A Homotopical Completion Procedure with Applications to Coherence of Monoids

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Abstract

One of the most used algorithm in rewriting theory is the Knuth-Bendix completion procedure which starts from a terminating rewriting system and iteratively adds rules to it, trying to produce an equivalent convergent rewriting system. It is in particular used to study presentations of monoids, since normal forms of the rewriting system provide canonical representatives of words modulo the congruence generated by the rules. Here, we are interested in extending this procedure in order to retrieve information about the low-dimensional homotopy properties of a monoid. We therefore consider the notion of *coherent presentation*, which is a generalization of rewriting systems that keeps track of the cells generated by confluence diagrams. We extend the Knuth-Bendix completion procedure to this setting, resulting in a *homotopical completion procedure*. It is based on a generalization of *Tietze transformations*, which are operations that can be iteratively applied to relate any two presentations of the same monoid. We also explain how these transformations can be used to remove useless generators, rules, or confluence diagrams in a coherent presentation, thus leading to a *homotopical reduction procedure*. Finally, we apply these techniques to the study of some examples coming from representation theory, to compute minimal coherent presentations for them: braid, plactic and Chinese monoids.

Introduction

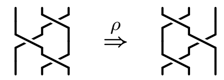
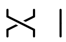
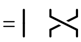
A monoid can be presented as the free monoid Σ_1^* on a set Σ_1 of *generators* quotiented by a congruence generated by a set of *relations* $\Sigma_2 \subseteq \Sigma_1^* \times \Sigma_1^*$. This data, called a *presentation* of the monoid, can be quite useful since it can provide a small description of it, from which various invariants can be computed, such as the homology of the monoid. For instance, the commutative monoid $\mathbb{N} \times \mathbb{N}$ admits the presentation $\langle \Sigma_1 \mid \Sigma_2 \rangle = \langle a, b \mid ba = ab \rangle$ with two generators and one relation. A way to show this result is to orient the relation as $ba \Rightarrow ab$ in order to obtain a *string rewriting system*. This rewriting system is easily checked to be terminating and confluent (there is no critical pair), and the normal forms are canonical representative of words modulo the congruence generated by the relation: here, normal forms are words of the form $a^m b^n$, which are in bijection with elements of the monoid $\mathbb{N} \times \mathbb{N}$.

The Knuth-Bendix completion. This recipe for constructing presentations does not always work as easily. In particular, the rewriting system obtained by orienting arbitrarily the relations has no reason to be convergent (i.e. both terminating and confluent). However, it was observed by Knuth and Bendix [16] that by adding rules to the rewriting system, one can sometimes complete it into a finite convergent one. The procedure that they have formulated in order to perform this completion in good cases (the procedure is not guaranteed to terminate) is one of the most used tool in rewriting theory.

Conference title on which this volume is based on.
Editors: Billy Editor, Bill Editors; pp. 1–20
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Tietze transformations. The starting point of the present work is the following observation: the Knuth-Bendix procedure operates by iteratively adding new relations, and this operation is a particular case of *Tietze transformation* [28]. These are basic operations that one can perform on a presentation, in such a way that they do not change the presented monoid, and one can always transform a presentation of a given monoid into another one by applying a series of such transformations; two presentations of the same monoid are thus called *Tietze-equivalent*. These transformations are of four kinds: add or remove a definable generator, and add or remove a derivable relation.

Adding generators. The Knuth-Bendix procedure only exploits one kind of transformations in order to complete a rewriting system: given a critical pair $v \overset{f}{\Leftarrow} u \overset{g}{\Rightarrow} w$ it adds a rule $h : v \Rightarrow w$ (or its converse), which is derivable since $h = f^{-1} \circ g$; in particular, adding it does not change the monoid presented by the rewriting system. Could the procedure be improved by also adding new generators during completion? On the theoretical level, an affirmative answer has been brought by Kapur and Narendran [14] who considered the usual Artin presentation Σ of the monoid $\mathbf{B}_3^+$ of positive braids with 3 strands (with its alternative graphical representation on the right):

$tst \overset{\rho}{\Rightarrow} sts$  with $s =$  and $t =$  (1)

They show that there exists no finite convergent string rewriting system, *with the same generators* s and t, that presents the monoid $\mathbf{B}_3^+$. However, they consider the string rewriting system Υ with three generators s, t and a new generator a (standing for the product st) and two relations $st \Rightarrow a$ and $sts \Rightarrow tst$. This rewriting system Υ is Tietze-equivalent to the rewriting system Σ, but applying the Knuth-Bendix completion procedure on it terminates, giving rise to the convergent rewriting system Υ' with s, t, a as generators, and rules

$$ta \overset{\alpha}{\Rightarrow} as, \qquad st \overset{\beta}{\Rightarrow} a, \qquad sas \overset{\gamma}{\Rightarrow} aa, \qquad saa \overset{\delta}{\Rightarrow} aat. \tag{2}$$

Thus, adding a superfluous generator has made completion possible. The reason why the completed rewriting system Υ' is Tietze-equivalent to the original rewriting system Σ can be understood by considering its four confluent critical branchings:

$$\begin{tikzcd} & aa \\ sta \arrow[ur, Rightarrow, "\beta a"] \arrow[dr, Rightarrow, "s\alpha"'] \arrow[r, phantom, "\Downarrow A"] & \\ & sas \arrow[uu, Rightarrow, "\gamma"'] \end{tikzcd}
\quad
\begin{tikzcd} & aat \\ sast \arrow[ur, Rightarrow, "\gamma t"] \arrow[dr, Rightarrow, "sa\beta"'] \arrow[r, phantom, "\Downarrow B"] & \\ & saa \arrow[uu, Rightarrow, "\delta"'] \end{tikzcd}
\quad
\begin{tikzcd} & aaas & \\ sasas \arrow[ur, Rightarrow, "\gamma as"] \arrow[dr, Rightarrow, "sa\gamma"'] \arrow[r, phantom, "\Downarrow C"] & & aata \arrow[ul, Rightarrow, "aa\alpha"'] \\ & saaa \arrow[ur, Rightarrow, "\delta a"'] & \end{tikzcd}
\quad
\begin{tikzcd} & aaaa & aaast \arrow[l, Rightarrow, "aaa\beta"'] \\ sasaa \arrow[ur, Rightarrow, "\gamma aa"] \arrow[dr, Rightarrow, "sa\delta"'] \arrow[r, phantom, "\Downarrow D"] & & \\ & saaat \arrow[r, Rightarrow, "\delta at"'] & aatat \arrow[uu, Rightarrow, "aa\alpha t"'] \end{tikzcd} \tag{3}$$

The cell $A : (\beta a) \Rrightarrow (s\alpha) \circ \gamma$ witnesses the fact that rule γ is superfluous since $\gamma = (s\alpha)^{-1} \circ (\beta a)$ and, similarly, the cell B proves that $\delta = (sa\beta)^{-1} \circ (\gamma t)$ is superfluous. Finally, the rule β witnesses the fact that the generator a is superfluous (it is equivalent to st). We are left with the rule α where a has been substituted by st, i.e. $\rho : tst \Rightarrow sts$ in (1). As we will see, this example is far from being isolated, thus justifying the use of Tietze transformations as a central concept to study existing extensions and refinements of completion procedures, such as Pedersen's morphocompletion [23], or to introduce new ones.

A homotopical completion procedure. The four diagrams in (3) are the generators of an equivalence relation between rewriting paths: two paths with the same source and the same target are equal up to those diagrams. The previous discussion shows the importance of keeping track of those higher-dimensional cells, which carry information about the rewriting system (for example, if one wants to compute invariants of monoids such as homology groups). This motivates the use of a generalization of the notion of presentation, called *coherent presentation*, which takes in account the higher-dimensional information contained in *homotopy generators* (the diagrams in (3)), and of a generalization of the notion of Tietze transformation to this setting. The *homotopical completion procedure* extends the Knuth-Bendix procedure into a tool for computing coherent presentations, by keeping track of homotopy generators created when adding new rules.

A homotopical reduction procedure. The additional information contained in coherent presentations can also help one to reduce a presentation by removing superfluous generators, rules and homotopy generators. For instance, we have already mentioned that the cell A in (3) indicates that the rule γ is superfluous. Similarly, the rule β indicates that the generator a is superfluous since it is equivalent to the product st, and we will see that superfluous homotopy generators in (3) can be also removed by computing critical triples of the rewriting system. All these operations of removing superfluous data from a presentation are again examples of Tietze transformations. Based on these, we introduce here a *homotopical reduction procedure* for coherent presentations which minimizes a coherent presentation, such as one obtained from our homotopical completion procedure. The general idea of this work is thus to give ways to mutate presentations using Tietze transformations in order to come up with presentations satisfying various properties: convergence, coherence, minimality, etc.

