Lectures on Algebraic Rewriting

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- December 2019 -

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Abstract Rewriting

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The principle of rewriting comes from combinatorial algebra. It was introduced by Thue when he considered systems of transformation rules on combinatorial objects such as strings, trees or graphs in order to solve the word problem, [Thu14]. Given a collection of objects and a system of transformation rules on these objects, the word problem is

INSTANCE: *given two objects*,

QUESTION: *can one of these objects be transformed to the other by means of a finite number of applications of the transformation rules?*

Dehn described the word problem for finitely presented groups, [Deh10] and Thue studied this problems for strings, which correspond to the word problem for finitely presented monoids, [Thu14]. Note that it was only much later, that the problem was shown to be undecidable, independently by Post [Pos47] and Markov [Mar47a, Mar47b]. Afterwards, the word problem have been considered in many contexts in algebra and in computer science.

Far beyond the precursor works on this decidabilty problem on strings, rewriting theory has been mainly developed in theoretical computer science, producing numerous variants corresponding to different syntaxes of the formulas being transformed: strings in a monoid, [BO93, GM18], paths in a graph, terms in an algebraic theory, [BN98, Klo92, Ter03], terms modulo, λ-terms, trees, Boolean circuits, [Laf03], graph grammars, etc. Rewriting appears also on various forms in algebra, for commutative algebras, [Buc65, Buc87], Lie algebras, [Shi62], with the notion of Gröbner-Shirshov bases, or associative algebras, [Bok76, Ber78, Mor94, Ufn95, GHM19] and operads, [DK10], as well as on topological objects, such as Reidemeister moves, knots or braids, [Bur01], or in higher-dimensional categories, [GM09, GM12a, Mim10, Mim14].

Many of the basic definitions and fundamental properties of these forms of rewriting can be stated on the most abstract version of rewriting that is given by a binary relation on set. In this chapter, we present the notion of abstract rewriting system and the main abstract rewriting properties used in these lectures. We refer the reader to [BN98, Klo92, Ter03] for a complete account on the abstract rewriting theory.

1.1. ABSTRACT REWRITING SYSTEMS

1.1.1. Abstract Rewriting Systems. An *abstract rewriting system*, ARS for short, is a data $(A, \to_I)$ made of a set A and a sequence $\to_I$ of binary relations on A indexed by a set I, that is,

$$\to_I = (\to_\alpha)_{\alpha \in I}, \quad \text{and} \quad \to_\alpha \subseteq A \times A.$$

The relation is called *reduction* or *rewrite* relation on A. An element (a, b) in the relation $\to$ will be denoted by $a \to b$, and we said that b is a *one-step reduct* of a, and that a is a *one-step expansion* of b. An element of $\to$ is called a *reduction step*. In most cases the elements of A have a syntactic or graphical nature (string, term, tree, graph, polynomial...). We will denoted by $\equiv$ the syntactical or graphical identity.

1.1.2. Reduction sequence. A *reduction sequence*, or *rewriting sequence*, with respect to a reduction relation $\to$ is a finite or infinite sequence of reduction steps

$$a_0 \to a_1 \to a_2 \to \ldots$$

If we have a reduction sequence

$$a \equiv a_0 \to a_1 \to a_2 \to \ldots \to a_{n-1} \to a_n \equiv b$$

we say that a *reduces to* b. The *length* of a finite reduction sequence is the number of its reduction steps.

1.1.3. Composition. Given two reduction relations $\to_1$ and $\to_2$ on A, their *composition* is denoted by $\to_1 \cdot \to_2$ and defined by

$$a \to_1 \cdot \to_2 b \quad \text{if} \quad a \to_1 c \to_2 b, \quad \text{for some } c \text{ in } A.$$

1.1.4. Notations. The identity relation is denoted by

$$\overset{0}{\to} = \{(a, a) \mid a \in A\}.$$

The inverse relation of $\rightarrow$ is denoted by $\leftarrow$, or by $\rightarrow^{-}$, and defined by:

$$\leftarrow \,=\, \{(b, a) \mid a \rightarrow b\}.$$

A relation is *reflexive* if $\xrightarrow{0} \subseteq \rightarrow$ and *transitive* if $\rightarrow \cdot \rightarrow \subseteq \rightarrow$. The reflexive closure of $\rightarrow$ is denoted by $\xrightarrow{\equiv}$ and defined by

$$\xrightarrow{\equiv} \,=\, \rightarrow \,\cup\, \xrightarrow{0}.$$

The symmetric closure of $\rightarrow$ is denoted by $\leftrightarrow$ and defined by

$$\leftrightarrow \,=\, \rightarrow \,\cup\, \leftarrow.$$

The transitive closure of $\rightarrow$ is denoted by $\xrightarrow{+}$ and defined by

$$\xrightarrow{+} \,\subseteq\, \bigcup_{i>0} \xrightarrow{i},$$

where $\xrightarrow{i+1} = \xrightarrow{i} \cdot \rightarrow$ for all $i > 0$. The reflexive and transitive closure of $\rightarrow$ is denoted by $\twoheadrightarrow$, or by $\rightarrow^{*}$, and defined by

$$\twoheadrightarrow \,=\, \xrightarrow{+} \cup \xrightarrow{0}.$$

The reflexive, transitive and symmetric closure of $\rightarrow$ is denoted by $\xleftrightarrow{*}$ and defined by

$$\xleftrightarrow{*} \,=\, (\leftrightarrow)^{*}.$$

In particular, we have $a \twoheadrightarrow b$ is there is a rewriting sequence from a to b and we have $a \xleftrightarrow{*} b$ if and only if there is a zig-zag of rewriting sequence from a to b:

$$a \equiv a_0 \leftrightarrow a_1 \leftrightarrow a_2 \leftrightarrow \ldots \leftrightarrow a_{n-1} \leftrightarrow a_n \equiv b.$$

The relation $\xleftrightarrow{*}$ is equal to the equivalence relation generated by $\rightarrow$.

1.1.5. Branchings and confluence pairs. A *branching* (resp. *local branching*) of the relation $\rightarrow$ is an element of the composition $\twoheadleftarrow \cdot \twoheadrightarrow$ (resp. $\leftarrow \cdot \rightarrow$). It is defined by a triple $a \twoheadleftarrow c \twoheadrightarrow b$ (resp. $a \leftarrow c \rightarrow b$) as pictured by the following diagram:

A *confluence pair* (resp. *local confluence pair*) of the relation $\rightarrow$ is an element of the composition $\twoheadrightarrow \cdot \twoheadleftarrow$ (resp. $\rightarrow \cdot \leftarrow$). It is defined by a triple $a \twoheadrightarrow d \twoheadleftarrow b$ as pictured by the following diagram:

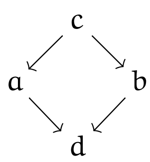
Note that the relations $\leftarrow \cdot \rightarrow$ and $\twoheadleftarrow \cdot \twoheadrightarrow$ are symmetric.

1.1.6. Commutation. Two relations $\rightarrow_1$ and $\rightarrow_2$ on A *commute* if

$$ {}_1\twoheadleftarrow \cdot \twoheadrightarrow_2 \subseteq \twoheadrightarrow_2 \cdot {}_1\twoheadleftarrow . $$

1.2. CONFLUENCE

1.2.1. Diamond property. A relation $\rightarrow$ has the *diamond property* if it commutes with itself. This means that for any local branching $a \leftarrow c \rightarrow b$ there exists a local confluence:



This property is hard to obtain in general. Let us give the main confluence patterns used in rewriting.

1.2.2. Confluence patterns. A reduction relation $\rightarrow$ is called

i) *Church-Rosser* if $\overset{*}{\leftrightarrow} \subseteq \twoheadrightarrow \cdot \twoheadleftarrow$.

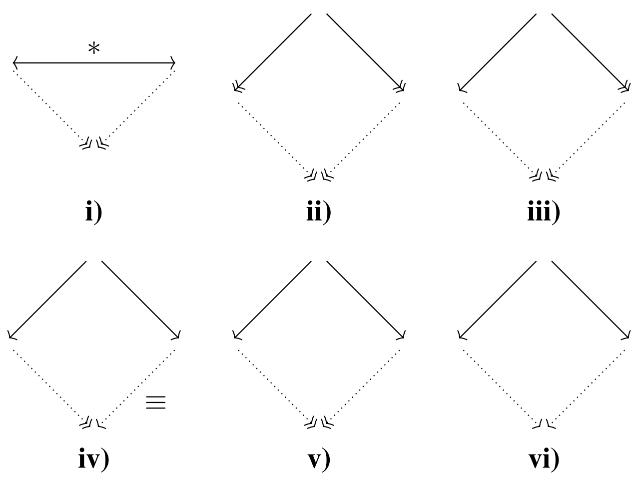
ii) *confluent* if the relation $\twoheadrightarrow$ commutes, that is $\twoheadleftarrow \cdot \twoheadrightarrow \subseteq \twoheadrightarrow \cdot \twoheadleftarrow$.

iii) *semi-confluent* if $\leftarrow \cdot \twoheadrightarrow \subseteq \twoheadrightarrow \cdot \twoheadleftarrow$.

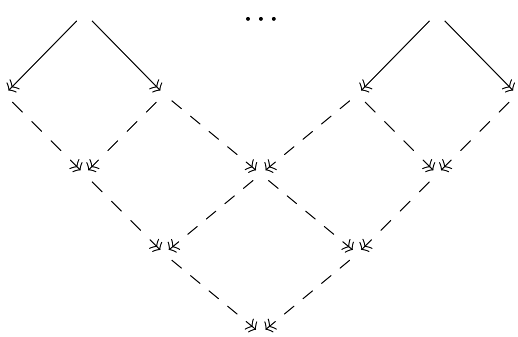
iv) *strongly-confluent* if $\leftarrow \cdot \rightarrow \subseteq \twoheadrightarrow \cdot \overset{\equiv}{\leftarrow}$.

v) *locally confluent* if $\leftarrow \cdot \rightarrow \subseteq \twoheadrightarrow \cdot \twoheadleftarrow$.

vi) *has the diamond property* if the relation $\rightarrow$ commutes, that is $\leftarrow \cdot \rightarrow \subseteq \rightarrow \cdot \leftarrow$.



1.2.3. Remark. The diamond property implies the Church-Rosser property, [New42, Theorem 1]. Note that in [New42] Newman called confluence the Church-Rosser property defined above. He showed that these properties coincide. Obviously, any Church-Rosser property is confluent, and the reverse implication is shown by the following diagram:



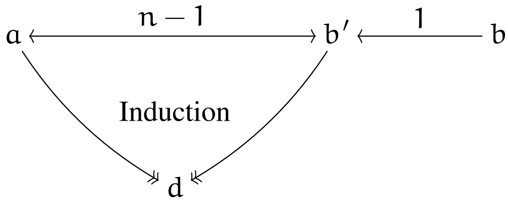
1.2.4. Proposition. *For an abstract rewriting system $(A, \to)$ the following conditions are equivalent*

i) $\to$ *is confluent,*

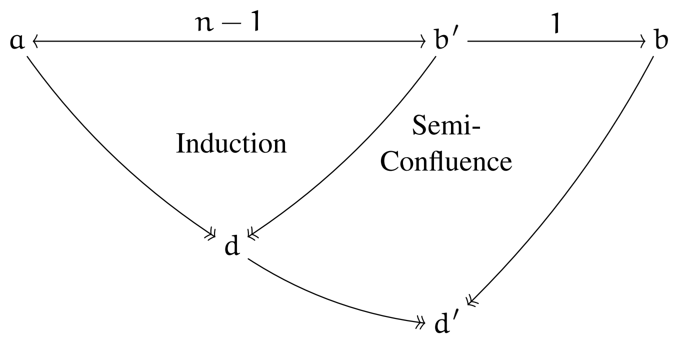
ii) $\to$ *is semi-confluent,*

iii) $\to$ *has the Church-Rosser property.*

Proof. Prove that **iii)** implies **i)**. Suppose that $\to$ is Church-Rosser. Given a branching $a \twoheadleftarrow c \twoheadrightarrow b$, we have $a \overset{*}{\leftrightarrow} b$. Hence by the Church-Rosser property, there is a confluence pair $a \twoheadrightarrow d \twoheadleftarrow b$, hence $\to$ is confluent. Obviously **i)** implies **ii)**. Prove that **ii)** implies **iii)**. Suppose that $\to$ is semi-confluent and let $a \overset{*}{\leftrightarrow} b$. Prove by induction on the length of the sequence of reductions between a and b, that there is a confluence pair $a \twoheadrightarrow d \twoheadleftarrow b$. This is obvious when the sequence is of length 0, that is $a \equiv b$, or when the sequence is of length 1, that is $a \to b$ or $a \leftarrow b$. Let consider a sequence of reductions $a \overset{n-1}{\leftrightarrow} b' \overset{1}{\leftrightarrow} b$. By induction hypothesis, there is a confluence pair $a \twoheadrightarrow d \twoheadleftarrow b'$. If $b \to b'$, that is



by induction, this gives a confluence pair $a \twoheadrightarrow d \twoheadleftarrow b$. In the other case, if $b' \to b$, by semi-confluence, there is a confluence pair $d \twoheadrightarrow d' \twoheadleftarrow b$:



hence, by induction we have a confluence pair $a \twoheadrightarrow d' \twoheadleftarrow b$. It follows that the relation $\rightarrow$ is Church-Rosser. ☐

1.2.5. Exercise. Prove that strong confluence implies confluence.

1.2.6. Exercise. Let A be a set and let $\rightarrow_1$ and $\rightarrow_2$ be two reduction relations on A.

1. Prove that the confluence of $\rightarrow_1$ and $\rightarrow_2$ does not imply the confluence of $\rightarrow_1 \cup \rightarrow_2$.

2. Prove that

$$\rightarrow_1 \subseteq \rightarrow_2 \subseteq \twoheadrightarrow_1 \quad \text{implies} \quad \twoheadrightarrow_1 = \twoheadrightarrow_2 .$$

3. Prove that if $\rightarrow_1 \subseteq \rightarrow_2 \subseteq \twoheadrightarrow_1$ and $\rightarrow_2$ is strongly confluent, then $\rightarrow_1$ is confluent.

4. Prove that if $\rightarrow_1$ and $\rightarrow_2$ are confluent and commute, then the relation $\rightarrow_1 \cup \rightarrow_2$ is also confluent.

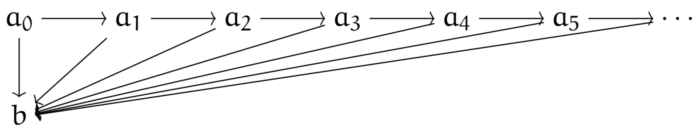
1.3. Normalisation

Let $(A, \rightarrow)$ be an abstract rewriting system.

1.3.1. Normal form. An element a in A is in *normal form*, or *irreductible*, with respect to $\rightarrow$ if there is no b in A such that $a \rightarrow b$. It is *reductible* if it is no irreductible. We denote by NF$(\rightarrow)$ the set of normal forms in A with respect to $\rightarrow$.

1.3.2. Normalizing. An element a in A is *(weakly) normalizing* if $a \twoheadrightarrow b$ for some b in NF$(\rightarrow)$. Then we say that a *has a normal form* b and b is called a *normal form of* a. The relation $\rightarrow$ is *(weakly) normalizing* if every element a in A is normalizing.

1.3.3. Termination. An element a in A is *strongly normalizing* if every reduction sequence starting from a is finite. The relation $\rightarrow$ is *strongly normalizing*, or *terminating*, or *noetherian* if every a in A is strongly normalizing. Any terminating relation is normalizing. Note that the converse is false as shown by the following abstract rewriting system



1.3.4. Convergence. We say that $\rightarrow$ is *convergent*, or *complete*, *canonical*, *uniquely terminating*, if $\rightarrow$ is confluent and terminating.

1.3.5. Normal form property. The relation $\rightarrow$ has the *normal form property* if for any a in A and any normal form b in A

$$a \overset{*}{\leftrightarrow} b \quad \text{implies} \quad a \twoheadrightarrow b.$$

The relation $\rightarrow$ has the *unique normal form property* if for all normal forms a and b in A

$$a \overset{*}{\leftrightarrow} b \quad \text{implies} \quad a \equiv b.$$

1.3.6. Semi-convergence. We say that $\rightarrow$ is *semi-convergent*, or *semi-complete*, if $\rightarrow$ has the unique normal form property and is normalizing. If $(A, \rightarrow)$ is semi-convergent, then every element a in A reduces to a unique normal form denoted by $\widehat{a}$.

1.3.7. Confluence and unicity of the normal form. If $\rightarrow$ is confluent, every element has at most one normal form. As an immediate consequence of the equivalence of the Church-Rosser property and the confluence property, Proposition 1.2.4, we have

1.3.8. Theorem. *For an abstract rewriting system $(A, \rightarrow)$ the following implications hold:*

i) *The normal form property implies the unique normal form property.*

ii) *If $\rightarrow$ is confluent then $\rightarrow$ has the normal form property.*

iii) *If $\rightarrow$ is semi-convergent then it is confluent.*

For a confluent abstract rewriting system $(A, \rightarrow)$, two elements a and b in A are equivalent if there are joignable: $a \twoheadleftarrow \cdot \twoheadrightarrow b$. The test of joignability may be not possible when the relation is not terminating. For example, how to test the joignability of $-n$ and n in the following example:

$$\cdots \longleftarrow -2 \longleftarrow -1 \longleftarrow 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots$$

Let us show that normalisation suffices to determine joignability.

If $\rightarrow$ is normalizing and confluent, every element a in A has a unique normal form denoted by $\widehat{a}$.

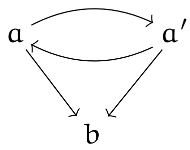
1.3.9. Theorem. *If $\rightarrow$ is normalizing and confluent then we have*

$$a \overset{*}{\leftrightarrow} b \qquad \textit{if and only if} \qquad \widehat{a} \equiv \widehat{b}.$$

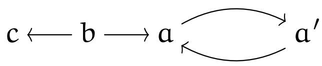
As a consequence of the previous result, for a normalizing and confluent abstract rewriting system $(A, \rightarrow)$ an equivalence test of two elements a and b in A is to check the syntactical equality of their normal forms $\widehat{a}$ and $\widehat{b}$. If the normal forms are computable and the syntactic identity is decidable then the equivalence is decidable.

1.3.10. Exercise. Prove Theorem 1.3.8 and Theorem 1.3.9.

1.3.11. Examples. The abstract rewriting system



is confluent, not terminating and admits a unique normal form. The abstract rewriting system



is not confluent, not terminating and admits a unique normal form.

1.3.12. Example. Let $A = \mathbb{N} - \{0, 1\}$. Consider the relation on A defined by the

$$\{(m, n) \mid m > n \text{ and } n \text{ divides } m\}.$$

Then m is in NF($\to$) if and only if m is prime. An element p is a normal form of m if and only if p is a prime factor of m. We have $m \twoheadrightarrow \cdot \twoheadleftarrow n$ if and only if m and n are note relatively prime. The transitive closure of $\to$ coincide woth $\to$ because $>$ and divide relations are already transitive. We have $\overset{*}{\leftrightarrow} = A \times A$ and $\to$ terminates and it is not confluent.

1.3.13. Example. Let $A = \{a, b\}^*$ be the free monoid on $\{a, b\}$. We consider the relation $\to$ defined by the set

$$\{(ubav, uabv) \mid u, v \in A\}.$$

Then an element of A is in normal form if and only if it is of the form $a^n b^m$ for $n, m \in \mathbb{N}$. Not that the relation $\to$ terminates and its confluent. Thus every element of A has a unique normal form and we have $w \twoheadrightarrow \cdot \twoheadleftarrow w'$ if and only if $w \overset{*}{\leftrightarrow} w'$ if and only if w and w' contain the same number of as and bs. We will see in the next chapter on string rewriting that such an abstract rewriting system can be specified by only one rewriting rule $ba \;\to\; ab$.

1.3.14. Exercise, [Jan88]. Consider the set $\mathbb{N} \times \mathbb{N}$ with the reduction relation $\to_1$ defined by $(x, y) \to_1 (x', y')$ if

$$\big((x' = x - 2) \text{ and } (y' = y \geqslant 1)\big) \text{ or } \big((x' = x + 2) \text{ and } (y' = y - 1)\big).$$

1. Show that $\to_1$ is terminating.

2. Show that $\to_1$ is not confluent.

3. Define a reduction relation $\to_2$ on $\mathbb{N} \times \mathbb{N}$ that is terminating, confluent and equivalent to $\to_1$, that is the relations $\overset{*}{\leftrightarrow}_1$ and $\overset{*}{\leftrightarrow}_2$ are equal.

1.3.15. Well-founded induction. The principle of induction for natural numbers ensures that a property $\mathcal{P}(n)$ holds for all natural numbers n if we can show that $\mathcal{P}(n)$ holds under the hypothesis that $\mathcal{P}(m)$ holds for all $m < n$. The principle is a consequence of the fact that there is no infinitely descending chain of natural numbers.

The *well-founded induction* principle for an abstract rewriting system $(A, \rightarrow)$ can be stated as follows. Given a property $\mathcal{P}$ on elements of A, then

$$\forall\, a \in A,\ \big(\,\forall\, b \in A,\ a \xrightarrow{+} b \ \text{ implies }\ \mathcal{P}(b)\,\big) \quad \text{implies}\ \ \mathcal{P}(a)$$

implies

$$\forall a \in A,\ \ \mathcal{P}(a).$$

With this principle, the property $\mathcal{P}(a)$ is proved for all elements a in A by proving that the property $\mathcal{P}(b)$ holds for any element b in A such that there is a rewriting sequence $a \twoheadrightarrow b$.

1.3.16. Theorem. *If $\rightarrow$ terminates then the principle of noetherian induction holds*

Proof. Suppose that the principle of induction does not hold, that is

$$\forall\, a \in A,\ \big(\,\forall\, b \in A,\ a \xrightarrow{+} b \ \text{ implies }\ \mathcal{P}(b)\,\big) \quad \text{implies}\ \ \mathcal{P}(a)$$

holds and that there exist an element c in A such that $\mathcal{P}(c)$ does not hold. Then there exists c' such that $c \xrightarrow{+} c'$ and $\mathcal{P}(c')$ does not hold. In this way, we construct an infinite reduction sequence starting on c. Hence, the reduction relation $\rightarrow$ does not terminate. □

Conversely, if the noetherian induction principe holds for an abstract rewriting system $(A, \rightarrow)$, then it terminates. It suffices to apply the induction principle to the property:

$$\mathcal{P}(a) \ \equiv\ (\text{there is no infinite reduction sequence starting on } a)\,.$$

1.3.17. Exercise. Let $(A, \rightarrow)$ be an abstract rewriting system. The relation $\rightarrow$ is called *finitely branching* if each element a of A has only finitely many direct successors, that is elements b such that $a \rightarrow b$. The relation is called *globally finite* if the relation $\xrightarrow{+}$ is finitely branching, that is each element a in A has only finitely many successors.

1. Suppose that the relation $\rightarrow$ is terminating and finitely branching. Prove that it is globally finite.

2. Show that it is not true that a finitely branching relation is terminating if it is globally finite.

A relation is *acyclic* if there is no element a in A such that $a \xrightarrow{+} a$.

3. Show that any acyclic relation is terminating if it is globally finite.

4. Show that a finitely branching and acyclic relation is terminating if and only if it is globally finite.

1.3.18. Exercise. Let $(A, \rightarrow)$ be an abstract rewriting system such that every element a in A has a unique irreducible descendant. Prove that the relation $\rightarrow$ is confluent.

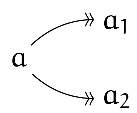
1.4. From local to global confluence

The local confluence does not generally imply confluence, however these properties are equivalent for terminating rewriting systems. This result is also due to Newman.

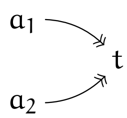
1.4.1. Theorem (Newman's lemma, [New42, Theorem 3]). *A terminating relation is confluent if it is locally confluent.*

A short proof by Noetherian induction is given by Huet in [Hue80]. Due to this proof, Newman's Lemma is also called the *diamond lemma*.

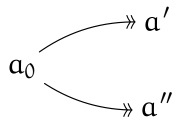
Proof. Suppose that $\rightarrow$ is locally confluent and terminating. We prove its confluence by Noetherian induction. Given a_0 in A, we suppose that for all a with $a_0 \twoheadrightarrow a$ and for all branching



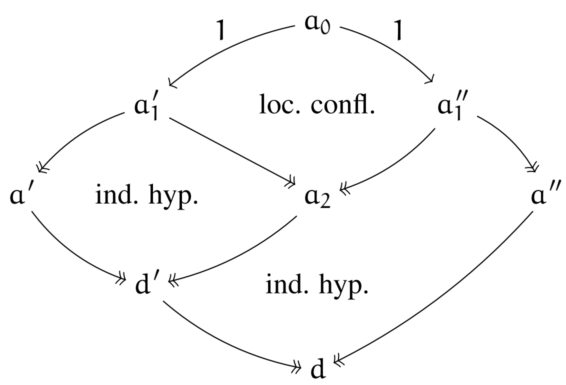
there exists a confluence



Let us consider a branching

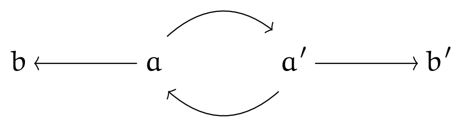


The cases $a' \equiv a_0$ or $a'' \equiv a_0$ are obvious. In the other case, the length of the reductions $a_0 \twoheadrightarrow a'$ and $a_0 \twoheadrightarrow a''$ are greater than 1:

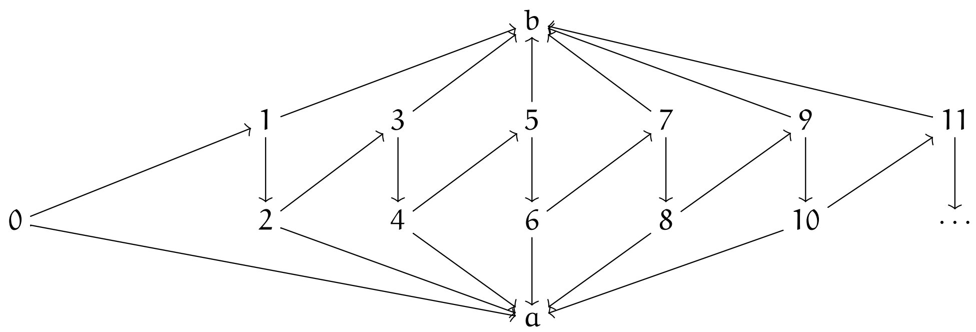


We conclude using the induction hypothesis and local confluence. □

1.4.2. Example, [Hue80]. The following examples illustrate that the requirement of noetherianity is necessary to prove confluence from local confluence. The following abstract rewriting system is locally confluent but not confluent



The following abstract rewriting system with $2n \to a$, $2n+1 \to b$ and $n \to n+1$ for all n in $\mathbb{N}$ without cycle is local confluent but not confluent:



It is locally confluent but not confluent.

CHAPTER
2

String Rewriting

Contents

A *string rewriting system*, SRS for short, historically called a *semi-Thue system*, is a rewriting system over a set of strings on an alphabet. String rewriting systems are Turing complete in the sense that they give a calculus that is equivalent to that of the Turing machine. String rewriting system appear in the formal language theory. They are also used in combinatorial algebra as a tool for presentation of semigroups, groups or monoids. For a fuller treatment on string rewriting systems we refer the reader to [BO93] and [Jan88].

In this chapter, string rewriting system will be describe in the categorical language of 2-polygraphs as in [GM18] and [GM12b, Section 4]. A 2-polygraph is a rewriting system over a set of paths of a given directed graph. String rewriting system is the particular case when the directed graph has only one vertex.

2.1. Preliminaries: one and two-dimensional categories

2.1.1. Categories. A *(small) category* (or 1*-category*) is a data **C** made of

i) a set $\mathbf{C}_0$, whose elements are called the 0*-cells of* **C**,

ii) for every 0-cells x and y of **C**, a set $\mathbf{C}(x, y)$, whose elements are called the 1*-cells from* x *to* y *of* **C**,

iii) for every 0-cells x, y and z of **C**, a map

$$\star_0^{x,y,z} \;:\; \mathbf{C}(x,y) \times \mathbf{C}(y,z) \;\to\; \mathbf{C}(x,z),$$

called the *composition* (or 0-composition) of **C**,

iv) for every 0-cell x, a specified element 1_x of $\mathbf{C}(x, x)$, called the *identity of* x.

The following relations are required to hold

v) the composition is associative, *i.e.*, for every 0-cells x, y, z and t and for every 1-cells $u \in \mathbf{C}(x, y)$, $v \in \mathbf{C}(y, z)$ and $w \in \mathbf{C}(z, t)$,

$$\star_0^{x,z,t}(\star_0^{x,y,z}(u,v), w) \;=\; \star_0^{x,y,t}(u, \star_0^{y,z,t}(v,w)),$$

vi) the identities are local units for the composition, *i.e.*, for every 0-cells x and y and for every 1-cell $u \in \mathbf{C}(x, y)$,

$$\star_0^{x,x,y}(1_x, u) \;=\; u \;=\; \star_0^{x,y,y}(u, 1_y).$$

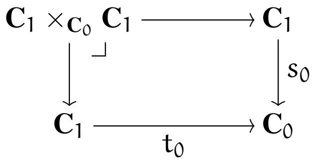
We write $u : x \to y$ to mean that u is in $\mathbf{C}(x, y)$. The 0-cell x is the *source* of u denoted by $s_0(u)$ and the 0-cell y is the *target* of u denoted by $t_0(u)$. The composition $\star_0^{x,y,z}(u, v)$ will be denoted by $u \star_0 v$, or simply by juxtaposition uv.

2.1.2. Monoids. A monoid M with product $\cdot$ and identity element 1_M corresponds to a category **M** with only one 0-cell, denoted by $\bullet$, and the 1-cells of $\mathbf{M}(\bullet, \bullet)$ are the elements of the monoid M. The identity arrow $1_\bullet$ of **M** is the identity element 1_M and the composition of $u \star_0 v$ of 1-cells in $\mathbf{M}(\bullet, \bullet)$ if the product $u \cdot v$ in the monoid M. The associativity and unitary properties of the composition, making **M** into a category, are induced by the corresponding properties of the product $\cdot$. In this way, any monoid can be thought of as a one-0-cell category and a category can be thought of as a "monoid with several 0-cells".

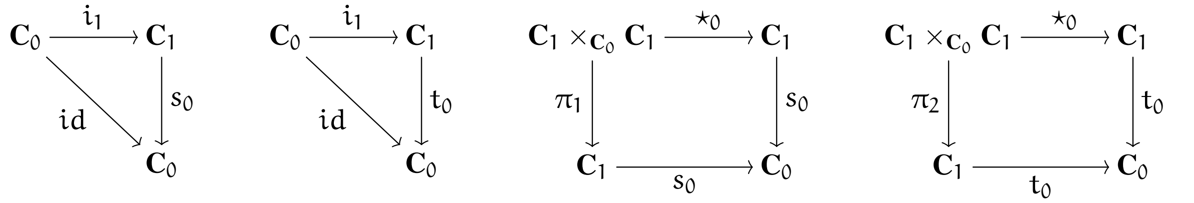
2.1.3. Internal definition. A category **C** can also be defined as an internal category in the category **Set** of sets. Explicitly, it is defined by a diagram in **Set**:

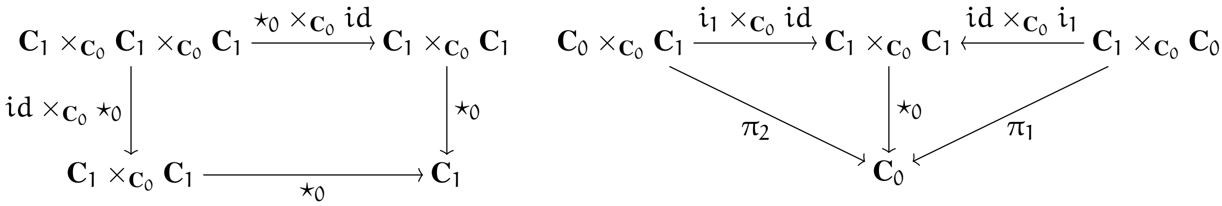
$$\mathbf{C}_0 \underset{i_1}{\overset{t_0}{\underset{s_0}{\leftleftarrows\!\!\!\!\rightarrow}}} \mathbf{C}_1 \overset{\star_0}{\longleftarrow} \mathbf{C}_1 \times_{\mathbf{C}_0} \mathbf{C}_1$$

where $\mathbf{C}_1 \times_{\mathbf{C}_0} \mathbf{C}_1$ is defined by the following pullback diagram in the category **Set**:



Elements of $\mathbf{C}_1 \times_{\mathbf{C}_0} \mathbf{C}_1$ are pairs (u, v) of 0-composable 1-cells u and v, that is satisfying $t_0(u) = s_0(v)$. The maps s_0, t_0 and $\star_0$ satisfy the axioms in such a way that the diagram above defines a category. Explicitly, the following diagrams commute in the category **Set**:





where π_1 and π_2 denote respectively first and second projection.

2.1.4. Product of categories. Given two categories $\mathbf{C}$ and $\mathbf{D}$, the *product category* $\mathbf{C} \times \mathbf{D}$ is defined as follows

i) the 0-cells are the pairs (x, y), where x is a 0-cell of $\mathbf{C}$ and y is a 0-cell of $\mathbf{D}$,

ii) the 1-cells are the pairs (u, v) where u is a 1-cell of $\mathbf{C}$ and v is a 1-cell of $\mathbf{D}$,

iii) the composition is component-wise: $(u, v)(u', v') = (uu', vv')$,

iv) the identities are the pairs of identities: $1_{(x,y)} = (1_x, 1_y)$.

