Two-dimensional Rewriting Techniques and Applications

Part II. Coherence and Rewriting

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Objective and Motivations

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▶ Use **two-dimensional rewriting techniques** to compute **homotopical properties** of presentations of monoids.

- ▷ String rewriting is **1-dimensional rewriting**.
- ▷ Homotopical properties of string rewriting systems are **2-dimensional**.

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- Prerequises from Part I of the lecture:
  - **Monoids**: presentations by **generators** and **relations**, the **word problem**.
  - **String rewriting systems** described as **1-dimensional rewriting systems**.
  - **Categories**: the category of rewriting paths.

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- Categorical interpretations
  - ▷ Burroni, 1993.
  - ▷ Lafont, 2003, Guiraud-Malbos, 2016.

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$$\{\, \mathbf{y_1}, \ldots, \mathbf{y_k} \,\},$$

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- Schreyer, 1980 : computation of linear syzygies by means of the **division algorithm**.

  ▷ Buchberger's completion algorithm for computing Gröbner bases allows the computation of the first syzygy module.

  ▷ The reduction to zero of the S-polynomial of two polynomials in a Gröbner basis gives a syzygy.

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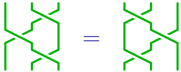
- Applications:
  - Explicit description of actions of a monoid on categories in representation theory.
  - Coherence theorems for monoids.
  - Algorithms in homological algebra.

Motivation II. Compute syzygies for presentations of monoids

- The **Artin monoid** $\mathbf{B}_3^+$ of braids on 3 strands.

$s =$  $t =$ 



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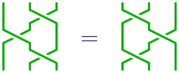
$s =$  $\qquad t =$  $\qquad$ 

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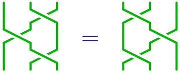
$s =$  $t =$  

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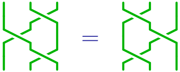

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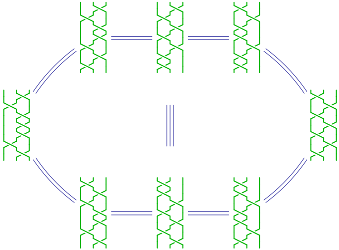

With this presentation two proofs of the same equality in $\mathbf{B}_3^+$ are equal.

Plan of the lecture

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I. String Rewriting and the Word Problem

- Strings are 1-dimensional
- String rewriting systems and 2-polygraphs
- String Rewriting and the Word Problem

II. Coherent presentations of monoids

- Coherent presentation
- Homotopical Squier Theorem

III. Homotopical completion-reduction procedure

- Homotopical completion-reduction
- Algebraic examples

Part I. String Rewriting and the Word Problem

String rewriting systems and 2-polygraphs

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- **2-polygraph** (**2-computad**) with only one 0-cell:

  - $\Sigma_0 = \{ \bullet \}$

  - Σ_1 set of **generators** : $\bullet = s_0(x) \xrightarrow{x} t_0(x) = \bullet$

  - Σ_1^* free monoid of **strings** :

$$\bullet \xrightarrow{x_1} \bullet \xrightarrow{x_2} \bullet \; \ldots \; \bullet \xrightarrow{x_k} \bullet$$

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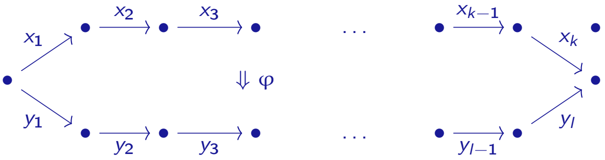
  - Σ_2 set of **rules** $\varphi : u \Rightarrow v$ with a globular shape:



    - 1-source of φ : $s_1(\varphi) = x_1x_2\ldots x_k$,
    - 1-target of φ : $t_1(\varphi) = y_1y_2\ldots y_l$.

Rewriting properties of 2-polygraphs

- A **1-polygraph** is an oriented graph (Σ_0, Σ_1)

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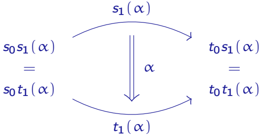
- A **2-polygraph** is a triple $\Sigma = (\Sigma_0, \Sigma_1, \Sigma_2)$ where
  - (Σ_0, Σ_1) is a 1-polygraph,
  - Σ_2 is a globular extension of the free category Σ_1^*.

$$\Sigma_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} \Sigma_1^* \underset{t_1}{\overset{s_1}{\leftleftarrows}} \Sigma_2$$



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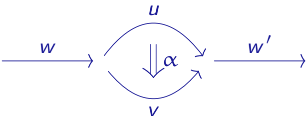

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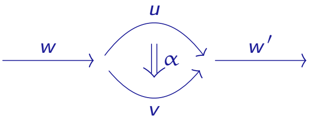

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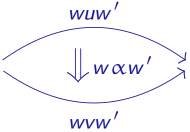

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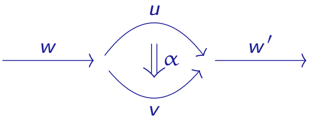

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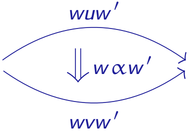

- A **rewriting sequence** of Σ is a finite or infinite sequence

$$u_1 \overset{f_1}{\Longrightarrow} u_2 \overset{f_2}{\Longrightarrow} \cdots \overset{f_{n-1}}{\Longrightarrow} u_n \overset{f_n}{\Longrightarrow} \cdots$$

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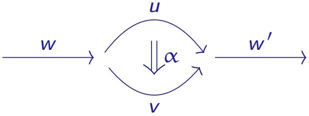

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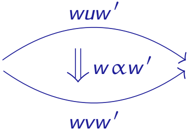

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- Rewriting sequences form a 2-category Σ_2^*.

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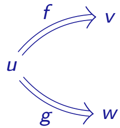

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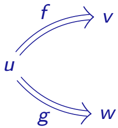

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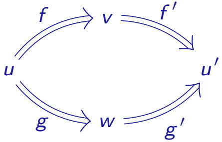

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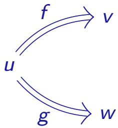

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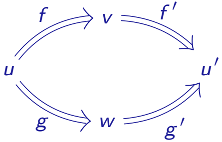

- Σ is **convergent** if it terminates and it is confluent.

Rewriting properties of 2-polygraphs

- A branching

$$\begin{array}{ccc} & \overset{f}{\Rightarrow} & v \\ u & & \\ & \underset{g}{\Rightarrow} & w \end{array}$$

is **local** if f and g are rewriting steps.