Coherent presentations to compute invariants of monoids. Minimal presentations obtained in this way exhibit invariants of the monoid, in the sense that even though constructed from a particular presentation of the monoid, their number of generators, rules and homotopy generators do not depend on the presentation, only on the monoid. In particular, when the monoid is presented by a finite convergent presentation, the corresponding coherent presentation always has a finite number of homotopy generators. This important result of Squier's theory [26, 25, 27], further studied in subsequent works [19, 18, 17, 12], has enabled him to construct a finitely presented decidable monoid with no finite convergent presentation: as a consequence, rewriting is not universal to decide the word problem in decidable monoids.

Applications in algebra and representation theory. Coherent presentations also appear as a fundamental structure in representation theory (in particular through the examples of Artin and plactic monoids). One of the motivations of the results presented here is to apply constructive rewriting methods to compute coherent presentations for these algebraic structures arising in geometry. For instance, Tits' theorem [29] states that an Artin group has a coherent presentation where coherence cells are given by its parabolic subgroups of rank 3 (a similar result holds for Artin monoids). The original proof relies on geometry, we give here a constructive methodology that has since been used to obtain a coherent presentation for any Artin monoid and group [9]. We also apply our completion methods to the plactic monoid, used in the representation theory of semisimple Lie algebras [21].

Contents of the paper. We introduce the notion of coherent presentation in Section 1, for which we formulate a homotopical completion-reduction procedure in Section 2, applied to various examples in Section 3. We should mention here that this works is part of a much larger general program aiming at studying the higher-dimensional properties of rewriting theory, see [10, 11, 22, 13, 12, 9] for example, based on Burroni's notion of polygraph [2].

1 Coherent presentations

The purpose of this section is to recall some classical material about rewriting systems and introduce some notions and notations from higher-dimensional rewriting. More details can be found in the mentioned references. In the following, we will assimilate the notion of string rewriting system to a presentation.

▸ **Definition 1.** A *presentation* $\Sigma = (\Sigma_1, \Sigma_2)$ consists of a set Σ_1 of *generators* and a set $\Sigma_2 \subseteq \Sigma_1^* \times \Sigma_1^*$ of *rewriting rules*. It is *finite* if both sets Σ_1 and Σ_2 are finite.

In the definition, Σ_1^* denotes the free monoid over Σ_1 (*words* over the alphabet Σ_1). We write $\rho : u \Rightarrow v$ for a rule $\rho = (u,v)$ in Σ_2. A word u *rewrites* to a word v, denoted by $u \Rightarrow_\Sigma v$, when there exist words w_1 and w_2 and a rule $\rho : u' \Rightarrow v'$ such that $u = w_1 u' w_2$ and $v = w_1 v' w_2$; the corresponding *rewriting step* is then denoted by $w_1 \rho w_2 : u \Rightarrow v$.

The *reduction graph* $\mathcal{G}_\Sigma$ of a presentation Σ is the graph whose vertices are words and whose edges are rewriting steps. We write $u \Rightarrow_\Sigma^* v$ when there exists a directed path from u to v in $\mathcal{G}_\Sigma$. A string rewriting system Σ is a *presentation* of a monoid $\mathbf{M}$ when $\mathbf{M}$ is isomorphic to the free monoid over Σ_1 quotiented by the congruence $\Leftrightarrow_\Sigma^*$ generated by the relations in Σ_2, i.e. $u \Leftrightarrow_\Sigma^* v$ whenever there exists a non-directed path from u to v in $\mathcal{G}_\Sigma$. We write $\langle a_1, \ldots, a_n \mid u_1 \Rightarrow v_1, \ldots, u_n \Rightarrow v_n \rangle$ for the monoid presented by the rewriting system Σ with $\Sigma_1 = \{a_1, \ldots, a_n\}$ and $\Sigma_2 = \{\rho_1 : u_1 \Rightarrow v_1, \ldots, \rho_n : u_n \Rightarrow v_n\}$.

1.1 Coherent presentations of monoids

We extend the notion of presentation in order to incorporate an equivalence between paths which is described by a set of homotopy generators, in the same way that rewriting rules specify an equivalence between words. In order to do so, we first have to explicitly consider the rewriting paths (and not only the convertibility relation, i.e. whether there exists a path between two words), which leads us to define the category generated by a rewriting system.

▸ **Definition 2.** Let Σ be a presentation. The *category* Σ_2^* *generated by* Σ has the words in Σ_1^* as objects, and the directed paths in $\mathcal{G}_\Sigma$ as morphisms, quotiented by the smallest congruence forgetting the order of rewriting steps at two disjoint positions, i.e. formally making the following diagrams commutative:

$$\begin{tikzcd}
& wu_2w'v_1w'' \arrow[dr, Rightarrow, "wu_2w'\sigma w''"] & \\
wu_1w'v_1w'' \arrow[ur, Rightarrow, "w\rho w'v_1w''"] \arrow[dr, Rightarrow, "wu_1w'\sigma w''"'] \arrow[rr, phantom, "="] & & wu_2w'v_2w'' \\
& wu_1w'v_2w'' \arrow[ur, Rightarrow, "w\rho w'v_2w''"'] &
\end{tikzcd}
\qquad \text{for all } \begin{cases} u_1 \overset{\rho}{\Rightarrow} u_2 \text{ and } v_1 \overset{\sigma}{\Rightarrow} v_2 \text{ in } \Sigma_2 \\ w,\ w' \text{ and } w'' \text{ in } \Sigma_1^*. \end{cases}$$

Equivalence of rewriting at disjoint positions is justified by local confluence (orthogonal branchings are never obstructions to confluence) and corresponds on the categorical side to the exchange axiom of 2-categories. The *groupoid* $\Sigma_2^\top$ *generated by the presentation* Σ is the category defined similarly, with inverses added for each morphism: morphisms are non-directed paths in $\mathcal{G}_\Sigma$ and we write $f^{-1} : v \Rightarrow u$ for a path $f : u \Rightarrow v$ taken backwards.

▸ **Example 3.** Consider the presentation Σ with $\Sigma_1 = \{s, t, a\}$ as generators, and whose rules in Σ_2 are the four rules of (2). The following composite of rewriting steps is a morphism in Σ_2^*: $(sa\gamma) \circ (\delta a) \circ (aa\alpha) : sasas \Rightarrow aaas$; it occurs in the border of the cell C in (3). Similarly, the composite $(s\alpha) \circ \gamma \circ (\beta a)^{-1} : sta \Rightarrow sta$ is a morphism in the groupoid $\Sigma_2^\top$, which is the border of the cell A in (3).

Given a morphism $f : u \Rightarrow v$ and words w_1, w_2 in Σ_1^*, we write $w_1 f w_2 : w_1 u w_2 \Rightarrow w_1 v w_2$ for the morphism f "extended by context". This enables us to equip the category Σ_2^* with a structure of monoidal category: given two morphisms $f : u \Rightarrow v$ and $f' : u' \Rightarrow v'$, we can define a morphism $f \otimes f' : (u \otimes u') \Rightarrow (v \otimes v')$ which corresponds to performing the two rewriting paths f and f' in parallel. Formally, the tensor product is defined on objects by concatenation ($u \otimes v = uv$) and on morphisms $f : u \Rightarrow v$ and $f' : u' \Rightarrow v'$ by $f \otimes f' = (fu') \circ (vf')$. This monoidal structure on the category Σ_2^* induces a shift in the dimension of objects. However, this category is a monoidal category, which equivalently amounts to say that it is a 2-category with only one 0-cell, the objects of Σ_2^* being the 1-cells and the morphisms of Σ_2^* being the 2-cells. From a diagrammatic point of view, this means that

$$\begin{tikzcd} ab \arrow[d, Rightarrow, "\rho"'] \\ cdcd \end{tikzcd} \qquad \text{should actually be drawn as} \qquad \begin{tikzcd}[column sep=small] & & {} \arrow[drr, "b"] & & \\ {} \arrow[urr, "a"] \arrow[dr, "c"'] & & \Downarrow\rho & & {} \\ & {} \arrow[r, "d"'] & {} \arrow[r, "c"'] & {} \arrow[ur, "d"'] & \end{tikzcd}$$

In the following, we adopt this convention for the dimension of cells, but we keep drawing diagrams as the one on the left, since those are closer to diagrams traditionally used in rewriting theory. This also applies to $\Sigma_2^\top$, which is also be considered as a 2-category (with invertible 2-cells) in the following.