2.1.5. Functors. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. A *functor* $F : \mathbf{C} \to \mathbf{D}$ is a data made of

i) a map $F_0 : \mathbf{C}_0 \to \mathbf{D}_0$,

ii) for every 0-cells x and y of $\mathbf{C}$, a map

$$F_{x,y} \;:\; \mathbf{C}(x, y) \;\to\; \mathbf{D}(F(x), F(y)),$$

such that the following relations are satisfied:

iii) for every 0-cells x, y and z and every 1-cells $u : x \to y$ and $v : y \to z$ of $\mathbf{C}$,

$$F_{x,z}(u \star_0 v) \;=\; F_{x,y}(u) \star_0 F_{y,z}(v),$$

iv) for every 0-cell x of $\mathbf{C}$,

$$F_{x,x}(1_x) \;=\; 1_{F(x)}.$$

We will write $F(x)$ for $F_0(x)$ and $F(u)$ for $F_{x,y}(u)$. A functor F is a *monomorphism* (resp. an *epimorphism*, resp. an *isomorphism*) if the map F_0 and each map $F_{x,y}$ is an injection (resp. a surjection, resp. a bijection).

2.1.6. Functors as morphisms of graphs. A functor $F : \mathbf{C} \to \mathbf{D}$ can be seen as a morphism of graphs

$$\begin{array}{ccc} \mathbf{C}_0 & \overset{s_0}{\underset{t_0}{\longleftleftarrows}} & \mathbf{C}_1 \\ {\scriptstyle F_0}\downarrow & & \downarrow{\scriptstyle F_1} \\ \mathbf{D}_0 & \overset{s_0}{\underset{t_0}{\longleftleftarrows}} & \mathbf{D}_1 \end{array}$$

where, for every 1-cell $u : x \to y$ of $\mathbf{C}$, the 1-cell $F_1(u)$ is defined as $F_{x,y}(u)$.

2.1.7. One-dimensional polygraphs. A 1-*polygraph* is a directed graph Σ, *i.e.*, a diagram of sets and maps

$$\Sigma_0 \overset{s_0}{\underset{t_0}{\longleftleftarrows}} \Sigma_1.$$

The elements of Σ_0 and Σ_1 are called the 0-cells and the 1-cells of Σ, respectively. If there is no confusion, we just write $\Sigma = (\Sigma_0, \Sigma_1)$. Note that the notion of 1-polygraph is equivalent to the notion of abstract rewriting system given in (1.1.1). A 1-polygraph is *finite* if it has finitely many 0-cells and 1-cells.

2.1.8. Free categories. If Σ is a 1-polygraph, the *free category over* Σ is the category denoted by Σ_1^* and defined as follows:

i) the 0-cells of Σ_1^* are the ones of Σ,

ii) the 1-cells of Σ_1^* from x to y are the finite paths of Σ, *i.e.*, the finite sequences

$$x \xrightarrow{u_1} x_1 \xrightarrow{u_2} x_2 \xrightarrow{u_3} \quad \cdots \quad \xrightarrow{u_{n-1}} x_{n-1} \xrightarrow{u_n} y$$

of 1-cells of Σ,

iii) the composition is given by concatenation,

iv) the identities are the empty paths.

If Σ has only one 0-cell, then the 1-cells of the free category Σ_1^* form the free monoid over the set Σ_1.

2.1.9. Generating 1-polygraph. Let $\mathbf{C}$ be a category. A 1-polygraph Σ *generates* $\mathbf{C}$ if there exists an epimorphism

$$\pi : \Sigma_1^* \longrightarrow \mathbf{C}$$

that is the identity on 0-cells. In that case, the 1-polygraph Σ has the same 0-cells as $\mathbf{C}$ and, for every 0-cells x and y of $\mathbf{C}$, the map

$$\pi : \Sigma_1^*(x, y) \longrightarrow \mathbf{C}(x, y)$$

is surjective. A category is *finitely generated* if it admits a finite generating 1-polygraph.

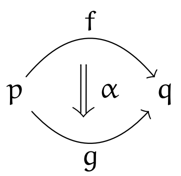
2.1.10. Spheres and cellular extensions of categories. A *sphere* of a category $\mathbf{C}$ is a pair $\gamma = (u, v)$ of parallel 1-cells of $\mathbf{C}$, that is, with the same source, $s_0(u) = s_0(v)$, and the same target, $t_0(u) = t_0(v)$. The 1-cell u is the *source of* γ and v is its *target*. A *cellular extension* of $\mathbf{C}$ is a set Γ equipped with a map from Γ to the set of spheres of $\mathbf{C}$. It is equivalent to the data of a set Γ with two maps

$$\mathbf{C} \underset{t_1}{\overset{s_1}{\leftleftarrows}} \Gamma.$$

satisfying the following *gobular relations*:

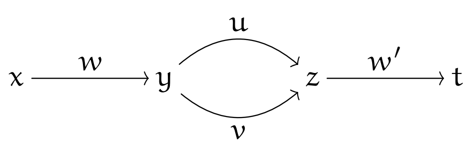
$$s_0 s_1 = s_0 t_1, \qquad t_0 s_1 = t_0 t_1.$$

An element of Γ will be graphically represented by a 2-cell with the following globular shape



that relates parallel 1-cells u and v in $\mathbf{C}$, also denoted by $u \overset{\alpha}{\Rightarrow} v$ or by $\alpha : u \Rightarrow v$.

2.1.11. Congruences. A *congruence* on a category $\mathbf{C}$ is an equivalence relation $\equiv$ on the parallel 1-cells of $\mathbf{C}$ that is compatible with the composition of $\mathbf{C}$, that is, for every 1-cells



of $\mathbf{C}$ such that $u \equiv v$, we have $wuw' \equiv wvw'$. If Γ is a cellular extension of $\mathbf{C}$, the *(Thue) congruence generated by* Γ is denoted by $\equiv_\Gamma$ and defined as the smallest congruence relation such that, if γ is in Γ, then $s_1(\gamma) \equiv_\Gamma t_1(\gamma)$.

2.1.12. Quotient categories. If $\mathbf{C}$ is a category and Γ is a cellular extension of $\mathbf{C}$, the *quotient of* $\mathbf{C}$ *by* Γ is the category denoted by $\mathbf{C}/\Gamma$ and defined as follows:

i) the 0-cells of $\mathbf{C}/\Gamma$ are the ones of $\mathbf{C}$,

ii) for every 0-cells x and y of $\mathbf{C}$, the set $\mathbf{C}/\Gamma(x,y)$ of 1-cell with source x and target y is the quotient of $\mathbf{C}(x,y)$ by the restriction of $\equiv_\Gamma$.

We will denote by

$$\pi_\Gamma : \mathbf{C} \longrightarrow \mathbf{C}/\Gamma$$

the canonical projection. We will denote by $\overline{u}^\Gamma$ for the image through π_Γ of a 1-cell u in $\mathbf{C}$. The superscript Γ in $\overline{u}^\Gamma$ will be omitted whenever ambiguity is not introduced.

2.1.13. Two-dimensional categories. A *(strict)* 2-*category* is a category enriched over the cartesian monoidal category **Cat** of categories. Explicitly, is a data $\mathbf{C}$ made of a set $\mathbf{C}_0$, whose elements are called the 0-*cells of* $\mathbf{C}$, and, for every 0-cells x and y of $\mathbf{C}$, a category $\mathbf{C}(x,y)$, whose 0-cells and 1-cells are respectively called the 1-*cells* and the 2-*cells from* x *to* y *of* $\mathbf{C}$. This data is equipped with the following algebraic structure:

i) for every 0-cells x, y and z of $\mathbf{C}$, a functor

$$\star_0^{x,y,z} \;:\; \mathbf{C}(x,y)\times\mathbf{C}(y,z) \;\to\; \mathbf{C}(x,z),$$

ii) for every 0-cell x, a specified 0-cell 1_x of the category $\mathbf{C}(x,x)$.

The following relations are required to hold:

iii) the composition is associative, *i.e.*, for every 0-cells x, y, z and t,

$$\star_0^{x,z,t}\circ(\star_0^{x,y,z}\times \mathrm{Id}_{\mathbf{C}(z,t)}) \;=\; \star_0^{x,y,t}\circ(\mathrm{Id}_{\mathbf{C}(x,y)}\times\star_0^{y,z,t}),$$

iv) the identities are local units for the composition, *i.e.*, for every 0-cells x and y,

$$\star_0^{x,x,y}\circ(1_x\times \mathrm{Id}_{\mathbf{C}(x,y)}) \;=\; \mathrm{Id}_{\mathbf{C}(x,y)} \;=\; \star_0^{x,y,y}\circ(\mathrm{Id}_{\mathbf{C}(x,y)},1_y).$$

2.1.14. Globular definition. A 2-category can, equivalently, be defined as a 2-graph

$$\mathbf{C}_0 \underset{t_0}{\overset{s_0}{\Longleftarrow}} \mathbf{C}_1 \underset{t_1}{\overset{s_1}{\Longleftarrow}} \mathbf{C}_2$$

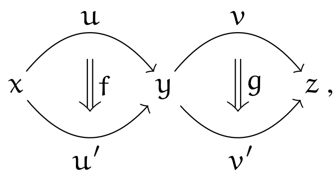
equipped with an additional algebraic structure. The definition of 2-graph requires that the source and target maps satisfy the globular relations:

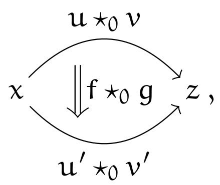
$$s_0\circ s_1 \;=\; s_0\circ t_1 \qquad\text{and}\qquad t_0\circ s_1 \;=\; t_0\circ t_1.$$

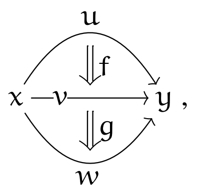
The 2-graph is equipped with two compositions, the 0-*composition* $\star_0$ and the 1-*composition* $\star_1$, respectively defined on 0-composable 1-cells and 2-cells, and on 1-composable 2-cells. We also have an inclusion of $\mathbf{C}_0$ into $\mathbf{C}_1$ given by the identities of the 2-category, and an inclusion of $\mathbf{C}_1$ into $\mathbf{C}_2$ induced by the identities of the hom-categories. Explicitly, we have the following operations:

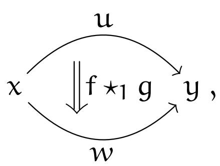
i) for every 1-cells $x \xrightarrow{u} y \xrightarrow{v} z$, a 0-composite 1-cell

$$u \star_0 v : x \to z,$$

ii) for every 2-cells  , a 0-composite 2-cell



iii) for every 2-cells  , a 1-composite 2-cell



iv) for every 0-cell x, an identity 1-cell

$$1_x : x \to x,$$

v) for every 1-cell $x \xrightarrow{u} y$, an identity 2-cell

$$1_u : u \to u.$$

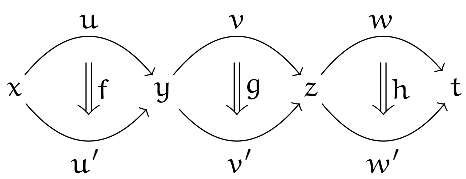
The 0-composition and the 1-composition satisfy the following relations:

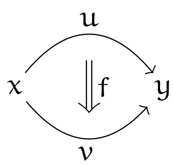
– for every 1-cells $x \xrightarrow{u} y \xrightarrow{v} z \xrightarrow{w} t$, $(u \star_0 v) \star_0 w = u \star_0 (v \star_0 w)$,

– for every 1-cell $x \xrightarrow{u} y$, $1_x \star_0 u = u = u \star_0 1_y$,

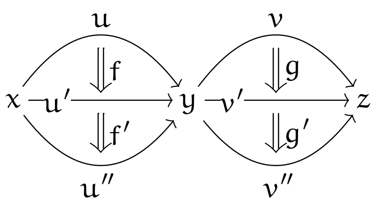
– for every 1-cells $x \xrightarrow{u} y \xrightarrow{v} z$, $1_{u \star_0 v} = 1_u \star_0 1_v$,

– for every 2-cells $u \overset{f}{\Longrightarrow} v \overset{g}{\Longrightarrow} w \overset{h}{\Longrightarrow} x$, $(f \star_1 g) \star_1 h = f \star_1 (g \star_1 h)$,

– for every 2-cells  , $(f \star_0 g) \star_0 h = f \star_0 (g \star_0 h)$,

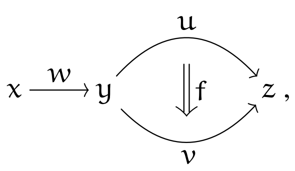
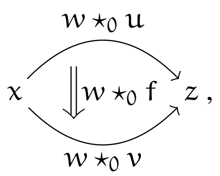
– for every 2-cell  , $1_x \star_0 f = f = f \star_0 1_y$,

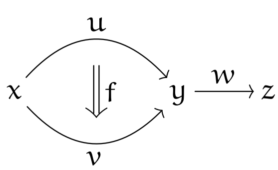
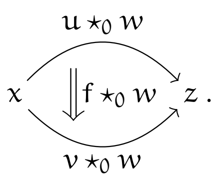
– for every 2-cell $u \overset{f}{\Longrightarrow} v$, $1_u \star_1 f = f = f \star_1 1_v$,

– for every 2-cells  , $(f \star_1 f') \star_0 (g \star_1 g') = (f \star_0 g) \star_1 (f' \star_0 g')$.

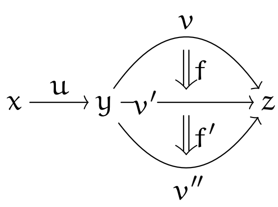
The last relation is usually called the *exchange relation* or the *interchange law* for the compositions $\star_0$ and $\star_1$. This globular definition of 2-categories is equivalent to the enriched one.

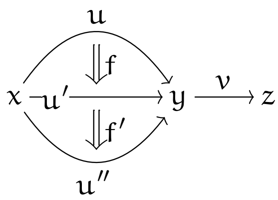
The 0-composition of 2-cells with identity 1-cells defines the *whiskering* operations:

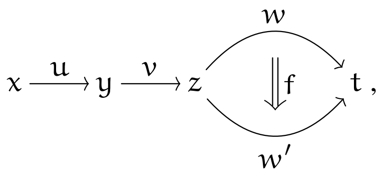
– for every  , the *left whiskering* is  ,

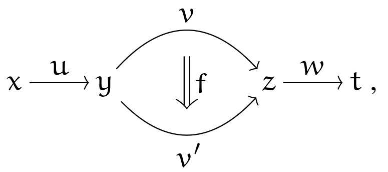
– for every  , the *right whiskering* is  .

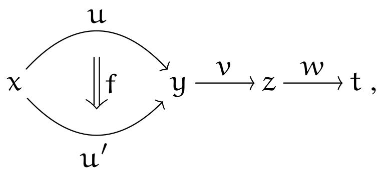
The left and right whiskering operations satisfy the following relations:

– for every  , $u \star_0 (f \star_1 f') = (u \star_0 f) \star_1 (u \star_0 f')$,

– for every  $(f \star_1 f') \star_0 v = (f \star_0 v) \star_1 (f' \star_0 v)$,

– for every  $(u \star_0 v) \star_0 f = u \star_0 (v \star_0 f)$,

– for every  $(u \star_0 f) \star_0 w = u \star_0 (f \star_0 w)$,

– for every  $(f \star_0 v) \star_0 w = f \star_0 (v \star_0 w)$,

As for categories, we usually write uv and fg instead of $u \star_0 v$ and $f \star_0 g$.

2.2. STRING REWRITING SYSTEMS

2.2.1. Two-dimensional polygraphs. A *2-polygraph* is a triple $\Sigma = (\Sigma_0, \Sigma_1, \Sigma_2)$ made of a 1-polygraph (Σ_0, Σ_1), often simply denoted by Σ_1, and a cellular extension Σ_2 of the free category Σ_1^*. In other terms, a 2-polygraph Σ is a 2-graph

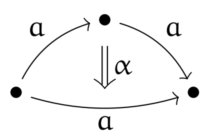
$$\Sigma_0 \overset{s_0}{\underset{t_0}{\leftleftarrows}} \Sigma_1^* \overset{s_1}{\underset{t_1}{\leftleftarrows}} \Sigma_2$$

whose 0-cells and 1-cells form a free category. The elements of Σ_k are called the k*-cells of* Σ and Σ is *finite* if it has finitely many cells in every dimension.

2.2.2. Example. The string rewriting system on the alphabet $\{a\}$ with only one rewriting rule $aa \to a$ is described by the 2-polygraph Σ, where

$$\Sigma_0 = \{\bullet\}, \qquad \Sigma_1 = \{a\}, \qquad \Sigma_2 = \{aa \overset{\alpha}{\Rightarrow} a\}.$$

The rule $aa \to a$ corresponds to the following globular 2-cell



2.2.3. Presentations of categories. If Σ is a 2-polygraph, the *category presented by* Σ is the category denoted by $\overline{\Sigma}$ and defined by

$$\overline{\Sigma} \;=\; \Sigma_1^*/\Sigma_2.$$

If **C** is a category, a *presentation of* **C** is a 2-polygraph Σ such that **C** is isomorphic to $\overline{\Sigma}$. In that case, the 1-cells of Σ are *the generators of* **C**, and the 2-cells of Σ are *the relations of* **C**.

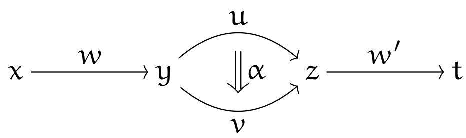
Two 2-polygraphs Σ and Υ are said to be *Tietze-equivalent* if they present isomorphic categories, that is there exists an isomorphism of categories $\overline{\Sigma} \simeq \overline{\Upsilon}$.

2.2.4. Free 2-categories. Let Σ be a 2-polygraph. The *free 2-category over* Σ is the 2-category denoted by Σ_2^* and defined as follows:

i) the 0-cells of Σ_2^* are the ones of Σ,

ii) for every 0-cells x and y of Σ, the category $\Sigma_2^*(x,y)$ is defined as

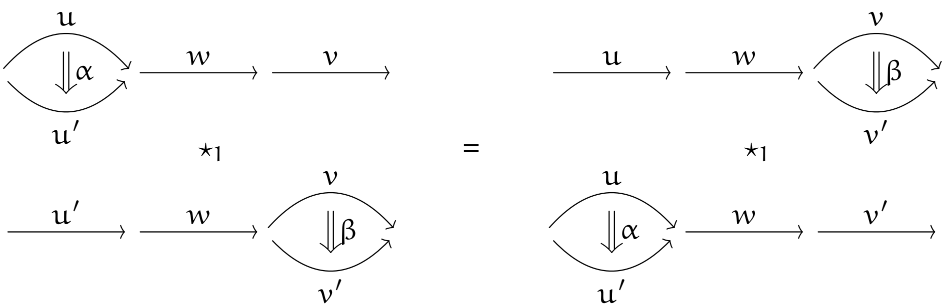
- the free category over the 1-polygraph
  - whose 0-cells are the 1-cells in $\Sigma_1^*(x,y)$,
  - whose 1-cells are the



  with $\alpha : u \Rightarrow v$ in Σ_2 and w and w' in Σ_1^*,
- quotiented by the congruence generated by the cellular extension made of all the possible

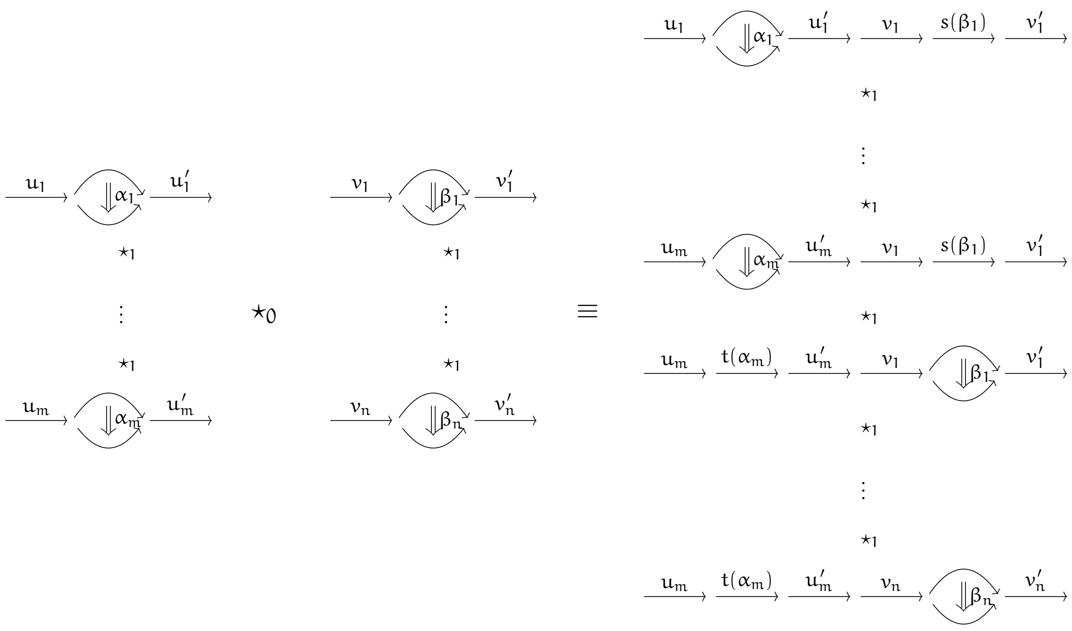
$$\alpha w v \star_1 u' w \beta \;\equiv\; u w \beta \star_1 \alpha w v',$$

for $\alpha : u \Rightarrow u'$ and $\beta : v \Rightarrow v'$ in Σ_2 and w in Σ_1^*:



iii) for every 0-cells x, y and z of Σ the composition functor is given by the concatenation on 1-cells and, on 2-cells, as follows:

$$\begin{aligned}
&(u_1\alpha_1u_1' \star_1 \cdots \star_1 u_m\alpha_m u_m') \;\star_0\; (v_1\beta_1v_1' \star_1 \cdots \star_1 v_n\beta_n v_n')\\
=\;\;& u_1\alpha_1u_1'v_1 s(\beta_1)v_1' \star_1 \cdots \star_1 u_m\alpha_m u_m' v_1 s(\beta_1) v_1'\\
&\star_1 u_m t(\alpha_m) u_m' v_1\beta_1 v_1' \star_1 \cdots \star_1 u_m t(\alpha_m) u_m' v_n \beta_n v_n'
\end{aligned}$$

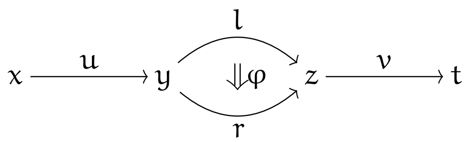


iv) for every 0-cell x, the identity 1-cell 1_x is the one of Σ_1^*.

By definition of the 2-category Σ_2^*, for every 1-cells u and v of Σ_1^*, we have $\overline{u} = \overline{v}$ in the quotient category $\overline{\Sigma}$ if, and only if, there exists a "zig-zag" sequence of 2-cells of Σ_2^* between them:

$$u \overset{f_1}{\Longrightarrow} u_1 \overset{g_1}{\Longleftarrow} v_1 \overset{f_2}{\Longrightarrow} u_2 \Longleftarrow (\cdots) \Longrightarrow u_{n-1} \overset{g_{n-1}}{\Longleftarrow} v_{n-1} \overset{f_n}{\Longrightarrow} u_n \overset{g_n}{\Longleftarrow} v.$$

2.2.5. Rewriting sequences. A *rewriting step* of a 2-polygraph Σ is a 2-cell of the free 2-category Σ_2^* with shape



where $\varphi : l \Rightarrow r$ is a generating 2-cell in Σ and u and v are 1-cells of Σ_1^*. Such a rewriting step will be denoted by $ulv \Rightarrow_{\Sigma_2} urv$. The subscript Σ_2 will be omitted whenever ambiguity is not introduced.

A *rewriting sequence of* Σ is a finite or infinite sequence

$$u_1 \Rightarrow_{\Sigma_2} u_2 \Rightarrow_{\Sigma_2} \ldots \Rightarrow_{\Sigma_2} u_n \Rightarrow_{\Sigma_2} \ldots$$

of rewriting steps. If Σ has a non-empty rewriting sequence from w to w', we say that w *rewrites into* w'. Let us note that every 2-cell f of the 2-category Σ_2^* decomposes into a finite rewriting sequence of Σ, this decomposition being unique up to exchange relations.

2.2.6. Leftmost reduction. Let Σ be a 2-polygraph. A reduction step $w \Rightarrow w'$ is *leftmost*, and we denote $w \Rightarrow^{\ell} w'$, if the two following conditions are satisfied

i) if $w = ulv$ and $w' = urv$ for some $l \Rightarrow r$ in Σ_2 with u and v in Σ_1^*,

ii) for any factorisation $w = u'l'r'$ for some $l' \Rightarrow r'$ in Σ_2, then ul is a proper prefix of $u'l'$ or $ul = u'l'$ and u is a prefix of u'.

2.2.7. Rewriting properties of 2-polygraphs. To any 2-polygraph Σ, we associate an abstract rewriting system whose elements are 1-cells in Σ_1^* and the reduction relations is the relation $\Rightarrow_{\Sigma_2}$. We say that a 2-polygraph has a rewriting property $\mathcal{P}$, such as normalisation, termination or confluence, if the associated abstract rewriting system $(\Sigma_1^*, \Rightarrow_{\Sigma_2})$ has the property $\mathcal{P}$. In particular, a 2-polygraph is confluent if and only if it is Church-Rosser and by Newman's lemma, Theorem 5.5.12, for a terminating 2-polygraph, local confluence and confluence are equivalent properties.

We will denote by Σ_1^{nf} the set of 1-cells of Σ_1^* in normal form with respect to Σ_2.

2.2.8. Theorem. *Let Σ be a terminating 2-polygraph. Then Σ is confluent if and only if the restriction of the canonical projection*

$$\pi : \Sigma_1^* \longrightarrow \overline{\Sigma}$$

to the irreducible 1-cells induces a bijection for any 0-cells x and y:

$$\widetilde{\pi}_{x,y} : \Sigma_1^{nf}(x,y) \xrightarrow{\sim} \overline{\Sigma}(x,y).$$

2.2.9. Exercise. Prove Theorem 2.2.8.

2.2.10. Termination order. A *termination order on* Σ is an order relation $\prec$ on parallel 1-cells of Σ_1^* such that the following three conditions are satisfied:

i) the composition of 1-cells of Σ_1^* is strictly monotone in both arguments, *i.e.*, $u' \prec u$ implies $vu'w \prec vuw$ for all composable 1-cells u, u', v and w in Σ_1^*,

ii) the relation is wellfounded, *i.e.*, every decreasing family $(u_n)_{n\in\mathbb{N}}$ of parallel 1-cells of Σ_1^* is stationary,

iii) for every 2-cell α of Σ_2, the strict inequality $t(\alpha) \prec s(\alpha)$ holds.

As a direct consequence of the definition, if a 2-polygraph admits a termination order, then it terminates.

2.2.11. Lexicographic order. A useful example of termination order is the *left degree-wise lexicographic order* (or *deglex* for short) generated by a given order on the 1-cells of Σ. It is defined by the following strict inequalities, where each x_i and y_j is a 1-cell of Σ:

$$x_1 \cdots x_p \;<\; y_1 \cdots y_q, \qquad \text{if } p < q,$$

$$x_1 \cdots x_{k-1} x_k \cdots x_p \;<\; x_1 \cdots x_{k-1} y_k \cdots y_p, \qquad \text{if } x_k < y_k.$$

The deglex order is total if and only if the original order on the set Σ_1 is total.

2.2.12. Reduced 2-polygraph. A 2-polygraph Σ is

i) *left-reduced* if for any $l \Rightarrow r$ in Σ_2 then l is irreducible with respect to $\Sigma_2 \setminus \{l \Rightarrow r\}$,

ii) *right-reduced* if for any $l \Rightarrow r$ in Σ_2 then r is irreducible with respect to Σ_2,

iii) *reduced* if it is left-reduced and right-reduced.

2.2.13. Exercise [Mét83], [Squ87, Theorem 2.4]. Show that every finite convergent 2-polygraph is Tietze equivalent to a finite reduced convergent 2-polygraph.

2.3. THE WORD PROBLEM

2.3.1. The word problem. The *word problem* for a 2-polygraph Σ is the following decision problem

INSTANCE: two 1-cells u and v in Σ_1^*.

QUESTION: Does $u \overset{*}{\Longleftrightarrow}_{\Sigma_2} v$?

There are finite string rewriting systems for which the word problem is algorithmically unsolvable. Hence, the word problem for finite string rewriting systems is undecidable in general. When a string rewriting system is finite and convergent, then its word problem is decidable by the normal form procedure.

2.3.2. Normal form procedure. Given a convergent 2-polygraph Σ, every 1-cell u of Σ_1^* has a unique normal form, denoted by $\widehat{u}$, so that we have $\overline{u} = \overline{v}$ in $\overline{\Sigma}$ if, and only if, $\widehat{u} = \widehat{v}$ holds in Σ_1^*. This defines a section

$$\overline{\Sigma} \rightarrowtail \Sigma_1^*$$

of the canonical projection $\Sigma_1^* \twoheadrightarrow \overline{\Sigma}$, mapping a 1-cell u of $\overline{\Sigma}$ to the unique normal form of its representative 1-cells in Σ_1^*, still denoted by $\widehat{u}$. As a consequence, a finite and convergent 2-polygraph Σ yields a decision procedure for the word problem of the category $\overline{\Sigma}$ it presents: the *normal-form procedure*:

```
Input: u, v two 1-cells of Σ1*.
begin
  reduce u to its normal form û with respect to Σ2 ;
  reduce v to its normal form v̂ with respect to Σ2 ;
  if û = v̂ then
    Accept
  else
    Reject
  end
end
```

Algorithm 1: Normal form procedure

Note that finiteness is used to test whether a given 1-cell u is a normal form or not, by examination of all the relations and their possible applications on u. Then, the equality $\overline{u} = \overline{v}$ holds in $\overline{\Sigma}$ if, and only if, the equality $\widehat{u} = \widehat{v}$ holds in Σ_1^*.

2.3.3. Complexity of the word problem for a finite 2-polygraph. For a finite convergent 2-polygraph Σ, consider a function $f_\Sigma : \mathbb{N} \to \mathbb{N}$ such that for any 1-cell u in Σ_1^*, the leftmost reduction sequence from u to its normal form contains at most $f_\Sigma(\ell(u))$ many steps. In [Boo82], Book proves that for a finite convergent and reduced 2-polygraph Σ, the normal form for a 1-cell u in Σ_1^* can be computed in time $O(\ell(u) + f_\Sigma(\ell(u)))$. As a consequence, if a 2-polygraph Σ is length-reducing and confluent, then its word problem is decidable in linear time.

2.3.4. Decidability of the word problem and Tietze invariance. It is well-known that the decidability of the word problem is an invariant property of finite presentations of monoids:

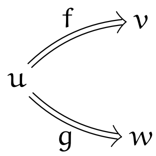
2.3.5. Proposition. *Let Σ and Υ two finite Tietze-equivalent 2-polygraphs. Then the word problem for Σ is decidable if and only if the word problem for Υ is decidable.*

We can thus talk about the decidability of the word problem in a finitely generated monoid. Finally, let us mention the following result obtained by Avenhaus and Madlener in [AM78a, AM78b] for presentations of groups, but the proof can be applied to presentation of monoids.

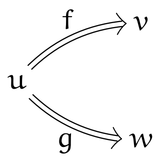
2.3.6. Theorem. *Let Σ and Υ be two Tietze-equivalent finite 2-polygraphs. If the word problem can be decided for Σ in time $O(f(n))$, then the word problem for Υ can be solved in time $O(f(c.n))$ for some constant natural number $c > 0$.*

2.4. Branchings

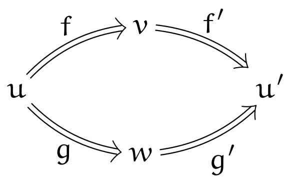
2.4.1. Branchings. Recall from (1.1.5), that a *branching* of Σ is a pair (f, g) of 2-cells of Σ_2^* with a common source, as in the following diagram



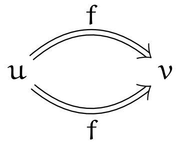
The 1-cell u is the *source* of this branching and the pair (v, w) is its *target*. A branching (f, g) is *local* if f and g are rewriting steps. A branching



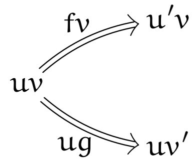
is *confluent* if there exist 2-cells f' and g' in Σ_2^*, as in the following diagram:



2.4.2. Local branchings. Local branchings belong to one of the three following families. The *aspherical* branchings have shape



where f is a rewriting step. The *orthogonal* branchings, also called *Peiffer* branchings, have shape



where $f : u \Rightarrow u'$ and $g : v \Rightarrow v'$ are rewriting steps. The *overlapping* branchings are the remaining local branchings.