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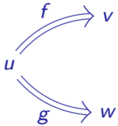

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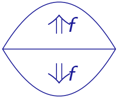
  - ▷ **aspherical** branchings have shape 

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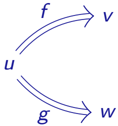

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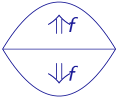
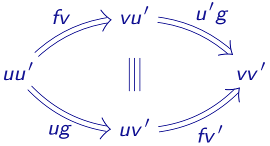


  ▷ **Peiffer** branchings have shape



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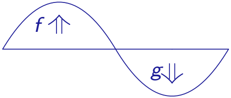

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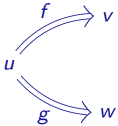

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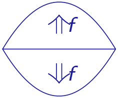

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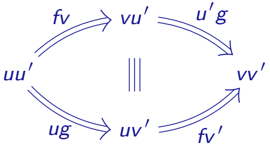

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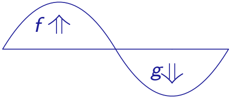
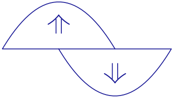


  ▷ **critical branchings** are all the other cases



Rewriting properties of 2-polygraphs

Example.

Consider the 2-polygraph

$$\langle\ s, t \mid tst \overset{\gamma_{st}}{\Longrightarrow} sts\ \rangle$$

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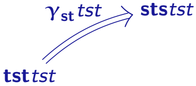

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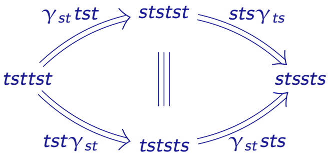

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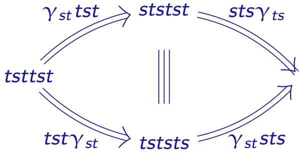

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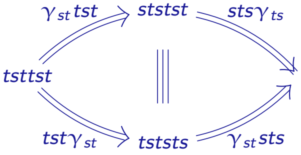

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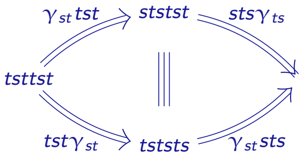

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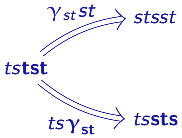

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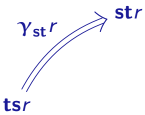

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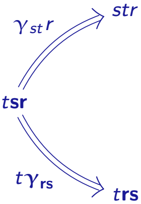

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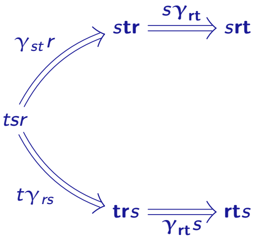

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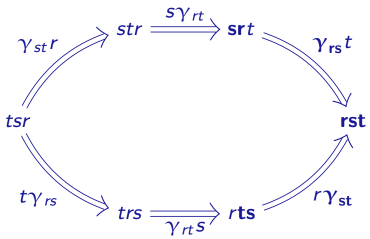

String Rewriting and the Word Problem

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▶ **Finite convergent presentations** give a method for solving the word problem algorithmically.

▷ Given a 2-polygraph Σ.

▷ Consider the monoid $\mathbf{M}$ **presented** by Σ, i.e., the quotient of the free monoid Σ_1^* by the congruence generated by Σ_2:

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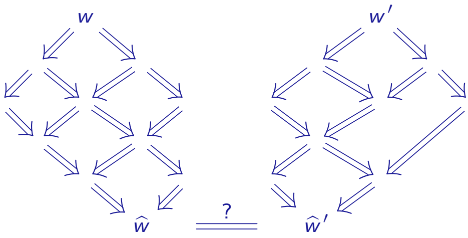
▷ **Normal form algorithm** for finite and convergent 2-polygraphs:



Fact. Monoids having a finite convergent presentation are decidable.

Finite Convergent Presentations

- Knuth-Bendix **completion procedure**, 1970.
  - **Input** : Σ a terminating 2-polygraph with a total termination order $\prec$.
  - The procedure will try to compute a 2-polygraph $\mathcal{KB}(\Sigma)$ such that
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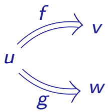
    - Picks a branching in $\mathcal{C}b$:



    - $\mathcal{C}b := \mathcal{C}b \setminus \{(f, g)\}$

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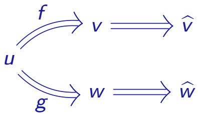
    - Reduce v to a normal form $\widehat{v}$ with respect to $\mathcal{KB}(\Sigma)_2$
    - Reduce w to a normal form $\widehat{w}$ with respect to $\mathcal{KB}(\Sigma)_2$



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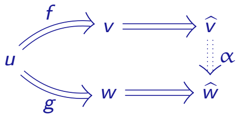
    - $\mathcal{KB}(\Sigma)_2 \;:=\; \mathcal{KB}(\Sigma)_2 \cup \{\,\alpha : \widehat{v} \Rightarrow \widehat{w}\,\}$:



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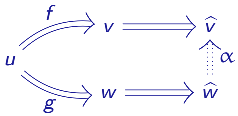

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- If the procedure stops, it returns the 2-polygraph $\mathcal{KB}(\Sigma)$.

- Otherwise, it builds an increasing sequence of 2-polygraphs, whose limit is denoted by $\mathcal{KB}(\Sigma)$.

Finite Convergent Presentations

- Finite convergent presentations.
  - ▷ If a monoid **M** admits a finite convergent presentation, then its word problem is decidable.
  - ▷ Knuth-Bendix, 1970, Nivat, 1972,
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Questions. (Book, 1985, Kapur-Narendran, 1985, Jantzen, 1985, ...)

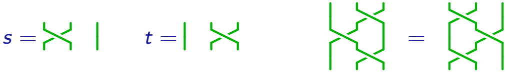
1. Does a finitely presented decidable monoid have a finite convergent presentation ?
2. Does rewriting is universal to decide the word problem in a finite presented monoid ?
3. What conditions a monoid must satisfy if it can be presented by a finite convergent rewriting system ?

Finite Convergent Presentations

Example. (Kapur-Narendran, 1985)

▶ Artin's presentation of monoid of **positive braids on 3 strands**:

$$\mathbf{B}_3^+ = \langle\, s, t \mid sts = tst \,\rangle$$



▷ $\mathbf{B}_3^+$ has a decidable word problem.

▷ There does not exist finite convergent presentation of $\mathbf{B}_3^+$ with two generators.

▷ But with three generators by adding a generator a standing for the product st.