We can now introduce the notion of coherent presentation, by enriching presentations with a suitable specified set of confluence 3-cells. Below, two 2-cells are said to be *parallel* when they have the same source and the same target 1-cells.

▸ **Definition 4.** A *(finite) extended presentation* $(\Sigma, s_2, t_2, \Sigma_3)$ consists of a (finite) presentation Σ, together with a (finite) set Σ_3 of 3-*cells* (or *homotopy generators*) and two maps $s_2, t_2 : \Sigma_3 \to \Sigma_2^\top$ associating, to each 3-cell A, parallel 2-cells of $\Sigma_2^\top$ which are respectively its *source* $s_2(A) : u \Rightarrow v$ and its *target* $t_2(A) : u \Rightarrow v$ (cf. the diagram on the right).

$$\begin{tikzcd}[column sep=large] u \arrow[r, Rightarrow, bend left=40, "s_2(A)"{name=U}] \arrow[r, Rightarrow, bend right=40, "t_2(A)"'{name=D}] & v \arrow[from=U, to=D, phantom, "\Downarrow A"] \end{tikzcd}$$

A *homotopy relation* on $\Sigma_2^\top$ is an equivalence relation $\equiv$ on parallel 2-cells which is stable under context and composition:
– for any f and g in $\Sigma_2^\top$ and any u and v in Σ_1^*, $f \equiv g$ implies $ufv \equiv ugv$,
– for any $h : u' \Rightarrow u$, $f, g : u \Rightarrow v$ and $k : v \Rightarrow v'$ in $\Sigma_2^\top$, $f \equiv g$ implies $h \circ f \circ k \equiv h \circ g \circ k$.
In particular, given an extended presentation Σ, we write $\equiv_{\Sigma_3}$ for the smallest homotopy relation containing Σ_3.

▸ **Definition 5.** A *(finite) coherent presentation* is a (finite) extended presentation Σ such that the homotopy relation generated by Σ_3 is the homotopy relation on $\Sigma_2^\top$ containing every pair of parallel 2-cells.

▸ **Example 6.** The presentation Σ of Example 3 can be extended into a coherent presentation with the three diagrams A, B and C of (3) as set of 3-cells. For instance, we have $s_2(A) = \beta a$ and $t_2(A) = s\alpha \circ \gamma$. Notice that, since the 2-cells of $\Sigma_2^\top$ are invertible, different choices for the source and target of 3-cells could still give a coherent presentation, such as $s_1(A) = (\beta a)^{-1} \circ (s\alpha)$ and $t_1(A) = \gamma^{-1}$.

In the same way that rewriting systems present monoids, a coherent presentation presents a 2-category. Namely, given a coherent presentation Σ, one can define a 2-category, denoted by $\Sigma_2^\top / \Sigma_3$, as the 2-category $\Sigma_2^\top$ whose 2-cells have been quotiented by the homotopy relation $\equiv_{\Sigma_3}$. Notice that this 2-category always has its 2-cells invertible.

1.2 Transformations of coherent presentations

Starting from a non-convergent presentation of a monoid, the Knuth-Bendix procedure provides a (convergent) presentation on the same set of generators, but a monoid can also admit other presentations with different sets of generators. The notion of Tietze transformation [28] describes elementary transformations (adding and removing definable generators or derivable rules) on presentations, leaving unchanged the presented monoid. Moreover, they are complete in the sense that two presentations of the same monoid are related by Tietze transformations. In [9], a corresponding notion has been introduced for extended presentations, defined as the composites of the following elementary transformations:

1. **add a generator** $\mathcal{T}_u^+$: for u in Σ_1^*, add x_u to Σ_1 and $\delta_u : u \Rightarrow x_u$ to Σ_2,
2. **add a relation** $\mathcal{T}_f^+$: for $f : u \Rightarrow v$ in $\Sigma_2^\top$, add $\chi_f : u \Rightarrow v$ to Σ_2 and $A_f : f \Rrightarrow \chi_f$ to Σ_3,
3. **add a 3-cell** $\mathcal{T}_{(f,g)}^+$: for $f \equiv_{\Sigma_3} g$ in $\Sigma_2^\top$, add $\Psi : f \Rrightarrow g$ to Σ_3,
4. **remove a generator** $\mathcal{T}_x^-$: for $\alpha : u \Rightarrow x$ in Σ_2, with $x \in \Sigma_1$ and $u \in (\Sigma_1 \setminus \{x\})^*$, remove x and α and replace x by u in the relations and 3-cells and α by 1_u in the 3-cells,
5. **remove a relation** $\mathcal{T}_\alpha^-$: for $A : f \Rrightarrow \alpha$ in Σ_3, with $\alpha \in \Sigma_2$ and $f \in (\Sigma_2 \setminus \{\alpha\})^*$, remove α and A and replace α by f in the 3-cells,
6. **remove a 3-cell** $\mathcal{T}_A^-$: for $A : f \Rrightarrow g$ in Σ_3 with $f \equiv_{\Sigma_3 \setminus \{A\}} g$, remove A.

1. $$\begin{tikzcd} u \arrow[r, Rightarrow, dotted, "\delta_u"] & x_u \end{tikzcd}$$
2. $$\begin{tikzcd} u \arrow[r, Rightarrow, bend left=60, "f"] \arrow[r, Rightarrow, dotted, bend right=60, "\chi_f"'] \arrow[r, phantom, "A_f \Downarrow"] & v \end{tikzcd}$$
3. $$\begin{tikzcd} u \arrow[r, Rightarrow, bend left=60, "f"] \arrow[r, Rightarrow, bend right=60, "g"'] \arrow[r, phantom, "\Psi \Downarrow"] & v \end{tikzcd}$$
4. $$\begin{tikzcd} u \arrow[r, Rightarrow, dotted, "\alpha"] & x \end{tikzcd}$$
5. $$\begin{tikzcd} u \arrow[r, Rightarrow, bend left=60, "f"] \arrow[r, Rightarrow, dotted, bend right=60, "\alpha"'] \arrow[r, phantom, "A \Downarrow"] & v \end{tikzcd}$$
6. $$\begin{tikzcd} u \arrow[r, Rightarrow, bend left=60, "f"] \arrow[r, Rightarrow, bend right=60, "g"'] \arrow[r, phantom, "B \Downarrow \; \Downarrow A"] & v \end{tikzcd}$$

A (finite) *Tietze transformation* is a (finite) composite of elementary Tietze transformations. The notion of Tietze-equivalence on presentations can be generalized to extended presentations: Σ and Υ are *Tietze-equivalent* extended presentations if they are Tietze-equivalent as presentations and when there is an equivalence of categories $(\Sigma_2^\top/\Sigma_3) \simeq (\Upsilon_2^\top/\Upsilon_3)$, see [9]. In particular, coherent presentations of a same monoid are Tietze-equivalent. As in the case of presentations, we have for extended presentations:

▸ **Theorem 7** ([9]). *Two (finite) extended presentations are Tietze-equivalent if, and only if, there exists a (finite) Tietze transformation between them.*

1.3 Computing coherent presentations

We investigate a method to construct of coherent presentations from convergent ones, based on Squier's theory. We suppose fixed a presentation Σ. A *branching* of Σ is a pair $(f : u \Rightarrow v, g : u \Rightarrow w)$ of 2-cells of Σ_2^* with a common source. Such a branching is *local* when f and g are both rewriting steps, *Peiffer* when it is of the form $(f'v, ug')$ or $(vf', g'u)$ for some rewriting steps $f' : u \Rightarrow u', g' : v \Rightarrow v'$, and *overlapping* when it is not Peiffer and f and g are distinct. A branching is *critical* when it is overlapping and minimal for the order generated on branchings by $(f, g) \preccurlyeq (ufv, ugv)$, for any words u and v. A branching $(f, g) : u \Rightarrow (v, w)$ is *confluent* when there exist a pair of 2-cells $f' : v \Rightarrow u'$ and $g' : w \Rightarrow u'$ in Σ_2^*. We say that Σ is (*locally*) *confluent* when all of its (local) branchings are confluent. We say that Σ is *convergent* when it terminates and it is confluent. By Newman's lemma, local confluence is equivalent to confluence of critical branchings for terminating rewriting systems. This result can be reformulated in the setting of coherent presentations as follows.