2.4.3. Critical branchings. Local branchings are compared by the order $\sqsubseteq$ generated by the relations

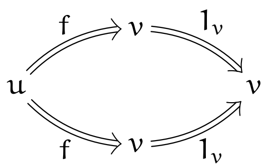
$$(f, g) \;\sqsubseteq\; \big(ufv, ugv\big)$$

given for any local branching (f, g) and any possible 1-cells u and v of Σ_1^*. An overlapping local branching that is minimal for the order $\sqsubseteq$ is called a *critical branching*, or a *critical pair*. Note that a 2-polygraph has two kinds of critical branchings, namely *inclusion* ones and *overlapping* ones, respectively corresponding to the two situations pictured on Figure 2.4.3:

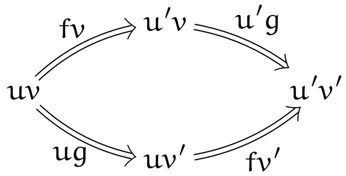
Figure 2.4.3: Critical branchings by inclusion and overlapping

2.4.4. Theorem (Critical pair theorem). *A 2-polygraph is locally confluent if, and only if, all its critical branchings are confluent.*

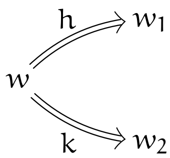
Proof. Every aspherical branching is confluent:



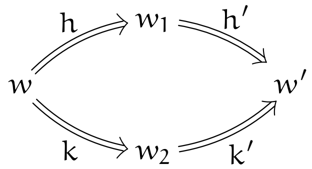
We also have confluence of every Peiffer local branching:



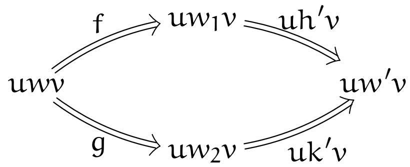
Finally, in the case of an overlapping but not minimal local branching (f, g), there exist factorisations $f = uhv$ and $g = ukv$ with



a critical branching of Σ. By hypothesis, the branching (h, k) is confluent:

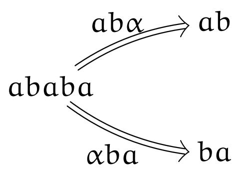


then so is (f, g):



□

2.4.5. Example. Consider the 2-polygraph Σ, with $\Sigma_1 = \{a, b\}$ and $\Sigma_2 = \{\alpha : aba \Rightarrow 1\}$. The 2-polygraph Σ is terminating since $\ell(u) > \ell(v)$ whenever $u \Rightarrow v$. The polygraphs admits one critical branching and this branching is not confluent:

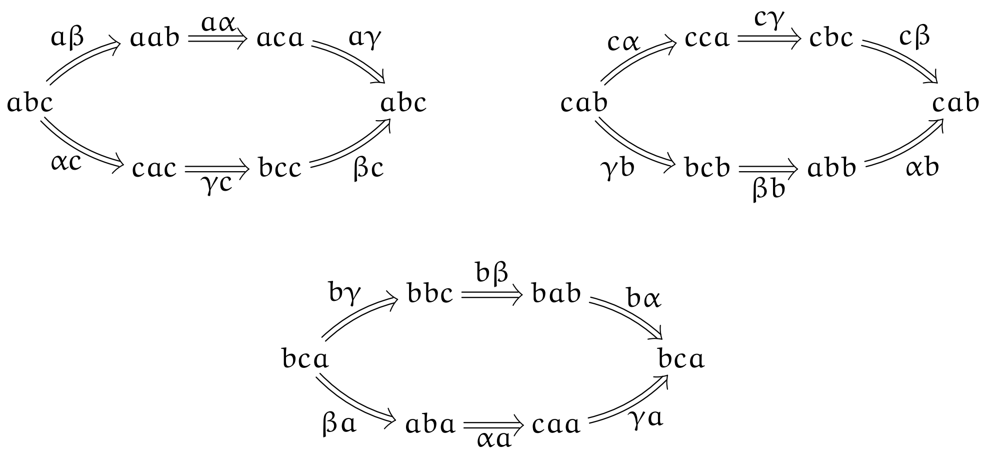


It follows that the 2-polygraph Σ is not confluent.

2.4.6. Example. Consider the 2-polygraph Σ, with $\Sigma_1 = \{a, b, c\}$ and

$$\Sigma_2 = \{\alpha : ab \Rightarrow ca,\ \beta : bc \Rightarrow ab,\ \gamma : ca \Rightarrow bc\}.$$

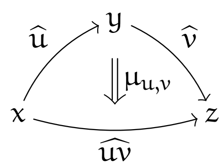
The 2-polygraph Σ is not terminating and local confluent with three confluent critical branchings:



2.4.7. Example: reduced standard presentation. Given a category **C**, we call *reduced standard polygraphic presentation* of **C**, the 2-polygraph Σ defined as follows:

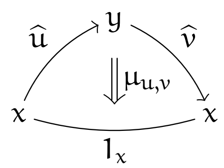
i) it has one 0-cell for each 0-cell of **C** and one 1-cell $\widehat{u} : x \to y$ for every non-identity 1-cell $u : x \to y$ of **C**,

ii) it has one 2-cell



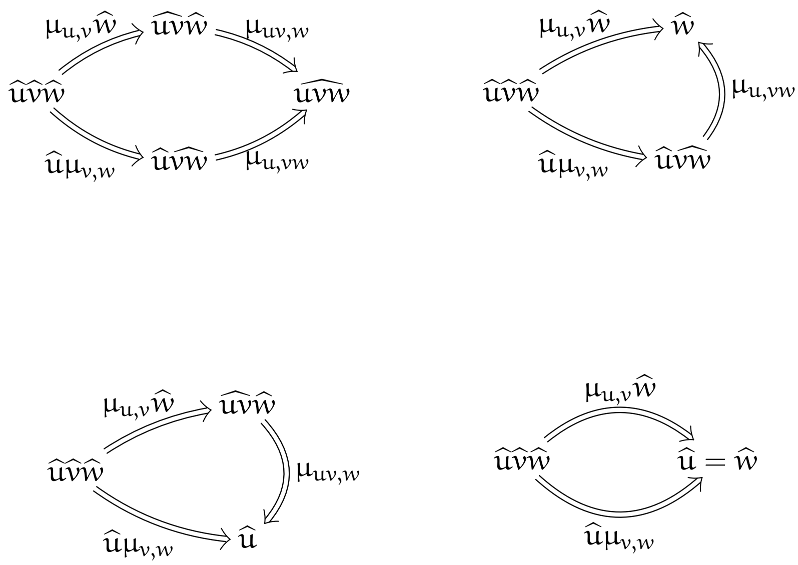
for every non-identity 1-cells $u : x \to y$ and $v : y \to z$ of **C** such that uv is not an identity,

iii) it has one 2-cell



for every non-identity 1-cells $u : x \to y$ and $v : y \to x$ of **C** such that $uv = 1_x$.

The 2-polygraph Σ is reduced and convergent. It has one critical branching $(\mu_{u,v}\widehat{w}, \widehat{u}\mu_{v,w})$ for every triple (u, v, w) of non-identity composable 1-cells in **C**. Each of these critical branchings is confluent, with four possible cases, depending on whether uv or vw is an identity or not:



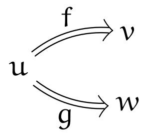
Following Theorem 2.4.4, one can decide whether a finite string rewriting system is convergent by checking confluence of critical branchings. If the set of rules is finite, there are only finitely many critical branchings. It thus can be tested whether every such branching is confluent. The result follows because, the rewriting system is locally confluent if and only if every critical branching is confluent.

2.4.8. Theorem ([Niv73]). *Let Σ be a finite terminating string rewriting system. Then, whether or not Σ is locally confluent, is decidable. Hence, it is decidable whether or not Σ is confluent.*

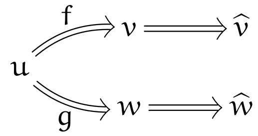
2.5. Completion

2.5.1. Knuth-Bendix's completion procedure. Let Σ be a terminating 2-polygraph, equipped with a total termination order $\prec$. A *Knuth-Bendix's completion of* Σ is a 2-polygraph $\mathcal{KB}(\Sigma)$ obtained by the following procedure.

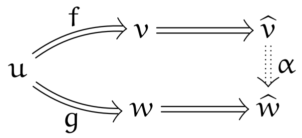
Input: Σ be a terminating 2-polygraph with a total termination order $\prec$.
$\mathcal{KB}(\Sigma) \leftarrow \Sigma$
$\mathcal{C}\mathrm{b} \leftarrow \{$ critical branchings of $\Sigma\}$
while $\mathcal{C}\mathrm{b} \neq \emptyset$ **do**
  Picks a branching in $\mathcal{C}\mathrm{b}$:



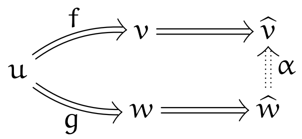
  $\mathcal{C}\mathrm{b} \leftarrow \mathcal{C}\mathrm{b} \setminus \{(f, g)\}$
  Reduce v to a normal form $\widehat{v}$ with respect to $\mathcal{KB}(\Sigma)_2$
  Reduce w to a normal form $\widehat{w}$ with respect to $\mathcal{KB}(\Sigma)_2$



  if $\widehat{v} \neq \widehat{w}$ **then**
    if $\widehat{v} > \widehat{w}$ **then**
      $\mathcal{KB}(\Sigma)_2 \leftarrow \mathcal{KB}(\Sigma)_2 \cup \{\alpha : \widehat{v} \Rightarrow \widehat{w}\}$:



    end
    if $\widehat{w} > \widehat{v}$ **then**
      $\mathcal{KB}(\Sigma)_2 \leftarrow \mathcal{KB}(\Sigma)_2 \cup \{\alpha : \widehat{w} \Rightarrow \widehat{v}\}$:



    end
  end
  $\mathcal{C}\mathrm{b} \leftarrow \mathcal{C}\mathrm{b} \cup \{$ critical branching created by $\alpha\}$
end

Algorithm 2: Knuth-Bendix completion procedure

If the procedure stops, it returns the 2-polygraph $\mathcal{KB}(\Sigma)$. Otherwise, it builds an increasing sequence of 2-polygraphs, whose limit is denoted by $\mathcal{KB}(\Sigma)$. Note that, if the starting 2-polygraph Σ is already convergent, then the Knuth-Bendix's completion of Σ is Σ.

2.5.2. Theorem ([KB70]). *The Knuth-Bendix's completion $\mathcal{KB}(\Sigma)$ of a 2-polygraph Σ is a convergent presentation of the category $\overline{\Sigma}$. Moreover, the 2-polygraph $\mathcal{KB}(\Sigma)$ is finite if, and only if, the 2-polygraph Σ is finite and if the Knuth-Bendix's completion procedure halts.*

2.5.3. Exercice. Find a finite convergent presentation of the monoid generated by two generators a and b and submitted to the relation $aba = 1$.

2.6. Existence of finite convergent presentations

When a string rewriting system is not convergent, one wishes to determine whether there exists a Tietze equivalent convergent string rewriting system. We can formulate the two following problems of existence of finite convergent presentations.

2.6.1. Problem.

INSTANCE: A finite string rewriting system (Σ_1, Σ_2).

QUESTION: Does (Σ_1, Σ_2) is Tietze equivalent to a finite convergent string rewriting system (Σ_1, Υ_2) ?

2.6.2. Problem.

INSTANCE: A finite string rewriting system (Σ_1, Σ_2).

QUESTION: Does (Σ_1, Σ_2) is Tietze equivalent to a finite convergent string rewriting system ?

2.6.3. Theorem ([BO84]). *The problems 2.6.1 and 2.6.2 are undecidable.*

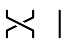
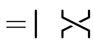
2.6.4. Existence of finite convergent presentations. The normal form procedure proves that, if a monoid admits a finite convergent presentation, then it has a decidable word problem. The converse implication was still an open problem in the middle of the eighties. Jantzen in [Jan82, Jan85] asked the following question.

2.6.5. Question. Does every finitely presented monoid with a decidable word problem admit a finite convergent presentation?

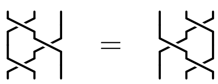
2.6.6. Example. In [KN85], Kapur and Narendran consider Artin's presentation of the monoid $\mathbf{B}_3^+$ of positive braids on three strands:

$$\Sigma = \langle\, s, t \mid sts \Rightarrow tst \,\rangle.$$

The generators s and t correspond to the following braids

$s =$  and $t =$ 

and the rule $sts \Rightarrow tst$ corresponds to the Yang-Baxter relation:



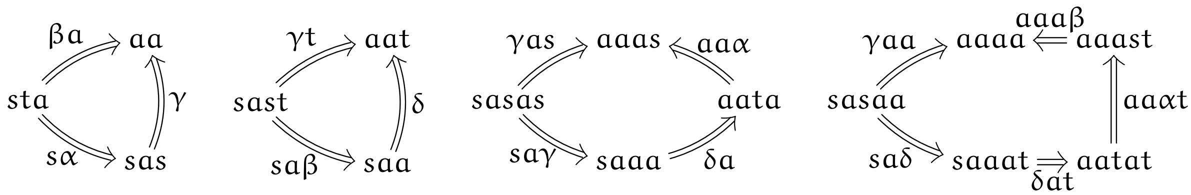
They proved that the word problem for $\mathbf{B}_3^+$ is decidable and that this monoid admits no finite convergent presentation on the two generators s and t. However, Bauer and Otto, [BO84], have found a finite convergent presentation of the monoid $\mathbf{B}_3^+$ by adjunction of a new generator a standing for the product st:

$$\Gamma = \langle\, s, t, a \mid \alpha : ta \Rightarrow as,\ \beta : st \Rightarrow a \,\rangle.$$

Indeed, this rewriting system can be completed by applying the *Knuth-Bendix completion procedure*, [KB70], into the following convergent presentation

$$\mathcal{KB}(\Gamma) = \langle\, s, t, a \mid \alpha : ta \Rightarrow as,\ \beta : st \Rightarrow a,\ \gamma : sas \Rightarrow aa,\ \delta : saa \Rightarrow aat \,\rangle \tag{2.6.7}$$

with the following four critical branchings:



(2.6.8)

As a consequence, the word problem for $\mathbf{B}_3^+$ is solvable by the normal form algorithm. The result of Kapur and Narendran shows that the existence of a finite convergent presentation depends on the specific presentation of the monoid, in particular on the chosen generators. In their example, by adding new letters in the alphabet it is possible to obtain a finite convergent string rewriting system. However, is it always possible to obtain such Tietze equivalent system by adding a finite set of letters? Thus, to provide the awaited negative answer to the open question, one would have to exhibit a monoid with a decidable word problem but with no finite convergent presentation on any possible set of generators.

2.6.9. Question. Which condition a monoid need to satisfy to admit a presentation by a finite convergent rewriting system?

Diekert solved the problem for the case of abelian groups. He derived a whole class of finite string rewriting systems presenting abelian groups with a decidable word problem which are not Tietze equivalent to finite convergent string rewriting systems on the same alphabet, [Die86]. Moreover, he constructed a necessary and sufficient conditions for the existence of convergent presentation for finitely generated abelian groups. However, the problem for general monoids was still open. At this point, new methods had to be introduced for a problem which seems to concern intrinsic properties of the presented monoid.

2.6.10. Exercise. Compute a convergent presentation of the monoid $\mathbf{B}_3^+$ with two generating 1-cells.

2.6.11. Exercise, [KN85]. Consider the monoid $\mathbf{B}_3^+$ of positive braids on three strands and the Artin's presentation $\Sigma = \langle s, t \mid \gamma : sts \Rightarrow tst \rangle$.

1. Show that the word problem is decidable for $\mathbf{B}_3^+$.

2. Show that for any $i \geqslant 0$ and any $j \geqslant 0$, the words $s^{i+1}t^{j+2}st$ and $tst^{i+2}s^{j+1}$ are equals in $\mathbf{B}_3^+$.

3. Denote by $[w]$ the equivalence class modulo the relation γ containing the word w. Prove that for any $n > 0$ the two following equalities hold

$$[t^n st] = \{\, t^{n-i} sts^i \mid 0 \leqslant i \leqslant n \,\}.$$
$$[tst^n] = \{\, s^j tst^{n-j} \mid 0 \leqslant j \leqslant n \,\}.$$

4. Show that there does not exist any finite convergent presentation of the monoid $\mathbf{B}_3^+$ with two generators s and t.

2.6.12. Example: plactic monoid. The structure of plactic monoids appeared in the combinatorial study of Young tableaux by Schensted [Sch61] and Knuth [Knu70]. The *plactic monoid* of rank $n > 0$, denoted by $\mathbf{P}_n$, is generated by the set $\{1, \ldots, n\}$ and subject to the *Knuth relations*:

$$zxy = xzy \quad \text{for} \quad 1 \leqslant x \leqslant y < z \leqslant n, \qquad yzx = yxz \quad \text{for} \quad 1 \leqslant x < y \leqslant z \leqslant n.$$

For instance, the monoid $\mathbf{P}_2$ is generated by $\{1, 2\}$ and submitted to the relations $211 = 121$ and $221 = 212$. These relations can be oriented with respect to the lexicographic order as follows

$$\eta_{1,1,2} : 211 \Rightarrow 121 \qquad \varepsilon_{1,2,2} : 221 \Rightarrow 212.$$

In this way, the Knuth presentation of the monoid $\mathbf{P}_2$ is convergent with a unique critical branching:

$$2211 \underset{\varepsilon_{1,2,2}1}{\overset{2\eta_{1,1,2}}{\rightrightarrows}} 2121 \; .$$

With respect to the lexicographic order, the Knuth presentation of the monoid $\mathbf{P}_3$ is not convergent, but it can be completed by adding 3 relations to get a convergent presentation with 27 critical branchings. For the monoid $\mathbf{P}_4$ we have 4 generators and 20 relations, and its completion is infinite. More generally, Kubat and Okniński showed in [KO14] that for rank $n > 3$, a finite convergent presentation of the monoid $\mathbf{P}_n$ cannot be obtained by completion of the Knuth presentation with the degree lexicographic order. Bokut, Chen, Chen and Li in [BCCL15], Cain, Gray and Malheiro in [CGM15], and Hage in [Hag15] for type C, constructed with independent methods a finite convergent presentation by adding column generators to the Knuth presentation.

CHAPTER
3

Coherent presentations and syzygies

Contents

The notion of *coherent presentation* extends those of presentation of a category by globular homotopy generators taking into account the relations amongst the relations. In this chapter we show how to compute a coherent presentation for a category using the completion procedure introduced in the previous chapter. The method follows a construction introduced by Squier in [SOK94] in his homotopical and homological study of finiteness conditions for finite convergence of finitely presented monoids. Many constructions presented in this chapter come from [GM18].

3.1. Introduction

3.1.1. Syzygies of Knuth's relations. Recall from (2.6.12), that for $n > 0$, the plactic monoid $\mathbf{P}_n$ is generated by the set $\{1, \ldots, n\}$ and subject to the Knuth relations:

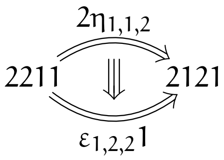
$$zxy = xzy \quad \text{for} \quad 1 \leqslant x \leqslant y < z \leqslant n, \qquad yzx = yxz \quad \text{for} \quad 1 \leqslant x < y \leqslant z \leqslant n.$$

We consider the problem of finding all independent irreducible algebraic relations amongst these relations. Such a relation is called a 2-*syzygy*, and we aim to to give an algorithmic method that computes all

2-syzygies of the presentation, and in particular a family of generators for these syzygies. For instance, the monoid $\mathbf{P}_2$ is generated by $\{1,2\}$ and submitted to the relations

$$\eta_{1,1,2} : 211 \Rightarrow 121, \qquad \varepsilon_{1,2,2} : 221 \Rightarrow 212.$$

There are two ways to prove the equality $2211 = 2121$ in the monoid $\mathbf{P}_2$, either by applying the first relation or the second relation. This two equalities are related by a syzygy:



We will prove that this syzygy generates all the syzygies of the above presentation. The proof is based on a categorical description of syzygies of such a presentation, and an extension of the Knuth-Bendix completion procedure given in (2.5.1), by keeping track of syzygies created when adding rules during the completion. The correctness of the procedure follows the coherent Squier theorem, [SOK94], which states that a convergent presentation of a monoid extended by the homotopy generators defined by the confluence diagrams induced by critical branchings forms a coherent convergent presentation.

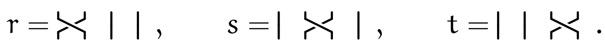
3.1.2. Positive braid monoids. Let us illustrate the notion of syzygy on the presentation of the braid monoid $\mathbf{B}_3^+$ studied in (2.6.6):

$$\langle\ s, t \ \mid\ \alpha : sts \Rightarrow tst\ \rangle.$$

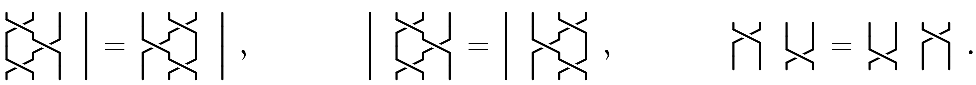
One proves that there is no nontrivial syzygy amongst the relations induce by the rule α. Now consider the braid monoid $\mathbf{B}_4^+$ on four strands with the following presentation:

$$\langle\ r, s, t \mid rsr = srs,\ sts = tst,\ rt = tr\ \rangle.$$

The generators corresponds to the following generating braids on four strands:

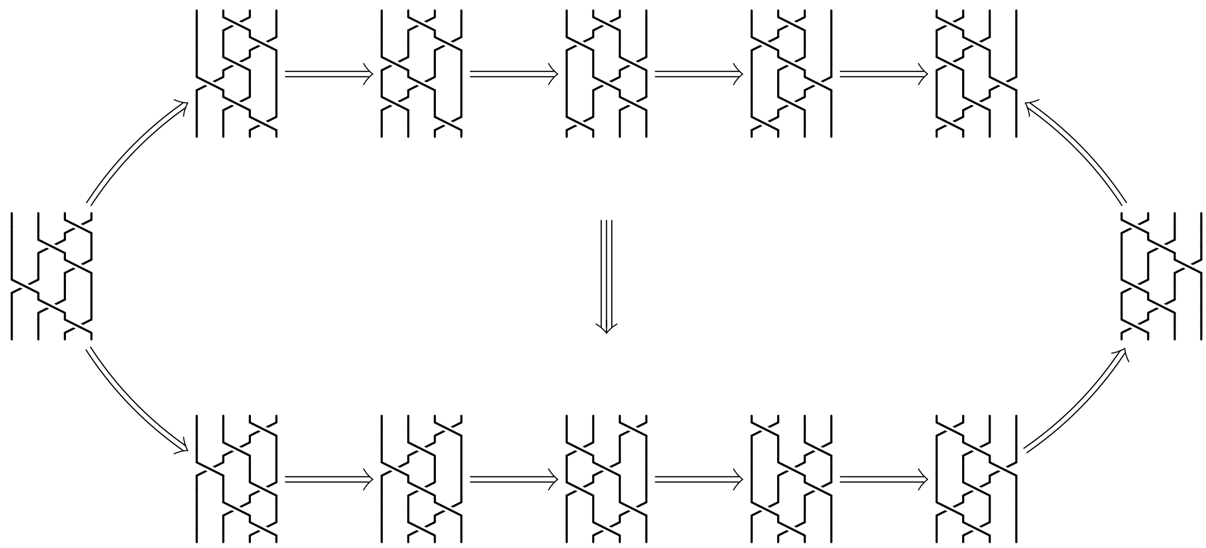


so that the relations read as follows:



In that case, one proves [Del97, GGM15], that all the syzygies are generated by the following

Zamolodchikov relation:



3.2. Categorical preliminaries

In this section, we recall the notion of 2-functor, $(2,1)$-category, and 3-category used in this chapter.

3.2.1. Two-dimensional functors. In (2.1.13), we have introduced the notion of 2-category as a category enriched over the cartesian monoidal category **Cat** of categories. A *(strict)* 2*-functor* between 2-categories is a functor enriched in categories. Explicitly, given two 2-categories $\mathbf{C}$ and $\mathbf{D}$. A 2*-functor* $F : \mathbf{C} \to \mathbf{D}$ is a data made of

i) a map $F_0 : \mathbf{C}_0 \to \mathbf{D}_0$,

ii) for every 0-cells x and y of $\mathbf{C}_0$, a functor

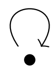
$$F_{x,y} \;:\; \mathbf{C}(x,y) \;\to\; \mathbf{D}(F_0(x), F_0(y)),$$

such that the following diagrams commute in the category **Cat**, for every 0-cells x, y, z in $\mathbf{C}_0$

$$\begin{array}{ccc} \mathbf{C}(x,y) \times \mathbf{C}(y,z) & \xrightarrow{\;\star_0^{x,y,z}\;} & \mathbf{C}(x,z) \\ {\scriptstyle F_{x,y} \times F_{y,z}} \downarrow & & \downarrow {\scriptstyle F_{x,z}} \\ \mathbf{D}(F_0(x), F_0(y)) \times \mathbf{D}(F_0(y), F_0(z)) & \xrightarrow[\star_0^{F_0(x),F_0(y),F_0(z)}]{} & \mathbf{D}(F_0(x), F_0(z)) \end{array}$$

$$\begin{array}{ccc} & \mathbf{1} & \\ \swarrow & & \searrow \\ \mathbf{C}(x,x) & \xrightarrow[F_{x,x}]{} & \mathbf{D}(F_0(x), F_0(x)) \end{array}$$

where **1** denotes the *terminal category*:



with a single 0-cell and a single 1-cell, and the downward arrows map the single 0-cell on identities 1-cells.

If there is no possible confusion, we will write $F(x)$ for $F_0(x)$ and $F(u)$ (resp. $F(\alpha)$) for $F_{x,y}(u)$ (resp. $F_{x,y}(\alpha)$), where u and α are 1-cells and 2-cells of **C** respectively.

The 2-categories and their 2-functors form a category that we will denote by 2**Cat**.

3.2.2. $(2,1)$-categories. A *(small) groupoid*, or $(1,0)$*-category*, is a 1-category **C** in which all 1-cells are isomorphisms, that is there is an *inverse* map $(-)^{-1} : \mathbf{C}_1 \to \mathbf{C}_1$ such that for any 1-cell u in $\mathbf{C}_1$, the following conditions hold:

$$uu^- = 1_{s_0(u)}, \qquad u^-u = 1_{t_0(u)}.$$

A $(2,1)$*-category* is a category enriched over the cartesian monoidal category **Gpd** of groupoids. That is, it is a 2-category $\mathbf{C}_2$, whose 2-cells are invertible for the 1-composition: for any 2-cell $f : u \Rightarrow v$, there exists a 2-cell $f^{-1} : v \Rightarrow u$, such that

$$f \star_1 f^- = 1_u, \qquad f^- \star_1 f = 1_v.$$

3.2.3. Free $(2,1)$-category. Given a 2-polygraph Σ, the *free* $(2,1)$*-category over* Σ is denoted by $\Sigma_2^\top$ and defined as the free 2-category generated by Σ, and whose every 2-cell is invertible. Explicitly, its set of 0-cells is Σ_0 and, for all 0-cells x and y, the groupoid $\Sigma_2^\top(x,y)$ is given as the quotient

$$\Sigma_2^\top(x,y) \;=\; \big(\Sigma \amalg \Sigma^-\big)^*(x,y) \big/ \mathrm{Inv}(\Sigma_2),$$

where:

i) the 2-polygraph Σ^- is defined from Σ by reversing its 2-cells, that is

$$\Sigma_2^- \;=\; \{\, t_1(\alpha) \Rightarrow s_1(\alpha) \mid \alpha \in \Sigma_2 \,\},$$

ii) the cellular extension $\mathrm{Inv}(\Sigma_2)$ contains the following two relations for every 2-cell α of Σ and all possible 1-cells u and v of Σ_1^* such that $s(u) = x$ and $t(v) = y$:

$$u\alpha v \star_1 u\alpha^- v \;\equiv\; 1_{us(\alpha)v} \qquad \text{and} \qquad u\alpha^- v \star_1 u\alpha v \;\equiv\; 1_{ut(\alpha)v}.$$

By definition of the $(2,1)$-category $\Sigma_2^\top$, for all 1-cells u and v of Σ_1^*, we have $\overline{u} = \overline{v}$ in the quotient category $\overline{\Sigma}$ if, and only if, there exists a 2-cell $f : u \Rightarrow v$ in the $(2,1)$-category $\Sigma_2^\top$.

3.2.4. Lemma. *Let* **C** *be a category and let* Σ *and* Υ *be two 2-polygraphs that present* **C**. *There exist two 2-functors*

$$F : \Sigma_2^\top \to \Upsilon_2^\top \qquad \textit{and} \qquad G : \Upsilon_2^\top \to \Sigma_2^\top$$

and, there exist two families of 2-cells

$$\big(\sigma_u : GF(u) \Rightarrow u\big)_{u \in \Sigma_1^*} \qquad \textit{and} \qquad \big(\tau_v : FG(v) \Rightarrow v\big)_{v \in \Upsilon_1^*}$$

in $\Sigma_2^\top$ *and* $\Upsilon_2^\top$ *respectively, such that the following conditions are satisfied:*

i) *the* 2*-functors* F *and* G *induce the identity through the canonical projections onto* $\mathbf{C}$*, that is the two following diagrams commute*

$$\begin{array}{ccc} \Sigma_2^\top & \xrightarrow{\pi_\Sigma} & \mathbf{C} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}_\mathbf{C}} \\ \Upsilon_2^\top & \xrightarrow[\pi_\Upsilon]{} & \mathbf{C} \end{array} \qquad\qquad \begin{array}{ccc} \Sigma_2^\top & \xrightarrow{\pi_\Sigma} & \mathbf{C} \\ {\scriptstyle G}\uparrow & & \uparrow{\scriptstyle \mathrm{Id}_\mathbf{C}} \\ \Upsilon_2^\top & \xrightarrow[\pi_\Upsilon]{} & \mathbf{C} \end{array}$$

ii) *the* 2*-cells* σ_u *and* τ_v *are functorial in* u *and* v*, that is*

$$\sigma_{uu'} = \sigma_u \sigma_{u'}, \qquad \sigma_{1_x} = 1_{1_x},$$

for any 1*-cells* u *and* u' *and* 0*-cell* x *and*

$$\tau_{vv'} = \tau_v \tau_{v'}, \qquad \tau_{1_y} = 1_{1_y},$$

for any 1*-cells* v *and* v' *and* 0*-cell* y.

Proof. We prove the existence of the functor F. The proof of the existence of the functor G is similar. For a 0-cell x, we set $F(x) = x$. If $a : x \to y$ is a 1-cell of Σ, we choose, in an arbitrary way, a 1-cell $F(a) : x \to y$ in Υ_1^* such that $\pi_\Upsilon F(a) = \pi_\Sigma(a)$. Then, we extend F to every 1-cell of Σ_1^* by functoriality. Let $\alpha : u \Rightarrow u'$ be a 2-cell of Σ_2. Since Σ is a presentation of the category $\mathbf{C}$, we have $\pi_\Sigma(u) = \pi_\Sigma(u')$, so that $\pi_\Upsilon F(u) = \pi_\Upsilon F(u')$ holds. Using the fact that Υ is a presentation of the category $\mathbf{C}$, we arbitrarily choose a 2-cell $F(\alpha) : F(u) \Rightarrow F(u')$ in the $(2,1)$-category $\Upsilon_2^\top$. Then, we extend F to every 2-cell of $\Sigma_2^\top$ by functoriality.

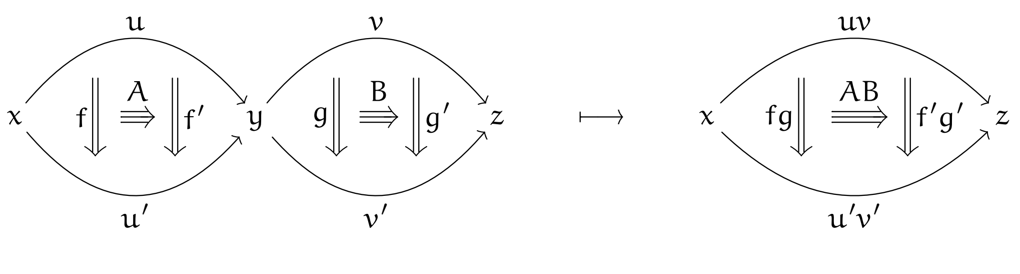
Now, let us define σ, the case of τ being symmetric. Let a be a 1-cell of Σ. By construction of F and G, we have:

$$\pi_\Sigma GF(a) \;=\; \pi_\Upsilon F(a) \;=\; \pi_\Sigma(a).$$

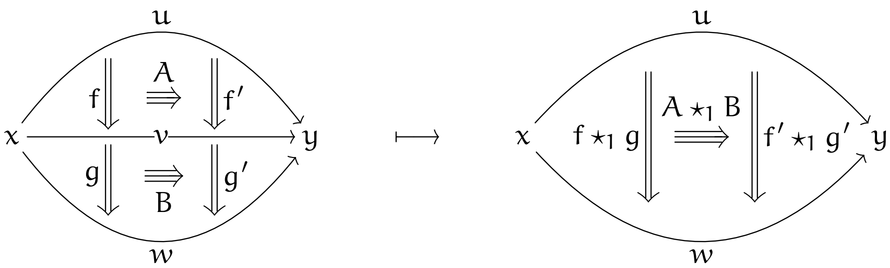
Since Σ is a presentation of $\mathbf{C}$, there exists a 2-cell $\sigma_a : GF(a) \Rightarrow a$ in $\Sigma_2^\top$. We extend σ to every 1-cell u of $\Sigma_2^\top$ by functoriality. □

3.2.5. 3**-categories.** The notion of 3-category is defined as the one of 2-category but by replacement of the hom-categories and the composition functors by hom-2-categories and composition 2-functors. A *(strict)* 3*-category* is a category enriched in the category 2**Cat** of 2-categories. In particular, in a 3-category, the 3-cells can be composed in three different ways:

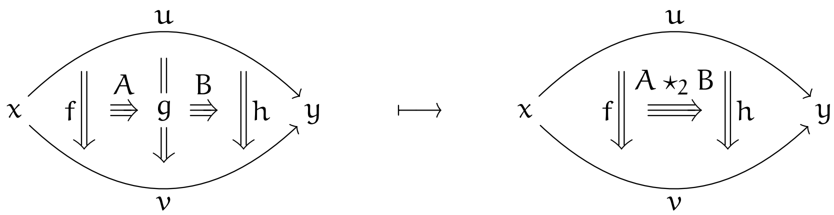
i) by $\star_0$, along their 0-dimensional boundary:



ii) by $\star_1$, along their 1-dimensional boundary:



iii) by $\star_2$, along their 2-dimensional boundary:



The compositions in a 3-category satisfy the *exchange relation*, for every $0 \leqslant i < j \leqslant 2$:

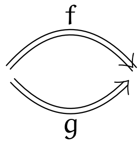
$$(A \star_i B) \star_j (A' \star_i B') = (A \star_j A') \star_i (B \star_j B').$$

3.2.6. $(3,1)$-categories. A $(3,1)$-*category* is a 3-category whose 2-cells are invertible for the composition $\star_1$ and whose 3-cells are invertible for the composition $\star_2$.