Finite Convergent Presentations

$$\Sigma = \langle\ s, t \quad |\ tst \Rightarrow sts$$

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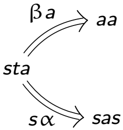

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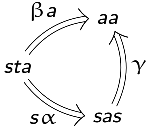

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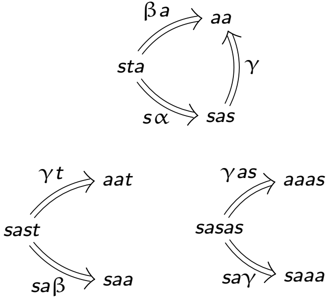

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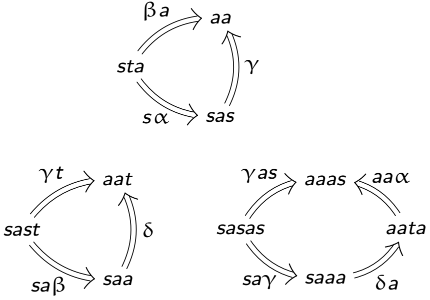

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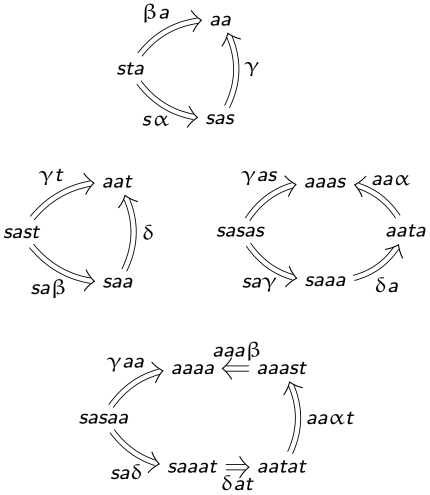

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Answears. (Squier, 1987)

1. No in general.
2. No.
3. Homological finiteness condition and **homotopical finiteness condition** (1994).

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▶ Generalization of finiteness conditions, Anick, 1987, Kobayashi, 1991, Brown, 1992.

Exercise I (Kapur-Narendran '85)

▶ Consider the monoid $\mathbf{B}_3^+$ of positive braids on three strands and the Artin's presentation

$$\langle s, t \mid \gamma : sts \Rightarrow tst \rangle.$$

1. Compute a convergent presentation of the monoid $\mathbf{B}_3^+$ with two generating 1-cells.

2. Show that the word problem is decidable for $\mathbf{B}_3^+$.

3. Show that for any $i \geqslant 0$ and any $j \geqslant 0$, the words

$$s^{i+1}t^{j+2}st \qquad \text{and} \qquad tst^{i+2}s^{j+1}$$

are equals in $\mathbf{B}_3^+$.

4. Denote by $[w]$ the equivalence class modulo the relation γ containing the word w. Prove that for any $n > 0$ the two following equalities hold

$$[t^n st] = \{\, t^{n-i}sts^i \mid 0 \leqslant i \leqslant n \,\}.$$
$$[tst^n] = \{\, s^j tst^{n-j} \mid 0 \leqslant j \leqslant n \,\}.$$

5. Show that there does not exist any finite convergent presentation of the monoid $\mathbf{B}_3^+$ with two generators s and t.

Part II. Coherent presentations of monoids

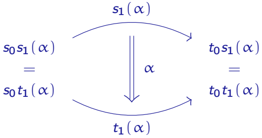
2-Polygraphs

- A **1-polygraph** is an oriented graph (Σ_0, Σ_1)

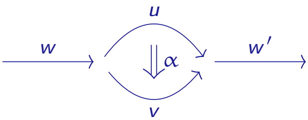
$$\Sigma_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} \Sigma_1$$

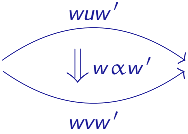
- A **2-polygraph** is a triple $\Sigma = (\Sigma_0, \Sigma_1, \Sigma_2)$ where
  - ▷ (Σ_0, Σ_1) is a 1-polygraph,
  - ▷ Σ_2 is a globular extension of the free category Σ_1^*.

$$\Sigma_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} \Sigma_1^* \underset{t_1}{\overset{s_1}{\leftleftarrows}} \Sigma_2$$



- A **rewriting step** is a 2-cell of the free 2-category Σ_2^* over Σ with shape





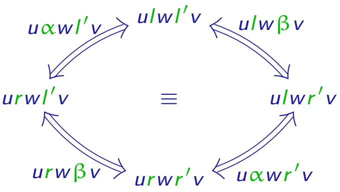
where $u \overset{\alpha}{\Longrightarrow} v$ is a 2-cell of Σ_2 and w, w' are 1-cells of Σ_1^*.

Homotopical Squier Theorem

- Σ a 2-polygraph.

- Denote by $\Sigma_2^{\top}$ the free **(2, 1)-category** on Σ, that is
  - free **category enriched in groupoid** on Σ,
  - free 2-category whose any 2-cell is invertible.

- Description of $\Sigma_2^{\top}$
  - 0-cells : Σ_0,
  - 1-cells strings in Σ_1^*,
  - 2-cells : reductions and their inverses $\Leftrightarrow$,
  - submited **Peiffer elements**:



for any 2-cells $l \overset{\alpha}{\Longrightarrow} r$ and $l' \overset{\beta}{\Longrightarrow} r'$.

$(3,1)$-Polygraphs

- A **$(3,1)$-polygraph** is a pair $\Sigma = (\Sigma_2, \Sigma_3)$ made of
  - a 2-polygraph Σ_2,
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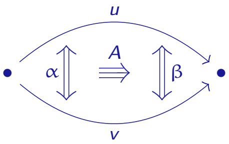

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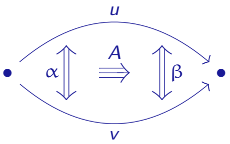

- Let $\mathbf{M}$ be a monoid.

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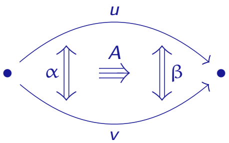

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- An **extended presentation** of $\mathbf{M}$ is a $(3,1)$-polygraph Σ such that

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Homotopical Squier Theorem

Definition. A **homotopy relation** on $\Sigma_2^\top$ is an equivalence relation $\equiv$ on parallel 2-cells stable under

- ▷ **context**: $f \equiv g$ implies $ufv \equiv ugv$,
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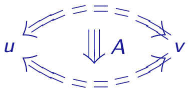
Definition. A **homotopy basis** is a cellular extension Σ_3 made of 3-cells



on spheres of $\Sigma_2^\top$ such that the homotopy relation generated by Σ_3 contains every pair of parallel 2-cells in $\Sigma_2^\top$.

Coherent presentations of categories

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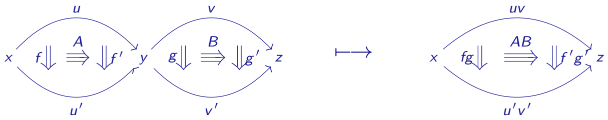

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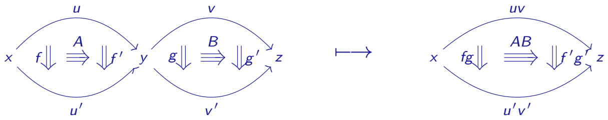
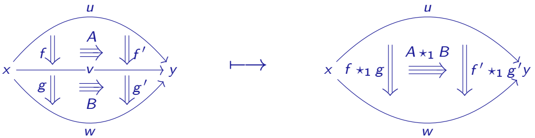


  - ▷ by $\star_1$, along their 1-dimensional boundary:



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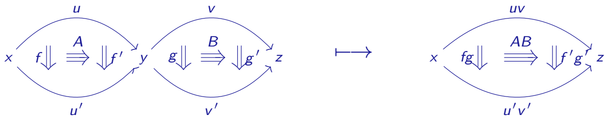

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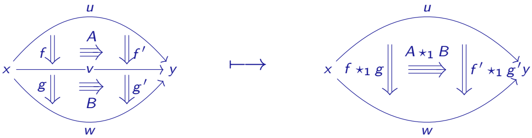
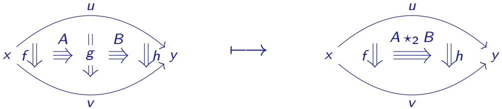


  - ▷ by $\star_2$, along their 2-dimensional boundary:



Examples

- The full coherent presentation contains all the 3-cells.