A *family of generating confluences of* Σ is a set of 3-cells over $\Sigma_2^\top$ that contains, for every critical branching (f,g) of Σ, one 3-cell whose shape is as in the diagram on the right. If Σ is confluent, it always admits at least one family of generating confluences. Given a convergent presentation Σ, we denote by $\mathcal{S}(\Sigma)$ the extended presentation obtained from Σ by adjunction of a chosen family of generating confluences of Σ. The presentation $\mathcal{S}(\Sigma)$ is only defined up to that choice, but two families of generating confluences give Tietze-equivalent extended presentations [12]. Squier proved in [27] the following result.

$$\begin{tikzcd}
& v \arrow[dr, Rightarrow, "f'"] \arrow[dd, phantom, "\Downarrow \sigma_{f,g}"] & \\
u \arrow[ur, Rightarrow, "f"] \arrow[dr, Rightarrow, "g"'] & & u' \\
& w \arrow[ur, Rightarrow, "g'"'] &
\end{tikzcd}$$

▸ **Theorem 8** (Squier's theorem)**.** *Let Σ be a (finite) convergent presentation of a monoid $\mathbf{M}$. The extended presentation $\mathcal{S}(\Sigma)$ is a (finite) coherent and convergent presentation of $\mathbf{M}$.*

Several examples of this construction are given in [18]. Squier proved that the property, for a finite presentation of a monoid $\mathbf{M}$, to be extendable into a finite coherent presentation is an invariant of $\mathbf{M}$, that is, one given finite presentation of $\mathbf{M}$ is extendable into a finite coherent presentation if, and only if, all of them are [27]. However, there are finitely presented decidable monoids with no finite coherent presentation (such an example was exhibited by Squier). For such a monoid, starting with a finite presentation, there is no hope to obtain a finite convergent presentation, by using the Knuth-Bendix procedure or other methods, with the same set of generators or another one. Conversely, if the Knuth-Bendix procedure terminates on a finite presentation, then it can be extended into a finite coherent presentation.

2 Homotopical completion and reduction procedures

As seen in Section 1.3, Squier's theorem extends a convergent presentation into a coherent one. With the Knuth-Bendix completion procedure, those are the two basic ingredients of the homotopical completion procedure we present, extended by a homotopical reduction procedure whose goal is to eliminate superfluous cells.

2.1 The homotopical completion procedure

This procedure, denoted by $\mathcal{HC}$, interleaves the Knuth-Bendix completion and Squier's theorem to produce a coherent and convergent presentation from a terminating presentation: it examines the critical branchings one by one, potentially adding 2-cells to reach a convergent presentation, but also 3-cells that tend towards forming a coherent presentation.

Let Σ be a terminating presentation, seen as an extended presentation with no 3-cell. Thereafter, we always assume that termination is due to a fixed total termination order. For every critical branching $(f,g) : u \Rightarrow (v,w)$ of Σ, the procedure $\mathcal{HC}$ computes 2-cells $f' : v \Rightarrow \widehat{v}$ and $g' : w \Rightarrow \widehat{w}$ in Σ_2^*, where $\widehat{v}$ and $\widehat{w}$ are some normal forms for v and w, respectively. There are two possibilities:

- if $\widehat{v} = \widehat{w}$, the dotted 3-cell A is added, as in situation (a),
- otherwise, for example if $\widehat{v} < \widehat{w}$, the Tietze transformation $\mathcal{T}^+_{g'^{-1} \circ g^{-1} \circ f \circ f'}$ is applied to add the dotted 2-cell χ and 3-cell A, as in (b).

(a)
$$\begin{tikzcd}
& v \arrow[dr, Rightarrow, "f'"] \arrow[dd, phantom, "\Downarrow A"] & \\
u \arrow[ur, Rightarrow, "f"] \arrow[dr, Rightarrow, "g"'] & & \widehat{v} = \widehat{w} \\
& w \arrow[ur, Rightarrow, "g'"'] &
\end{tikzcd}$$

(b)
$$\begin{tikzcd}
& v \arrow[r, Rightarrow, "f'"] \arrow[dd, phantom, "\Downarrow A"] & \widehat{v} \\
u \arrow[ur, Rightarrow, "f"] \arrow[dr, Rightarrow, "g"'] & & \\
& w \arrow[r, Rightarrow, "g'"'] & \widehat{w} \arrow[uu, dotted, "\chi"']
\end{tikzcd}$$

The adjunction of new 2-cells can create new critical branchings: the $\mathcal{HC}$ procedure iterates this operation until it reaches, potentially after an infinite time, a stable extended presentation $\mathcal{HC}(\Sigma)$. From a computational point of view, an application of Squier's theorem to the result of the Knuth-Bendix completion on Σ would require to compute again all the critical branchings explored during completion, when the $\mathcal{HC}$ procedure computes 3-cells during completion. The properties of the Knuth-Bendix procedure and Squier's theorem induce the following result.

▸ **Theorem 9.** *Let Σ be a terminating presentation of a monoid $\mathbf{M}$. The extended presentation $\mathcal{HC}(\Sigma)$ is a coherent and convergent presentation of $\mathbf{M}$ and it is finite if, and only if, the presentation Σ is finite and the homotopical completion procedure terminates.*

▸ **Example 10.** The Kapur-Narendran presentation

$$\mathbf{B}_3^+ = \langle s, t, a \mid \alpha : ta \Rightarrow as,\ \beta : st \Rightarrow a \rangle$$

has two non-confluent critical branchings, resulting in the adjunction of the 2-cells γ and δ as in (2) and the 3-cells A and B as in (3). The 2-cells γ and δ generate two new critical branchings that are confluent: the $\mathcal{HC}$ procedure adds two extra 3-cells C and D and terminates with this finite coherent and convergent presentation of the monoid $\mathbf{B}_3^+$.

2.2 An optimized homotopical completion procedure

The $\mathcal{HC}$ procedure computes a coherent and convergent presentation that contains, in general, superfluous 3-cells, in the sense that they are not necessary to relate all the parallel 2-cells. To eliminate them, we apply a *homotopical reduction* mechanism in dimension 3: it computes the critical triple branchings to produce relations between 3-cells and to eliminate some of them by Tietze transformations. A *critical triple branching* (f, g, h) is a triple of distinct rewriting steps with common source, such that each one overlaps with at least one of the other two, and that is minimal for the order $\preccurlyeq$ generated by relations $(f, g, h) \preccurlyeq (ufv, ugv, uhv)$ for every such triples (f, g, h) and words u, v.

Let Σ be a convergent and coherent presentation. The *homotopical reduction in dimension* 3 builds, for each critical triple branching (f, g, h) of Σ, a 4-cell Ω with shape

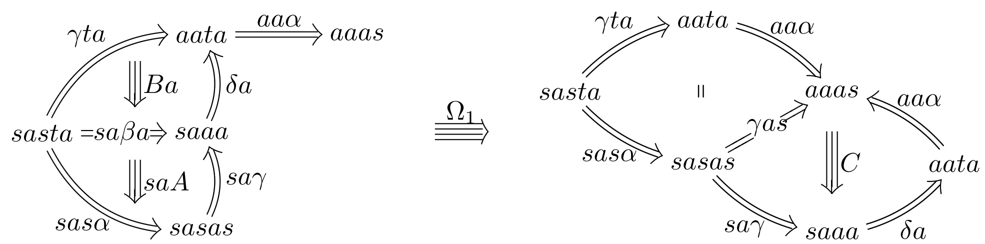
$$\begin{tikzcd}
 & v \arrow[rr, Rightarrow, "f_1'"] \arrow[d, phantom, "\Downarrow A"] & & x' \arrow[dr, Rightarrow, "h''"] \arrow[d, phantom, "\Downarrow C'"] & \\
u \arrow[ur, Rightarrow, "f"] \arrow[rr, Rightarrow, "g"] \arrow[dr, Rightarrow, "h"'] & {} & w \arrow[ur, Rightarrow, "g_1'"] \arrow[dr, Rightarrow, "g_2'"'] & {} & \widehat{u} \\
 & x \arrow[rr, Rightarrow, "h_2'"'] \arrow[u, phantom, "\Downarrow B"] & & v' \arrow[ur, Rightarrow, "f''"'] &
\end{tikzcd}
\overset{\Omega}{\Rrightarrow}
\begin{tikzcd}
 & v \arrow[rr, Rightarrow, "f_1'"] \arrow[dr, Rightarrow, "f_2'"] \arrow[d, phantom, "\Downarrow C"] & & x' \arrow[dr, Rightarrow, "h''"] \arrow[d, phantom, "B' \Downarrow"] & \\
u \arrow[ur, Rightarrow, "f"] \arrow[dr, Rightarrow, "h"'] & {} & w' \arrow[rr, Rightarrow, "g''"] & {} & \widehat{u} \\
 & x \arrow[rr, Rightarrow, "h_2'"'] \arrow[ur, Rightarrow, "h_1'"] & & v' \arrow[ur, Rightarrow, "f''"'] \arrow[u, phantom, "A' \Downarrow"] &
\end{tikzcd}$$

as follows. We consider the branching (f, g) and use confluence to get f_1' and g_1' and, then, coherence to build the 3-cell A. We proceed similarly with the branchings (g, h) and (f, h). Then, for the branching (f_1', f_2'), we use convergence to get g'' and h'' with $\widehat{u}$ as common target, and the 3-cell B' by coherence. We do the same operation with (h_1', h_2') to get A'. Finally, we get the 3-cell C' by coherence. The source and the target of Ω are made of generating 3-cells of Σ in context: they have shape uXv where X is a generating 3-cell and u and v are words. If one of those generating 3-cell appears only once and in an empty context ($u = v = 1$), then Ω is used as a definition of X in terms of the other 3-cells: X is removed by a Tietze transformation.