3.2.7. Exercise. Show that in a $(3,1)$-category, all the 3-cells are invertible for the composition $\star_1$.

3.3. Coherent presentation of categories

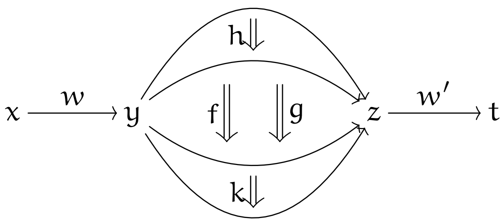
3.3.1. Cellular extension of 2-categories. Let $\mathbf{C}$ be a 2-category. A 2-*sphere* of $\mathbf{C}$ is a pair (f,g) of parallel 2-cells of $\mathbf{C}$, that is such that $s_1(f) = s_1(g)$ and $t_1(f) = t_1(g)$:



A *cellular extension* of the 2-category $\mathbf{C}$ is a set Γ equipped with a map from Γ to the set of 2-spheres of $\mathbf{C}$. It is equivalent to the data of a set Γ with two maps

$$\mathbf{C}_2 \overset{s_2}{\underset{t_2}{\leftleftarrows}} \Gamma$$

satisfying the globular relations $s_1 s_2 = s_1 t_2$ and $t_1 s_2 = t_1 t_2$. A congruence on the 2-category **C** is an equivalence relation $\equiv$ on the parallel 2-cells of **C** such that, for every cells



of **C**, if $f \equiv g$, then

$$w \star_0 (h \star_1 f \star_1 k) \star_0 w' \;\equiv\; w \star_0 (h \star_1 g \star_1 k) \star_0 w'.$$

If Γ is a cellular extension of **C**, the congruence generated by Γ is denoted by $\equiv_\Gamma$ and defined as the smallest congruence such that, if Γ contains a 3-cell $\gamma : f \Rrightarrow g$, then $f \equiv_\Gamma g$. The quotient 2-category of a 2-category **C** by a congruence relation $\equiv$ is the 2-category, denoted by $\mathbf{C}/\equiv$, whose 0-cells and 1-cells are those of **C** and the 2-cells are the equivalence classes of 2-cells of **C** modulo the congruence $\equiv$.

3.3.2. Acyclicity. A cellular extension Γ of a 2-category **C** is called *acyclic* if for every parallel 2-cells f and g of **C**, we have $f \equiv_\Gamma g$, that is, the equality $\overline{f} = \overline{g}$ holds in the quotient 2-category $\mathbf{C}/\equiv_\Gamma$. For instance, the set of 2-spheres of **C** forms an acyclic extension of **C**. In the literature, an acyclic extension of **C** is also called an *homotopy basis* of **C**.

3.3.3. $(3,1)$-polygraphs. A $(3,1)$-polygraph is a data $(\Sigma_0, \Sigma_1, \Sigma_2, \Sigma_3)$ made of a 2-polygraph $(\Sigma_0, \Sigma_1, \Sigma_2)$ and a cellular extension Σ_3 of the free $(2,1)$-category $\Sigma_2^\top$ over Σ_2, as summarised in the following diagram:

$$\Sigma_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} \Sigma_1^* \underset{t_1}{\overset{s_1}{\leftleftarrows}} \Sigma_2^\top \underset{t_2}{\overset{s_2}{\leftleftarrows}} \Gamma_3$$

3.3.4. Coherent presentations. A *coherent presentation* of a 1-category **C** is a $(3,1)$-polygraph $(\Sigma_0, \Sigma_1, \Sigma_2, \Sigma_3)$ such that the 2-polygraph $(\Sigma_0, \Sigma_1, \Sigma_2)$ is a presentation of **C** and Σ_3 is an acyclic cellular extension of the $(2,1)$-category $\Sigma_2^\top$.

3.3.5. Free $(3,1)$-categories. Given a $(3,1)$-polygraph Σ, the *free* $(3,1)$*-category over* Σ is denoted by $\Sigma_3^\top$ and defined as follows:

i) its underlying 2-category is the free $(2,1)$-category $\Sigma_2^\top$,

ii) its 3-cells are all the formal compositions by $\star_0$, $\star_1$ and $\star_2$ of 3-cells of Σ_3, of their inverses and of identities of 2-cells, up to associativity, identity, exchange and inverse relations.

In particular, we get that Σ_3 is an acyclic extension of $\Sigma_2^\top$ if, and only if, for every pair (f, g) of parallel 2-cells of $\Sigma_2^\top$, there exists a 3-cell $A : f \Rrightarrow g$ in $\Sigma_3^\top$.

3.4. FINITE DERIVATION TYPE

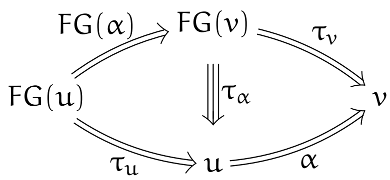
3.4.1. 2-polygraphs of finite derivation type. A 2-polygraph Σ is of *finite derivation type*, FDT for short, if it is finite and if the free $(2,1)$-category $\Sigma_2^\top$ admits a finite acyclic cellular extension. A category $\mathbf{C}$ is said to be of *finite derivation type* if it admits a finite coherent presentation. Let us prove now that this property does not depend on this presentation provide is finite. The proof is based on the following theorem, that allows transfers of acyclic cellular extensions of two $(2,1)$-categories that present the same category.

3.4.2. Homotopy bases transfer theorem. Given a category $\mathbf{C}$ a category, we consider two presentations Σ and Υ of $\mathbf{C}$. By Lemma 3.2.4, there exist 2-functors

$$F : \Sigma_2^\top \to \Upsilon_2^\top \qquad \text{and} \qquad G : \Upsilon_2^\top \to \Sigma_2^\top$$

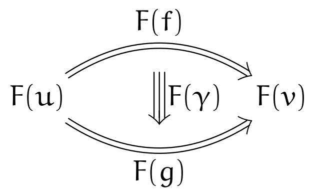
and for every 1-cell v of Υ_2^*, there exists a 2-cell $\tau_v : FG(v) \Rightarrow v$ in $\Upsilon_2^\top$ that satisfy the conditions given in Lemma 3.2.4.

Let define the cellular extension τ_Υ of the $(2,1)$-category $\Upsilon_2^\top$ that contains one 3-cell



for every 2-cell $\alpha : u \Rightarrow v$ of Υ_2.

Given a cellular extension Γ of the $(2,1)$-category $\Sigma_2^\top$, we denote by $F(\Gamma)$ the cellular extension of $\Sigma_2^\top$ that contains one 3-cell



for every 3-cell $\gamma : f \Rrightarrow g$ of Γ.

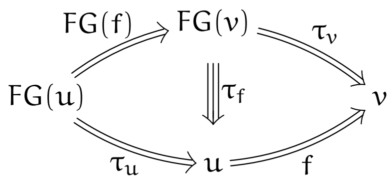
Using these notations, we can formulate the following result, called the *acyclicity transfer theorem* in [GM18].

3.4.3. Theorem. *If Γ is an acyclic cellular extension of the $(2,1)$-category $\Sigma_2^\top$, then the cellular extension*

$$\Delta = F(\Gamma) \sqcup \tau_\Upsilon$$

is an acyclic cellular extension of the $(2,1)$-category $\Upsilon_2^\top$.

Proof. Let us define, for every 2-cell $f : u \Rightarrow v$ of $\Upsilon_2^\top$, a 3-cell τ_f of the free $(3,1)$-category $\Delta_3^\top$ with the following shape:



We extend the notation τ_α, where α is a 2-cell of Υ_2 in a functorial way, according to the following formulas:

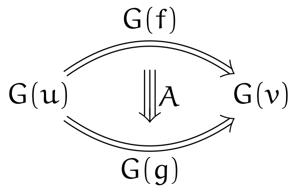
$$\tau_{1_u} \;=\; 1_{\tau_u}, \qquad \tau_{fg} \;=\; \tau_f \tau_g, \qquad \tau_{f^-} \;=\; FG(f)^- \star_1 \tau_f^- \star_1 f^-,$$

$$\tau_{f \star_1 g} \;=\; \big(FG(f) \star_1 \tau_g\big) \star_2 \big(\tau_f \star_1 g\big).$$

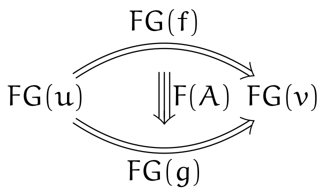
One checks that the 3-cells τ_f are well-defined, *i.e.*, that their definition is compatible with the relations on 2-cells, such as the exchange relation:

$$\tau_{fg \star_1 hk} \;=\; \tau_{(f \star_1 h)(g \star_1 k)}.$$

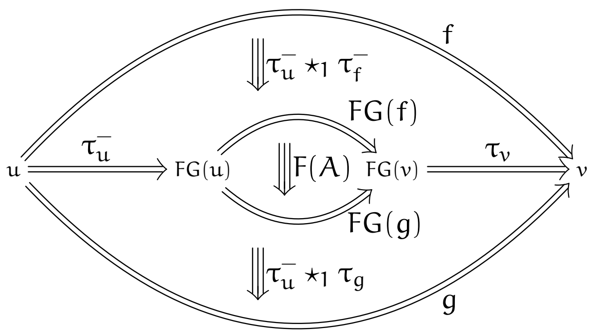
Now, let us consider parallel 2-cells $f, g : u \Rightarrow v$ of $\Upsilon_2^\top$. The 2-cells $G(f)$ and $G(g)$ are parallel in $\Sigma_2^\top$ so that, since Γ is an acyclic cellular extension of $\Sigma_2^\top$, there exists a 3-cell



in $\Gamma_3^\top$. An application of F to A gives the 3-cell



which, by definition of the cellular extension Δ and functoriality of F, is in $\Delta_2^\top$. Using the 3-cells $F(A)$, τ_f and τ_g, we get the following 3-cell from f to g in $\Delta_3^\top$:



This concludes the proof that $\Delta = F(\Gamma) \amalg \tau_{\Upsilon}$ is an acyclic cellular extension of the $(2,1)$-category $\Upsilon_2^{\top}$. □

We deduce from Theorem 3.4.3 that the finite derivation type property is Tietze invariant for finite 2-polygraphs:

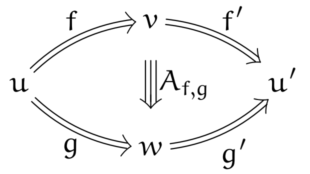
3.4.4. Theorem ([SOK94, Theorem 4.3]). *Let Σ and Υ be two Tietze-equivalent finite 2-polygraphs. Then Σ is of finite derivation type if and only if Υ is of finite derivation type.*

The result of the following exercise is useful to prove that a presentation admits no finite acyclic cellular extensions.

3.4.5. Exercise. Let Σ be a 2-polygraph and let Γ be an acyclic cellular extension of the free $(2,1)$-category $\Sigma_2^{\top}$. Show that if $\Sigma_2^{\top}$ admits a finite acyclic cellular extension, then there exists a finite subset of Γ that is an acyclic cellular extension of $\Sigma_2^{\top}$.

3.5. Coherence from convergence

3.5.1. Generating confluences. Squier's completion procedure provides a way to extend a convergent presentation of a 1-category **C** into a coherent presentation of **C**. We fix a convergent 2-polygraph Σ. A *family of generating confluences of* Σ is a cellular extension of the free $(2,1)$-category $\Sigma_2^{\top}$ that contains exactly one 3-cell



for every critical branching (f, g) of Σ.

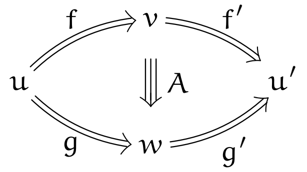
Note that, if Σ is confluent, it always admits a family of generating confluences. However, such a family is not necessarily unique, since the 3-cell $A_{f,g}$ can be directed in the reverse way and, for a given branching (f, g), we can have several possible 2-cells f' and g' with the required shape. Later, we will define the notion of normalisation strategies that provide a deterministic way to construct a family of generating confluences.

3.5.2. Squier's completion for convergent presentations. A *Squier's completion* of a convergent 2-polygraph Σ is the $(3,1)$-polygraph denoted by $\mathcal{S}(\Sigma)$ and defined by $\mathcal{S}(\Sigma) = (\Sigma, \Gamma)$, where Γ is a chosen family of generating confluences of Σ. The first proof of the following result is due to Squier, [SOK94], in the case where the category **C** is a monoid. We present the proof given in [GM18] in the language of polygraphs.

3.5.3. Theorem ([SOK94, Theorem 5.2]). *For every convergent presentation Σ of a category* **C***, Squier's completion of Σ is coherent presentation of* **C**.

Proof. We proceed in three steps.

3.5.4. Step 1. We prove that, for every local branching $(f,g) : u \Rightarrow (v,w)$ of Σ, there exist 2-cells $f' : v \Rightarrow u'$ and $g' : w \Rightarrow u'$ in Σ_2^* and a 3-cell $A : f \star_1 f' \Rrightarrow g \star_1 g'$ in $\mathcal{S}(\Sigma)_3^\top$, as in the following diagram:

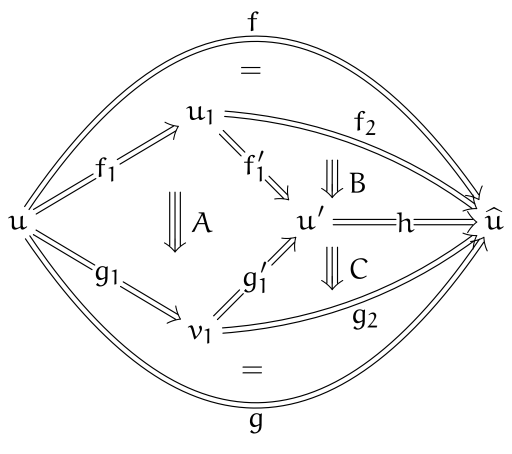


As we have seen in the study of confluence of local branchings, in the case of an aspherical or Peiffer branching, we can choose f' and g' such that $f \star_1 f' = g \star_1 g'$: an identity 3-cell is enough to link them. Moreover, if we have an overlapping branching (f,g) that is not critical, we have $(f,g) = (uhv, ukv)$ with (h,k) critical. We consider the 3-cell $\alpha : h \star_1 h' \Rrightarrow k \star_1 k'$ of $\mathcal{S}(\Sigma)$ corresponding to the critical branching (h,k) and we conclude that the following 2-cells f' and g' and 3-cell A satisfy the required conditions:

$$f' = uh'v \qquad g' = uk'v \qquad A = u\alpha v.$$

3.5.5. Step 2. We prove that, for every parallel 2-cells f and g of Σ_2^* whose common target is a normal form, there exists a 3-cell from f to g in $\mathcal{S}(\Sigma)_3^\top$. We proceed by noetherian induction on the common source u of f and g, using the termination of Σ. Let us assume that u is a normal form: then, by definition, both 2-cells f and g must be equal to the identity of u, so that $1_{1_u} : 1_u \Rrightarrow 1_u$ is a 3-cell of $\mathcal{S}(\Sigma)_3^\top$ from f to g.

Now, let us fix a 1-cell u with the following property: for any 1-cell v such that u rewrites into v and for any parallel 2-cells $f, g : v \Rightarrow \widehat{v} = \widehat{u}$ of Σ_2^*, there exists a 3-cell from f to g in $\mathcal{S}(\Sigma)_3^\top$. Let us consider parallel 2-cells $f, g : u \Rightarrow \widehat{u}$ and let us prove the result by progressively constructing the following composite 3-cell from f to g in $\mathcal{S}(\Sigma)_3^\top$:

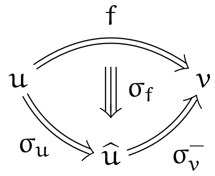


Since u is not a normal form, we can decompose $f = f_1 \star_1 f_2$ and $g = g_1 \star_1 g_2$ so that f_1 and g_1 are rewriting steps. They form a local branching (f_1, g_1) and we build the 2-cells f_1' and g_1', together with the 3-cell A as in the first part of the proof. Then, we consider a 2-cell h from u' to $\widehat{u}$ in Σ_2^*, that must exist by confluence of Σ and since $\widehat{u}$ is a normal form. We apply the induction hypothesis to the parallel 2-cells f_2 and $f_1' \star_1 h$ in order to get B and, symmetrically, to the parallel 2-cells $g_1' \star_1 h$ and g_2 to get C.

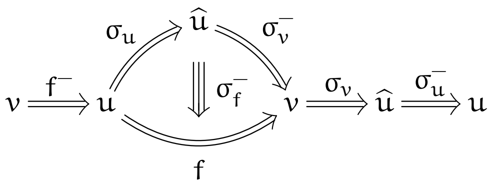
3.5.6. Step 3. We prove that every 2-sphere of $\Sigma_2^\top$ is the boundary of a 3-cell of $\mathcal{S}(\Sigma)_3^\top$. First, let us consider a 2-cell $f : u \Rightarrow v$ in Σ_2^*. Using the confluence of Σ, we choose 2-cells

$$\sigma_u : u \Rightarrow \widehat{u} \qquad \text{and} \qquad \sigma_v : v \Rightarrow \widehat{v} = \widehat{u}$$

in Σ_2^*. By construction, the 2-cells $f \star_1 \sigma_v$ and σ_u are parallel and their common target $\widehat{u}$ is a normal form. Thus by Step 2, there exists a 3-cell in $\mathcal{S}(\Sigma)_3^\top$ from $f \star_1 \sigma_v$ to σ_u or, equivalently, a 3-cell σ_f from f to $\sigma_u \star_1 \sigma_v^-$ in $\mathcal{S}(\Sigma)_3^\top$, as in the following diagram:



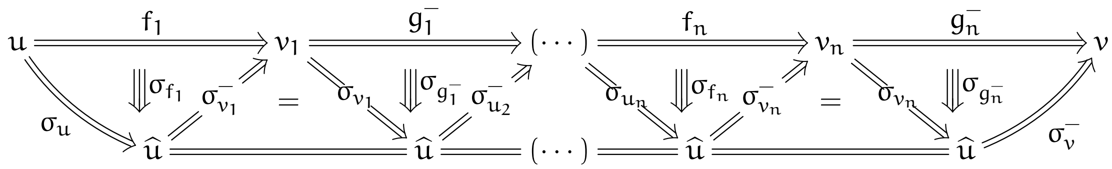
Moreover, the free $(3,1)$-category $\mathcal{S}(\Sigma)_3^\top$ contains a 3-cell σ_{f^-} from f^- to $\sigma_v \star_1 \sigma_u^-$, given as the following composite:



Now, let us consider a general 2-cell $f : u \Rightarrow v$ of $\Sigma_2^\top$. By construction of the free $(2,1)$-category $\Sigma_2^\top$, the 2-cell f can be decomposed into a "zig-zag", that is non-unique in general,

$$u \overset{f_1}{\Longrightarrow} v_1 \overset{g_1^-}{\Longrightarrow} u_2 \overset{f_2}{\Longrightarrow} (\cdots) \overset{g_{n-1}^-}{\Longrightarrow} u_n \overset{f_n}{\Longrightarrow} v_n \overset{g_n^-}{\Longrightarrow} v$$

where each f_i and g_i is a 2-cell of Σ_2^*. We define σ_f as the following composite 3-cell of $\mathcal{S}(\Sigma)_3^\top$, with source f and target $\sigma_u \star_1 \sigma_v^-$:

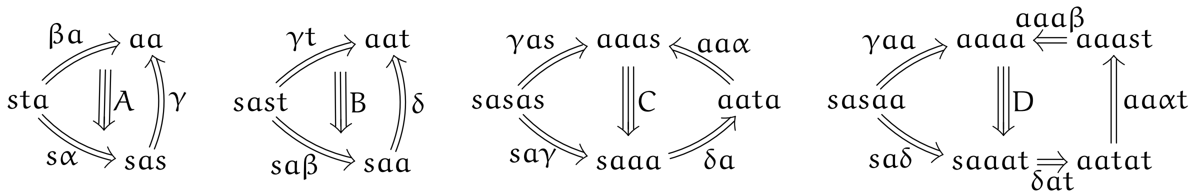


We proceed similarly for any other 2-cell $g : u \Rightarrow v$ of $\Sigma_2^\top$, to get a 3-cell σ_g from g to $\sigma_u \star_1 \sigma_v^-$ in $\mathcal{S}(\Sigma)_3^\top$. Thus, the composite $\sigma_f \star_2 \sigma_g^-$ is a 3-cell of the free $(3,1)$-category $\mathcal{S}(\Sigma)_3^\top$ from f to g, concluding the proof. □

Theorem 3.5.3 is extended to higher-dimensional polygraphs in [GM09, Proposition 4.3.4].

3.5.7. Theorem ([SOK94, Theorem 5.3]). *If a monoid admits a finite convergent presentation, then it is of finite derivation type.*

3.5.8. Example. Consider the convergent presentation $\mathcal{KB}(\Gamma)$ of the braid monoid $\mathbf{B}_3^+$ given in (2.6.7). It has four critical branchings given in (2.6.8). We deduce an acyclic extension of the $(2,1)$-category $\mathcal{KB}(\Gamma)^\top$, with the following 3-cells:



(3.5.9)

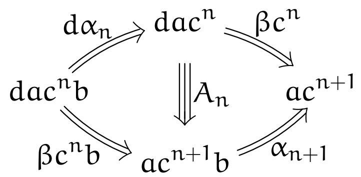
3.5.10. Example [LP91]. Consider the following 2-polygraph:

$$\Sigma = \langle\ a, b, c, d \ \mid\ \alpha : ab \Rightarrow a,\ \beta : da \Rightarrow ac\ \rangle.$$

The 2-polygraph Σ is not convergent and can be completed into the following infinite but convergent polygraph

$$\mathcal{KB}(\Sigma) = \langle\ a, b, c, d \ \mid\ \alpha_n : ac^n b \Rightarrow ac^n,\ n \in \mathbb{N}, \beta : da \Rightarrow ac\ \rangle,$$

with an infinity of confluent critical branchings:



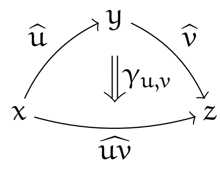
By Theorem 3.5.3, the 2-polygraph $\mathcal{KB}(\Sigma)$ can be extended into a coherent presentation of the monoid $\overline{\Sigma}$ presented by Σ with infinitely many 3-cells A_n, for n in $\mathbb{N}$.

Now, consider the following 2-polygraph

$$\Gamma = \langle\ a, b, c, d \ \mid\ \alpha : ab \Rightarrow a,\ \gamma : ac \Rightarrow da\ \rangle.$$

It presents the monoid $\overline{\Sigma}$ and it is convergent with no critical branching. It follows that it forms a coherent presentation of the monoid $\overline{\Sigma}$ with no 3-cell.

3.5.11. Exercise. The *standard presentation* of a category $\mathbf{C}$ is the 2-polygraph $\mathrm{Std}_2(\mathbf{C})$ defined as follows. The 0-cells and 1-cells of $\mathrm{Std}_2(\mathbf{C})$ are the ones of $\mathbf{C}$, with $\widehat{u}$ denoting a 1-cell u of $\mathbf{C}$ when seen as a 1-cell of $\mathrm{Std}_2(\mathbf{C})$. The 2-polygraph $\mathrm{Std}_2(\mathbf{C})$ contains a 2-cell



for all 1-cells $u : x \to y$ and $v : y \to z$ of **C**, and a 2-cell

$$x \underset{\widehat{1}_x}{\overset{1_x}{\underset{\Downarrow \iota_x}{\rightrightarrows}}} x$$

for every 0-cell x of **C**.

Extend this 2-polygraph into a coherent presentation of the category **C**.

3.5.12. Exercise. Let us consider the monoid **M** presented by the 2-polygraph

$$\Sigma \;=\; \langle\, x, y \;\mid\; \alpha : xyx \Rightarrow yy \,\rangle.$$

1. Prove that Σ terminates.

2. Complete Σ into a coherent presentation of the monoid **M**.

3.5.13. Squier's example. In [Squ87], Squier defines, for every $k \geqslant 1$, the monoid $\mathbf{S}_k$ presented by the 2-polygraph

$$\langle\, a, b, t, x_1, \ldots, x_k, y_1, \ldots, y_k \;\mid\; (\alpha_n)_{n\in\mathbb{N}},\ (\beta_i)_{1\leqslant i\leqslant k},\ (\gamma_i)_{1\leqslant i\leqslant k},\ (\delta_i)_{1\leqslant i\leqslant k},\ (\varepsilon_i)_{1\leqslant i\leqslant k} \,\rangle$$

with

$$at^nb \overset{\alpha_n}{\Longrightarrow} 1, \quad x_ia \overset{\beta_i}{\Longrightarrow} atx_i, \quad x_it \overset{\gamma_i}{\Longrightarrow} tx_i, \quad x_ib \overset{\delta_i}{\Longrightarrow} bx_i, \quad x_iy_i \overset{\varepsilon_i}{\Longrightarrow} 1.$$

In [SOK94], Squier proves the following finiteness properties for the monoid $\mathbf{S}_1$. With similar arguments, the result extends to every monoid $\mathbf{S}_k$, for $k \geqslant 1$.

3.5.14. Theorem ([SOK94, Theorem 6.7, Corollary 6.8]). *For every $k \geqslant 1$, the monoid $\mathbf{S}_k$ satisfies the following properties:*

i) *it is finitely presented,*

ii) *it has a decidable word problem,*

iii) *it is not of finite derivation type,*

iv) *it admits no finite convergent presentation.*

This result shows in particular, that the property of being decidable is not sufficient for finitely presented monoids to have a finite convergent presentation or to have finite derivation type. Let us prove the result in the case of the monoid $\mathbf{S}_1$, with the following infinite presentation:

$$\Sigma^{Sq_1} \;=\; \langle\, a, b, t, x, y \;\mid\; (\alpha_n)_{n\in\mathbb{N}},\ \beta,\ \gamma,\ \delta,\ \varepsilon \,\rangle$$

whose rules are defined by

$$at^nb \overset{\alpha_n}{\Longrightarrow} 1, \quad xa \overset{\beta}{\Longrightarrow} atx, \quad xt \overset{\gamma}{\Longrightarrow} tx, \quad xb \overset{\delta}{\Longrightarrow} bx, \quad xy \overset{\varepsilon}{\Longrightarrow} 1.$$

We will denote by $\gamma_n : xt^n \Rightarrow t^n x$ the 2-cell of $(\Sigma^{Sq_1})_2^*$ defined by induction on n as follows:

$$\gamma_0 = 1_x \qquad \text{and} \qquad \gamma_{n+1} = \gamma t^n \star_1 t\gamma_n.$$

For every n, we write $f_n : xat^b \Rightarrow at^{n+1}bx$ the 2-cell of $(\Sigma^{Sq_1})_2^*$ defined as the following composite:

$$xat^nb \overset{\beta t^n b}{\Longrightarrow} atxt^nb \overset{at\gamma_n b}{\Longrightarrow} at^{n+1}xb \overset{at^{n+1}\delta}{\Longrightarrow} at^{n+1}bx.$$

3.5.15. Exercise.

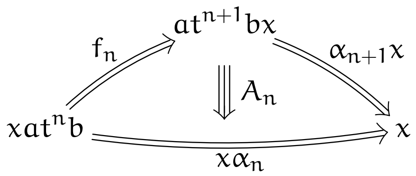
1. Show that the monoid $\mathbf{S}_1$ admits the following finite presentation:

$$\langle\, a, b, t, x, y \mid \alpha_0, \beta, \gamma, \delta, \varepsilon \,\rangle.$$

2. Show that the monoid $\mathbf{S}_1$ has a decidable word problem.

3.5.16. Exercise, [GM18].

1. Show that the 2-polygraph Σ^{Sq_1} is convergent and Squier's completion of Σ^{Sq_1} contains a 3-cell A_n for every natural number n with the following shape:



2. Show that the monoid $\mathbf{S}_1$ is not of finite derivation type.

3.5.17. Exercise, [LP91, Laf95]. Consider the monoid $\mathbf{M}$ presented by the following 2-polygraph:

$$\langle\, a, b, c, d, d' \mid \alpha_0 : ab \Rightarrow a,\ \beta : da \Rightarrow ac,\ \gamma : d'a \Rightarrow ac \,\rangle.$$

Show that the monoid $\mathbf{M}$ admits a finite presentation, it has a decidable word problem, yet it is not of finite derivation type and, as a consequence, it does not admit a finite convergent presentation.

CHAPTER
4

Two-dimensional homological syzygies

Contents

In this chapter we present the result obtained by Squier relating the finite-convergence of a string rewriting system with the homotopical type left-FP_3, [Squ87]. The constructions developed in this chapter come from [GM18].

4.1. Preliminaries on modules

In this section, we fix a ring R. We will say "R-module" of "module" for "left R-module". All the notions presented are defined in the same manner for right R-modules since every right R-module is a left R^{op}-module, where R^{op} is the opposite ring. We will say "homomorphism" for a homomorphism of left R-modules. We refer the reader to [Lan02] or to [Rot09] for a deeper presentation and the proofs of the results given in this preliminary part on modules.

4.1.1. Exact sequences. Two homomorphisms of modules

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

are *exact at* M if $\operatorname{Im} f = \ker g$. A sequence of homomorphisms

$$\cdots \longrightarrow M_{n+1} \overset{d_{n+1}}{\longrightarrow} M_n \overset{d_n}{\longrightarrow} M_{n-1} \longrightarrow \cdots$$

is *exact* if each adjacent pair of homomorphisms is exact.

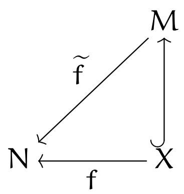
4.1.2. Examples. If $0 \to M \overset{f}{\to} M'$ is exact, then the map f is injective. If $M \overset{f}{\to} M' \to 0$ is exact, then the map f is surjective. If the sequence $0 \to M \overset{f}{\to} M' \to 0$ is exact, then the map f is an isomorphism. If the sequence $M' \overset{f}{\to} M \overset{g}{\to} M''$ is exact with f surjective and g injective, then $M = 0$.

4.1.3. Free modules. A R-module M is *free* if it is a direct sum of copies of R. If $M = \coprod_{i \in I} Rx_i$, with $R \simeq Rx_i$, the the set $\{x_i \mid i \in I\}$ is called a *basis* of M. It follows that each element x in M has a unique decomposition

$$x = \sum_{i \in I} \lambda_i x_i,$$

where $\lambda_i \in R$ and almost all λ_i are zero.

4.1.4. Proposition. *Let* $X = \{x_i \mid i \in I\}$ *be a basis of a free module* M. *For any module* N *and any map* $f : X \longrightarrow N$, *there is a unique map* $\widetilde{f} : M \longrightarrow N$ *extending* f, i.e., *such that the following diagram commutes:*



4.1.5. Proposition. *Let* X *be a set. There exists a free* R*-module having* X *as a basis.*

4.1.6. Proposition. *Every* R*-module is a quotient of a free* R*-module.*

Proposition 4.1.6 says that any R-module M may be described by *generators* and *relations* in the following way. Given a free R-module F with basis X and given

$$f : F \longrightarrow M$$

be a surjective homomorphism of R-modules, we say that X is a set of *generators of* M and the kernel $\ker f$ is called its submodule of *relations*.

4.1.7. Finitely generated modules. A R-module M is *finitely generated* if there is a finite subset $\{x_1, x_2, ..., x_n\}$ of M such that for all x in M, there exist $r_1, r_2, ..., r_n$ in R with $x = r_1x_1 + r_2x_2 + ... + r_nx_n$. Then the set $\{x_1, x_2, ..., x_n\}$ is referred to as a generating set for M. The finite generators need not be a basis, since they need not be linearly independent over R. A R-module M is finitely generated if and only if there is a surjective homomorphism:

$$R^n \longrightarrow M$$

for some n. That is, M is a quotient of a free module of finite rank.

4.1.8. Proposition. *Let* F*,* M *and* N *be left* R*-modules. If* F *is free,* $\varepsilon : M \longrightarrow N$ *is a surjective homomorphism and* $f : F \longrightarrow N$ *is any homomorphism, then there exists a homomorphism* $\widetilde{f} : F \longrightarrow M$ *such that following diagram commutes*

$$\begin{array}{ccccc} & & F & & \\ & \overset{\widetilde{f}}{\swarrow} & \downarrow f & & \\ M & \xrightarrow[\varepsilon]{} & N & \longrightarrow & 0 \end{array}$$

As a consequence of Proposition 4.1.8, for any free R-module, the functor $\mathrm{Hom}_R(F, -)$ is exact.