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- Free monoid : no relation, an empty homotopy basis.

Examples

- Free commutative monoid of rank 3:
  - the full coherent presentation:

$$\left\langle\; r, s, t \;\middle|\; sr \overset{\gamma_{rs}}{\Longrightarrow} rs,\; ts \overset{\gamma_{st}}{\Longrightarrow} st,\; tr \overset{\gamma_{rt}}{\Longrightarrow} rt \;\middle|\; \begin{array}{c}\text{all the}\\ \text{3-cells}\end{array} \;\right\rangle$$

Examples

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  - A homotopy basis can be made with only one 3-cell

$$\langle\ r, s, t \mid sr \overset{\gamma_{rs}}{\Longrightarrow} rs,\ ts \overset{\gamma_{st}}{\Longrightarrow} st,\ tr \overset{\gamma_{rt}}{\Longrightarrow} rt \mid Z_{r,s,t}\ \rangle$$

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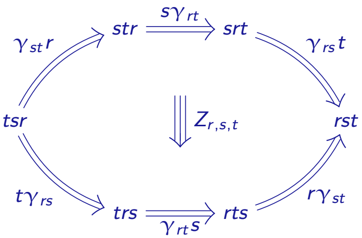
where the 3-cell $Z_{r,s,t}$ is the **permutohedron**



Examples

▶ **Artin's coherent presentation** of the monoid $\mathbf{B}_3^+$

$s =$  $t =$ 

 $=$ 

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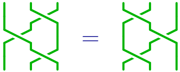
$s =$  $t =$  

▷ The homotopy basis is empty.

$$\langle\, s, t \mid tst \overset{\gamma_{st}}{\Longrightarrow} sts \mid \emptyset \,\rangle$$

Homotopical Squier's Theorem

Homotopical Squier's Theorem: objective

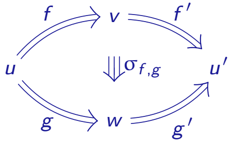
▶ A method to compute a coherent presentation starting from a convergent presentation.

▶ **Squier's completion procedure** provides a way to extend a convergent presentation of a monoid **M** into a coherent presentation.

▶ Given a convergent 2-polygraph Σ.

▶ We compute a coherent presentation whose 3-cells are **generating confluences**,

▷ that is, one 3-cell:

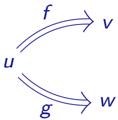


▷ for every critical branching (f, g) of Σ.

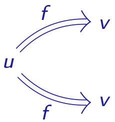
Homotopical Squier's Theorem

Branchings

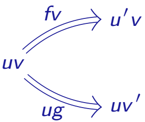
- Let Σ be a 2-polygraph.
- A **branching** of Σ is a pair (f, g) of 2-cells of Σ_2^* with a common source:



- A branching (f, g) is **local** when f and g are rewriting steps.
- Local branchings are
  - **aspherical**



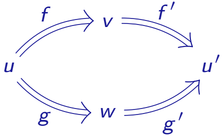
  - **Peiffer**



  - or **overlapping**.

Generating confluences

- A branching $(f, g) : u \Rightarrow (v, w)$ is **confluent** when there exist 2-cells $f' : v \Rightarrow u'$ and $g' : w \Rightarrow u'$ in Σ_2^* such that



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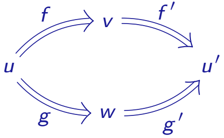
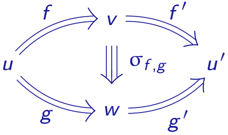


- A family of **generating confluences** of Σ is a cellular extension of $\Sigma_2^\top$ that contains exactly one 3-cell



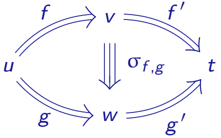
for every critical branching (f, g) of Σ.

- If Σ is confluent, it always admit a family of generating confluences.
- However, such a family is not necessarily unique, since
  - ▷ the 3-cell $\sigma_{f,g}$ can be directed in the reverse way,
  - ▷ for a given branching (f, g), we can have several possible 2-cells f' and g' with the required shape.

Homotopical Squier's Theorem

Theorem. [Squier, 1994]

For a convergent presentation Σ of a monoid $\mathbf{M}$, the $(3,1)$-polygraph obtained from Σ by adjunction of a generating confluence



for every critical branching (f, g) is a coherent presentation of $\mathbf{M}$.

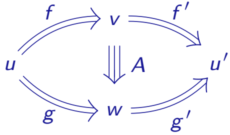
Homotopical Squier's Theorem: proof

Generating confluences

- Let Σ be a convergent 2-polygraph.
- Let Γ be a family of generating confluences of Σ.

Lemma 1.

For every local branching $(f, g) : u \Rightarrow (v, w)$ of Σ, there exist 2-cells f' and g' in Σ_2^* and a 3-cell A in $\Gamma^\top$, as in the following diagram:



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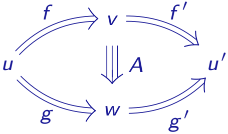
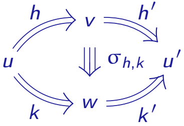


Proof.

▷ For aspherical or Peiffer branching, choose f' and g'such that $f \star_1 f' = g \star_1 g'$ and A is identity.

▷ An overlapping branching (f, g) that is not critical is of the form $(f, g) = (uhv, ukv)$ with (h, k) critical.

▷ Consider a generating confluence



in Γ.

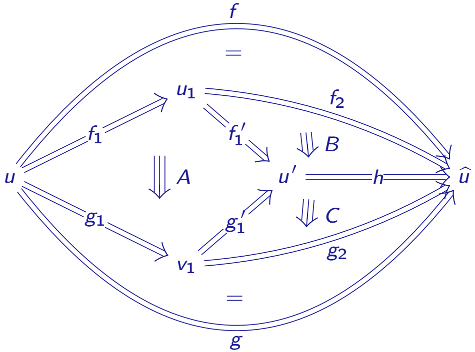
▷ We set $A = u\sigma_{h,k}v$, $f' = uh'v$ and $g' = kuk'v$.

Generating confluences

Lemma 2.

For every parallel 2-cells f and g of Σ_2^* whose common target is a normal form, there exists a 3-cell from f to g in $\Gamma^\top$.