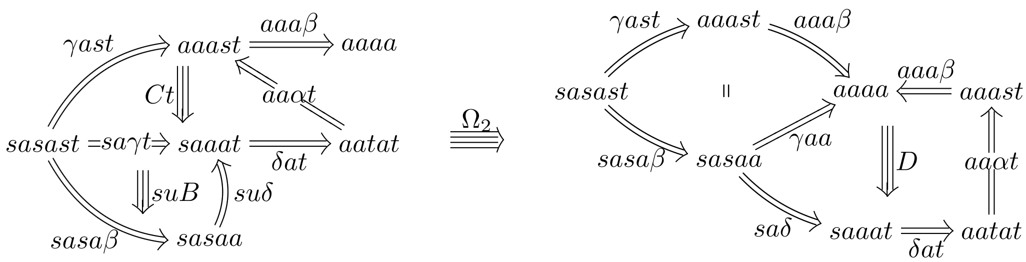
A coherent and convergent presentation on which no 3-cell can be removed by homotopical reduction in dimension 3 is called *reduced*. The *optimized homotopical completion procedure* $\overline{\mathcal{HC}}$ applies homotopical reduction in dimension 3 after $\mathcal{HC}$. Since the procedure acts by Tietze transformations only, we get:

▸ **Theorem 11.** *Let Σ be a terminating presentation of a monoid* $\mathbf{M}$*. The extended presentation $\overline{\mathcal{HC}}(\Sigma)$ is a reduced coherent and convergent presentation of* $\mathbf{M}$*, that is finite if, and only if, the presentation Σ is finite and the homotopical completion procedure $\mathcal{HC}$ terminates.*

▸ **Example 12.** After the $\mathcal{HC}$ procedure is applied to the Kapur-Narendran presentation of the monoid $\mathbf{B}_3^+$, we have four critical triple branchings, overlapping on the words *sasta*, *sasast*, *sasasas* and *sasasaa*. On *sasta*, we get the 4-cell



This 4-cell proves that C is superfluous in the coherent presentation: it appears only once in the boundary of Ω_1, in an empty context (unlike A and B). Then, we consider the critical triple branching with source *sasast*:



For the same reasons, the 4 cell Ω_2 removes D, leaving only the 3-cells A and B to form a reduced coherent and convergent presentation of the monoid $\mathbf{B}_3^+$. The other two critical triple branchings on words *sasasas* and *sasasaa* do not generate any other relation. Indeed, since the relations are weight-homogeneous (they relate words with the same weight, where s and t have weight 1 and a has weight 2), all the words that occur in the 4-cells corresponding to *sasasas* and *sasasaa* have weight 10 and 11, respectively. Since A and B have respective weights 4 and 5, their potential occurrences in those 4-cells must be in non-empty contexts.

2.3 The homotopical completion-reduction procedure

After the $\overline{\mathcal{HC}}$ procedure, we get a reduced coherent and convergent presentation of the considered monoid. Its underlying presentation is, in general, not minimal since homotopical completion has potentially adjoined superfluous 2-cells to get confluence. However, for each of those extra 2-cells, a 3-cell has also been added to fill the corresponding confluence diagram: a Tietze transformation can be used to remove both of them.

Given a coherent presentation Σ, we call *homotopical reduction in dimension* 2 the following process. For each 3-cell A of Σ_3, its source and target are made of reduction steps $u\alpha v$, where α is a generating 2-cell and u and v are words. If one such α appears only once and in an empty context ($u = v = 1$), both α and A are removed by a Tietze transformation. On the special case of $\overline{\mathcal{HC}}(\Sigma)$, every superfluous 2-cell appears once and in an empty context in the boundary of its associated 3-cell. The *homotopical completion-reduction procedure* $\mathcal{HCR}$ applies homotopical reduction in dimension 2 after $\overline{\mathcal{HC}}$. Since the procedure acts by Tietze transformations only, we get:

▸ **Theorem 13.** *Let Σ be a terminating presentation of a monoid $\mathbf{M}$. The extended presentation $\mathcal{HCR}(\Sigma)$ is a coherent presentation of $\mathbf{M}$, whose underlying presentation is contained in Σ, and it is finite if, and only if, the homotopical completion procedure terminates.*

▸ **Example 14.** After the $\overline{\mathcal{HC}}$ procedure is applied to the Kapur-Narendran presentation of the monoid $\mathbf{B}_3^+$, we have a coherent presentation with three generators s, t and a, four 2-cells α, β, γ and δ and two 3-cells A and B, corresponding to the adjunction of γ and δ respectively. They are removed by the $\mathcal{HCR}$ procedure, yielding a coherent presentation of $\mathbf{B}_3^+$ with the 2-cells α and β only, and no 3-cell. Informally, for two words on $\{s, t, a\}$ that represent the same element in the monoid $\mathbf{B}_3^+$, there is only one proof of their equality modulo α and β.

2.4 Completion and reduction on generators

As proved by Kapur and Narendran [14], the introduction of superfluous generators can be necessary for completion to terminate. These generators can of course be added by hand before the completion, but we briefly indicate here a possible heuristic, based on algebraic properties observed on the examples in Section 3. Indeed, in those cases, it always helps completion to add generators of the *quasicenter* of each submonoid. More precisely, for a given presentation Σ of a monoid $\mathbf{M}$, we seek minimal elements u of Σ_1^* such that $uX = Xu$ holds in $\mathbf{M}$ for a maximal subset X of Σ_1. Such a property is possible to observe during completion: one computes the products ux and xu for u a word of bounded length and x a generator. If $uX = Xu$ for a set X of generators, one adds a new generator (u) and a relation $u \Rightarrow (u)$. Moreover, the cardinal of X seems to determine a way to extend to (u) the termination order used for completion (see 3.3).

Whether the generators have been added before or during homotopical completion, one can remove them at the end. Indeed, each superfluous generator (u) comes with a defining relation $\alpha : u \Rightarrow (u)$, so that a Tietze transformation removes both of them (and replaces (u) by u and α by the identity in the boundary of the other 2-cells and 3-cells). Applied to the result of the $\mathcal{HCR}$ completion on the Kapur-Narendran presentation of the monoid $\mathbf{B}_3^+$, this contracts the obtained coherent presentation (with no 3-cell) to Artin presentation of the monoid $\mathbf{B}_3^+$, proving that this is also a coherent presentation with no 3-cell.

3 Applications

The results mentioned in this section were obtained with the help of a prototype implementation; an online version (unfortunately much slower than the offline one) is available[1].

[1] `http://www.pps.univ-paris-diderot.fr/~smimram/rewr/`

3.1 The braid monoid

The Artin presentation. The *monoid* $\mathbf{B}_n^+$ *of positive braids on* n *strands* is defined by

$$\mathbf{B}_n^+ = \langle s_1, \ldots, s_{n-1} \mid s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ for } 1 \leq i < n-1 \text{ and } s_i s_j = s_j s_i \text{ for } |i-j| \geq 2 \rangle \quad (4)$$

This presentation, called *Artin presentation*, is known to be minimal, so that one wants to compute a minimal coherent presentation of the monoid $\mathbf{B}_n^+$ by extending it. In [29], Tits proved a result that implies that a coherent presentation is given by 3-cells whose boundaries are in one of the Artin submonoid of rank 3 of $\mathbf{B}_n^+$, i.e. the boundary of each 3-cell is made of copies of the three relations involving only three given distinct generators. As a direct consequence, Artin presentation with no 3-cell is a coherent presentation of the monoid $\mathbf{B}_3^+$, but this result fails to say anything about $\mathbf{B}_4^+$ and does not give an explicit description of the coherence cells of $\mathbf{B}_n^+$ for $n \geq 5$. Unfortunately, homotopical completion cannot be used either in practice because it does not terminate on Artin presentation: indeed, as proved by Kapur and Narendran, any orientation of a relation $sts = tst$ generates a relation $stsst^k = ts^{k+1}ts$ for every $k \geq 1$ that must be contained in every convergent presentation [14].