4.1.9. Projective modules. A projective module is a module which behaves as the free module F in Proposition 4.1.8. More explicitly, a R-module P is *projective* if whenever $\varepsilon : M \to N$ is a surjective homomorphism and $f : P \longrightarrow N$ is any homomorphism, there exists a homomorphism $\widetilde{f} : P \longrightarrow M$ making the following diagram commutative:

$$\begin{array}{ccccc} & & P & & \\ & \overset{\widetilde{f}}{\swarrow} & \downarrow f & & \\ M & \xrightarrow[\varepsilon]{} & N & \longrightarrow & 0 \end{array}$$

In particular, any free module is projective.

The following result gives several ways to characterise projective modules.

4.1.10. Proposition. *The following conditions are equivalent for a* R*-module* P*:*

i) P *is projective,*

ii) *if* $f : M \to P$ *is a surjective homomorphism, then there exists* $h : P \longrightarrow M$ *such that* $fh = \mathrm{Id}_P$*,*

iii) *if* $f : M \to P$ *is a surjective homomorphism, then* $M \simeq P \oplus \ker f$*,*

iv) *the functor* $\mathrm{Hom}_R(P, -)$ *is exact, that is for any exact sequence* $0 \to M' \to M \to M'' \to 0$*, the induced sequence* $0 \to \mathrm{Hom}_R(P, M') \to \mathrm{Hom}_R(P, M) \to \mathrm{Hom}_R(P, M'') \to 0$ *is exact,*

v) P *is a summand of a free module, that is there exists a free* R*-module* F *such that* $F \simeq P \oplus Q$*, for some* R*-module* Q[1]*.*

4.1.11. Proposition (Schanuel's Lemma). *Given exact sequences of* R*-modules*

$$0 \longrightarrow K_1 \longrightarrow P_1 \longrightarrow M \longrightarrow 0,$$

$$0 \longrightarrow K_2 \longrightarrow P_2 \longrightarrow M \longrightarrow 0,$$

where P_1 *and* P_2 *are projective. Then* $K_1 \oplus P_2 \simeq K_2 \oplus P_1$*.*

[1] by the equivalence, the R-module Q is necessarily projective.

4.1.12. Exercise. Prove Proposition 4.1.11.

4.1.13. Proposition (Generalised Schanuel's Lemma). *Given exact sequences of* R*-modules*

$$0 \longrightarrow K \longrightarrow P_k \longrightarrow P_{k-1} \longrightarrow \quad \cdots \quad \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

$$0 \longrightarrow L \longrightarrow Q_k \longrightarrow Q_{k-1} \longrightarrow \quad \cdots \quad \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0,$$

where all the P_i *and* Q_i *are projective. Let*

$$P_{odd} = \bigoplus_{i \text{ odd}} P_i, \qquad P_{even} = \bigoplus_{i \text{ even}} P_i,$$

and

$$Q_{odd} = \bigoplus_{i \text{ odd}} Q_i, \qquad Q_{even} = \bigoplus_{i \text{ even}} Q_i.$$

Then the following properties hold

i) *If* k *is even, then* $K \oplus Q_{even} \oplus P_{odd} \simeq L \oplus Q_{odd} \oplus P_{even}$*,*

ii) *If* k *is odd then* $K \oplus Q_{odd} \oplus P_{even} \simeq L \oplus Q_{even} \oplus P_{odd}$*.*

Let us mention a consequence of the Proposition 4.1.13.

4.1.14. Corollary. *Given exact sequences of* R*-modules*

$$0 \longrightarrow K \longrightarrow P_k \longrightarrow P_{k-1} \longrightarrow \quad \cdots \quad \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

$$0 \longrightarrow L \longrightarrow Q_k \longrightarrow Q_{k-1} \longrightarrow \quad \cdots \quad \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0,$$

where all the P_i *and* Q_i *are finitely generated and projective, then the* R*-module* K *is finitely generated if and only if* L *is finitely generated.*

4.1.15. Exercise. Prove Proposition 4.1.13.

4.1.16. Chain's complex. A *(chain) complex* of R-modules is a sequence $(M_n)_{n\in\mathbb{N}}$ of R-modules, together with a sequence $(d_n)_{n\in\mathbb{N}}$ of homomorphisms

$$\cdots \quad \longrightarrow M_n \xrightarrow{d_n} M_{n-1} \longrightarrow \quad \cdots \quad \longrightarrow M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0$$

such that we have the inclusion

$$\operatorname{Im} d_{n+1} \subseteq \ker d_n$$

for all n, or equivalently, the relation $d_n d_{n+1} = 0$ holds for all n. The map d_n are called *boundary maps*.

4.1.17. Resolutions. A *resolution* of a R-module M is an exact sequence of R-modules

$$\cdots \longrightarrow M_n \xrightarrow{d_n} M_{n-1} \longrightarrow \cdots \longrightarrow M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

From the definition, the homomorphism ε is surjective and

$$\operatorname{Im} d_1 = \ker \varepsilon, \quad \text{and} \quad \operatorname{Im} d_{n+1} = \ker d_n, \quad \text{for all } n.$$

Such a resolution is called *projective* (resp. *free*) if all the modules M_n are projective (resp. free). Given a natural number n, a *partial resolution of length* n *of* **M** is defined in a similar way but with a bounded sequence $(M_k)_{0\leqslant k\leqslant n}$ of R-modules:

$$M_n \xrightarrow{d_n} M_{n-1} \longrightarrow \cdots \longrightarrow M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

4.1.18. Proposition. *Every* R*-module* M *has a free resolution.*

4.1.19. Exercise. Prove Proposition 4.1.18.

4.1.20. Contracting homotopies. Given a complex of R-modules

$$\cdots \longrightarrow M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \longrightarrow \cdots \longrightarrow M_1 \xrightarrow{d_1} M_0 \xrightarrow{\varepsilon} M \longrightarrow 0 \tag{4.1.21}$$

a method to prove that such a complex is a resolution of M is to construct a *contracting homotopy*, that is a sequence of homomorphisms of $\mathbb{Z}$-modules

$$\cdots \longleftarrow M_{n+1} \xleftarrow{i_{n+1}} M_n \xleftarrow{i_n} M_{n-1} \longleftarrow \cdots \longleftarrow M_1 \xleftarrow{i_1} M_0 \xleftarrow{i_0} M$$

satisfying the following equalities

$$\begin{aligned} \varepsilon i_0 &= \mathrm{Id}_M, \\ d_1 \iota_1 + \iota_0 \varepsilon &= \mathrm{Id}_{M_0}, \\ d_{n+1} i_{n+1} + i_n d_n &= \mathrm{Id}_{M_n}, \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

Indeed, in that case, the first equality proves that the homomorphism ε is surjective. Moreover, for every natural number n and every x in $\ker d_n$, the equality $d_{n+1} i_{n+1}(x) = x$ holds, proving that x is in $\operatorname{Im} d_{n+1}$, so that $\ker d_n \subseteq \operatorname{Im} d_{n+1}$ holds. As a consequence, the complex (4.1.21) is a resolution of the R-module M.

4.2. Monoids of finite homological type

Let **M** be a monoid. We denote by $\mathbb{Z}\mathbf{M}$ the ring generated by **M**, that is, the free abelian group over **M**, equipped with the canonical extension of the product of **M**:

$$\Big(\sum_{u\in\mathbf{M}} \lambda_u u\Big)\Big(\sum_{v\in\mathbf{M}} \lambda_v v\Big) = \sum_{u,v\in\mathbf{M}} \lambda_u\lambda_v uv = \sum_{w\in\mathbf{M}} \sum_{uv=w} \lambda_u\lambda_v w,$$

with λ_u, λ_v in $\mathbb{Z}$. The *trivial* $\mathbb{Z}\mathbf{M}$*-module* is the abelian group $\mathbb{Z}$ equipped with the trivial action $un = n$, for every u in **M** and n in $\mathbb{Z}$.

4.2.1. Homological type left-FP_n. A monoid $\mathbf{M}$ is of *homological type left-*FP_n, for a natural number n, if there exists a partial resolution of length n of the trivial $\mathbb{Z}\mathbf{M}$-module $\mathbb{Z}$ by projective, finitely generated $\mathbb{Z}\mathbf{M}$-modules:

$$P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \mathbb{Z} \longrightarrow 0.$$

A monoid $\mathbf{M}$ is of *homological type left-*FP_∞ if there exists a resolution of $\mathbb{Z}$ by projective, finitely generated $\mathbb{Z}\mathbf{M}$-modules.

4.2.2. Proposition. *Let $\mathbf{M}$ be a monoid and let n be a natural number. The following assertions are equivalent:*

i) *The monoid $\mathbf{M}$ is of homological type left-*FP_n.

ii) *There exists a free, finitely generated partial resolution of the trivial $\mathbb{Z}\mathbf{M}$-module $\mathbb{Z}$ of length n*

$$F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow \mathbb{Z} \longrightarrow 0.$$

iii) *For every $0 \leqslant k < n$ and every projective, finitely generated partial resolution of the trivial $\mathbb{Z}\mathbf{M}$-module $\mathbb{Z}$ of length k*

$$P_k \xrightarrow{d_k} P_{k-1} \xrightarrow{d_{k-1}} \cdots \longrightarrow P_0 \xrightarrow{d_0} \mathbb{Z} \longrightarrow 0,$$

the $\mathbb{Z}\mathbf{M}$-module $\ker d_k$ *is finitely generated.*

4.2.3. Exercise. Show Proposition 4.2.2 using Proposition 4.1.14.

4.2.4. Homological type FP_0. The *augmentation map* of a monoid $\mathbf{M}$ is the ring homomorphism

$$\varepsilon : \mathbb{Z}\mathbf{M} \to \mathbb{Z}$$

defined by

$$\varepsilon\Big(\sum_{u\in\mathbf{M}} \lambda_u u\Big) = \sum_{u\in\mathbf{M}} \lambda_u.$$

The ring homomorphism ε is extended to a homomorphism of $\mathbb{Z}\mathbf{M}$-modules in the obvious way. If we consider the homomorphism of $\mathbb{Z}$-modules $i_0 : \mathbb{Z} \to \mathbb{Z}\mathbf{M}$ defined by $i_0(1) = 1$, we have $\varepsilon i_0 = \mathrm{Id}_{\mathbb{Z}}$. Hence the homomorphism ε is surjective. It follows that

4.2.5. Proposition. *Every monoid $\mathbf{M}$ is of homological type* FP_0.

4.2.6. Remark. Every R-module admits a free resolution. In particular, given a monoid $\mathbf{M}$, there exists a resolution

$$\cdots \longrightarrow F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \longrightarrow \;\cdots\; \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} \mathbb{Z} \longrightarrow 0$$

of the trivial $\mathbb{Z}\mathbf{M}$-module $\mathbb{Z}$ by free $\mathbb{Z}\mathbf{M}$-modules. We can build such a resolution by setting $F_0 = \mathbb{Z}\mathbf{M}$ and $d_0 = \varepsilon$. Let F_1 be the free $\mathbb{Z}\mathbf{M}$-module generated by $\ker \varepsilon$, and let $d_1 : F_1 \to \mathbb{Z}\mathbf{M}$ be the canonical homomorphism induced by the homomorphism $\ker \varepsilon \to F_0$. Then, for any $n \geqslant 2$, F_n is the free $\mathbb{Z}\mathbf{M}$-module generated by $\ker d_{n-1}$, and the homomorphism $d_n : F_n \to F_{n-1}$ is induced by the homomorphism $\ker d_{n-1} \to F_{n-1}$.

Note that in this way, the obtained resolution is too big in general. In the rest of this section, we show how to construct a partial resolution which is more "economic" in the sense that the free modules are generated by a reduced number of generators.

4.2.7. Normalisation strategies. Given a monoid $\mathbf{M}$, we consider a presentation of $\mathbf{M}$ by a 2-polygraph Σ with a single 0-cell $\bullet$. Let $\pi : \Sigma_1^* \to \mathbf{M}$ be the canonical projection. We will write $\overline{u}$ instead of $\pi(u)$. We consider a section

$$\mathbf{M} \to \Sigma_1^*$$

of π, *i.e.*, we choose, for every 1-cell u of $\mathbf{M}$, a 1-cell $\widehat{u}$ of Σ_1^* such that $\pi(\widehat{u}) = u$. In general, we cannot assume that the chosen section is functorial, that is $\widehat{uv} = \widehat{u}\widehat{v}$ holds in Σ_1^*. However, we will assume that $\widehat{1_\bullet} = 1_\bullet$ holds. Given a 1-cell u of Σ_1^*, we simply write $\widehat{u}$ for $\widehat{\overline{u}}$.

Such a section being fixed, a *normalisation strategy for* Σ is a map

$$\sigma : \Sigma_1^* \to \Sigma_2^\top$$

that sends every 1-cell u of Σ_1^* to a 2-cell

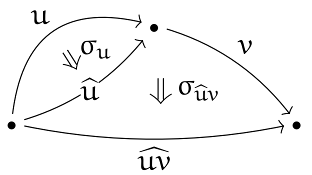
$$\sigma_u : u \Rightarrow \widehat{u}$$

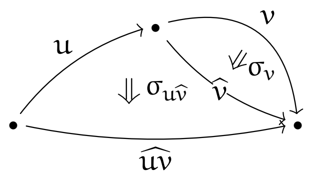
of the free $(2,1)$-category $\Sigma_2^\top$, such that $\sigma_{\widehat{u}} = 1_{\widehat{u}}$ holds for every 1-cell u of Σ_1^*.

4.2.8. Left and right normalisation strategies. Let Σ be a 2-polygraph, with a chosen section. A normalisation strategy σ for Σ is a *left* one (resp. a *right* one) if it satisfies

$$\sigma_{uv} = (\sigma_u \star_0 v) \star_1 \sigma_{\widehat{u}v}, \qquad \big(\text{resp. } \sigma_{uv} = (u \star_0 \sigma_v) \star_1 \sigma_{u\widehat{v}}\,\big). \tag{4.2.9}$$

That is

$\sigma_{uv} =$ 

(resp. $\sigma_{uv} =$ 

).

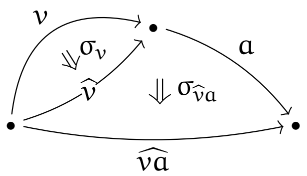
4.2.10. Proposition. *Any 2-polygraph admits a left (resp. right) normalisation strategy.*

Proof. Let Σ be a 2-polygraph with a chosen section. Prove that Σ admits a left normalisation strategy $\sigma : \Sigma_1^* \to \Sigma_2^\top$. The proof of the existence of a right normalisation strategies is similar.

Let us arbitrarily choose a 2-cell $\sigma_{\widehat{u}a} : \widehat{u}a \Rightarrow \widehat{ua}$ in $\Sigma_2^\top$, for every 1-cell u of Σ_1^* and every 1-cell a of Σ_1, such that $\widehat{ua} \neq \widehat{u}a$. Then, we extend σ into a left normalisation strategy for Σ by setting $\sigma_{\widehat{u}} = 1_{\widehat{u}}$, for any u in Σ_1^*, and for $u \neq \widehat{u}$ by setting

$$\sigma_u = \sigma_v a \star_1 \sigma_{\widehat{v}a}$$

if $u = va$ with v in Σ_1^* and a in Σ_1:

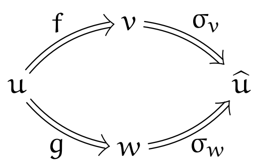


The relations $\sigma_{1_\bullet} = 1_{1_\bullet}$ and (4.2.9) are immediate consequences of the definition of the map σ. □

4.2.11. Leftmost and rightmost normalisation strategies. If Σ is a reduced 2-polygraph, then, for every 1-cell u of Σ_1^*, the set of rewriting steps with source u can be ordered from left to right: for two rewriting steps $f = v\alpha v'$ and $g = w\beta w'$ with source u, we have $f \prec g$ if the length of v is strictly smaller than the length of w. If Σ is finite, then the order $\prec$ is total and the set of rewriting steps of source u is finite. Hence, this set contains a smallest element λ_u and a greatest element ρ_u, respectively called the *leftmost* and the *rightmost rewriting steps on* u. If, moreover, the 2-polygraph Σ terminates, the iteration of λ (resp. ρ) yields a normalisation strategy σ called the *leftmost* (resp. *rightmost*) *normalisation strategy of* Σ:

$$\sigma_u = \lambda_u \star_1 \sigma_{t(\lambda_u)} \qquad (\text{resp. } \sigma_u = \rho_u \star_1 \sigma_{t(\rho_u)}).$$

The leftmost and rightmost normalisation strategies give a way to make constructive some of the results we present here. For example, when Σ is convergent they provide a deterministic choice of a confluence diagram



for every branching (f, g) of Σ.

4.2.12. Exercice. Prove (by noetherian induction) that the leftmost (resp. rightmost) normalisation strategy of Σ is a left (resp. right) normalisation strategy.

4.2.13. Presentations and partial resolutions of length 2. Let $\mathbf{M}$ be a monoid and let Σ be a presentation of $\mathbf{M}$. Let us define a partial resolution of length 2 of $\mathbb{Z}$ by free $\mathbb{Z}\mathbf{M}$-modules

$$\mathbb{Z}\mathbf{M}[\Sigma_2] \xrightarrow{d_2} \mathbb{Z}\mathbf{M}[\Sigma_1] \xrightarrow{d_1} \mathbb{Z}\mathbf{M} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

The $\mathbb{Z}\mathbf{M}$-modules $\mathbb{Z}\mathbf{M}[\Sigma_1]$ and $\mathbb{Z}\mathbf{M}[\Sigma_2]$ are the free $\mathbb{Z}\mathbf{M}$-modules over Σ_1 and Σ_2, respectively: they contain the formal sums of elements denoted by $u[x]$, where u is an element of $\mathbf{M}$ and x is a 1-cell of Σ or a 2-cell of Σ. Let us note that $\mathbb{Z}\mathbf{M}$ is isomorphic to the free $\mathbb{Z}\mathbf{M}$-module over the singleton Σ_0. The map ε is the augmentation map defined in (4.2.4) and the boundary maps are defined, on generators, by

$$d_1([x]) = \overline{x} - 1 \qquad d_2([\alpha]) = [s_1(\alpha)] - [t_1(\alpha)].$$

The map d_2 is called the *Reidemester-Fox Jacobian of* Σ. In the definition of d_2, the bracket $[\cdot]$ is extended to the 1-cells of Σ_1^* thanks to the relation

$$[1] \ = \ 0 \qquad \text{and} \qquad [uv] \ = \ [u] + \overline{u}[v], \tag{4.2.14}$$

for all 1-cells u and v of Σ_1.

4.2.15. Lemma. *For any* u *in* Σ_1^**, we have* $d_1(u) = \overline{u} - 1$.

Proof. We prove the relation by induction on the length of u. For the unit, we have $d_1[1] = d_1(0) = 0$ and $\overline{1} - 1 = 0$. Then, for a composite 1-cell uv such that the result holds for both u and v, we get

$$d_1[uv] \ = \ d_1[u] + \overline{u}d_1[v] \ = \ \overline{u} - 1 + \overline{uv} - \overline{u} \ = \ \overline{uv} - 1.$$

□

4.2.16. Proposition. *Let* $\mathbf{M}$ *be a monoid and let* Σ *be a presentation of* $\mathbf{M}$*. The sequence of* $\mathbb{Z}\mathbf{M}$*-modules*

$$\mathbb{Z}\mathbf{M}[\Sigma_2] \xrightarrow{d_2} \mathbb{Z}\mathbf{M}[\Sigma_1] \xrightarrow{d_1} \mathbb{Z}\mathbf{M} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0 \tag{4.2.17}$$

is a partial free resolution of length 2 *of* $\mathbb{Z}$.

Proof. We first note that the sequence is a chain complex. Indeed, the augmentation map is surjective by definition. Moreover, we have

$$\varepsilon d_1[x] \ = \ \varepsilon(\overline{x}) - \varepsilon(1) \ = \ 1 - 1 \ = \ 0,$$

for every 1-cell x of Σ_1. The relation $d_1 d_2 = 0$ is consequence of Lemma 4.2.15. Indeed, we have

$$d_1 d_2[\alpha] \ = \ d_1[s(\alpha)] - d_1[t(\alpha)] \ = \ \overline{s(\alpha)} - \overline{t(\alpha)} \ = \ 0,$$

for every 2-cell α of Σ_2, where the last equality comes from $\overline{s(\alpha)} = \overline{t(\alpha)}$, that holds since Σ is a presentation of the monoid $\mathbf{M}$.

The rest of the proof consists in defining contracting homotopies i_0, i_1, i_2:

$$\mathbb{Z}\mathbf{M}[\Sigma_2] \underset{i_2}{\overset{d_2}{\rightleftarrows}} \mathbb{Z}\mathbf{M}[\Sigma_1] \underset{i_1}{\overset{d_1}{\rightleftarrows}} \mathbb{Z}\mathbf{M} \underset{i_0}{\overset{\varepsilon}{\rightleftarrows}} \mathbb{Z}$$

We choose a representative $\widehat{u}$ in Σ_1^* for every element u of $\mathbf{M}$, with $\widehat{1}_x = 1_x$ for every 0-cell x of Σ, and we fix a normalisation strategy σ for Σ. Then we define the homomorphisms of $\mathbb{Z}$-modules i_0, i_1 and i_2 by setting

$$i_0(1) \ = \ 1, \qquad i_1(u) \ = \ [\widehat{u}], \qquad i_2(u[x]) \ = \ [\sigma_{\widehat{u}x}], \tag{4.2.18}$$

for any u in **M** and x in Σ_1.

In (4.2.18), the element $[\sigma_{\widehat{u}x}]$ is defined using an extension of the bracket notation $[\cdot]$ on 2-cells of Σ_2 into a map

$$[\cdot] : \Sigma_2^\top \to \mathbb{Z}\mathbf{M}[\Sigma_2]$$

thanks to the relations

$$[1_u] \;=\; 0, \qquad [ufv] \;=\; \overline{u}[f] \qquad \text{and} \qquad [f \star_1 g] \;=\; [f] + [g],$$

for all 1-cells u and v and 2-cells f and g of $\Sigma_2^\top$ such that the composites ufv and $f \star_1 g$ are defined.

First, we have $\varepsilon i_0 = \mathrm{Id}_{\mathbb{Z}}$. Next, for every u in **M**, we have $i_0\varepsilon(u) = 1$ and

$$d_1 i_1(u) \;=\; d_1[\widehat{u}] = u - 1.$$

Thus $d_1 i_1 + i_0\varepsilon = \mathrm{Id}_{\mathbb{Z}\mathbf{M}}$. Finally, we have, on the one hand,

$$i_1 d_1(u[x]) \;=\; i_1(u\overline{x} - u) \;=\; [\widehat{ux}] - [\widehat{u}]$$

and, on the other hand,

$$d_2 i_2(u[x]) \;=\; d_2[\sigma_{\widehat{u}x}] \;=\; [\widehat{u}x] - [\widehat{ux}] \;=\; u[x] + [\widehat{u}] - [\widehat{ux}].$$

For this equality, we check that $d_2[f] = [s(f)] - [t(f)]$ holds for every 2-cell f of $\Sigma_2^\top$ by induction on the size of f. Hence we have $d_2 i_2 + i_1 d_1 = \mathrm{Id}_{\mathbb{Z}\mathbf{M}[\Sigma_1]}$, thus concluding the proof. □

From Proposition 4.2.16, we deduce the following result:

4.2.19. Theorem. *The following properties hold.*

i) *Every monoid is of homological type left-*FP_0.

ii) *Every finitely generated monoid is of homological type left-*FP_1.

iii) *Every finitely presented monoid is of homological type left-*FP_2.

4.2.20. Examples. Let us consider the monoid **M** presented by the 2-polygraph

$$\Sigma \;=\; \langle\; a, c, t \;\mid\; \alpha_n : at^{n+1} \Rightarrow ct^n \,,\; n \in \mathbb{N} \;\rangle.$$

The monoid **M** is finitely generated and, thus, it is of homological type left-FP_1. However, for every natural number n, we have

$$\begin{aligned} d_2[\alpha_{n+1}] \;&=\; [at^{n+2}] - [ct^{n+1}], \\ &=\; [at^{n+1}] + \overline{at^{n+1}}[t] - [ct^n] - \overline{ct^n}[t], \\ &=\; d_2[\alpha_n] + (\overline{at^{n+1}} - \overline{ct^n})[t]. \end{aligned}$$

The equality $\overline{at^{n+1}} = \overline{ct^n}$ holds in **M** by definition, yielding $d_2[\alpha_{n+1}] = d_2[\alpha_n]$. As a consequence, the $\mathbb{Z}\mathbf{M}$-module $\ker d_2$ is generated by the elements $[\alpha_n] - [\alpha_0]$. Since the $\mathbb{Z}\mathbf{M}$-module $\ker d_1$ is equal to Im d_2, hence isomorphic to $\mathbb{Z}\mathbf{M}[\Sigma_2]/\ker d_2$, it follows that $\ker d_1$ is generated by $[\alpha_0]$ only, so that, by Lemma 4.2.2, the monoid $\mathbf{M}$ is of homological type left-FP$_2$. This can also be obtained by simply observing that $\mathbf{M}$ admits the finite presentation $\langle a, c, t \mid \alpha_0 \rangle$.

Now, let us consider the monoid $\mathbf{M}$ presented by the 2-polygraph

$$\Sigma \;=\; \big\langle\; a, b, t \;\;\big|\;\; \alpha_n : at^n b \Rightarrow 1 \,,\; n \in \mathbb{N} \;\big\rangle.$$

The monoid $\mathbf{M}$ is of homological type left-FP$_1$, but not left-FP$_2$. This is proved by showing that $\ker d_1$ is not finitely generated as a $\mathbb{Z}\mathbf{M}$-module, which is tedious by direct computation in this case. Another way to conclude is to extend the partial resolution of Proposition 4.2.16 by one dimension: it will then be sufficient to compute Im d_3, which is trivial in this case because Σ has no critical branching, so that $\ker d_2 = 0$ and, as a consequence, $\ker d_1$ is isomorphic to $\mathbb{Z}\mathbf{M}[\Sigma_2]$. Convergent presentations provide a method to obtain such a length-three partial resolution.

4.3. Squier's homological theorem

4.3.1. Coherent presentations and partial resolutions of length 3. Let $\mathbf{M}$ be a monoid and let Σ be a coherent presentation of $\mathbf{M}$. Let us extend the partial resolution (4.2.17) into the resolution of length 3

$$\mathbb{Z}\mathbf{M}[\Sigma_3] \xrightarrow{\;d_3\;} \mathbb{Z}\mathbf{M}[\Sigma_2] \xrightarrow{\;d_2\;} \mathbb{Z}\mathbf{M}[\Sigma_1] \xrightarrow{\;d_1\;} \mathbb{Z}\mathbf{M} \xrightarrow{\;\varepsilon\;} \mathbb{Z} \longrightarrow 0,$$

where the $\mathbb{Z}\mathbf{M}$-module $\mathbb{Z}\mathbf{M}[\Sigma_3]$ is the free $\mathbb{Z}\mathbf{M}$-module over Σ_3, formed by the linear combination of elements $u[\gamma]$, with u in $\mathbf{M}$ and γ a 3-cell of Σ_3. The boundary map d_3 is defined, for every 3-cell γ of Σ_3, by

$$d_3[\gamma] \;=\; [s_2(\gamma)] - [t_2(\gamma)].$$

The bracket notation $[\cdot]$ defined on 3-cells of Σ_3 can be extended into a map

$$[\cdot] : \Sigma_3^\top \to \mathbb{Z}\mathbf{M}[\Sigma_3]$$

thanks to the relations

$$[uAv] \;=\; \overline{u}[A], \qquad [A \star_1 B] \;=\; [A] + [B], \qquad [A \star_2 B] \;=\; [A] + [B],$$

for all 1-cells u and v and 3-cells A and B of $\Sigma_3^\top$ such that the composites are defined. In particular, the latter relation implies $[1_f] = 0$ for every 2-cell f of $\Sigma_2^\top$. We check, by induction on the size, that $d_3[A] = [s_2(A)] - [t_2(A)]$ holds for every 3-cell A of $\Sigma_3^\top$.

4.3.2. Proposition. *Let* $\mathbf{M}$ *be a monoid and let* Σ *be a coherent presentation of* $\mathbf{M}$. *The sequence of* $\mathbb{Z}\mathbf{M}$*-modules*

$$\mathbb{Z}\mathbf{M}[\Sigma_3] \xrightarrow{\;d_3\;} \mathbb{Z}\mathbf{M}[\Sigma_2] \xrightarrow{\;d_2\;} \mathbb{Z}\mathbf{M}[\Sigma_1] \xrightarrow{\;d_1\;} \mathbb{Z}\mathbf{M} \xrightarrow{\;\varepsilon\;} \mathbb{Z} \longrightarrow 0$$

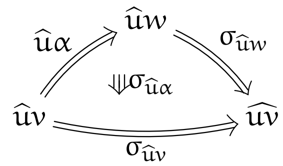
is a partial free resolution of length 3 *of* $\mathbb{Z}$.

Proof. We proceed with the same notations as the ones of the proof of Proposition 4.2.16, with the extra hypothesis that σ is a left normalisation strategy for Σ. This implies that $i_2(u[v]) = [\sigma_{\widehat{uv}}]$ holds for all u in $\mathbf{M}$ and v in Σ_1^*, by induction on the length of v.

We have $d_2 d_3 = 0$ because $s_1 s_2 = s_1 t_2$ and $t_1 s_2 = t_1 t_2$. Then, we define the following homomorphism of $\mathbb{Z}$-modules

$$\begin{array}{rcl} \mathbb{Z}\mathbf{M}[\Sigma_2] & \xrightarrow{i_3} & \mathbb{Z}\mathbf{M}[\Sigma_3] \\ u[\alpha] & \longmapsto & [\sigma_{\widehat{u}\alpha)}] \end{array}$$

where $\sigma_{\widehat{u}\alpha}$ is a 3-cell of $\Sigma_3^\top$ with the following shape, with $v = s(\alpha)$ and $w = t(\alpha)$:



Let us note that such a 3-cell necessarily exists in $\Sigma_3^\top$ because Σ_3 is an acyclic extension of the free $(2,1)$-category $\Sigma_2^\top$. Then we have, on the one hand,

$$i_2 d_2(u[\alpha]) \;=\; i_2(u[v] - u[w]) \;=\; [\sigma_{\widehat{u}v}] - [\sigma_{\widehat{u}w}]$$

and, on the other hand,

$$\begin{aligned} d_3 i_3(u[\alpha]) \;&=\; [\widehat{u}\alpha \star_1 \sigma_{\widehat{u}w}] - [\sigma_{\widehat{u}v}], \\ &=\; u[\alpha] + [\sigma_{\widehat{u}w}] - [\sigma_{\widehat{u}v}]. \end{aligned}$$

Hence $d_3 i_3 + i_2 d_2 = \mathrm{Id}_{\mathbb{Z}\mathbf{M}[\Sigma_2]}$, concluding the proof. □

4.3.3. Remark. The proof of Proposition 4.3.2 uses the fact that Σ_3 is an acyclic extension to produce, for every 2-cell α of Σ_2 and every u in $\mathbf{M}$, a 3-cell $\sigma_{\widehat{u}\alpha}$ with the required shape. The hypothesis on Σ_3 could thus be modified to only require the existence of such a 3-cell in $\Sigma_3^\top$. It is proved in [GM12b] that this implies that Σ_3 is an acyclic extension of the free $(2,1)$-category $\Sigma_2^\top$.

From Proposition 4.3.2, we deduce

4.3.4. Theorem ([CO94, Theorem 3.2], [Laf95, Theorem 3], [Pri95]). *Let* $\mathbf{M}$ *be a finitely presented monoid. If* $\mathbf{M}$ *is of finite derivation type, then it is of homological type left-*FP_3.

By Theorem 3.5.7, this implies

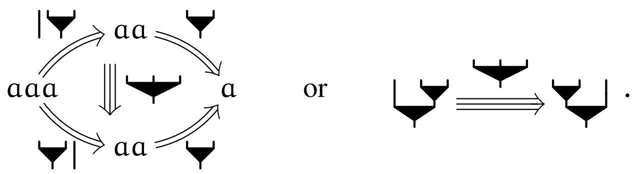
4.3.5. Theorem ([Squ87, Theorem 4.1]). *If a monoid admits a finite convergent presentation, then it is of homological type left-*FP_3.