Proof. By Noetherian induction on the common source of f and g.



Homotopical Squier's Theorem

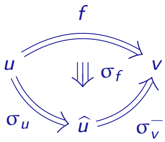
Proposition. Let Σ be a convergent 2-polygraph. Every family Γ of generating confluences of Σ is a homotopy basis of $\Sigma_2^\top$.

Proof.

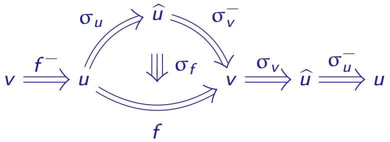
- Consider a 2-cell $f : u \Rightarrow v$ in Σ_2^*.
- Using the confluence, choose 2-cells

$$\sigma_u \;:\; u \Rightarrow \widehat{u} \qquad \text{and} \qquad \sigma_v \;:\; v \Rightarrow \widehat{v} = \widehat{u} \qquad \text{in} \qquad \Sigma_2^*.$$

- By Lemma 2, there exists a 3-cell



- Moreover, the $(3,1)$-category $\Gamma^\top$ contains a 3-cell $\sigma_{f^-} : f^- \Rrightarrow \sigma_v \star_1 \sigma_u^-$, given as the composite:



Homotopical Squier's Theorem

Proposition. Let Σ be a convergent 2-polygraph. Every family Γ of generating confluences of Σ is a homotopy basis of $\Sigma_2^\top$.

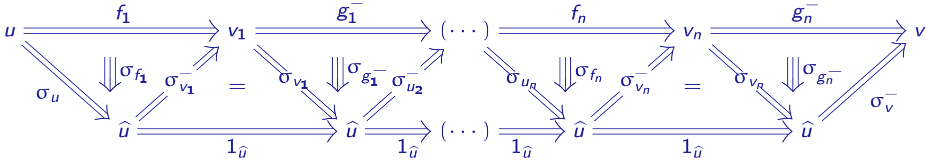
Proof.

▷ Consider a 2-cell $f : u \Rightarrow v$ in $\Sigma_2^\top$. It can be decomposed into a "zig-zag" sequence

$$u \overset{f_1}{\Longrightarrow} v_1 \overset{g_1^-}{\Longrightarrow} u_2 \overset{f_2}{\Longrightarrow} (\cdots) \overset{g_{n-1}^-}{\Longrightarrow} u_n \overset{f_n}{\Longrightarrow} v_n \overset{g_n^-}{\Longrightarrow} v$$

where each f_i and g_i is a 2-cell of Σ_2^*.

▷ We construct a 3-cell of $\Gamma^\top$, with source f and target $\sigma_u \star_1 \sigma_v^-$:



▷ We proceed similarly for any 2-cell $g : u \Rightarrow v$ of $\Sigma_2^\top$, to get a 3-cell from g to $\sigma_u \star_1 \sigma_v^-$.

▷ Thus, the composite is a 3-cell of $\Gamma^\top$ from f to g.

Finite derivation type

Definition. Σ has **finite derivation type** (**FDT**) if

i) Σ is finite,

ii) $\Sigma_2^\top$ has a finite homotopy basis Σ_3.

$$\Sigma_0 \overset{t_0}{\underset{s_0}{\leftleftarrows}} \Sigma_1^* \overset{t_1}{\underset{s_1}{\leftleftarrows}} \Sigma_2^\top \overset{t_2}{\underset{s_2}{\leftleftarrows}} \Sigma_3$$

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Theorem. (Squier, 1994)

i) Property FDT is Tietze invariant for finite rewriting systems.

ii) A monoid having a finite convergent rewriting system has FDT.

Example. (Squier, 1994) The monoid

$$\mathbf{S}_1 = \langle\, a, b, t, x, y \mid at^n b \Rightarrow 1,\ xa \Rightarrow atx,\ xt \Rightarrow tx,\ xb \Rightarrow bx,\ xy \Rightarrow 1 \,\rangle.$$

▷ has a decidable word problem,

▷ does not have finite derivation type.

▶ Hence, the monoid $\mathbf{S}_1$ does not have a finite convergent presentation,

Part III. Homotopical completion-reduction procedure

Tietze transformations

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- Two 2-polygraphs are **Tietze-equivalent** if they present the same monoid.

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▷ **remove a generator**: for a generating 2-cell α in Σ_2 with $x \in \Sigma_1$, remove x and α

$$u \overset{\cancel{\alpha}}{\Longrightarrow} \cancel{x}$$

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- We generalize this notion to $(3,1)$-polygraphs.

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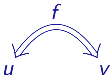

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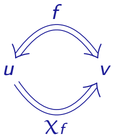

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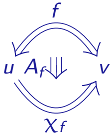

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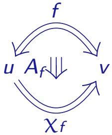

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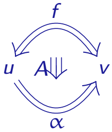

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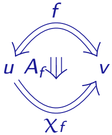

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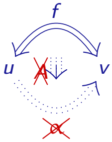

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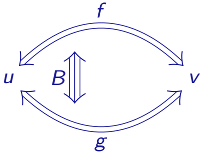

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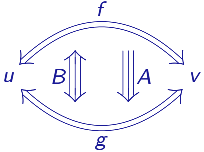

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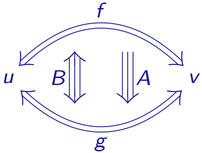

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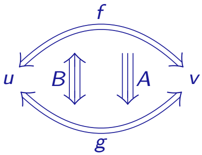

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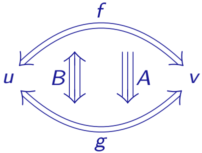

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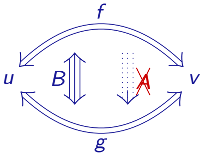

Tietze transformations

Theorem. [Gaussent-Guiraud-Malbos, 2015]

If

- ▷ Σ is a coherent presentation of a monoid $\mathbf{M}$,
- ▷ $\mathcal{T}$ is a composition of elementary Tietze transformations,

then

- ▷ $\mathcal{T}(\Sigma)$ is a coherent presentation of $\mathbf{M}$.

Homotopical completion-reduction procedure

- **Homotopical completion** combines completion procedure and Squier's theorem.

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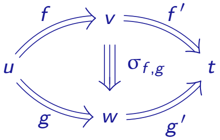
▶ **Homotopical completion** combines completion procedure and Squier's theorem.

▶ The **Knuth-Bendix procedure** computes a convergent presentation from a terminating presentation, Knuth-Bendix, 1970.

▶ The **Squier theorem** constructs a coherent presentation of a monoid $\mathbf{M}$ from a convergent presentation of $\mathbf{M}$ by adjunction of a generating confluence



for any critical branching (f, g), Squier, 1994.

Homotopical completion procedure

Let Σ be a terminating 2-polygraph (with a total termination order).