The Kapur-Narendran presentation. As far as we know, the adjunction of the superfluous generator for $\mathbf{B}_3^+$, as seen in the introduction, has not been studied for $\mathbf{B}_n^+$ with $n > 3$. There are several possible generalizations, but we define the *Kapur-Narendran presentation* of $\mathbf{B}_n^+$ as the one obtained from Artin presentation by adjunction of superfluous generators corresponding to a Coxeter element for each Artin submonoid of $\mathbf{B}_n^+$, namely all the products $s_{i_1} \cdots s_{i_k}$ for every $1 \leq i_1 < \cdots < i_k < n$. Our experiments lead to positive results for the cases $n = 4$ and $n = 5$, see Table 1. We have also tested the Kapur-Narendran presentation on well-known generalizations of the braids monoids known as Artin monoids and got a finite coherent and convergent presentation for the Artin monoids of types B_2, B_3, B_4 and F_4 (the braid monoid $\mathbf{B}_n^+$ is the Artin monoid of type A_{n-1}). An open question is to determine if the Kapur-Narendran presentation yields a finite coherent and convergent presentation for any braid monoid and, more generally, for other types of Artin monoids.

The Garside presentation. The Kapur-Narendran presentation is contained in a bigger presentation called the *Garside presentation* [8]. For $\mathbf{B}_3^+$, the Garside presentation is obtained from Artin one by adjunction of superfluous generators (st), (ts) and (sts) corresponding to products st, ts and sts respectively: the element sts is the generator of the quasicenter of $\mathbf{B}_3^+$ and the elements s, t, st and ts are all its divisors. On the Garside presentation, the homotopical completion procedure produces a finite coherent and convergent presentation with five generators, twelve relations and 24 3-cells; the corresponding normal forms are known as *Deligne's normal forms* [5]. Deligne has proved in [6] that this coherent presentation of $\mathbf{B}_3^+$ can be reduced to one with six relations

$$st \Rightarrow (st) \quad ts \Rightarrow (ts) \quad s(ts) \Rightarrow (sts) \quad t(st) \Rightarrow (sts) \quad (st)s \Rightarrow (sts) \quad (ts)t \Rightarrow (sts)$$

and two 3-cells:

$$\begin{tikzcd}[column sep=small] & (st)s \arrow[dr, Rightarrow] \arrow[dd, phantom, "\Downarrow"] & \\ sts \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & (sts) \\ & s(ts) \arrow[ur, Rightarrow] & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & (ts)t \arrow[dr, Rightarrow] \arrow[dd, phantom, "\Downarrow"] & \\ tst \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & (sts) \\ & t(st) \arrow[ur, Rightarrow] & \end{tikzcd}$$

The homotopical completion-reduction, applied to the Garside presentation, gives a new, constructive proof of this result [9]. In fact, it goes even further: the 3-cells are used to remove the relations $(st)s \Rightarrow (sts)$ and $(ts)t \Rightarrow (sts)$ and, then, the relations $st \Rightarrow (st)$, $ts \Rightarrow (ts)$ and $s(ts) \Rightarrow (sts)$ remove the generators (st), (ts) and (sts). This leaves the generators s and t, the relation $t(st) \Rightarrow (sts)$, projected onto $tst \Rightarrow sts$, and no coherence cell, yielding another proof that Artin presentation with no 3-cell is a coherent presentation of $\mathbf{B}_3^+$.

The Garside presentation exists for every monoid $\mathbf{B}_n^+$ and, more generally, for every Artin monoid: in the spherical case (such as $\mathbf{B}_n^+$), its generators are made of the generator of the quasicenter of every Artin submonoid, plus all of its divisors. On this presentation, the homotopical completion-reduction procedure also applies, extending Deligne's result to non-spherical Artin monoids. Moreover, in [9], Gaussent and the first two authors apply homotopical reduction further to get an explicit coherent presentation of every Artin monoid, thus, in particular, giving a constructive proof of Tits's result. For the particular case of $\mathbf{B}_4^+$, the homotopical completion-reduction procedure gives a (minimal) coherent presentation made of Artin presentation with exactly one coherence cell, known as the *Zamolodchikov relation*:

$$
\begin{tikzcd}
& stsrst \arrow[r, Rightarrow] & strsrt \arrow[r, Rightarrow] & srtstr \arrow[r, Rightarrow] \arrow[dd, phantom, "\Downarrow"] & srstsr \arrow[r, Rightarrow] & rsrtsr & \\
tstrst \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & & & & & rstrsr \arrow[ul, Rightarrow] \\
& tsrtst \arrow[r, Rightarrow] & tsrsts \arrow[r, Rightarrow] & trsrts \arrow[r, Rightarrow] & rtstrs \arrow[r, Rightarrow] & rstsrs \arrow[ur, Rightarrow] &
\end{tikzcd}
$$

The Brieskorn-Saito presentation. For the monoid $\mathbf{B}_3^+$, it is defined by the adjunction to Artin presentation of a generator (sts) for sts [1]. This presentation is known in general for Artin monoids and obtained by adjunction of the generator (when it exists) of the quasicenter of each Artin submonoid. Those generators produce special normal forms that, up to our knowledge, are not yet linked to a convergent presentation. Contrarily to Garside's generators, Brieskorn-Saito's generators come in a finite number for every Artin monoid, motivating the research for a finite convergent presentation on those generators to give a solution to the still-open word problem for general Artin groups. Our experiments show that, on the Brieskorn-Saito presentation, the homotopical completion procedure gives a finite coherent and convergent presentation for the monoids $\mathbf{B}_3^+$, $\mathbf{B}_4^+$ and $\mathbf{B}_5^+$, but also for other Artin monoid such as the ones of type B_3 and, interestingly, of type $\tilde{A}_2$: this last example is an Artin monoid of *affine* type, for which the Garside presentation is infinite.

3.2 The plactic monoid

The Knuth presentation. The *plactic monoid* $\mathbf{P}_n$ *of rank* n is given by the Knuth presentation:

$$\mathbf{P}_n = \langle x_1, \dots, x_n \mid x_j x_i x_k = x_j x_k x_i \text{ for } i < j \le k \text{ and } x_i x_k x_j = x_k x_i x_j \text{ for } i \le j < k \rangle.$$

This monoid originates in the work of Schensted [24], Knuth [15] and Lascoux and Schützenberger [20]. It has found several applications, such as in representation theory [21] because of its strong connection to Young tableaux: semistandard Young tableaux correspond to elements of the plactic monoid and Schensted's insertion algorithm gives a way to compute normal forms for the Knuth presentation of the plactic monoid.

$$\begin{tikzcd} bbaa \arrow[rr, bend left, Rightarrow, "\beta a"] \arrow[rr, bend right, Rightarrow, "b\alpha"'] & & baba \end{tikzcd}$$

In the case $n = 2$, the Knuth presentation has two generators $x_1 = a$ and $x_2 = b$ and two relations $\alpha : baa \Rightarrow aba$ and $\beta : bba \Rightarrow bab$. This terminating presentation (for the deglex order generated by $a < b$ for example) is already convergent: the homotopical completion procedure yields a homotopy basis with exactly one coherence cell depicted on the right. Moreover, the convergent presentation has no critical triple branching: hence the computed coherent presentation of the monoid $\mathbf{P}_2$ is minimal.

In the case $n = 3$, with generators a, b and c, the Knuth presentation has eight relations, three pairs corresponding to the three plactic submonoids over two of the three generators, plus two relations involving all three generators: $\gamma : cab \Rightarrow acb$ and $\delta : bca \Rightarrow bac$. For the monoid $\mathbf{P}_3$, the Knuth presentation is not confluent anymore (on words *cbba*, *ccbaa*, *ccbab*) and homotopical completion adds three more relations: $\varepsilon : cbab \Rightarrow bcba$, $\varphi : cbaba \Rightarrow cacba$ and $\psi : cbcba \Rightarrow cbacb$. At the end, we get a finite coherent and convergent presentation with 27 3-cells, corresponding to all the critical branchings. The presentation also has 29 triple critical branchings, and homotopical reduction uses four of them to eliminate of four 3-cells. Then, the removal of the three extra relations and their corresponding coherence cells, added by completion, yields a homotopy basis with 20 coherence cells for the Knuth presentation of the monoid $\mathbf{P}_3$.

For higher values of n, the homotopical completion procedure cannot succeed on the Knuth presentation. Indeed, as in the case of braid monoids, a proof similar to the one of Kapur and Narendran for the monoid $\mathbf{B}_3^+$ shows that the infinite family of relations $cbc^k dca = cbac^k dc$, for every natural number k, must be part of any convergent presentation of the monoid $\mathbf{P}_n$.

