1)}$. The 3-cell γ in $R^{*(1)}$ does not overlap with $\alpha_\pm$ or $\beta_\pm$, so the only critical branchings we have to consider are those of the 3-polygraph of permutations described in [17, 5.4.4], with either upward or downward oriented strands.

6.3.12. Decreasing order operator for E-normalization. The 3-polygraph R is left-disjoint from E, since no caps and cups 2-cells appear in the sources of the generating 3-cells of R. Following 5.2.6, we prove that $\mathrm{Irr}(E)$ is E-normalizing with respect to ${}_E R$ using a decreasing order operator Φ for E compatible with R.

6.3.13. Lemma. *Let E and R be the 3-polygraphs defined above. There exists a decreasing operator order Φ for E compatible with R.*

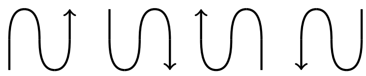
Proof. For any 1-cells p and q in R_1^*, we set $m(p, q) = 2$ and for any 2-cell u of source p and target q in R_2^*, $\Phi_{p,q}(u) = (\mathrm{ldot}(u), I(u))$ where:

- **i)** $\mathrm{ldot}(u)$ counts the number of left-dotted caps and cups, adding for such cap and cup the number of dots on it. In particular, for any n in $\mathbb{N}^*$, we have

  ldot() = ldot() $:= n + 1$

  for both orientations of strands.

ii) $I(u)$ counts the number of instances of one of the following 2-cells of R_2^* in u:



For any 3-cell $u \Rrightarrow v$ in E, we have $\Phi(u) > \Phi(v)$ and that $\Phi(u, u) = (0, 0)$ for any u in $\mathrm{Irr}(E)$. Moreover, Φ is compatible with R because rewritings with respect to R do not make the dot 2-cell move around a cup or a cap, or create sources of isotopies. $\square$

As a consequence of Theorem 5.2.2, we deduce an acyclic square extension of the pair of $(3, 2)$-categories $(E^\top, {}_E R^\top)$. This square extension is made of the ten elements given by the diagrams of the homotopy basis for the 3-polygraph of permutations from [17, Section 5.4.4] for both upward and downward orientations of strands and the 16 elements given by the diagrams of the homotopy basis or the 3-polygraph of pearls in [17, Section 5.5.3] for both orientations of strands form.

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