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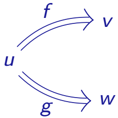

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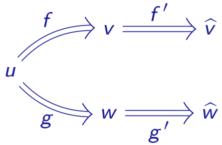

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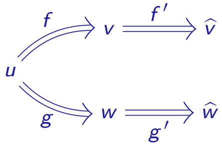

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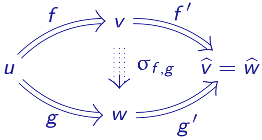

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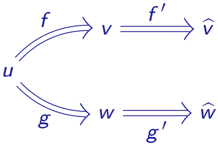

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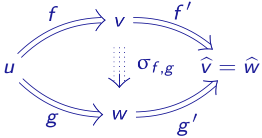
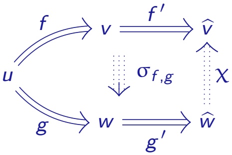


▷ if $\widehat{v} < \widehat{w}$, add a 2-cell χ and a 3-cell $\sigma_{f,g}$



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- A prototype implementation of homotopical completion-reduction procedure
  - `http://www.pps.univ-paris-diderot.fr/~smimram/rewr/`

Homotopical completion procedure

Example. The **Kapur-Narendran's presentation** of $\mathbf{B}_3^+$, obtained from Artin's presentation by coherent adjunction of the element st

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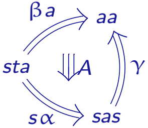

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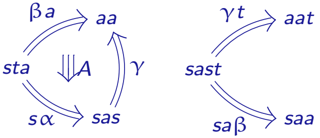

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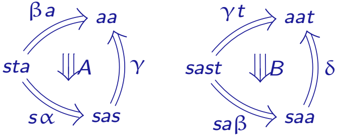

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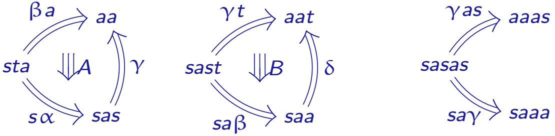

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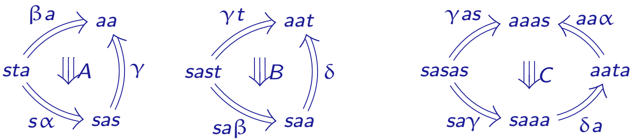

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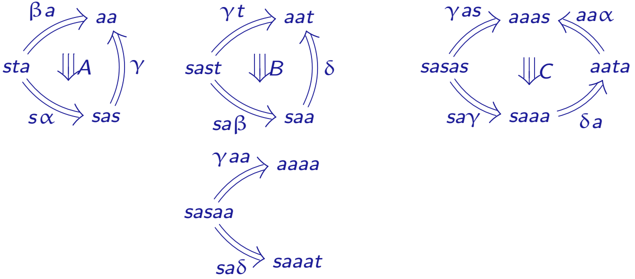

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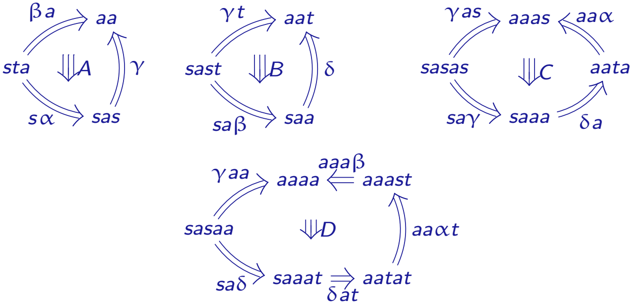

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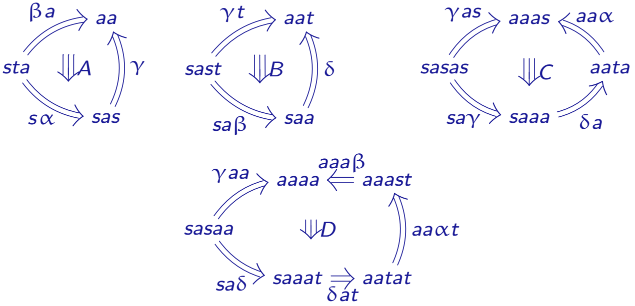

However. The extended presentation $\mathcal{S}(\Sigma_2^{\mathrm{KN}})$ obtained is bigger than necessary.

Homotopical completion-reduction procedure

INPUT: A terminating 2-polygraph Σ.

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Homotopical completion-reduction procedure

INPUT: A terminating 2-polygraph Σ.

Step 1. Compute the homotopical completion $\mathcal{S}(\Sigma)$ (convergent and coherent).

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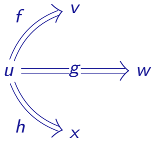

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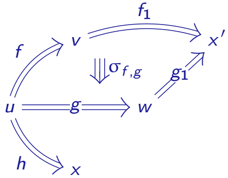

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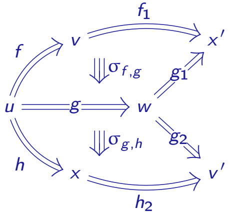

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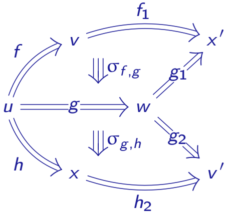

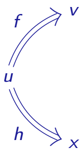

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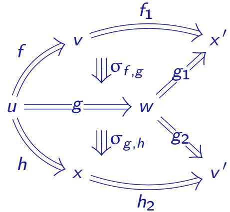

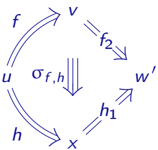

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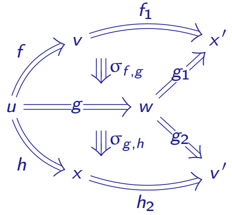

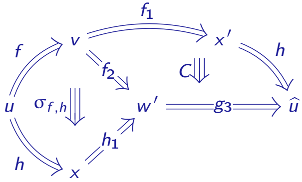

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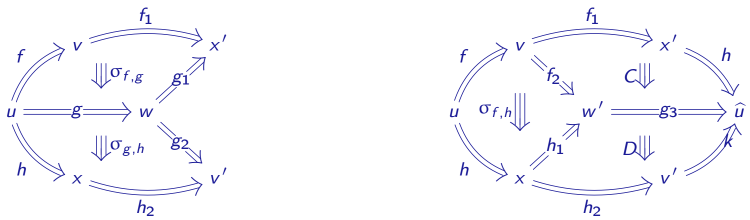

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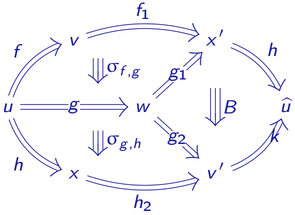

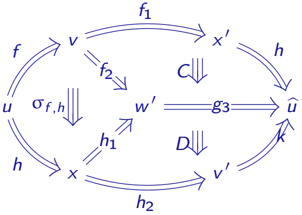

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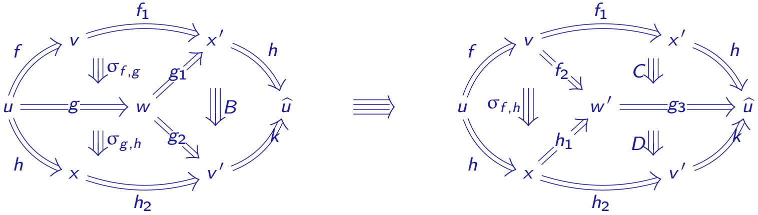

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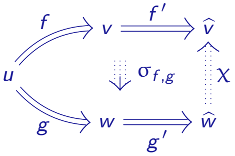

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We obtain the **homotopical completion-reduction** $\mathcal{R}(\Sigma)$ of the terminating 2-polygraph Σ.