4.3.6. Example. Let us consider the monoid **M** with the convergent presentation

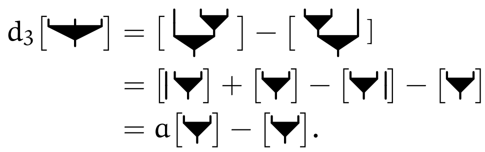
$$\langle\, a \mid \mu : aa \Rightarrow a \,\rangle.$$

With the leftmost normalisation strategy σ, we get, writing the 2-cell μ as a string diagram :

The presentation has exactly one critical branching, whose corresponding generating confluence can be written in the two equivalent ways



The $\mathbb{Z}\mathbf{M}$-module $\ker d_2$ is generated by



4.4. HOMOLOGY OF MONOIDS WITH INTEGRAL COEFFICIENTS

4.4.1. Morphism of resolutions. Let **M** be a monoid. Consider two free resolutions of the trivial $\mathbb{Z}\mathbf{M}$-module $\mathbb{Z}$ by $\mathbb{Z}\mathbf{M}$-modules

$$\mathcal{F}: \quad \cdots \longrightarrow F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

$$\mathcal{F}': \quad \cdots \longrightarrow F'_{n+1} \xrightarrow{d'_{n+1}} F'_n \xrightarrow{d'_n} F'_{n-1} \longrightarrow \cdots \longrightarrow F'_1 \xrightarrow{d'_1} F'_0 \xrightarrow{\varepsilon'} \mathbb{Z} \longrightarrow 0$$

A *homomorphism of resolutions* $f : \mathcal{F} \to \mathcal{F}'$ is a family of homomorphisms $f = (f_n : F_n \to F'_n)_{n\in\mathbb{N}}$ making the following diagrams commutative

$$\begin{array}{ccccccccccccccc}
\cdots & \longrightarrow & F_{n+1} & \xrightarrow{d_{n+1}} & F_n & \xrightarrow{d_n} & F_{n-1} & \longrightarrow & \cdots & \longrightarrow & F_1 & \xrightarrow{d_1} & F_0 & \xrightarrow{\varepsilon} & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \mathrm{Id}_{\mathbb{Z}}} & & \\
\cdots & \longrightarrow & F'_{n+1} & \xrightarrow{d'_{n+1}} & F'_n & \xrightarrow{d'_n} & F'_{n-1} & \longrightarrow & \cdots & \longrightarrow & F'_1 & \xrightarrow{d'_1} & F'_0 & \xrightarrow{\varepsilon'} & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

4.4.2. Homotopy of resolutions. Given two homomorphisms of resolutions $f, g : \mathcal{F} \to \mathcal{F}'$ given by

$$\begin{array}{ccccccccccccccc}
\cdots & \longrightarrow & F_{n+1} & \xrightarrow{d_{n+1}} & F_n & \xrightarrow{d_n} & F_{n-1} & \longrightarrow & \cdots & \longrightarrow & F_1 & \xrightarrow{d_1} & F_0 & \xrightarrow{\varepsilon} & \mathbb{Z} \longrightarrow 0 \\
 & & g_{n+1}\downarrow\ \downarrow f_{n+1} & & g_n\downarrow\ \downarrow f_n & & g_{n-1}\downarrow\ \downarrow f_{n-1} & & & & g_1\downarrow\ \downarrow f_1 & & g_0\downarrow\ \downarrow f_0 & & \downarrow \mathrm{Id}_{\mathbb{Z}} \\
\cdots & \longrightarrow & F'_{n+1} & \xrightarrow{d'_{n+1}} & F'_n & \xrightarrow{d'_n} & F'_{n-1} & \longrightarrow & \cdots & \longrightarrow & F'_1 & \xrightarrow{d'_1} & F'_0 & \xrightarrow{\varepsilon'} & \mathbb{Z} \longrightarrow 0
\end{array}$$

We say that f is *homotopic* to g if there exists a family of homomorphisms $h = (h_n : F_n \to F'_{n+1})_{n\in\mathbb{Z}}$ such that

$$f_0 - g_0 = d'_1 h_0,$$
$$f_n - g_n = d'_{n+1} h_n + h_{n-1} d_n,$$

for all $n \geqslant 1$. It is easy to see that homotopy is an equivalence relation on the set of homomorphisms of resolutions from $\mathcal{F}$ to $\mathcal{F}'$.

4.4.3. Proposition. *Between two free resolutions, there exists a homomorphism. Moreover, two such homomorphisms are homotopic.*

4.4.4. Exercise. Prove Proposition 4.4.3.

4.4.5. Homology with integral coefficients. Let $\mathbf{M}$ be a monoid. To a free resolution of the trivial $\mathbb{Z}\mathbf{M}$-module $\mathbb{Z}$ by left $\mathbb{Z}\mathbf{M}$-modules

$$\cdots \longrightarrow F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

we associate the following complex of $\mathbb{Z}$-modules

$$\cdots \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}\mathbf{M}} F_{n+1} \xrightarrow{\widetilde{d}_{n+1}} \mathbb{Z} \otimes_{\mathbb{Z}\mathbf{M}} F_n \xrightarrow{\widetilde{d}_n} \mathbb{Z} \otimes_{\mathbb{Z}\mathbf{M}} F_{n-1} \longrightarrow \cdots \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}\mathbf{M}} F_1 \xrightarrow{\widetilde{d}_1} \mathbb{Z} \otimes_{\mathbb{Z}\mathbf{M}} F_0$$

where $\widetilde{d}_n = \mathrm{Id} \otimes d_n$. Note that the $\mathbb{Z}$-module $\mathbb{Z} \otimes_{\mathbb{Z}\mathbf{M}} F_n$ is obtained from F_n by trivialising the action of $\mathbf{M}$, that is F_n quotiented by all relations $ux = x$ for $u \in \mathbf{M}$ and $x \in F_n$. In particular, if $F_n = \mathbb{Z}\mathbf{M}[X]$, then $\mathbb{Z} \otimes_{\mathbb{Z}\mathbf{M}} F_n = \mathbb{Z}[X]$ is the free $\mathbb{Z}$-module on X. We obtain a chain complex, because $d_n d_{n+1} = 0$ induces that $\widetilde{d}_n \widetilde{d}_{n+1} = 0$.

We define the n*-th homology group* of $\mathbf{M}$ with integral coefficient $\mathbb{Z}$ as the quotient $\mathbb{Z}$-module:

$$H_n(\mathbf{M}, \mathbb{Z}) = \ker(\widetilde{d}_n)/\mathrm{Im}\,(\widetilde{d}_{n+1}),$$

with the convention that $d_0 = 0$. For any monoid $\mathbf{M}$, we have $H_0(\mathbf{M}, \mathbb{Z}) \simeq \mathbb{Z}$.

4.4.6. Proposition. *For $n \geqslant 0$, the group $H_n(\mathbf{M}, \mathbb{Z})$ does not depend on a particular choice of a free resolution, but only on the monoid $\mathbf{M}$ itself.*

4.4.7. Exercise. Prove the Proposition 4.4.6.

4.4.8. Proposition. *If a monoid* $\mathbf{M}$ *is of homological type left-*FP_n *for all* $n \geqslant 0$*, then the groups* $H_n(\mathbf{M}, \mathbb{Z})$ *are all finitely generated.*

In particular, we have the following consequence that gives a necessary condition for a monoid to have a finite convergent presentation.

4.4.9. Corollary. *If a monoid admits a finite convergent presentation, then the group* $H_3(\mathbf{M}, \mathbb{Z})$ *is finitely generated.*

4.4.10. Exercise. Consider the monoid $\mathbf{M}$ presented by the following 2-polygraph:

$$\langle\, a, b, c \mid \alpha_n : ac^n b \Rightarrow 1,\ n \in \mathbb{N} \,\rangle.$$

1. Compute homology groups $H_1(\mathbf{M}, \mathbb{Z})$ and $H_2(\mathbf{M}, \mathbb{Z})$.

2. Show that $\mathbf{M}$ is a finitely generated monoid which cannot be finitely presented.

4.4.11. Exercise. Consider the monoid $\mathbf{M}$ presented by the following 2-polygraph:

$$\langle\, a, b, c, d \mid ab \Rightarrow a,\ da \Rightarrow ac \,\rangle.$$

Compute the homology groups $H_n(\mathbf{M}, \mathbb{Z})$, for $n = 1, 2, 3$.

4.4.12. Exercise. Consider the monoid $\mathbf{M}$ presented by the following 2-polygraph:

$$\langle\, a, b, c, d, d' \mid ab \Rightarrow a,\ da \Rightarrow ac,\ d'a \Rightarrow ac \,\rangle.$$

Compute the homology groups $H_n(\mathbf{M}, \mathbb{Z})$, for $n = 1, 2, 3$.

4.5. Historical notes

4.5.1. Homological finiteness condition. Jantzen in [Jan82, Jan85] asked the following question: does every finitely presented monoid with a decidable word problem admit a finite convergent presentation? At the end of the eighties, using a homological argument Squier answered the Jantzen question negatively by showing that there are finitely presented monoids with a decidable word problem which do not have a finite convergent presentation, [SO87, Squ87]. He linked the existence of a finite convergent presentation for a finitely presented monoid to the homological type left-FP_3 property, Theorem 4.3.5. He showed that a monoid needs to satisfy this invariant to have a finite convergent presentation. Giving examples, recalled in Example 3.5.13, of finitely presented monoids that have a decidable word problem and that do not have homological type left-FP_3, he proved that there are finitely presented monoids with a decidable word problem that cannot be presented by a finite convergent string rewriting system. However, it still remains open to characterize the class of monoids with a decidable word problem and having a finite convergent presentation. Squier result leads to the following question: is the homological finiteness condition left-FP_3 sufficient for a finitely presented monoid with a decidable word problem to admit a finite convergent presentation?

4.5.2. Homotopical finiteness condition. Squier answered this question negatively in an another article. In [SOK94], he related the existence of a finite convergent presentation to a new finiteness condition of finitely presented monoids, called *finite derivation type*, Definition 3.4.1. This property is a natural extension of the properties of being finitely generated and finitely presented. Squier defined the finite derivation type for a monoid as a finiteness property on a 2-dimensional combinatorial complex associated to a presentation of the monoid. Note that this complex was defined independently by Kilibarda, [Kil97], and Pride, [Pri95]. Squier proved that the finite derivation type property is an invariant property for finitely presented monoids, Theorem 3.4.4. As a consequence, the property finite derivation type can be defined for monoids independently of a considered presentation: a monoid is of *finite derivation type* if its finite presentations are of finite derivation type. The proof given by Squier is based on Tietze transformations. Finally, Squier proved that, if a monoid admits a finite convergent presentation, then it is of finite derivation type, Theorem 3.5.7. This result corresponds to a "homotopical" version of Newman's Lemma 5.5.12 for string rewriting systems. Squier used this result to give another proof that there exist finitely presented monoids with a decidable word problem that do not admit a finite convergent presentation. Moreover, he showed that the homological finiteness condition left-FP_3 is not sufficient for a finitely presented monoid with a decidable word problem to admit a finite convergent presentation. Indeed, he showed that the finitely presented monoid $\mathbf{S}_1$ given in Example 3.5.13 has a decidable word problem and is of homological type left-FP_3, but it is not of finite derivation type, and, thus, it does not admit a finite convergent presentation.

The article [SOK94] concludes with the following question: for finitely presented monoids does the property of having finite derivation type implies the existence of a finite convergent presentation? The answer is negative, indeed there exist finitely presented groups of homological type left-FP_3 that have undecidable word problems, [Mil92]. Since for finitely presented groups the property of having finite derivation type is equivalent to the homological type left-FP_3, [CO96], it follows that a finitely presented group can have an undecidable word problem even if it has finite derivation type. Hence in general the finite derivation property is not sufficient for the existence of a finite convergent presentation.

4.5.3. Extensions of Squier's finiteness conditions. By his results, Squier has opened a homological direction and a homotopical one, in the quest for a complete characterisation of the existence of finite convergent presentations of monoids. In the homological direction, it has been shown that a finitely presented monoid admitting a finite convergent presentation satisfies the more restrictive condition *homological type left*-FP_∞, Definition 4.2.1. Further proofs of the following result can be found in the literature.

4.5.4. Theorem ([Ani86, Kob90, Gro90, Bro92]). *If a monoid admits a finite convergent presentation, then it is of homological type left*-FP_∞.

The proofs are based on distinct ways to describe the n-fold critical branchings of a convergent rewriting system. Note that the converse implication of this result is false in general. By this fact, there were numerous finiteness conditions introduced with the goal to have a sufficient condition for the finite-convergence, [WP00, KO01, KO02, KO03, PO04, MPP05, GM13]. However, all these conditions were necessarily but not sufficient. The characterization of the class of finitely presented monoids having a presentation by a finite convergent rewriting system is still an open problem.

Beyond this problem, the methods initiated by Squier have opened the way to homotopical and homological analysis of rewriting systems. Moreover, it was shown in [Ani86, Kob90, Gro90, Bro92, Mal03, GHM19] that this methods highlight the way to compute "effectively" free resolutions for groups, monoids, associative algebras or small categories using rewriting.

Finally, the question of putting all this work in a higher-categorical framework was posed by Lafont and Métayer, [Laf95, Mét03, LM09]. In particular, is it possible to describe in the higher-categorical framework the constructions developed in [Ani86, Kob90, Gro90, Bro92]:

4.5.5. Question ([LM09, LMW10]). Is it true that a monoid presented by a finite convergent rewriting system always has a finite cofibrant approximation in the folk model structure on ∞-categories?

We will see that in fact the higher-dimensional strict categories constitute a natural setting for the analysis of rewriting systems.

CHAPTER
5

Linear rewriting

Contents

We must be careful when we rewrite in a linear structure defined over a field. For example, consider a rewriting system over a ring or an algebra. We expect that the rewriting rules are compatible with the linear structure in the following way. For a rewriting rule

$$f \to g$$

relating two elements of an algebra on a ground field $\mathbb{K}$, then for any scalar λ in $\mathbb{K}$ we would like the reduction:

$$\lambda f \to \lambda g,$$

and for any other element h of the algebra, we would like the following reduction:

$$f + h \to g + h.$$

Taken together, these two reductions lead to losing termination of rewriting. Indeed, in that case from the rule $f \to g$, we deduce the reductions $-f \to -g$ and $-f + (f + g) \to -g + (f + g)$. Finally, we deduce the following reduction

$$g \to f.$$

As a consequence, the system will never terminate. Further to this remark, it is necessary to adapt the notion of rewriting system to linear situations. In the example presented above the reduction $-f + (f + g) \to -g + (f + g)$ appears as the source of the nontermination problem.

There are two ways to solve this problem. The most well-known method is to choose an orientation of the rules induced by a *monomial order*, which is well-founded by definition, see 5.4.1. This approach is used in various paradigms of linear rewriting as recalled in Chapter 6. In this chapter, we present the categorical description of linear rewriting that extends to associative algebras the notion of 2-polygraph, with an appropriated notion of reduction. The constructions given in this chapter come from [GHM19].

The ground field will be denoted by $\mathbb{K}$. We denote by **Vect** the category of vector spaces over $\mathbb{K}$ and linear maps. This category is a monoidal category with the tensor product over $\mathbb{K}$ of vector spaces, denoted by $\otimes$. We will denote by **Alg** the category of (unital associative) algebras over $\mathbb{K}$.

5.1. Linear 2-polygraphs

We have seen in (2.1.2) that a category can be thought of as a "monoid with several 0-cells". Similarly, the notion of 1-algebroid describes the concept of associative algebra with several 0-cells.

5.1.1. Algebroids. A 1-*algebroid* over a ground field $\mathbb{K}$ is a category enriched over the monoidal category **Vect**. Explicitly, a 1-algebroid $\mathbf{A}$ is specified by the following data:

i) a set $\mathbf{A}_0$ of 0-*cells*, that we will denote by p, q...

ii) for every 0-cells p and q, a vector space $\mathbf{A}(p, q)$, whose elements are the 1-*cells* of $\mathbf{A}$, with *source* p and *target* q, that we will denote by f, g...

iii) for every 0-cells p, q and r, a linear map

$$\star_0 : \mathbf{A}(p, q) \otimes \mathbf{A}(q, r) \longrightarrow \mathbf{A}(p, r)$$

called the 0-*composition* of $\mathbf{A}$ and whose image on $f \otimes g$ is denoted by $f \star_0 g$ or fg. This composition is *associative*, that is the relation:

$$(f \star_0 g) \star_0 h = f \star_0 (g \star_0 h),$$

holds for any 0-composable 1-cells f, g and h, and *unitary*, that is, for any 0-cell p, there is a 1-cell 1_p such that for any 1-cell f in $\mathbf{A}(p, q)$, the following relation holds

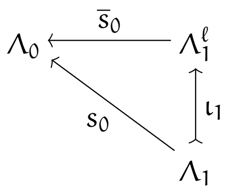
$$1_p \star_0 f = f \star_0 1_q = f.$$

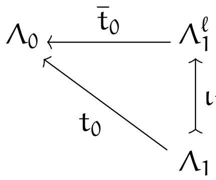
A 1-cell f with source p and target q will be graphically represented by

$$p \xrightarrow{\ f\ } q$$

5.1.2. Remarks. A 1-*algebra* is a 1-algebroid with a single one 0-cell, that can be identified to an algebras over $\mathbb{K}$. The notion of 1-algebroid was first introduced by Mitchell as *ring with several objects* called $\mathbb{K}$-*category* in [Mit72], terminology *linear category* appear also in the literature. A small $\mathbb{Z}$-category is called a *ringoid* and a one-0-cell ringoid is a ring.

5.1.3. Free 1-algebroid. The free 1-algebroid on a 1-polygraph $\Lambda = (\Lambda_0, \Lambda_1)$ is the 1-algebroid, denoted by Λ_1^ℓ, whose set of 0-cells is Λ_0, and for any 0-cells p and q, $\Lambda_1^\ell(p,q)$ is the free vector space on $\Lambda_1^*(p,q)$. In other words, any 1-cell in the space $\Lambda_1^\ell(p,q)$ is a linear combination of paths from p to q generated by the 1-polygraph Λ. If Λ_0 has only one 0-cell, Λ_1^ℓ is the free algebra with basis Λ_1. The source and target maps s_0 and t_0 of the 1-polygraph Λ are extended into maps on Λ_1^ℓ, denoted by $\overline{s}_0$ and $\overline{t}_0$, in a natural way making the following two diagrams commutative:



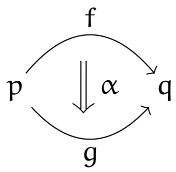


where ι_1 denotes the inclusion of 1-cells of Λ_1 in the free algebroid Λ_1^ℓ.

5.1.4. Two-dimensional linear polygraphs. A *cellular extension* of the 1-algebroid Λ_1^ℓ is a set Λ_2 equipped with two maps

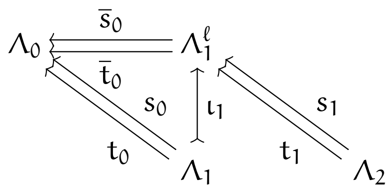
$$\Lambda_1^\ell \underset{t_1}{\overset{s_1}{\leftleftarrows}} \Lambda_2$$

such that, for every α in Λ_2, the pair $(s_1(\alpha), t_1(\alpha))$ is a 1-sphere in Λ_1^ℓ, that is, the following *globular relations* hold $s_0 s_1(\alpha) = s_0 t_1(\alpha)$ and $t_0 s_1(\alpha) = t_0 t_1(\alpha)$. As in the non linear situation of (2.1.10), an element of the cellular extension Λ_2 will be graphically represented by a 2-cell with the following globular shape



that relates parallel 1-cells f and g in Λ_1^ℓ, also denoted by $f \overset{\alpha}{\Rightarrow} g$ or by $\alpha : f \Rightarrow g$.

We define a *linear 2-polygraph* as a triple $(\Lambda_0, \Lambda_1, \Lambda_2)$, where (Λ_0, Λ_1) is a 1-polygraph and Λ_2 is a cellular extension of the free 1-algebroid Λ_1^ℓ:



The elements of Λ_2 are called the 2-*cells* of Λ, or the *rewriting rules* of Λ.

In the sequel, we will consider polygraphs with one 0-cell denoted $\bullet$.

5.1.5. The ideal of a linear 2-polygraph. Given a linear 2-polygraph Λ. We denote by $I(\Lambda)$ the two-sided ideal of the free algebra Λ_1^ℓ generated by the following set of 1-cells

$$\{s_1(\alpha) - t_1(\alpha) \mid \alpha \in \Lambda_2\}.$$

The ideal $I(\Lambda)$ is made of the linear combinations

$$\sum_{i=1}^{p} \lambda_i u_i (s_1(\alpha_i) - t_1(\alpha_i)) v_i,$$

for pairwise distinct 2-monomials $u_1\alpha_1 v_1, \ldots, u_p \alpha_p v_p$ of Λ_1^ℓ, and nonzero scalars $\lambda_1, \ldots, \lambda_p$.

5.1.6. Presentations of algebras. The *algebra presented* by a linear 2-polygraph Λ, and denoted by $\overline{\Lambda}$, is the quotient of the free algebra Λ_1^ℓ by the two-sided ideal $I(\Lambda)$. We denote by $\overline{f}$ the image of a 1-cell f of Λ_1^ℓ through the canonical projection

$$\pi : \Lambda_1^\ell \twoheadrightarrow \mathbf{A}$$

We say that a linear 2-polygraph Λ is a *presentation* of an algebra $\mathbf{A}$ if the algebra presented by Λ is isomorphic to $\mathbf{A}$. Two linear 2-polygraphs are said to be *Tietze equivalent* if they present isomorphic algebras.

5.1.7. First toy example. Here our first toy example that we will use through this chapter:

$$\Lambda = \langle\, x, y, z \mid xyz \overset{\gamma}{\Rightarrow} x^3 + y^3 + z^3 \,\rangle.$$

The free 1-algebroid generated by $\Lambda_1 = \{x, y, z\}$ is the free algebra $\mathbb{K}\langle x, y, z\rangle$. The algebra presented by the linear 2-polygraph Λ is the quotient of the algebra $\mathbb{K}\langle x, y, z\rangle$ by the two-sided ideal generated by the 1-cell $xyz - x^3 - y^3 - z^3$.

5.1.8. Other toy examples. We will consider the two following Tietze equivalent linear 2-polygraphs:

$$\langle\, x, y \mid x^2 \Rightarrow yx \,\rangle, \qquad \langle\, x, y \mid yx \Rightarrow x^2 \,\rangle.$$

5.1.9. 2-algebras. We define a 2-*algebra* $\mathbf{A}$ as an internal 1-category in the category **Alg**. Explicitly, it is defined by a diagram

$$\mathbf{A}_1 \underset{i_2}{\overset{t_1}{\underset{s_1}{\leftleftarrows\!\!\!\rightarrow}}} \mathbf{A}_2 \xleftarrow{\ \star_1\ } \mathbf{A}_2 \times_{\mathbf{A}_1} \mathbf{A}_2 \tag{5.1}$$

where $\mathbf{A}_2 \times_{\mathbf{A}_1} \mathbf{A}_2$ is the algebra defined by the following pullback diagram in the category **Alg**:

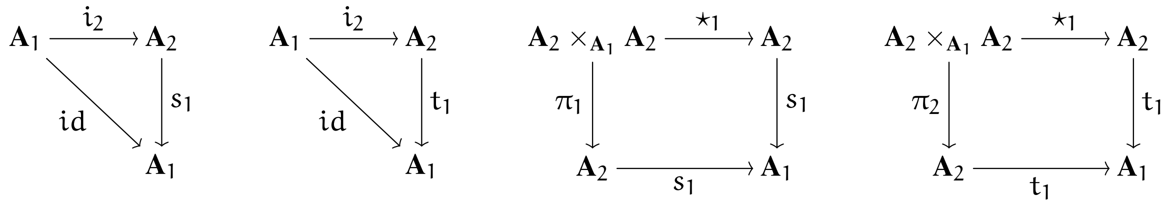
$$\begin{array}{ccc} \mathbf{A}_2 \times_{\mathbf{A}_1} \mathbf{A}_2 & \longrightarrow & \mathbf{A}_2 \\ \big\downarrow & \lrcorner & \big\downarrow s_1 \\ \mathbf{A}_2 & \xrightarrow[t_1]{} & \mathbf{A}_1 \end{array}.$$

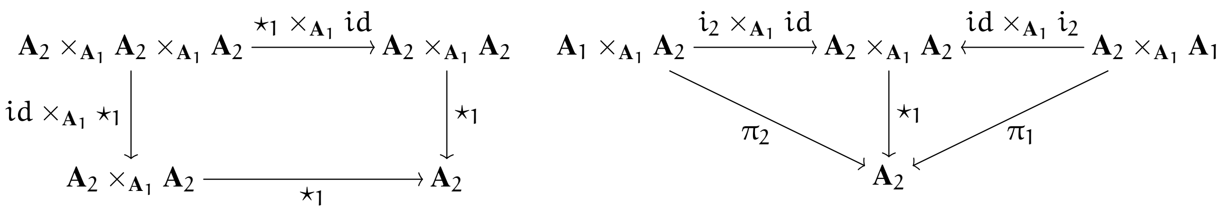
Elements of the algebra $\mathbf{A}_2 \times_{\mathbf{A}_1} \mathbf{A}_2$ are pairs (a, a') of 1-*composable* 2-cells a and a', that is satisfying $t_1(a) = s_1(a')$. We denote by ab the product of two 2-cells a and b in the algebra $\mathbf{A}_2$. The linear structure and the product in the algebra $\mathbf{A}_2 \times_{\mathbf{A}_1} \mathbf{A}_2$ are given by setting

$$\begin{aligned} (a, a') + (b, b') &= (a + b, a' + b'), \\ \lambda(a, a') &= (\lambda a, \lambda a'), \\ (a, a')(b, b') &= (ab, a'b'), \end{aligned}$$

for all pair of 1-composable 2-cells (a, a') and (b, b') and scalar λ in $\mathbb{K}$.

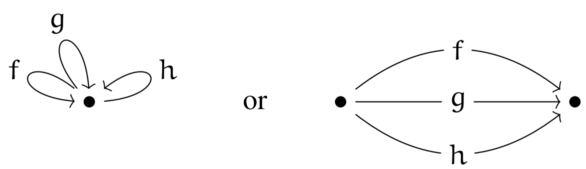
The morphisms of algebras s_1, t_1 and $\star_1$ satisfy the axioms in such a way that Diagram (5.1) defines a 1-category. Explicitly, the following diagrams commute in the category **Alg**:



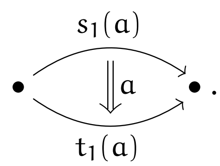


where π_1 and π_2 denote respectively first and second projection.

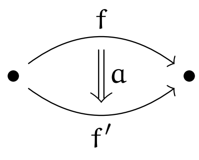
5.1.10. Notations. For a 1-cell f, the identity 2-cell $i_2(f)$ is denoted by 1_f, or f if there is no possible confusion. The 1-composite $\star_1(a, a')$ of 1-composable 2-cells a and a', will be denoted by $a \star_1 a'$. Elements of the algebra $\mathbf{A}_1$, called 1-*cells of* $\mathbf{A}$, are graphically pictured as follows



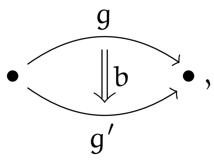
The elements of $\mathbf{A}_2$, called 2-*cells of* $\mathbf{A}$ are graphically represented by



Given 2-cells



and



,

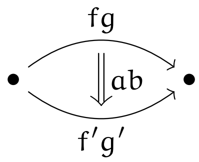
the source and target maps s_1 and t_1 being morphisms of algebras, we have

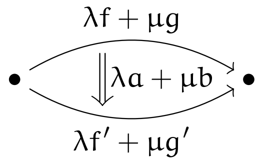
$$s_1(ab) = s_1(a)s_1(b), \qquad \text{and} \qquad t_1(ab) = t_1(a)t_1(b),$$

and for any scalars λ and μ in $\mathbb{K}$, we have

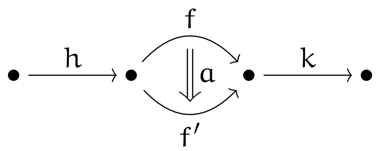
$$s_1(\lambda a + \mu b) = \lambda s_1(a) + \mu s_1(b), \qquad \text{and} \qquad t_1(\lambda a + \mu b) = \lambda t_1(a) + \mu t_1(b).$$

Hence



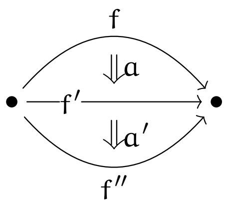


Given 1-cells h, f, f' and k in $\mathbf{A}_1$ and a 2-cell a in $\mathbf{A}_2$ such that

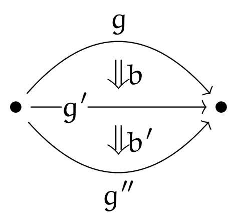


we will denote by $hak : hfk \Rightarrow hf'k$ the 0-composite $1_h \star_0 a \star_0 1_k$.

5.1.11. Properties of 1-composition. Given 1-composable 2-cells:



and



in $\mathbf{A}_2 \star_{\mathbf{A}_1} \mathbf{A}_2$, the 1-composition $\star_1$ being linear, $a \star_1 a' + b \star_1 b'$ is a 2-cell from $f + g$ to $f' + g'$ and we have

$$\begin{aligned}(a + b) \star_1 (a' + b') &= \star_1(a + b, a' + b'),\\ &= \star_1(a, a') + \star_1(b, b'),\\ &= a \star_1 a' + b \star_1 b'.\end{aligned}$$

Furthermore, for any scalar λ in $\mathbb{K}$, $\lambda(a \star_1 a')$ is a 2-cell from λf to $\lambda f''$ and we have

$$(\lambda a) \star_1 (\lambda a') = \lambda(a \star_1 a').$$

Finally, the compatibility with the product induces that $\star_1((a,a')(b,b')) = \star_1(ab, a'b')$. Hence, we have

$$(a \star_1 a')(b \star_1 b') = ab \star_1 a'b'. \tag{5.2}$$

Relation (5.2) corresponds to the *exchange law* in the 2-algebra **A** between the 1-composition and the product.

5.1.12. Remarkable identities in a 2-algebra. The following properties hold in a 2-algebra **A**

i) for any 1-composable 2-cells a and a' in **A**, we have

$$a \star_1 a' = a + a' - t_1(a), \tag{5.3}$$

ii) any 2-cell a in **A** is invertible for the $\star_1$-composition, and its inverse is given by

$$a^- = -a + s_1(a) + t_1(a). \tag{5.4}$$

iii) for any 2-cells a and b in **A**, we have

$$ab = as_1(b) + t_1(a)b - t_1(a)s_1(b) = s_1(a)b + at_1(b) - s_1(a)t_1(b). \tag{5.5}$$

Relation (5.3) is a consequence of the linearity of the 1-composition $\star_1$. Indeed, for any (a, a') in $\mathbf{A}_2 \times_{\mathbf{A}_1} \mathbf{A}_2$, we have

$$\begin{aligned} a \star_1 a' &= (a - s_1(a') + s_1(a')) \star_1 (t_1(a) - t_1(a) + a'), \\ &= a \star_1 t_1(a) - s_1(a') \star_1 t_1(a) + s_1(a') \star_1 a', \\ &= a - t_1(a) + a'. \end{aligned}$$

5.1.13. Exercice. Show identities (5.4) and (5.5).

5.1.14. The free 2-algebra on a linear 2-polygraph. The *free 2-algebra over a linear 2-polygraph* Λ is the 2-algebra, denoted by Λ_2^ℓ, defined as follows. In dimension 1, it is the free 1-algebra Λ_1^ℓ over Λ_1. For dimension 2, we consider the following diagram in the category of Λ_1^ℓ-bimodule

$$\Lambda_1^\ell \underset{i_2}{\overset{t_1}{\underset{s_1}{\leftleftarrows\!\!\rightarrow}}} \Lambda_2^{\mathcal{M}},$$

where Λ_1^ℓ is seen as Λ_1^ℓ-bimodule, $\Lambda_2^{\mathcal{M}}$ is the Λ_1^ℓ-bimodule $\left(\Lambda_1^\ell \otimes \mathbb{K}\Lambda_2 \otimes \Lambda_1^\ell\right) \oplus \Lambda_1^\ell$ and where the linear maps s_1, t_1 and i_2 are defined by:

$$s_1(f\alpha g) = fs_1(\alpha)g, \qquad t_1(f\alpha g) = ft_1(\alpha)g \qquad \text{and} \qquad s_1(h) = t_1(h) = i_2(h) = h,$$

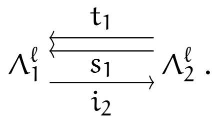
for all 2-cell α in Λ_2, and 1-cells f, g, h in Λ_1^ℓ. The quotient of the Λ_1^ℓ-bimodule $\Lambda_2^{\mathcal{M}}$ by the equivalence relation generated by

$$as_1(b) + t_1(a)b - t_1(a)s_1(b) \sim s_1(a)b + at_1(b) - s_1(a)t_1(b),$$

for all a and b in $\Lambda_1^\ell \otimes \mathbb{K}\Lambda_2 \otimes \Lambda_1^\ell$, has a structure of algebra, denoted by Λ_2^ℓ, and whose product is given by

$$ab = as_1(b) + t_1(a)b - t_1(a)s_1(b).$$

One proves that the source and target maps are compatible with this quotient, so giving a structure of 2-algebra:



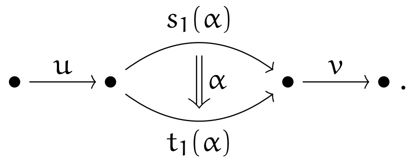
5.1.15. Exercise. Let Λ be a linear 2-polygraph. Given 1-cells f and g in Λ_1^ℓ, show that the 1-cell $f - g$ belongs to $I(\Lambda)$ if and only if there exists a 2-cell $a : f \Rightarrow g$ in Λ_2^ℓ. As a consequence, the algebra presented by Λ is obtained by identifying in Λ_1^ℓ all the 1-cells $s_1(a)$ and $t_1(a)$, for every 2-cell a in Λ_2^ℓ.

5.1.16. Monomials. A *monomial* in the free 2-algebra Λ_2^ℓ is a 1-cell of the free monoid Λ_1^* over Λ_1. The set monomials of Λ_2^ℓ, also denoted by Λ_1^*, forms a linear basis of the free algebra Λ_1^ℓ. As a consequence, every nonzero 1-cell f of Λ_1^ℓ can be uniquely written as a linear combination of pairwise distinct monomials $u_1, \ldots, u_p$:

$$f = \lambda_1 u_1 + \ldots + \lambda_p u_p$$

with $\lambda_i \in \mathbb{K} \setminus \{0\}$, for all $i = 1, \ldots, p$. The set of monomials $\{u_1, \ldots, u_p\}$ will be called the *support* of f and denoted by $\mathrm{Supp}(f)$.