The column presentation. The analogy with Young tableaux leads to introduce a finite number of superfluous generators to the Knuth presentation of $\mathbf{P}_n$, representing all the possible columns in semistandard Young tableaux: one generator $(x_{i_k} \cdots x_{i_1})$ for every possible $1 < k \leq n$ and $1 \leq i_1 < \cdots < i_k \leq n$, together with the corresponding defining relation $x_{i_k} \cdots x_{i_1} \Rightarrow (x_{i_k} \cdots x_{i_1})$. The column generators have an important property in plactic monoids: indeed, the center (and the quasicenter) of the plactic monoid $\mathbf{P}_n$ is generated by exactly one element: $x_n \cdots x_1$. Thus the column generators for $\mathbf{P}_n$ are exactly the generators of the quasicenters of all the plactic submonoids of $\mathbf{P}_n$.

From the column presentation, homotopical completion yields a finite coherent and convergent presentation of $\mathbf{P}_n$ (as in [3]). In particular, for the monoid $\mathbf{P}_4$, we get the following construction. Starting with the Knuth presentation with four generators and 20 relations, we add the eleven column generators (*ba*, *ca*, *cb*, *da*, *db*, *dc*, *cba*, *dba*, *dca*, *dcb*, *dcba*) and the corresponding relations to get 15 generators and 31 relations. Homotopical completion results in a finite coherent and convergent presentation with 115 relations and 621 3-cells. Then, homotopical reduction in dimension 3 uses the triple critical branchings to reduce the number of 3-cells to 212. The removal of the 84 relations and 3-cells added during homotopical completion, then of the eleven superfluous generators and their defining relations, finally produces a coherent presentation made of the Knuth presentation of the monoid $\mathbf{P}_4$ and 128 3-cells.

3.3 The Chinese monoid

The standard presentation. The *Chinese monoid* $\mathbf{Ch}_n$ *of rank* n is defined by

$$\mathbf{Ch}_n = \langle x_1, \ldots, x_n \mid x_j x_k x_i = x_k x_i x_j = x_k x_j x_i \text{ for } i \leq j \leq k \rangle.$$

It is a variant of the plactic monoid discovered in [7]. For $n = 2$, the Chinese monoid coincides with the plactic monoid $\mathbf{P}_2$: its standard presentation (with the orientation $baa \Rightarrow aba$ and $bba \Rightarrow bab$) is convergent and, with the same 3-cell as $\mathbf{P}_2$, forms a coherent presentation of $\mathbf{Ch}_2$. For higher values of n, the presentation of $\mathbf{Ch}_n$ (with the orientation $x_k x_i x_j \Rightarrow x_j x_k x_i$ and $x_k x_j x_i \Rightarrow x_j x_k x_i$) is not convergent anymore, but it can be finitely completed (without change of generators) by adjunction of the relations $x_k x_j x_k x_i \Rightarrow x_k x_i x_k x_j$ for $1 \le i < j < k \le n$. The homotopical completion-reduction yields a coherent presentation made of the standard presentation extended with 12 3-cells for $\mathbf{Ch}_3$, 56 for $\mathbf{Ch}_4$ and 176 for $\mathbf{Ch}_5$.

The quasicentral presentation. The (quasi)center of the monoid $\mathbf{Ch}_n$ is generated by the element $x_n x_1$ [4]. Thus, the generators of the quasicenters of all the Chinese submonoids of $\mathbf{Ch}_n$ are exactly the elements $x_j x_i$ for $1 \le i < j \le n$. Our experiments, conducted up to $n = 5$, show that the adjunction of those elements as superfluous generators still allow completion to reach a finite convergent presentation. Moreover, the obtained finite convergent presentation gives a rewriting-based procedure to compute the *column normal form* [4]. For the completion, a special order has to be chosen (corresponding to the column normal form), such as a weight lexicographic order, where each x_i has weight 1 and $(x_j x_i)$ has weight 2, and with an order on generators that satisfies $(x_l x_i) > (x_k x_j)$ if $i \le j \le k \le l$ with $i \ne j$ or $k \ne l$. This last inequality can be determined automatically from the fact that $(x_k x_i)$ commutes with $l - i$ elements and $(x_k x_j)$ with $k - j$ and, by assumption, we have $l - i > k - j$.

Coherent presentations						
Monoid	Presentation	Gen.	Rel.	Rel. comp.	Hom. gen.	Hom. gen. red.
$\mathbf{B}_3^+$	Artin	2	1	$\infty^\dagger$	$\infty^\dagger$	$0^\dagger$
	Kapur-Narendran	3	2	4	4	2
	Brieskorn-Saito	3	2	4	6	2
	Garside	5	4	12	24	8
$\mathbf{B}_4^+$	Artin	3	3	$\infty^\dagger$	$\infty^\dagger$	$1^\dagger$
	Kapur-Narendran	7	7	47	356	31
	Brieskorn-Saito	7	7	46	378	35
$\mathbf{B}_5^+$	Artin	4	6	$\infty^\dagger$	$\infty^\dagger$	$4^\dagger$
	Kapur-Narendran	15	17	692	48260	?
	Brieskorn-Saito	15	17	598	28384	?
$\mathbf{P}_2 = \mathbf{Ch}_2$	Knuth	2	2	2	1	1
	Column	3	3	3	1	1
$\mathbf{P}_3$	Knuth	3	8	11	27	23
	Column	7	12	22	42	30
$\mathbf{P}_4$	Knuth	4	20	$\infty^\dagger$	$\infty^\dagger$	$?^\dagger$
	Column	15	31	115	621	212
$\mathbf{P}_5$	Column	31	66	531	6893	?

Table 1 Results of experiments indicating, for various sets of generators, the number of generators, relations (before and after completion), and homotopy generators (before and after homotopy reduction by 4-cells) of the completed rewriting system. Values marked "$\dagger$" arise from theoretical computations, and "?" indicate computations too big to be performed in reasonable time with our prototype implementation.

Conclusion

We have generalized the Knuth-Bendix completion procedure to coherent presentations, which has enabled us to formulate a reduction procedure. Some practical outcomes have been investigated, providing constructive results about presentations of braid, plactic and Chinese monoids. These procedures have been implemented in a proof-of-concept software, and much work remain to be done in order to better understand the structures put to use and how to efficiently manipulate them. The idea of adding superfluous generators seems very promising, but we have only been able to provide heuristics to do so which have to be refined and supported by more experiments. Finally, the approach developed here handles generators, relations and homotopy generators uniformly; its likely extension to higher dimensions will be investigated in future works, in relation with methods for constructing minimal presentations of algebraic structures.

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A Prototype tool

A.1 Completion of the monoid B_3^+

We give below an example of usage of our prototype tool[2] to compute a reduced coherent presentation of the braid monoid $\mathbf{B}_3^+$.

Generate the Artin monoid $\mathbf{B}_3^+$ (which is of Artin type A_2).

```
# artin a2
a,b | A : bab -> aba
We have 1 generator and 1 rule.
```

Perform a Knuth-Bendix completion.

```
# kbc
Timeout!
```

Add the Kapur-Narendran generator.

```
# rgen ba
a,b,ba | A : bab -> aba, B : ba -> (ba)
We have 3 generators and 2 rules.
```

Perform a Knuth-Bendix completion.

```
# kbc
a,b,ba | B : ba -> (ba), D : (ba)b -> a(ba), E : a(ba)a -> (ba)(ba), G : (ba)(ba)a -> b(ba)(ba)
We have 3 generators and 4 rules.
```

Compute homotopy generators.

```
# hbasis
ba(ba)a --B(ba)a--> (ba)(ba)a --G--> b(ba)(ba)
ba(ba)a --bE--> b(ba)(ba)

(ba)ba --Da--> a(ba)a --E--> (ba)(ba)
(ba)ba --(ba)B--> (ba)(ba)

a(ba)a(ba)a --E(ba)a--> (ba)(ba)(ba)a --(ba)G--> (ba)b(ba)(ba) --D(ba)(ba)--> a(ba)(ba)(ba)
a(ba)a(ba)a --a(ba)E--> a(ba)(ba)(ba)

(ba)(ba)a(ba)a --G(ba)a--> b(ba)(ba)(ba)a --b(ba)G--> b(ba)b(ba)(ba)
    --bD(ba)(ba)--> ba(ba)(ba)(ba) --B(ba)(ba)(ba)--> (ba)(ba)(ba)(ba)
(ba)(ba)a(ba)a --(ba)(ba)E--> (ba)(ba)(ba)(ba)

We have 4 homotopy generators.
```