Theorem. [Gaussent-Guiraud-Malbos, 2015]
For every terminating presentation Σ of a monoid $\mathbf{M}$, the homotopical completion-reduction $\mathcal{R}(\Sigma)$ is a coherent presentation of $\mathbf{M}$.

▶ Note that $\mathcal{R}(\Sigma)$ is not convergent in general.

The homotopical completion-reduction procedure

Example.

$$\Sigma_2^{\mathrm{KN}} = \langle\, s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a \,\rangle$$

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- There are four critical triple branchings, overlapping on

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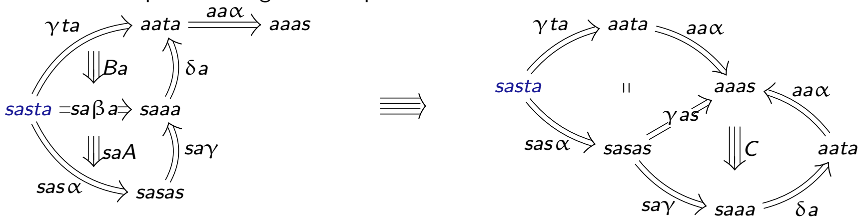
$$sasta,\; sasast,\; sasasas,\; sasasaa.$$

  - Critical triple branching on $sasta$ proves that C is redundant:



$$C = sas\alpha^{-1} \star_1 (Ba \star_1 aa\alpha) \star_2 (saA \star_1 \delta a \star_1 aa\alpha)$$

The homotopical completion-reduction procedure

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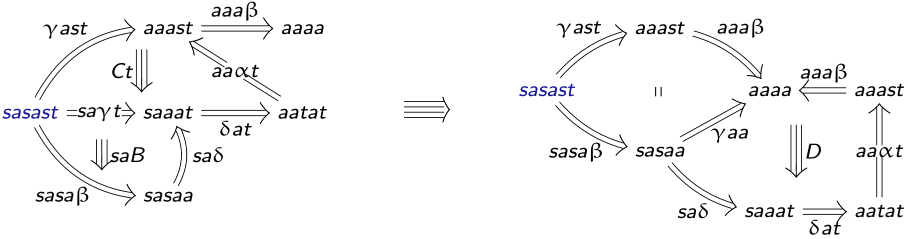

$$D = sasa\beta^{-1} \star_1 \big((Ct \star_1 aaa\beta) \star_2 (saB \star_1 \delta at \star_1 aa\alpha t \star_1 aaa\beta) \big)$$

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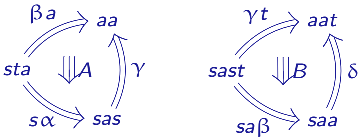

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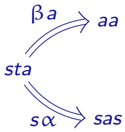

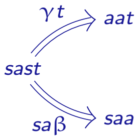

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- ▷ The rule $st \overset{\beta}{\Longrightarrow} a$ is collapsible and the generator a is redundant.

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$$\langle\, s, t, \cancel{a} \mid tst \overset{\alpha}{\Longrightarrow} sts, \cancel{st \overset{\beta}{\Longrightarrow} a}, \cancel{sas \overset{\gamma}{\Longrightarrow} aa}, \cancel{saa \overset{\delta}{\Longrightarrow} aat} \mid \cancel{A}, \cancel{B}, \cancel{C}, \cancel{D} \,\rangle$$

- Artin's coherent presentation:

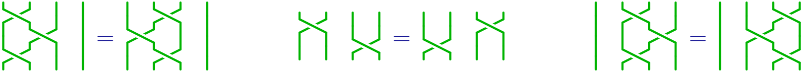
$$\mathcal{R}(\Sigma_2^{\mathrm{KN}}) = \langle\, s, t \mid tst \overset{\alpha}{\Longrightarrow} sts \mid \emptyset \,\rangle$$

Exercice II

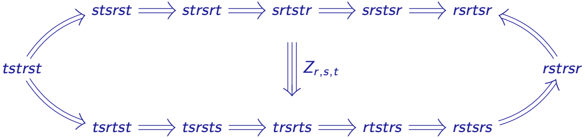
▶ Consider the **Artin monoid** $\mathbf{B}_4^+$ of braids on 4 strands.

▷ given by the **Artin presentation**

$$\langle\, r, s, t \;\mid\; rsr \Rightarrow srs,\;\; rt \Rightarrow tr,\;\; tst \Rightarrow sts \,\rangle$$



▶ Show that this presentation can be extended into a coherent presentation with only one 3-cell



▷ It is called **Zamolodchikov relation** (Deligne, 1997).

Algebraic examples

Artin monoids: Garside's presentation

- Let $\mathbf{W}$ be a **Coxeter group**

$$\mathbf{W} = \langle\ S \quad \Big| \quad s^2 = 1,\ \ \langle ts \rangle^{m_{st}} = \langle st \rangle^{m_{st}}\ \rangle$$

where $\langle ts \rangle^{m_{st}}$ stands for the word $tsts\ldots$ with m_{st} letters.

Artin monoids: Garside's presentation

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▶ **Artin's presentation** of the Artin monoid $\mathbf{B}^+(\mathbf{W})$

$$\mathrm{Art}_2(\mathbf{W}) = \langle\ S \quad \Big| \quad \langle ts \rangle^{m_{st}} \ = \ \langle st \rangle^{m_{st}}\ \rangle$$

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Examples.

- If $\mathbf{W} = \mathbf{S}_n$, the Artin monoid $\mathbf{B}^+(\mathbf{W})$ is the monoid $\mathbf{B}_n^+$ of braids on n strands.

Artin monoids: Garside's presentation

▶ **Garside's extended presentation** of the Artin monoid $\mathbf{B}^+(\mathbf{W})$

▷ 1-cells:

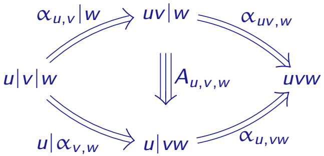
$$\mathrm{Gar}_1(\mathbf{W}) = \mathbf{W} \setminus \{1\}$$

▷ 2-cells:

$$\mathrm{Gar}_2(\mathbf{W}) = \{\ u|v \overset{\alpha_{u,v}}{\Longrightarrow} uv \ \text{ whenever } l(uv) = l(u) + l(v)\ \}$$

where uv is the product in $\mathbf{W}$ and $u|v$ is the product in the free monoid over $\mathbf{W}$.