5.1.17. 2-monomials. A 2-*monomial* of a free 2-algebra Λ_2^ℓ is a 2-cell of Λ_2^ℓ with shape $u\alpha v$, where α is a 2-cell in Λ_2, and u and v are monomials in Λ_1^*:



By construction of the free 2-algebra Λ_2^ℓ, and by freeness of Λ_1^ℓ, every non-identity 2-cell a of Λ_2^ℓ can be written as a linear combination of pairwise distinct 2-monomials $a_1, \ldots, a_p$ and of an 1-cell h of Λ_1^ℓ:

$$a = \lambda_1 a_1 + \ldots + \lambda_p a_p + h. \tag{5.6}$$

5.1.18. Exercise. Prove that the decomposition in (5.6) is unique up to the following relations

$$as_1(b) + t_1(a)b - t_1(a)s_1(b) = s_1(a)b + at_1(b) - s_1(a)t_1(b), \tag{5.7}$$

for all 2-monomials a and b in Λ_2^ℓ.

5.1.19. Monomial linear 2-polygraphs. A linear 2-polygraph Λ is *left-monomial* if, for every 2-cell α of Λ_2, the source $s_1(\alpha)$ is a monomial in $\Lambda_1^* \setminus \mathrm{Supp}(t_1(\alpha))$. Note that a non-left monomial linear 2-polygraph would produce useless ambiguity only due to the linear structure.

A linear 2-polygraph Λ is *monomial* if it is left-monomial and for every 2-cell α of Λ_2, $t_1(\alpha) = 0$ holds. A *monomial algebra* is an algebra admitting a presentation by a monomial linear 2-polygraph.

5.1.20. Exercise. Show that any linear 2-polygraph is Tietze equivalent to a left-monomial linear 2-polygraph.

5.1.21. Examples. The linear 2-polygraph Λ given in Example 5.1.7 is left-monomial. The linear 2-polygraph $\langle\ x, y \mid x^2 + y^2 \Rightarrow 2xy\ \rangle$ is not left-monomial, but it is Tietze equivalent to the following left-monomial 2-polygraph:

$$\Lambda' = \langle\, x, y \mid xy \overset{\alpha'}{\Rightarrow} \frac{1}{2}(x^2 + y^2)\, \rangle.$$

The linear 2-polygraphs $\langle\, x \mid x^2 \Rightarrow 0\, \rangle$ and $\langle\, x, y \mid xy \Rightarrow 0\, \rangle$ are monomials.

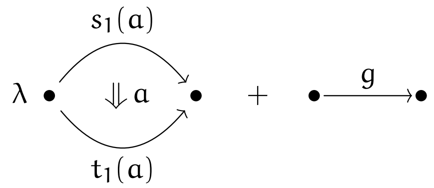
5.1.22. Degrees and length. For monomials u and v in Λ_1^*, we denote by $\deg_v u$ the number of different occurrences of the monomial v in the monomial u. For instance $\deg_{x^2} x^4 = 3$ and $\deg_y x^4 = 0$. For a subset M of monomials in Λ_1^*, we denote

$$\deg_M u = \sum_{v \in M} \deg_v u.$$

The *length* of a monomial u in Λ_1^*, denoted by $\ell(u)$, is equal to $\deg_{\Lambda_1} u$.

5.2. Linear rewriting steps

5.2.1. Elementary 2-cells. Let Λ be a linear 2-polygraph. An *elementary* 2-cell of the free 2-algebra Λ_2^ℓ is a 2-cell of Λ_2^ℓ with shape



where a is a 2-monomial, g is a 1-cell of Λ_1^ℓ and λ is a nonzero scalar in $\mathbb{K}$.

5.2.2. Example. With the polygraph Λ' of Example 5.1.21, the 2-cell

$$2x\alpha'y + y^3 : 2x^2y^2 \Rightarrow x^3y + xy^3 - y^3$$

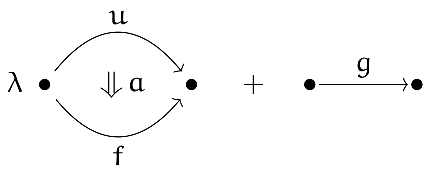
is elementary and the 2-cell

$$x\alpha' + \alpha'y : x^2y + xy^2 \Rightarrow \frac{1}{2}(x^3 + xy^2 + x^2y + y^3)$$

is not elementary.

5.2.3. Exercise. Show that any 2-cell in a free 2-algebra Λ_2^ℓ can be decomposed into a 1-composition of elementary 2-cells of Λ_2^ℓ

5.2.4. Rewriting steps. Let Λ be a left-monomial linear 2-polygraph. A *rewriting step of* Λ is an elementary 2-cell



of Λ_2^ℓ such that λ is a nonzero scalar and u is not in the support of g.

5.2.5. Examples. For the linear 2-polygraph given in Example 5.1.7, the 2-cell

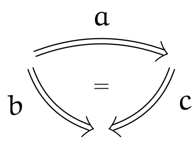
$$3x\gamma - 3xz^3 : 3x^2yz - 3xz^3 \Longrightarrow 3x^4 + 3xy^3$$

is a rewriting step. For a linear 2-polygraph having a rule $\alpha : u \Rightarrow f$, the 2-cell

$$-\alpha + (u+f) : -u + (u+f) \Longrightarrow -f + (u+f)$$

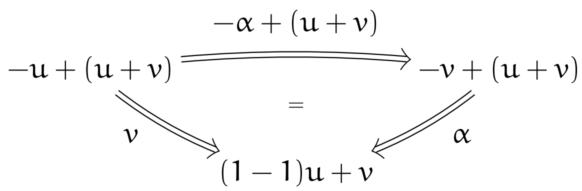
is not a rewriting step because the monomial u appears in the context $u + f$.

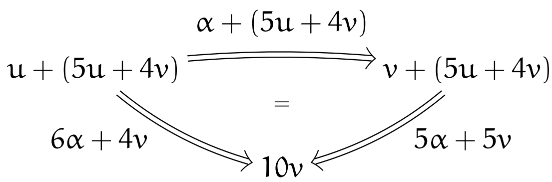
5.2.6. Exercise. Let Λ be a left-monomial linear 2-polygraph and let a be an elementary 2-cell of the 2-algebra Λ_2^ℓ. Show that a can be factorised in the 2-algebra Λ_2^ℓ into



where b and c are either identities of rewriting steps.

5.2.7. Example. Let Λ be a linear 2-polygraph and let $\alpha : u \Rightarrow v$ be a 2-cell of Λ_2. The 2-cell $-\alpha + (u+v)$ and $\alpha + (5u + 4v)$ are not rewriting steps of Λ. They can be decomposed respectively as follows:





5.2.8. Rewriting sequences. A 2-cell a of Λ_2^ℓ is *positive*, or a *rewriting sequence*, if it is an identity or a 1-composite

$$f_0 \overset{a_1}{\Rightarrow} f_1 \Rightarrow \cdots \Rightarrow f_{k-1} \overset{a_k}{\Rightarrow} f_k$$

of rewriting steps of Λ.

5.2.9. Reduced cells. A 1-cell f of Λ_1^ℓ is called *reduced*, or *irreducible*, *with respect to* Λ_2, if there is no rewriting step of Λ with source f. As a consequence, a 1-cell is reduced if and only if it is the zero 1-cell of Λ_1^ℓ, or a linear combination of reduced monomials in Λ_1^*. The reduced 1-cells of Λ_1^ℓ form a vector subspace of Λ_1^ℓ, denoted by Λ_1^{nf}. Since Λ is left-monomial, the set of reduced monomials of Λ_1^*, denoted by Λ_1^{irm}, forms a basis of the vector space Λ_1^{nf}.

We denote by $s_1(\Lambda)$ the set of *redex* of a reduced left-monomial linear 2-polygraph Λ defined by

$$s_1(\Lambda) = \{s_1(\alpha) \mid \alpha \text{ in } \Lambda_2\}.$$

In [Ani86], a redex is called an *obstruction*. The number of possible application of rules of Λ_2 to a monomial u is $\deg_{s_1(\Lambda)} u$.

5.2.10. Reduced linear 2-polygraphs. We say that a linear 2-polygraph Λ is *left-reduced* if, for every 2-cell α in Λ_2, the 1-cell $s_1(\alpha)$ is reduced with respect to $\Lambda_2 \setminus \{\alpha\}$. We say that Λ is *right-reduced* if, for every 2-cell α of Λ, the 1-cell $t_1(\alpha)$ is reduced. The linear polygraph Λ is *reduced* if it is both left-reduced and right-reduced.

5.2.11. Exercise. Show that any left-monomial linear 2-polygraph is Tietze equivalent to a reduced left-monomial linear 2-polygraph.

5.2.12. Normal forms. If f is a 1-cell of Λ_1^ℓ, a *normal form for* f *with respect to* Λ_2 is a reduced 1-cell g of Λ_1^ℓ such that there exists a positive 2-cell $a : f \Rightarrow g$ in Λ_2^ℓ.

5.3. TERMINATION FOR LINEAR 2-POLYGRAPHS

We recall the notion of rewrite relation for linear 2-polygraphs from [GHM19]. Let us fix a left-monomial linear 2-polygraph Λ.

5.3.1. Termination. The *rewrite relation of* Λ is the binary relation, denoted by $\prec_\Lambda$ on the set of monomial Λ_1^* defined by

- **i)** $w \prec_\Lambda u$ for every 2-cell $\alpha : u \Rightarrow f$ of Λ_2 and every monomial w in $\mathrm{Supp}(f)$,
- **ii)** $u' \prec_\Lambda u$ implies $vu'w \prec_\Lambda vuw$ for all monomials u, u', v and w of Λ_1^*.

We say that Λ *terminates* if its rewrite relation $\prec_\Lambda$ is wellfounded, that is, there is no infinite descending chains in Λ_1^*:

$$u_1 \succ_\Lambda u_2 \succ_\Lambda u_3 \succ_\Lambda \ldots \succ_\Lambda u_n \succ_\Lambda u_{n+1} \succ_\Lambda \ldots$$

5.3.2. Example. Consider the linear 2-polygraph $\Lambda = \langle\, x, y \mid xy \overset{\alpha}{\Rrightarrow} x^2 + y^2 \,\rangle$. We have $x^2 \prec_\Lambda xy$ and $y^2 \prec_\Lambda xy$. It follows that

$$x^2y \succ_\Lambda xy^2 \succ_\Lambda x^2y.$$

Hence the relation $\prec_\Lambda$ is not a wellfounded and the polygraph is not terminating. Note that, we have an infinite sequence of rewriting steps:

$$x^2y \overset{x\alpha}{\Rrightarrow} x^3 + xy^2 \xRightarrow{x^3 + \alpha y} x^3 + y^3 + x^2y \Rrightarrow \ldots$$

5.3.3. The rewrite relation on 1-cells. The rewrite relation $\prec_\Lambda$ is extended to the 1-cells of Λ_1^ℓ by setting, for any 1-cells f and g, $g \prec_\Lambda f$ if the following two conditions hold

- **i)** there exists a monomial w in $\mathrm{Supp}(f)$ which is not in $\mathrm{Supp}(g)$,
- **ii)** for any monomial v in $\mathrm{Supp}(g) \setminus \mathrm{Supp}(f)$, there exists a monomial u in $\mathrm{Supp}(f) \setminus \mathrm{Supp}(g)$, such that $v \prec_\Lambda u$

5.3.4. Proposition. *The rewrite relation $\prec_\Lambda$ is wellfounded on 1-cells if and only if it is wellfounded on monomials.*

If Λ terminates, then for every rewriting step a of Λ, we have $t_1(a) \prec_\Lambda s_1(a)$. This implies that the 2-algebra Λ_2^ℓ contains no infinite sequence of pairwise 1-composable rewriting steps

$$f_0 \overset{a_1}{\Rrightarrow} f_1 \Rrightarrow \cdots \Rrightarrow f_{k-1} \overset{a_k}{\Rrightarrow} f_k \Rrightarrow \cdots$$

so that every 1-cell of Λ_1^ℓ admits at least one normal form with respect to Λ_2.

5.4. Monomial orders

5.4.1. Monomial orders. A total order $\prec$ on the set of monomials Λ_1^* is a *monomial order* if the following conditions are satisfied

- **i)** $\prec$ is a well-order, that is, there is no infinite descending chains in Λ_1^*.

$$u_1 \succ u_2 \succ u_3 \succ \ldots \succ u_n \succ u_{n+1} \succ \ldots$$

- **ii)** $\prec$ is compatible with the multiplicative structure on monomials, that is

$$u \prec u' \quad \text{implies} \quad vuw \prec vu'w,$$

for all monomials u, u', v and w in Λ_1^*.

5.4.2. Example. Given a total order relation $\prec$ on Λ_1, we define the *left degree-wise lexicographic order generated by* $\prec$, or *deglex order generated by* $\prec$, as the order $\prec_{\text{deglex}}$ on Λ_1^* that compare two monomials first by degree and then lexicographically. It is defined by

- **i)** $y_1 \ldots y_p \prec_{\text{deglex}} x_1 \ldots x_q$, if $p < q$,
- **ii)** $y_1 \ldots y_{j-1} y_j \ldots y_p \prec_{\text{deglex}} y_1 \ldots y_{j-1} x_j \ldots x_p$, if $y_j \prec x_j$.

5.4.3. Exercise. Show that the order $\prec_{\text{deglex}}$ is a monomial order.

5.4.4. Exercise. Explain why the pure lexicographic order is not a monomial order. Show that it is neither a well-order nor compatible with the product of monomials.

5.4.5. Polygraph compatible with a monomial order. A linear 2-polygraph Λ is say to be *compatible with a monomial order* $\prec$ if for every 2-cell $\alpha : u \Rightarrow f$ of Λ_2, then $w \prec u$ for any monomial w in the support of f. The monomial order $\prec$ is thus a well-founded rewrite relation for Λ. It follows that any linear 2-polygraph compatible with a monomial order is terminating. The converse is false in general as we will see in Exercise 5.4.7.

5.4.6. Example. Consider the linear 2-polygraph $\Lambda = \langle\, x, y \mid x^2 \overset{\alpha}{\Rightarrow} xy - y^2 \,\rangle$. It is Tietze equivalent to the linear 2-polygraph of Example 5.3.2, but it is terminating. Indeed, having $xy \prec x^2$ and $y^2 \prec x^2$, the linear 2-polygraph Λ is compatible with the deglex order $\prec_{\text{deglex}}$ induced by $y \prec x$, hence it is terminating. An other way to prove that Λ is terminating, is to count the number of occurrence of x in monomials. For any u in Λ_1^*, let denote by $A(u)$ the number of occurrence of x in u. To prove that the linear 2-polygraph Λ terminates, it is sufficient to check that, for every rewriting step $a : s_1(a) \Rightarrow f$, we have $A(s_1(a)) > A(v)$, for any monomial v in $\mathrm{Supp}(f)$.

5.4.7. Exercise. Show that the linear 2-polygraph Λ given in Example 5.1.7 is terminating. Show that Λ is not compatible with a monomial order.

5.4.8. Exercise, [Ber78, Exercice 5.2.1.]. Examine termination of the linear 2-polygraph $\langle\, x, y \mid \alpha \,\rangle$ in each of the following situations

$$x^2y \overset{\alpha}{\Rightarrow} yx, \qquad yx \overset{\alpha}{\Rightarrow} x^2y, \qquad x^2y^2 \overset{\alpha}{\Rightarrow} yx, \qquad yx \overset{\alpha}{\Rightarrow} x^2y^2.$$

5.4.9. Noetherian induction. Let us recall the principle of noetherian induction for terminating rewriting systems, see [Hue80] for more details. Let Λ be a left-monomial terminating linear 2-polygraph. Given a property $\mathcal{P}(f)$ of the 1-cells f of Λ_1^ℓ. In order to show that $\mathcal{P}(f)$ holds for any 1-cell f of Λ_1^ℓ, it suffices to show that

- **i)** $\mathcal{P}(f)$ holds for f reduced with respect to Λ_2,
- **ii)** $\mathcal{P}(f)$ holds under the assumption that $\mathcal{P}(g)$ is hold for every $g \prec_\Lambda f$.

5.4.10. Leading terms. Let Λ_1^ℓ be a free algebra over a set Λ_1 and let $\prec$ be a monomial order on Λ_1^ℓ. For a nonzero 1-cell f of Λ_1^ℓ, the *leading monomial of* f *with respect to* $\prec$ is the monomial of f, denoted by $\mathrm{lm}(f)$, such that $w \prec \mathrm{lm}(f)$, for any monomial w in the support of f. The *leading coefficient of* f is the coefficient $\mathrm{lc}(f)$ of $\mathrm{lm}(f)$ in f, and the *leading term of* f is the 1-cell $\mathrm{lt}(f) = \mathrm{lc}(f)\,\mathrm{lm}(f)$ of Λ_1^ℓ. We also define $\mathrm{lt}(0) = \mathrm{lc}(0) = \mathrm{lm}(0) = 0$.

Note that for any 1-cells f and g in Λ_1^ℓ, we have $f \prec g$ if and only if either $\mathrm{lm}(f) \prec \mathrm{lm}(g)$ or ($\mathrm{lm}(f) = \mathrm{lm}(g)$ and $f - \mathrm{lt}(f) \prec g - \mathrm{lt}(g)$). The following property

$$\mathrm{lt}(fg) = \mathrm{lt}(f)\,\mathrm{lt}(g),$$

for any 1-cells f and g is also useful.

5.4.11. Leading polygraph. Given a monomial order $\prec$ on Λ_1^ℓ and a nonzero 1-cell g in Λ_1^ℓ, we define the 2-cell:

$$\alpha_{g,\prec} : \mathrm{lm}(g) \implies \mathrm{lm}(g) - \frac{1}{\mathrm{lc}(g)} g.$$

For any set $\mathcal{G}$ of nonzero 1-cells in Λ_1^ℓ, the *leading* 2*-polygraph* associated to $\mathcal{G}$ with respect to $\prec$ is the linear 2-polygraph $\Lambda(\mathcal{G}, \prec)$ whose set of 1-cells is Λ_1 and

$$\Lambda(\mathcal{G}, \prec)_2 = \{\alpha_{g,\prec} \mid g \in \mathcal{G}\}.$$

By definition, the leading polygraph $\Lambda(\mathcal{G} \prec)$ is compatible with the monomial order $\prec$.

A monomial w in Λ_1^* is *$\mathcal{G}$-reduced with respect to the monomial order* $\prec$ if it reduced with respect to $\Lambda(\mathcal{G}, \prec)_2$, that is, there is no factorisation $w = u\,\mathrm{lm}(g)v$, with u and v monomials in Λ_1^* and g in $\mathcal{G}$. A set $\mathcal{G}$ of 1-cells is *reduced with respect to the monomial order* $\prec$ if for any 1-cell g in $\mathcal{G}$, any monomial in the support of g is $(\mathcal{G} \setminus \{g\})$-reduced.

5.5. Confluence and convergence

5.5.1. Suppose that Λ is a terminating left-monomial linear 2-polygraph. Every 1-cell f of Λ_1^ℓ admits at least a normal form $\widetilde{f}$. That is, $\widetilde{f}$ is reduced and there exists a positive 2-cell $a : f \Rightarrow \widetilde{f}$ in Λ_2^ℓ. As a consequence, we have a decomposition

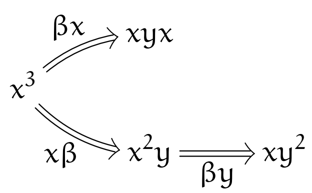
$$f \;=\; \widetilde{f} + (f - \widetilde{f}),$$

with $\widetilde{f}$ in Λ_1^{nf} and $f - \widetilde{f}$ in $I(\Lambda)$ by Exercice 5.1.15. It follows that the vector space Λ_1^ℓ admits the following decomposition

$$\Lambda_1^\ell = \Lambda_1^{\mathrm{nf}} + I(\Lambda). \tag{5.8}$$

In this section we show that the decomposition (5.8) is direct if and only if the polygraph Λ is confluent.

5.5.2. Example. Note that the decomposition (5.8) is not direct in general. Indeed, consider the linear 2-polygraph $\Lambda = \langle\, x, y \mid x^2 \overset{\beta}{\Rightarrow} xy \,\rangle$. It is terminating thanks to the deglex order generated by $x > y$. Consider the two following reduction sequences reducing the 1-cell x^3:

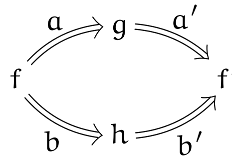


Thus the 1-cell

$$xyx - xy^2 = -(x^2 - xy)x + x(x^2 - xy) + (x^2 - xy)y$$

is both in Λ_1^{nf} and $I(\Lambda)$. It follows that the sum $\Lambda_1^{\mathrm{nf}} + I(\Lambda)$ is not direct.

5.5.3. Branchings and confluence. Let Λ be a left-monomial linear 2-polygraph. A *branching of* Λ is a non-ordered pair (a, b) of positive 2-cells of Λ_2^ℓ with a common *source* $s_1(a) = s_1(b)$. A branching (a, b) is *local* if both a and b are rewriting steps of Λ. A branching (a, b) of Λ is *confluent* if there exist positive 2-cells a' and b' of Λ as in the following diagram



We say that Λ is *confluent* (resp. *locally confluent*) if every branching (resp. local branching) of Λ is confluent. An immediate consequence of the confluence property is that every 1-cell of Λ_1^ℓ admits at most one normal form.

5.5.4. Proposition. *Let Λ be a terminating left-monomial linear 2-polygraph. The following conditions are equivalent.*

i) *Λ is confluent.*

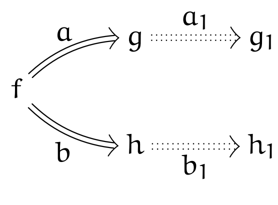
ii) *Every 1-cell of $I(\Lambda)$ admits 0 as a normal form with respect to Λ_2.*

iii) *The vector space Λ_1^ℓ admits the direct decomposition $\Lambda_1^\ell = \Lambda_1^{nf} \oplus I(\Lambda)$.*

Proof. **i)** $\Rightarrow$ **ii)**. Let f be a 1-cell in the ideal $I(\Lambda)$, then there exists a 2-cell $a : f \Rightarrow 0$ in Λ_2^ℓ. The polygraph Λ being confluent, the 1-cells f and 0 have the same normal form. Finally, 0 beaing reduced, this implies that 0 is a normal form for f.

ii) $\Rightarrow$ **iii)**. Prove that $\Lambda_1^{\mathrm{nf}} \cap I(\Lambda) = 0$. If f is in Λ_1^{nf}, then f is reduced and, thus, admits itself as normal form. If f is in $I(\Lambda)$, then f admits 0 as a normal form by **ii)**. Hence $\Lambda_1^{\mathrm{nf}} \cap I(\Lambda) = 0$.

iii) $\Rightarrow$ **i)**. Consider a branching (a, b) of Λ with $a : f \Rightarrow g$ and $b : f \Rightarrow h$. Since Λ terminates, each of g and h admits at least one normal form. Hence, there exist positive 2-cells a_1 and b_1 in Λ_2^ℓ:



with g_1 and h_1 reduced. It follows that $g_1 - h_1$ is also reduced. Moreover, the 2-cell $(a \star_1 a_1)^- \star_1 (b \star b_1)$ has g_1 as source and h_1 as target. This implies that $g_1 - h_1$ is also in $I(\Lambda)$. As $\Lambda_1^{\mathrm{nf}} \cap I(\Lambda) = 0$, we have $g_1 - h_1 = 0$, hence the branching (a, b) is confluent. □

5.5.5. Convergence. We say that a left-monomial linear 2-polygraph Λ is *convergent* if it terminates and it is confluent. In that case, every 1-cell f of Λ_1^ℓ has a unique normal form, denoted by $\widehat{f}$, such that $\overline{f} = \overline{g}$ holds in $\overline{\Lambda}$ if and only if $\widehat{f} = \widehat{g}$ holds in Λ_1^ℓ.

As a consequence, if Λ is a convergent presentation of an algebra $\mathbf{A}$, the assignment of every 1-cell f of $\mathbf{A}$ to the normal form $\widehat{f}$, defines a section $\iota : \mathbf{A} \longrightarrow \Lambda_2^\ell$ of the canonical projection $\pi : \Lambda^\ell \longrightarrow \mathbf{A}$. The section ι is a linear map, i.e., it satisfies $\widehat{\lambda f + \mu g} = \lambda\widehat{f} + \mu\widehat{g}$, and it preserves the identities because Λ terminates.

5.5.6. Exercise. Show that the section ι is not a morphism of algebras in general.

5.5.7. Theorem. *Let* **A** *be an algebra and* Λ *be a convergent presentation of* **A**. *The set* Λ_1^{irm} *of reduced monomials is a linear basis of* **A**. *Moreover, the vector space* Λ_1^{nf} *equipped with the product defined by* $f \cdot g = \widehat{fg}$, *for any* 1*-cells* f *and* g *in* Λ_1^{nf}, *is an algebra isomorphic to* **A**.

Proof. Suppose that Λ is a convergent linear 2-polygraph. By Proposition 5.5.4 the following sequence of vector spaces is exact:

$$0 \longrightarrow \mathrm{I}(\Lambda) \rightarrowtail \Lambda_1^{\ell} \twoheadrightarrow \Lambda_1^{\mathrm{nf}} \longrightarrow 0$$

The vector space Λ_1^{nf} admits Λ_1^{irm} as a basis, hence Λ_1^{irm} forms a basis of the vector space underlying the quotient algebra $\Lambda_1^{\ell}/\mathrm{I}(\Lambda)$, that is the algebra **A**. The polygraph Λ being convergent, any 1-cell of Λ_1^{ℓ} has a unique normal form, hence the product defined by $f \cdot g = \widehat{fg}$ is associative. Indeed, for any 1-cells f, g and h, we have

$$(f \cdot g) \cdot h = \widehat{fg} \cdot h = \widehat{\widehat{fg}h} = \widehat{f\widehat{gh}} = f \cdot \widehat{gh} = f \cdot (g \cdot h).$$

It follows that this product equips Λ_1^{nf} with a structure of algebra in such a way that Λ_1^{nf} is isomorphic to the algebra **A**. ☐

5.5.8. Exercise. Compute a linear basis of the algebra presented by $\langle x, y \mid xy = x^2 \rangle$.

5.5.9. Exercise. Compute a linear basis for the symmetric algebra on k variables presented by

$$\langle\, x_1, \ldots, x_k \mid x_i x_j \overset{\tau_{ij}}{\Rightarrow} x_j x_i \mid 1 \leqslant i < j \leqslant k \,\rangle$$

and for the skew-polynomial algebra on k variables presented by

$$\langle\, x_1, \ldots, x_k \mid x_i x_j \overset{\tau_{ij}}{\Rightarrow} q_i^j x_j x_i \mid 1 \leqslant i < j \leqslant k \,\rangle,$$

where the q_i^j are scalars in $\mathbb{K}$.

5.5.10. Exercise: Poincaré-Birkhoff-Witt theorem, [Bok76, §1.], [Ber78, Theorem 3.1]. Consider an ordered bases $x_1 \prec x_2 \prec \ldots \prec x_k$ of a Lie algebra $\mathfrak{g}$. Consider the following ideals of the free tensor algebra $\mathrm{T}(\mathfrak{g})$ over $\mathfrak{g}$:

$$\begin{aligned} \mathrm{I} &= \langle\, x_j x_i - x_i x_j \mid 1 \leqslant i < j \leqslant k \,\rangle, \\ \mathrm{J} &= \langle\, x_j x_i - x_i x_j + [x_i, x_j] \mid 1 \leqslant i < j \leqslant k \,\rangle. \end{aligned}$$

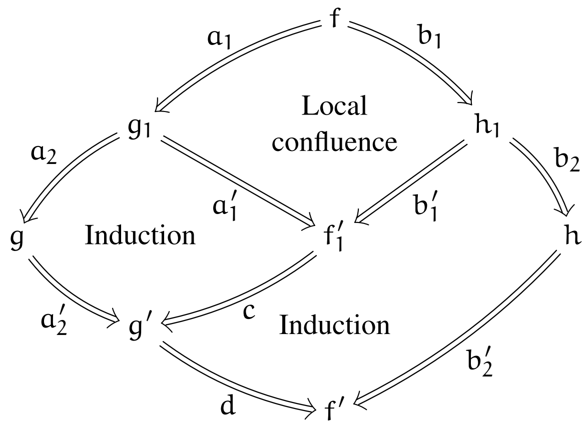
Show that the symmetric algebra $\mathbf{S}\mathfrak{g} = \mathrm{T}(\mathfrak{g})/\mathrm{I}$ and the enveloping algebra $\mathbf{U}\mathfrak{g} = \mathrm{T}(\mathfrak{g})/\mathrm{J}$ are isomorphic as vector spaces.

5.5.11. From local to global confluence. The Newman lemma, also called the diamond lemma, states that for terminating rewriting systems local confluence and confluence are equivalent properties. This result was proved by Newman in [New42] for abstract rewriting systems. A short and simple proof of this result was given by Huet in [Hue80] using the principle of noetherian induction. Let us recall the arguments of this proof for linear 2-polygraphs.

5.5.12. Theorem (Newman's Lemma). *Let Λ be a terminating left-monomial linear 2-polygraph. Then Λ is confluent if and only if it is locally confluent.*

Proof. One implication is trivial. Suppose Λ locally confluent and prove that it is confluent at every 1-cell f of Λ_1^ℓ. We proceed by noetherian induction on f. If f is reduced, the only branching with source f is $(1_f, 1_f)$ which is confluent.

Suppose that f is a nonreduced 1-cell of Λ_1^ℓ and such that Λ is confluent at every 1-cell $g \prec f$. Consider a branching (a, b) of Λ with source f. If a or b is an identity, then (a, b) is confluent. Otherwise, we prove that the branching (a, b) is confluent by induction. Since a and b are not identities, they admit decompositions $a = a_1 \star_1 a_2$ and $b = b_1 \star_1 b_2$ where a_1 and b_1 are rewriting steps, and a_2 and b_2 are positive 2-cells. By local confluence, the local branching (a_1, b_1) is confluent. Hence there exist positive 2-cells a_1' and b_1' as indicated in the following diagram



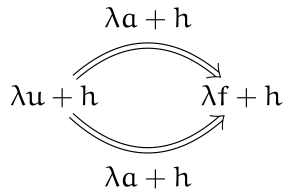
We have $g_1 \prec_\Lambda f$ and $h_1 \prec_\Lambda f$. Then we apply the induction hypothesis on the branching (a_2, a_1') to get positive 2-cells a_2' and c, and, then, to the branching $(b_1' \star_1 c, b_2)$ to get positive 2-cells d and b_2', which complete the proof. □

5.6. The Critical Branchings Theorem

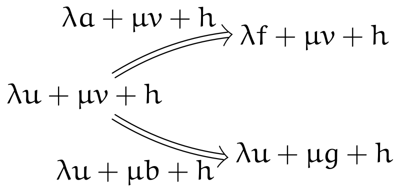
5.6.1. Local branchings. A case analysis leads to a partition of the local branchings of a left-monomial linear 2-polygraph Λ into the following four families:

i) *Aspherical* branchings, for all 2-monomial $a : u \Rightarrow f$ of Λ_2^ℓ, nonzero scalar λ, and 1-cell h of Λ_1^ℓ

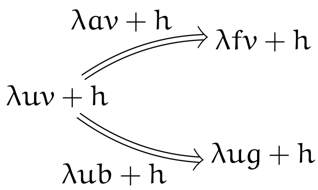
such that the monomial u is not in the support of h:



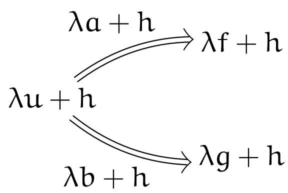
ii) *Additive* branchings, for all 2-monomials $a : u \Rightarrow f$ and $b : v \Rightarrow g$ of Λ_2^ℓ, nonzero scalars λ and μ, and 1-cell h of Λ_1^ℓ such that the monomials u and v are not in the support of h:



iii) *Peiffer* branchings, for all 2-monomials $a : u \Rightarrow f$ and $b : v \Rightarrow g$ of Λ_2^ℓ, nonzero scalar λ, and 1-cell h of Λ_1^ℓ such that the monomial uv is not in the support of h:



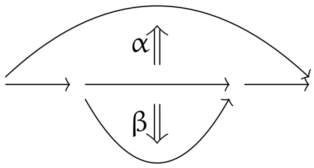
iv) *Overlapping* branchings, for all 2-monomials $a : u \Rightarrow f$ and $b : u \Rightarrow g$ of Λ_2^ℓ such that the branching (a, b) is neither aspherical nor Peiffer, and all nonzero scalar λ and 1-cell h of Λ_1^ℓ such that the monomial u is not in the support of h:

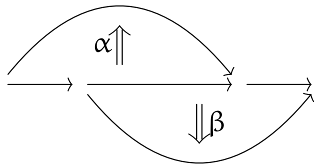


5.6.2. Critical branchings. A *critical branching of a left-monomial linear* 2*-polygraph* Λ is an overlapping branchings, as defined in 5.6.1, with $\lambda = 1$ and $h = 0$, and that is minimal for the relation on branchings defined by

$$(a, b) \sqsubseteq (waw', wbw') \quad \text{for any } w \text{ and } w' \text{ in } \Lambda_1^*.$$

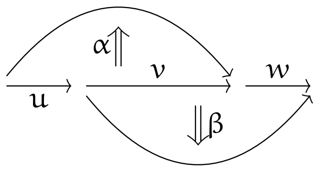
By case analysis on the source of critical branchings, they must have one of the following two shapes





with α, β in Λ_2. When the linear 2-polygraph Λ is reduced, the first case cannot occur since, otherwise, the monomial $s_1(\alpha)$ would be reducible by β.