Compute a reduced set of homotopy generators.

```
# rhbasis
We remove
a(ba)a(ba)a --E(ba)a--> (ba)(ba)(ba)a --(ba)G--> (ba)b(ba)(ba) --D(ba)(ba)--> a(ba)(ba)(ba)
a(ba)a(ba)a --a(ba)E--> a(ba)(ba)(ba)
with
```

---

[2] http://www.pps.univ-paris-diderot.fr/~smimram/rewr/

```
--Da(ba)a-->
--(ba)B(ba)a-->
--(ba)bE-->

We remove
(ba)(ba)a(ba)a --G(ba)a--> b(ba)(ba)(ba)a --b(ba)G--> b(ba)b(ba)(ba)
     --bD(ba)(ba)--> ba(ba)(ba)(ba) --B(ba)(ba)(ba)--> (ba)(ba)(ba)(ba)
(ba)(ba)a(ba)a --(ba)(ba)E--> (ba)(ba)(ba)(ba)
with
--B(ba)a(ba)a-->
--bE(ba)a-->
--ba(ba)E-->

The reduced set of homotopy generators is

ba(ba)a --B(ba)a--> (ba)(ba)a --G--> b(ba)(ba)
ba(ba)a --bE--> b(ba)(ba)

(ba)ba --Da--> a(ba)a --E--> (ba)(ba)
(ba)ba --(ba)B--> (ba)(ba)

We have 2 homotopy generators.
```

A.2 Completion of the monoid B_4^+

Generate the Artin monoid $\mathbf{B}_4^+$ *(which is of Artin type* A_3*).*

```
# artin a3
a,b,c | A : bab -> aba, B : cbc -> bcb, C : ca -> ac
We have 3 generators and 3 rules.
```

Add the Kapur-Narendran generators.

```
# rgensw cba
a,b,c,cb,ca,ba,cba | A : bab -> aba, B : cbc -> bcb, C : ca -> ac, D : cb -> (cb),
                     E : ca -> (ca), F : ba -> (ba), G : cba -> (cba)
We have 7 generators and 7 rules.
```

Perform a Knuth-Bendix completion.

```
# kbc
a,b,c,cb,ca,ba,cba | D : cb -> (cb), E : ca -> (ca), F : ba -> (ba), J : ac -> (ca),
                     K : (cb)a -> (cba), L : (ba)b -> a(ba), M : (cb)c -> b(cb),
                     N : (cb)(ca) -> b(cba), O : b(cb)b -> (cb)(cb), P : a(ba)a -> (ba)(ba),
                     ...
We have 7 generators and 47 rules.
```

Compute homotopy generators.

```
# hbasis
cba --Da--> (cb)a --K--> (cba)
cba --cF--> c(ba) --V--> (cba)

cb(cb)b --D(cb)b--> (cb)(cb)b --M1--> c(cb)(cb)
cb(cb)b --cO--> c(cb)(cb)
```

```
cb(cba)(ba) --D(cba)(ba)--> (cb)(cba)(ba) --S2--> c(cba)(cba)
cb(cba)(ba) --cC1--> c(cba)(cba)

[...]

We have 356 homotopy generators.
```

Compute a reduced set of homotopy generators.

```
# rhbasis
We remove
(cba)(ba)(cb)(ba)(ba)(cba)(ba) --J3(ba)(cba)(ba)--> (cb)(ba)(cb)(cba)(ba)(cba)(ba)
    --(cb)(ba)S2(cba)(ba)--> (cb)(ba)c(cba)(cba)(cba)(ba)
    --(cb)T(cba)(cba)(cba)(ba)--> (cb)b(ca)(cba)(cba)(cba)(ba)
    --(cb)b(ca)A3--> (cb)b(ca)(cb)(cba)(cba)(cba)
(cba)(ba)(cb)(ba)(ba)(cba)(ba) --(cba)(ba)(cb)V2--> (cba)(ba)(cb)b(ba)(cba)(cba)
    --O2(ba)(cba)(cba)--> (cb)(ba)(cb)(cb)(ba)(cba)(cba)
    --(cb)(ba)U1(cba)(cba)--> (cb)(ba)c(cb)(cba)(cba)(cba)
    --(cb)T(cb)(cba)(cba)(cba)--> (cb)b(ca)(cb)(cba)(cba)(cba)
with
--J3(ba)(cb)(ba)b-->
--(cba)(ba)(cb)I3b-->
--(cba)(ba)(cb)(ba)(ba)(cb)L-->

We remove
(cba)(ba)(cb)(ba)(cba)(cba)(ba) --J3(cba)(cba)(ba)--> (cb)(ba)(cb)(cba)(cba)(cba)(ba)
    --(cb)(ba)(cb)A3--> (cb)(ba)(cb)(cb)(cba)(cba)(cba)
(cba)(ba)(cb)(ba)(cba)(cba)(ba) --(cba)(ba)(cb)E3--> (cba)(ba)(cb)b(cba)(cba)(cba)
    --O2(cba)(cba)(cba)--> (cb)(ba)(cb)(cb)(cba)(cba)(cba)
with
--J3(ba)(cba)(ba)b-->
--(cba)(ba)(cb)V2b-->
--(cba)(ba)(cb)(ba)(ba)(cba)L-->

We remove
(cba)(ba)(cb)(ba)(ba)(cb)(ba) --J3(ba)(cb)(ba)--> (cb)(ba)(cb)(cba)(ba)(cb)(ba)
    --(cb)(ba)S2(cb)(ba)--> (cb)(ba)c(cba)(cba)(cb)(ba)
    --(cb)T(cba)(cba)(cb)(ba)--> (cb)b(ca)(cba)(cba)(cb)(ba)
    --(cb)b(ca)(cba)E1(ba)--> (cb)b(ca)(cba)(ba)(cba)(ba)
    --(cb)b(ca)W2--> (cb)b(ca)(cb)(ba)(cba)(cba)
(cba)(ba)(cb)(ba)(ba)(cb)(ba) --(cba)(ba)(cb)I3--> (cba)(ba)(cb)b(ba)(cb)(cba)
    --O2(ba)(cb)(cba)--> (cb)(ba)(cb)(cb)(ba)(cb)(cba)
    --(cb)(ba)U1(cb)(cba)--> (cb)(ba)c(cb)(cba)(cb)(cba)
    --(cb)T(cb)(cba)(cb)(cba)--> (cb)b(ca)(cb)(cba)(cb)(cba)
    --(cb)b(ca)(cb)E1(cba)--> (cb)b(ca)(cb)(ba)(cba)(cba)
with
--J3(ba)(cb)ba-->
--(cba)(ba)(cb)P2a-->
--(cba)(ba)(cb)(ba)(ba)(cb)F-->

[...]

The reduced set of homotopy generators is

cba --Da--> (cb)a --K--> (cba)
cba --cF--> c(ba) --V--> (cba)
```

```
cb(cb)b --D(cb)b--> (cb)(cb)b --M1--> c(cb)(cb)
cb(cb)b --cO--> c(cb)(cb)

cb(cba)a --D(cba)a--> (cb)(cba)a --R2--> c(cba)(ca)
cb(cba)a --cD1--> c(cba)(ca)

[...]

We have 28 homotopy generators.
```

A.3 Other examples

Other examples mentioned in this article can be tested similarly as follows.

The plactic monoid (e.g. $\mathbf{P}_4$*).*

```
# plactic 4
a,b,c,d | A : baa -> aba, B : caa -> aca, C : daa -> ada, D : cab -> acb, E : dab -> adb,
          F : dac -> adc, G : cbb -> bcb, H : dbb -> bdb, I : dbc -> bdc, J : dcc -> cdc,
          K : bba -> bab, L : bca -> bac, M : bda -> bad, N : cca -> cac, O : cda -> cad,
          P : dda -> dad, Q : ccb -> cbc, R : cdb -> cbd, S : ddb -> dbd, T : ddc -> dcd
We have 4 generators and 20 rules.
```

The Chinese monoid (e.g. $\mathbf{Ch}_4$*).*

```
# chinese 4
a,b,c,d | A : bba -> bab, B : baa -> aba, C : cca -> cac, D : caa -> aca, E : dda -> dad,
          F : daa -> ada, G : ccb -> cbc, H : cbb -> bcb, I : ddb -> dbd, J : dbb -> bdb,
          K : ddc -> dcd, L : dcc -> cdc, M : cba -> bca, N : cab -> bca, O : dba -> bda,
          P : dab -> bda, Q : dca -> cda, R : dac -> cda, S : dcb -> cdb, T : dbc -> cdb
We have 4 generators and 20 rules.
```