▷ $\mathrm{Gar}_3(\mathbf{W})$ made of one 3-cell



for every u, v, w in $\mathbf{W} \setminus \{1\}$ such that the lengths can be added.

Theorem. [Gaussent-Guiraud-Malbos, 2015]

$\mathrm{Gar}_3(\mathbf{W})$ is a coherent presentation the Artin monoid $\mathbf{B}^+(\mathbf{W})$

Proof. By homotopical completion-reduction of the 2-polygraph $\mathrm{Gar}_2(\mathbf{W})$.

Artin monoids: Artin's coherent presentation

Theorem. [Gaussent-Guiraud-M., 2015]

The Artin monoid $\mathbf{B}^+(\mathbf{W})$ admits the coherent presentation $\mathrm{Art}_3(\mathbf{W})$ made of

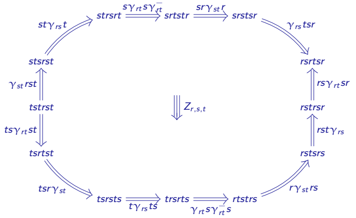
▷ Artin's presentation

$$\mathrm{Art}_2(\mathbf{W}) = \langle\ S \quad \big| \quad \langle ts \rangle^{m_{st}} = \langle st \rangle^{m_{st}}\ \rangle$$

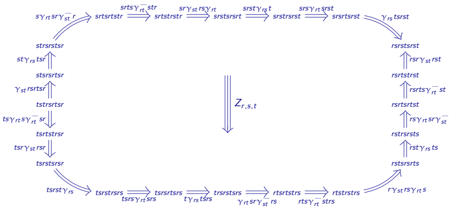
▷ one 3-cell $Z_{r,s,t}$ for every $t > s > r$ in S such that the subgroup $\mathbf{W}_{\{r,s,t\}}$ is finite.

Artin monoids: Zamolodchikov $Z_{r,s,t}$ according to Coxeter type

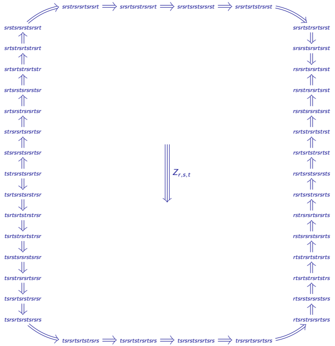
Type A_3



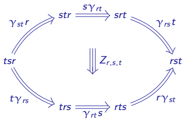
Type B_3



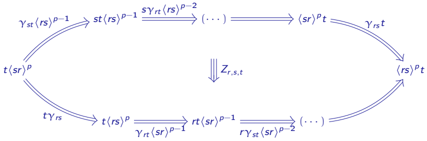
Type H_3



Type $A_1 \times A_1 \times A_1$



Type $I_2(p) \times A_1$, $p \geqslant 3$,



Plactic monoids

▶ **Knuth's presentation** of the plactic monoid $\mathbf{P}_n$

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  - Generators:

$$\mathrm{Knuth}_1(n) = \{ 1, \ldots, n \}$$

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  - **Knuth relations**:

$$\mathrm{Knuth}_2(n) = \left\{ \begin{array}{ll} zxy \Rightarrow xzy & \text{for all } 1 \leqslant x \leqslant y < z \leqslant n \\ yzx \Rightarrow yxz & \text{for all } 1 \leqslant x < y \leqslant z \leqslant n \end{array} \right\}$$

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$$\mathrm{Knuth}_2(n) = \left\{ \begin{array}{ll} zxy \Rightarrow xzy & \text{for all } 1 \leqslant x \leqslant y < z \leqslant n \\ yzx \Rightarrow yxz & \text{for all } 1 \leqslant x < y \leqslant z \leqslant n \end{array} \right\}$$

▶ For $n \geqslant 4$, there is no finite completion of $\mathrm{Knuth}_2(n)$ on $\mathrm{Knuth}_1(n)$ compatible with the degree lexicographic order, Kubat-Okniński, 2014.

Plactic monoids: column presentation

- **Column presentation** of the plactic monoid $\mathbf{P}_n$, Cain-Gray-Malheiro, 2015.

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  - add **columns** as generators:

$$c_u = x_p \ldots x_2 x_1 \in \mathrm{Knuth}_1^*(n) \quad \text{such that} \quad x_p > \ldots > x_2 > x_1.$$

$$\mathrm{Col}_1(n) = \left\{ \, c_u \;\middle|\; u \text{ is a column} \, \right\}$$

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▷ 2-cells: $\mathrm{Col}_2(n)$ is the set of 2-cells

$$c_u c_v \overset{\alpha_{u,v}}{\Longrightarrow} c_w c_{w'}$$

such that

▷ u and v are columns,

▷ the planar representation of the Schensted tableau $P(uv)$ is not the juxtaposition of columns u and v and where w and w' are respectively the left and right columns of $P(uv)$.

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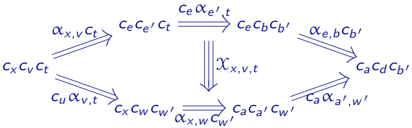
1	1	1	2	2	3	4
2	2	3	3	4	6	
4	5	6	6			
6	7					

Plactic monoids: column presentation

Theorem. [Hage-Malbos, 2015]

For $n \geqslant 2$, the 2-polygraph $\mathrm{Col}_2(n)$ can be extended into a coherent presentation of the plactic monoid $\mathbf{P}_n$, whose 3-cells are of the following form



with x in $\mathrm{Knuth}_1(n)$ and v, t are columns.

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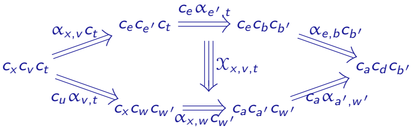

with x in $\mathrm{Knuth}_1(n)$ and v, t are columns.

Proof.

By homotopical completion-reduction of the 2-polygraph $\mathrm{Col}_2(n)$.

Exercise III

1. Compute a coherent presentation of the plactic monoid $\mathbf{P}_2$.

[Hint. There are two ways to prove that $2211 = 2121$ in $\mathbf{P}_2$.]

2. Complete the 2-polygraph $\mathrm{Knuth}_2(3)$ that presents the plactic monoid $\mathbf{P}_3$ into a coherent presentation.

[Hint. $\mathrm{Knuth}_2(3)$ can be completed with 3 relations and 27 3-cells.]

3. (Kubat-Okniński, 2014) Prove that for $n \geqslant 4$, there is no finite completion of the 2-polygraph $\mathrm{Knuth}_2(n)$ on $\mathrm{Knuth}_1(n)$ compatible with the degree lexicographic order.

4. Compute a coherent presentation of the plactic monoid $\mathbf{P}_4$.

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