5.6.3. Exercise. Let Λ be a reduced linear 2-polygraph. Show that for any critical branching



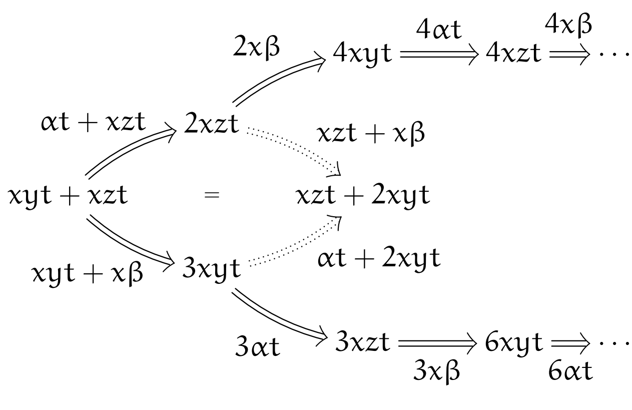
the monomial u, v and w are reduced and cannot be identities or null.

5.6.4. Critical branching lemma. By the Newman lemma 5.5.12, for terminating rewriting systems, local confluence and confluence are equivalent properties. It turns out that one can decide whether a rewriting system is convergent by checking local confluence. For string rewriting systems, that is 2-polygraphs, the critical branching lemma states that local confluence is equivalent to the confluence of all critical branching, see [GM18, 3.1.5] for details. For linear 2-polygraphs the critical branching lemma given in [GHM19] differs from the case of 2-polygraphs. Indeed, in the linear setting the termination hypothesis is required. Moreover, nonoverlapping branchings may be non confluent as illustrated by the following example in which an additive branching is nonconfluent.

5.6.5. Example. Some local branchings can be nonconfluent without termination, even if critical confluence holds. Indeed, consider for instance the following linear 2-polygraph

$$\langle\, x, y, z, t \mid xy \overset{\alpha}{\Rrightarrow} xz,\ zt \overset{\beta}{\Rrightarrow} 2yt \,\rangle$$

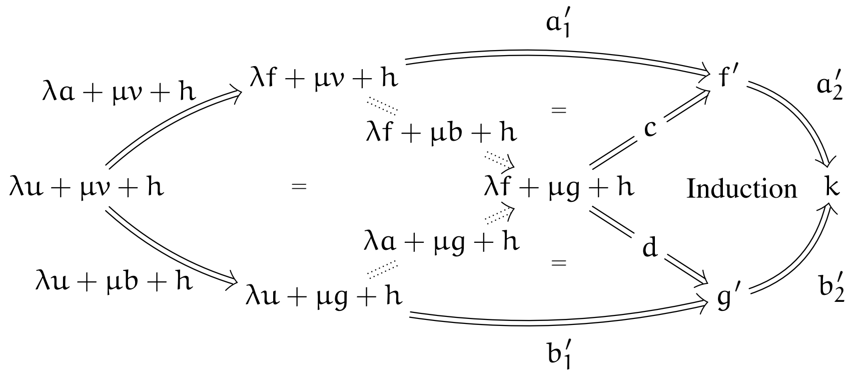
has no critical branching, but it has a nonconfluent additive branching:



5.6.6. If a linear 2-polygraph Λ is terminating and with any critical branching confluent, we can show that such an additive branching is confluent by noetherian induction on the sources of the branchings. Let consider an additive branching $(\lambda u+\mu v+h, \lambda u+\mu g+h)$ as in (5.6.1) and suppose that Λ is locally confluent at every $g \prec_\Lambda \lambda u + \mu v + h$. By linearity of the 1-composition, the following equation

$$(\lambda a + \mu v + h) \star_1 (\lambda f + \mu b + h) = (\lambda u + \mu b + h) \star_1 (\lambda a + \mu g + h)$$

holds in the free 2-algebra Λ_2^ℓ:



Note that the dotted 2-cells $\lambda a + \mu g + h$ and $\lambda f + \mu b + h$ may be not positive in general. Indeed, the monomial u can be in the support of g or the monomial v can be in the support of f, as illustrated in Example 5.6.5. However, those 2-cells are elementary, hence there exist, see Exercise 5.2.6, positive 2-cells a_1', b_1', c and d that satisfy

$$a_1' = (\lambda f + \mu b + h) \star_1 c \qquad \text{and} \qquad b_1' = (\lambda a + \mu g + h) \star_1 d.$$

We have $f \prec_\Lambda u$ and $g \prec_\Lambda v$, hence $\lambda f + \mu g + h \prec_\Lambda \lambda u + \mu v + h$. Thus, the branching (c, d) is confluent by induction hypothesis, yielding the positive 2-cells a_2' and b_2'.

In this way, one shows that under terminating hypothesis, all local branching given in (5.6.1) are confluent if all critical branching are confluent.

5.6.7. Theorem (Critical branching lemma). *A terminating left-monomial linear 2-polygraph is locally confluent if and only if all its critical branchings are confluent.*

As consequence of the critical branching lemma and of the Newman lemma 5.5.12, a terminating left-monomial linear 2-polygraph is confluent if all its critical branchings are confluent. In particular a terminating left-monomial 2-polygraph with no critical branching is convergent.

5.6.8. Example. The linear 2-polygraph given in Example 5.1.7 is terminating, see Exercise 5.4.7. Moreover, it does not have critical branching, hence it is convergent.

5.6.9. The Knuth-Bendix completion procedure. The completion procedure for terminating 2-polygraphs given in (2.5.1) can be adapted to linear 2-polygraphs as follows. Let Λ be a left-monomial linear 2-polygraph compatible with a monomial order $\prec$ on Λ_1^*. A *Knuth-Bendix completion* of Λ is a linear

2-polygraph $\mathcal{KB}(\Lambda)$ obtained by the following procedure that examines the confluence of the set of critical branchings.

Input: Λ be a left-monomial linear 2-polygraph compatible with a monomial order $\prec$ on Λ_1^*.
$\mathcal{KB}(\Lambda) := \Lambda$
$\mathcal{C}\mathrm{b} := \{$ critical branchings with respect to $\Lambda_2 \}$
while $\mathcal{C}\mathrm{b} \neq \emptyset$ **do**

  Picks a branching in $\mathcal{C}\mathrm{b}$:

$$\begin{array}{ccc} & \overset{f}{\Rightarrow} & v \\ u & & \\ & \underset{g}{\Rightarrow} & w \end{array}$$

  $\mathcal{C}\mathrm{b} := \mathcal{C}\mathrm{b} \setminus \{(f, g)\}$
  Reduce v to a normal form $\widehat{v}$ with respect to $\mathcal{KB}(\Lambda)_2$
  Reduce w to a normal form $\widehat{w}$ with respect to $\mathcal{KB}(\Lambda)_2$

$$\begin{array}{ccccc} & \overset{f}{\Rightarrow} & v & \Longrightarrow & \widehat{v} \\ u & & & & \\ & \underset{g}{\Rightarrow} & w & \Longrightarrow & \widehat{w} \end{array}$$

  $g = \widehat{v} - \widehat{w}$
  if $g \neq 0$ **then**
    $\mathcal{KB}(\Lambda)_2 := \mathcal{KB}(\Lambda)_2 \cup \{ \alpha_{g,\prec} : \mathrm{lm}(g) \Rightarrow \mathrm{lm}(g) - \frac{1}{\mathrm{lc}(g)} g \}$
    $\mathcal{C}\mathrm{b} := \mathcal{C}\mathrm{b} \cup \{$ critical branching created by $\alpha_{g,\prec} \}$
  end

end

If the procedure stops, it returns a finite convergent left-monomial linear 2-polygraph $\mathcal{KB}(\Lambda)$. Otherwise, it builds an increasing sequence of left-monomial linear 2-polygraphs, whose limit is also denoted by $\mathcal{KB}(\Lambda)$. Note that, if the starting linear 2-polygraph Λ is convergent, then the Knuth-Bendix completion of Λ is Λ itself. The linear 2-polygraph $\mathcal{KB}(\Lambda)$ obtained by this procedure depends on the order of examination of the critical branchings. Finally, since all the operations of adding new rules performed by the procedure are Tietze transformations, the linear 2-polygraph $\mathcal{KB}(\Lambda)$ is Tietze-equivalent to Λ.

5.6.10. Exercice. Prove that the following linear 2-polygraph has a nonconfluent Peiffer branching

$$\langle\, x, y, z \mid xy \overset{\alpha}{\Rightarrow} 2x,\ yz \overset{\beta}{\Rightarrow} z \,\rangle.$$

5.6.11. Weyl algebras. Let $\mathbb{K}$ be a field of characteristic zero. The *Weyl algebra* of dimension n over $\mathbb{K}$ is the algebra presented by the linear 2-polygraph whose 1-cells are

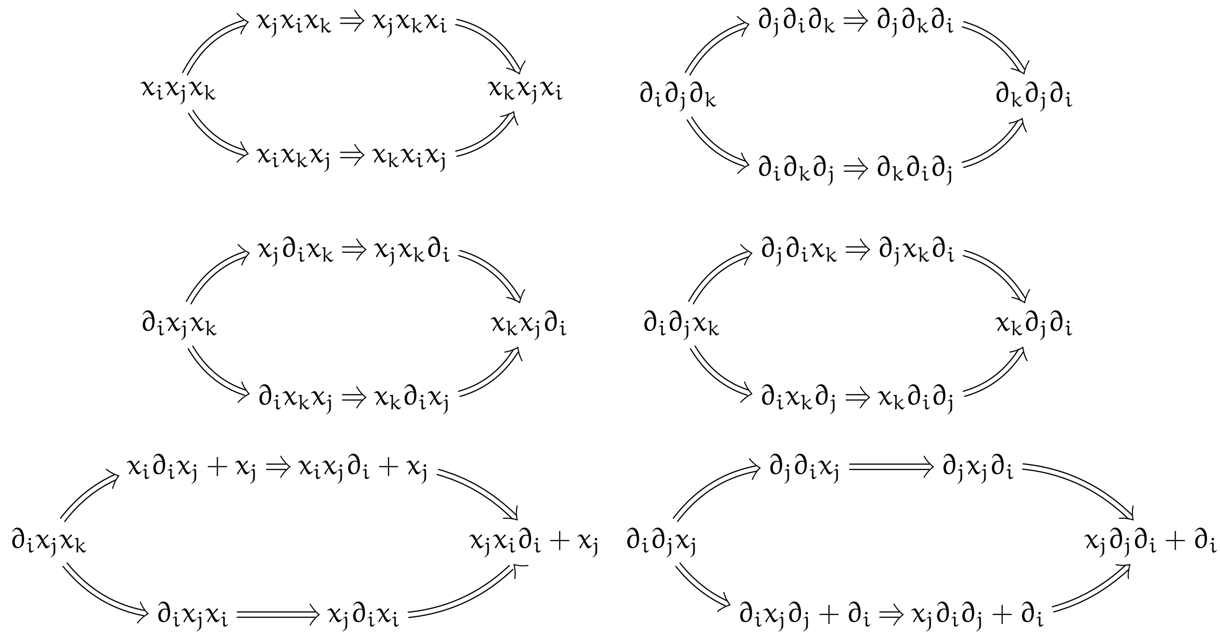
$$x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$$

and with the following 2-cells:

$$x_i x_j \Rightarrow x_j x_i, \quad \partial_i \partial_j \Rightarrow \partial_j \partial_i, \quad \partial_i x_j \Rightarrow x_j \partial_i, \qquad \text{for any } 1 \leqslant i < j \leqslant n,$$

$$\partial_i x_i \Rightarrow x_i \partial_i + 1, \qquad \text{for any } 1 \leqslant i \leqslant n.$$

This polygraph is convergent with the following six families of confluent critical branchings:



where $1 \leqslant i < j \leqslant n$.

5.6.12. Exercice. In his seminal paper on the diamond lemma, Bergman point out that he was first led to the ideas of his paper with the following American Mathematical Monthly Advanced Problem 5082, [Ber78, 2.1.].

> *Let* R *be a ring in which, if either* $x + x = 0$ *or* $x + x + x = 0$*, it follows that* $x = 0$*. Suppose that* a, b, c *and* $a + b + c$ *are all idempotents in* R*. Does it follows that* $ab = 0$*?*

Solve this problem. $\big[$Hints. Consider the following linear 2-polygraph:

$$\Lambda = \langle\, a, b, c \mid a^2 \Rightarrow a,\; b^2 \Rightarrow b,\; c^2 \Rightarrow c,\; ba \Rightarrow -ab - bc - cb - ac - ca \,\rangle.$$

1/ List all critical branchings of Λ. 2/ Compute a convergent left-monomial linear 2-polygraph $\mathcal{KB}(\Lambda)$ by applying the Knuth-Bendix completion procedure to Λ. 3/ List all irreducible monomials with respect to $\mathcal{KB}(\Lambda)_2$. 4/ Conclude that $ab \neq 0$.$\big]$

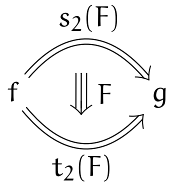
5.7. Coherent presentations of algebras

In this last section, we recall from [GHM19] the notion of coherent presentation for an algebra as a presentation of the algebra extended by a family of generating syzygies. We explain how to generate syzygies when the presentation is convergent.

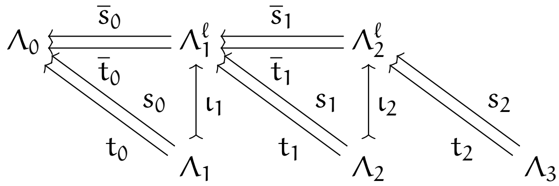
5.7.1. Linear 3-polygraph. Let Λ be a linear 2-polygraph. A *cellular extension* of the free 2-algebroid Λ_2^ℓ is a set Λ_3 equipped with maps

$$\Lambda_2^\ell \underset{t_2}{\overset{s_2}{\Longleftarrow}} \Lambda_3$$

such that, for every F in Λ_3, the pair $(s_2(F), t_2(F))$ is a 2-sphere in Λ_2^ℓ, that is, $s_1s_2(F) = s_1t_2(F)$ and $t_1s_2(F) = t_1t_2(F)$ hold in Λ_2^ℓ. The elements of Λ_3 are the 3-*cells* of the cellular extension and graphically represented by



A *linear* 3-*polygraph* is a data $(\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3)$, where $(\Lambda_0, \Lambda_1, \Lambda_2)$ is a linear 2-polygraph and Λ_3 is a cellular extension of the free 2-algebroid Λ_2^ℓ:



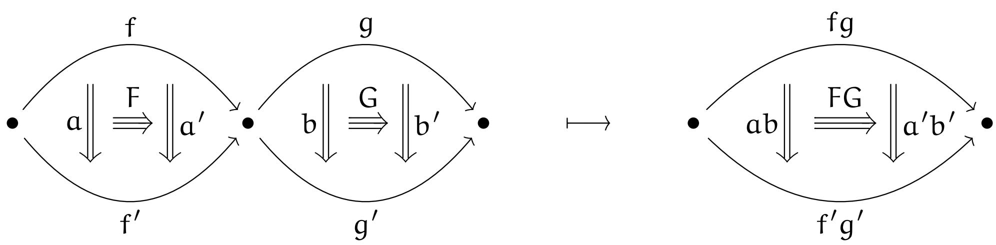
5.7.2. Three-dimensional algebras. We define a 3-*algebra* as an internal 2-category in the category **Alg**:

$$\mathbf{A}_1 \underset{t_1}{\overset{s_1}{\Longleftarrow}} \mathbf{A}_2 \underset{t_2}{\overset{s_2}{\Longleftarrow}} \mathbf{A}_3$$

In particular, the algebras $\mathbf{A}_1$ and $\mathbf{A}_2$ with composition $\star_1 : \mathbf{A}_2 \times_{\mathbf{A}_1} \mathbf{A}_2 \to \mathbf{A}_2$ form a 2-algebra. The 3-cells can be composed in two different ways:

$$\star_1 : \mathbf{A}_3 \times_{\mathbf{A}_1} \mathbf{A}_3 \to \mathbf{A}_3 \qquad \star_2 : \mathbf{A}_3 \times_{\mathbf{A}_2} \mathbf{A}_3 \to \mathbf{A}_3$$

by $\star_1$ along their 1-dimensional boundary, and by $\star_2$ along their 2-dimensional boundary as pictured in (3.2.5). The source and target maps s_1, s_2 and t_1, t_2 being morphisms of algebras, the product of 3-cells F and G satisfies:



These compositions and the product satisfy remarkable properties similar to those given in (5.1.12) for 2-algebras.

5.7.3. Free 3-algebras. The *free 3-algebra over a linear 3-polygraph* Λ is constructed similarly to the free 2-algebra given in (5.1.14). It is the 3-algebra, denoted by Λ_3^ℓ, whose underlying 2-algebra is the free 2-algebra Λ_2^ℓ, and its 3-cells are all the formal 1-composition, 2-composition and product of 3-cells of Λ_3, of identities of 2-cells, up to associativity, identity, exchange and inverse relations, see [GHM19] for more details.

5.7.4. Coherent presentations of algebras. A *coherent presentation* of an algebra **A** is a linear 3-polygraph Λ such that

i) the linear 2-polygraph $(\Lambda_0, \Lambda_1, \Lambda_2)$ is a presentation of **A**,

ii) Λ_3 is a *homotopy basis* of the free 2-algebra Λ_2^ℓ, that is, a cellular extension

$$\Lambda_2^\ell \underset{t_2}{\overset{s_2}{\Leftarrow\!\!\Leftarrow}} \Lambda_3$$

such that for every 2-sphere (a, b) of the free 2-algebra Λ_2^ℓ, there exists a 3-cell A in the free 3-algebra Λ_3^ℓ such that $s_2(A) = a$ and $t_2(A) = b$.

5.7.5. Squier's completion. Let Λ be a left-monomial linear 2-polygraph. Suppose that all critical branching of Λ are confluent. For every critical branching (a, b) in Λ, we choose two positive 2-cells a' and b' making the branching confluent:

$$\xymatrix{ & g \ar@2[dr]^{a'} \ar@3[dd]^{F_{(a,b)}} & \\ f \ar@2[ur]^{a} \ar@2[dr]_{b} & & f' \\ & h \ar@2[ur]_{b'} & } \tag{5.7.6}$$

For any such a confluent branching, we consider a 3-cell $F_{(a,b)} : a \star_1 a' \Rrightarrow b \star_1 b'$. The set of such 3-cells

$$\Lambda_3 = \{ F_{(a,b)} \mid (a, b) \text{ is a critical branching } \}$$

forms a cellular extension of the free 2-algebra Λ_2^ℓ. The linear 3-polygraph $(\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3)$ is a *Squier's completion* of Λ. When the polygraph is confluent, there exists such a Squier's completion. However, the cellular extension Λ_3 is not unique in general. Indeed, the 3-cells can be directed in the reverse way and a branching (a, b) can have several possible positive 2-cells a' and b' making the branching confluent.

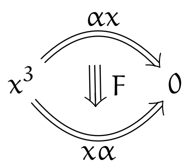
The following result is a formulation of the Squier Lemma, [SOK94], in the setting of linear 2-polygraphs.

5.7.7. Theorem ([GHM19, Thm. 4.3.2]). *Let* **A** *be an algebra and let* Λ *be a convergent left-monomial presentation of* **A**. *Any Squier's completion of* Λ *is a coherent presentation of* **A**.

5.7.8. Linear oriented syzygies. Let Λ be presentation of an algebra **A**. Any nontrivial 2-sphere (a, b) in the free 2-algebra Λ_2^ℓ is called a *linear oriented* 3-*syzygy* of the presentation Λ. If Λ is extended into a coherent presentation (Λ, Λ_3) of the algebra **A**, the quotient 2-algebra Λ_2^ℓ/Λ_3 is *aspherical*, that is, for any 2-sphere (a, b) in Λ_2^ℓ/Λ_3, we have $a = b$. In other words, the cellular extension Λ_3 forms a generating set of linear 3-syzygies of the presentation Λ. Theorem 5.7.7 say that, when the presentation Λ is convergent the 3-cells defined by confluence diagrams of the critical branchings, as in (5.7.6), form a family of generator for 3-syzygies.

5.7.9. Exercice. Let $\{F_1, \ldots, F_k\}$ be a generating set for linear 3-syzygies of a linear 2-polygraph Λ. Prove that $\{F_1^-, \ldots, F_k^-\}$ is also a generating set for linear 3-syzygies of Λ.

5.7.10. Example. The linear 2-polygraph $\langle\, x \mid x^2 \overset{\alpha}{\Rrightarrow} 0 \,\rangle$ has one critical branching



which is confluent. The polygraph being convergent the 3-cell $F : \alpha x \Rrightarrow x\alpha$ generates all linear 3-syzygies of this presentation.

5.7.11. Example. Consider the algebra **A** presented by the linear 2-polygraph

$$\Lambda = \langle\, x, y, z \mid xyz \overset{\gamma}{\Rrightarrow} x^3 + y^3 + z^3 \,\rangle$$

given in Example 5.1.7. It does not have critical branching, hence any Squier's completion of Λ is empty. As a consequence, Λ can be extended into a coherent presentation with an empty homotopy basis. That is, there is no 3-syzygy for this presentation.

The linear 2-polygraph $\langle\, x, y, z \mid \alpha_f, \beta \,\rangle$ considered in Example 6.3.7 is Tietze equivalent to Λ, convergent and compatible with a monomial order. It has three critical branchings, as shown in Example 6.3.7. It can be extended into a coherent presentation of **A** with three generating 3-syzygies.

5.7.12. Exercise. Give an explicit description of the 3-cells of a coherent presentation on the linear 2-polygraph Λ' of Example 5.7.11.

5.7.13. Exercise. Compute a coherent presentation for the algebras presented by the following linear 2-polygraphs

1) $\langle\, x, y \mid xyx \Rightarrow y^2 \,\rangle$.

2) $\langle\, x, y, z \mid yz \overset{\alpha}{\Rrightarrow} -x^2,\ zy \overset{\beta}{\Rrightarrow} -\lambda^{-1}x^2 \,\rangle$, where $\lambda \in \mathbb{K} \setminus \{0, 1\}$, see [PP05, 4.3].

5.7.14. Exercise. Compute a minimal coherent presentation for the algebra presented by the linear 2-polygraph $\langle\, x \mid x^3 = 0 \,\rangle$.

CHAPTER
6

Paradigms of linear rewriting

Contents

In this chapter, we survey several approaches in linear rewriting. The most well-known is given by the Gröbner basis theory for ideals in commutative polynomial rings introduced by Buchberger in [Buc65]. A subset G of an ideal I in the polynomial ring $\mathbb{K}[x]$ of commutative polynomials is a *Gröbner basis* of I with respect to a given monomial order $\prec$, if the leading term ideal of I is generated by the set of leading monomials of G, that is

$$\langle \operatorname{lt}_{\prec}(\mathrm{I}) \rangle = \langle \operatorname{lt}_{\prec}(\mathrm{G}) \rangle.$$

Buchberger introduced the notion of *S-polynomial* to describe the obstructions to local confluence and gave an algorithm for computation of Gröbner bases, [Buc65, Buc06], see also [Buc87] for an historical account. Any ideal I of a commutative polynomial ring $\mathbb{K}[x]$ has a finite Gröbner basis. Indeed, the Buchberger algorithm on a finite family of generators of an ideal I always terminates and returns a Gröbner basis of the ideal I.

Shirshov introduced in [Shi62] an algorithm to compute a linear basis of a Lie algebra defined by generators and relations. He used the notion of *composition* of elements in a free Lie algebra, that corresponds to the notion of S-polynomial in the work of Buchberger. He gave an algorithm to compute bases in free algebras having the computational properties of the Gröbner bases. He proved that irreducible elements for such a basis forms a linear basis of the Lie algebra. This result is called now the *Composition Lemma* for Lie algebras.

Subsequently, the Gröbner basis theory has been developed for other types of algebras, such as associative algebras by Bokut in [Bok76] and by Bergman in [Ber78]. They prove Newman's Lemma for rewriting systems in free associative algebras compatible with a monomial order stating that local confluence and confluence are equivalent properties. This result was called *Composition Lemma* by Bokut and *Diamond Lemma* for ring theory by Bergman, see also [Mor94, Ufn95]. In general, the Buchberger algorithm does not terminate for ideals in a noncommutative polynomial ring $\mathbb{K}\langle x\rangle$. Indeed, its termination would give a decision procedure of the undecidable word problem. Even if the ideal is finitely generated it may not have a finite Gröbner basis. However, when $\mathbb{K}$ is a field an infinite Gröbner basis can be computed, [Mor94, Ufn98].

Note that ideas in the spirit of the Gröbner basis approach appear in several others works. Let us mention works by Hironaka in [Hir64] and Grauert in [Gra72] that compute bases of ideals in rings of power series having analogous properties to Gröbner bases but without a constructive method for computing such bases. In [Coh65], Cohn gave a method to decide the word problem by a normal form algorithm based on a confluence property. Finally, Janet [Jan20], Thomas [Tho37] and Pommaret [Pom78] developed the notion of involutive bases that are particular cases of Gröbner bases in the context of partial differential algebra. We refer the reader to [IM19] for an historical account on involutives bases and their applications to algebraic analysis of linear partial differential systems. Much more recently, Gröbner basis theory was developed in various noncommutative contexts such as Weyl algebras, see [SST00], or operads [DK10].

6.1. Composition Lemma

6.1.1. Compositions in free Lie algebras. Shirshov introduced in [Shi62] an algorithm to compute a linear basis of a Lie algebra defined by generators and relations. He used the notion of *composition* of elements in a free Lie algebra, that corresponds to the notion of S*-polynomial* in the work of Buchberger, [Buc65]. This work remained unknown outside the USSR and the two theories were developed in parallel. The algorithm *completes* a given set of elements in a free algebra by adding all nontrival compositions. This algorithm corresponds to the completion algorithm given by Knuth-Bendix for term rewriting systems, [KB70], and by Buchberger for commutative polynomials, [Buc65]. The Shirshov completion constructs a set, that may be infinite, such that every composition of its elements is trivial. Such a subset is called a *Lie Gröbner-Shirshov basis*. The key result in [Shi62] states that the set of irreducible elements for a Gröbner-Shirshov basis S forms a linear basis of the Lie algebra with defining relations S. This result is called now the *Composition-Diamond Lemma* for Lie algebras. For a recent account of the theory of Gröbner-Shirshov we refer the reader to [BC14].

In this subsection we summarize without proofs an analogue of Shirshov's composition-diamond lemma for associative algebras given by Bokut in [Bok76].

6.1.2. Compositions. Let Λ_1^ℓ be a free algebra over a set Λ_1 and let $\prec$ be a monomial order on Λ_1^ℓ. Bokut introduced in [Bok76] the notion of composition of elements of a free associative algebra as follows. Given two 1-cells f and g in Λ_1^ℓ and a monomial w in Λ_1^*. There are two kinds of compositions:

i) if $w = \mathrm{lm}(f)v = u\,\mathrm{lm}(g)$ with $\ell(\mathrm{lm}(f)) + \ell(\mathrm{lm}(g)) > \ell(w)$, for some monomials u and v in Λ_1^*, then the 1-cell

$$(f,g)_w = \frac{1}{\mathrm{lc}(f)} f v - \frac{1}{\mathrm{lc}(g)} u g$$

is called the *intersection composition* of f and g with respect to w.

ii) if $w = \mathrm{lm}(f) = u\,\mathrm{lm}(g)v$, for some monomials u and v in Λ_1^*, then the 1-cell

$$(f,g)_w = \frac{1}{\mathrm{lc}(f)} f - \frac{1}{\mathrm{lc}(g)} u g v$$

is called the *inclusion composition* of f and g with respect to w.

A composition $(f,g)_w$ can also be called an *S-polynomial* of f and g with respect to w. A composition $(f,g)_w$ is either zero or satisfy $(f,g)_w \prec w$. Moreover the composition $(f,g)_w$ is in the ideal $\langle f, g \rangle$ generated by f and g. Note that a composition $(f,g)_w$ depends on the two polynomials f and g as well as the monomial w. Indeed, in some cases two polynomials f and g may overlap with different combinations creating several compositions.

6.1.3. Gröbner-Shirshov's bases. Let $\mathcal{G}$ be a set of nonzero 1-cells in Λ_1^ℓ. Given a monomial w in Λ_1^*, a 1-cell h is *trivial modulo* $(\mathcal{G}, w)$ if there exists a decomposition

$$h = \sum_{i \in I} \lambda_i u_i g_i v_i,$$

with λ_i in $\mathbb{K}$, u_i, v_i in Λ_1^* and g_i in $\mathcal{G}$ such that $u_i\,\mathrm{lm}(g_i)v_i \prec w$.

A set $\mathcal{G}$ set of nonzero 1-cells in Λ_1^ℓ is a *Gröbner-Shirshov's basis*, *GS basis* for short, with respect to the monomial ordering $\prec$ if every composition $(f,g)_w$ of 1-cells in **G** is trivial modulo $(\mathcal{G}, w)$. A GS-basis $\mathcal{G}$ is *minimal* if there is no inclusion composition with elements of $\mathcal{G}$. A minimal GS-basis $\mathcal{G}$ is called *closed under composition* in [Bok76]. A GS-basis $\mathcal{G}$ is *reduced* if the set $\mathcal{G}$ is reduced with respect to the monomial order $\prec$.

6.1.4. Exercise. Let $\mathcal{G}$ be a minimal Gröbner-Shirshov basis in a free algebra Λ_1^ℓ. Suppose that there exists a decomposition

$$w = u_1\,\mathrm{lm}(g_1)v_1 = u_2\,\mathrm{lm}(g_2)v_2,$$

with $u_1, v_1, u_2, v_2 \in \Lambda_1^*$ and $g_1, g_2 \in \mathcal{G}$. Show that $u_1 g_1 v_1 - u_2 g_2 v_2$ is trivial modulo $(\mathcal{G}, w)$.

6.1.5. Theorem (The Composition Lemma, [Bok76, Proposition 1 & Corollary 1]). *Let Λ_1^ℓ be a free algebra and let $\prec$ be a monomial order on Λ_1^ℓ. Let $\mathcal{G}$ be a set of 1-cells in Λ_1^ℓ and let I be the ideal generated by $\mathcal{G}$. Denote by* **A** *the algebra given by the quotient of the free algebra Λ_1^ℓ by the ideal I. The following conditions are equivalent.*

i) *$\mathcal{G}$ is a GS-basis.*

ii) *For any f in I, there exists a decomposition $\mathrm{lm}(f) = u\,\mathrm{lm}(g)v$ for some u, v in Λ_1^* and g in $\mathcal{G}$.*

iii) *The set of $\mathcal{G}$-reduced monomial forms a linear basis of the algebra* **A**.

6.2. REDUCTION OPERATORS

Yet another approach of rewriting in associative algebras were developed by Bergman in [Ber78]. With a functional description of linear rewriting reductions he obtained an equivalent result of the composition lemma 6.1.5.

6.2.1. Reduction operators. Given Λ_1^ℓ a free algebra over a set Λ_1, he defines a *reduction system* as a set S of pairs $\sigma = (w_\sigma, f_\sigma)$, where w_σ is a monomial of Λ_1^ℓ and f_σ is a 1-cell of Λ_1^ℓ. Given σ in S and two monomials u, v in Λ_1^*, he considers the linear map $r_{u\sigma v} : \Lambda_1^\ell \longrightarrow \Lambda_1^\ell$ defined by

$$r_{u\sigma v}(w) = \begin{cases} uf_\sigma v & \text{if } w = uw_\sigma v, \\ w & \text{otherwise.} \end{cases}$$

The endomorphism $r_{u\sigma v}$ is called *reduction by* σ. Note that this notion of reduction corresponds to the notion of rewriting step given in (5.2.4).

A 1-cell f in Σ_1^ℓ is *irreducible under* S if every reduction by elements of S acts trivially on f, that is $uw_\sigma v$ is not in the support of f, for any σ in S and monomials u, v in Σ_1^*. As in the case of linear 2-polygraphs, we denote by Λ_1^{nf} the vector subspace of Λ_1^ℓ of all irreducible 1-cells of Λ_1^ℓ.

6.2.2. Reduction-unique. Bergman introduced the notion of confluence for reduction systems as follows. A finite sequence of reductions $r_1, \ldots, r_n$ is *final* on a 1-cell f, if the 1-cell $r_n \ldots r_1(f)$ is irreducible. A 1-cell f of Λ_1^ℓ is *reduction-finite* if for any infinite sequence $(r_n)_{n\geqslant 1}$ of reductions, r_i acts trivially on $r_{i-1} \ldots r_1(f)$ for a sufficiently large i. A 1-cell f is *reduction-unique* if it is reduction-finite and if its images under all final sequences of reduction are the same. This common image is denoted by $r_S(f)$. A reduction system S is *reduction-unique* if all 1-cells of Λ_1^ℓ are reduction-unique under S.

6.2.3. Exercise, [Ber78, Lemma 1.1.].

1) Show that the set of reduction-unique 1-cells of Λ_1^ℓ forms a subspace of Λ_1^ℓ denoted by Λ_1^{ru} and that $r_S : \Lambda_1^{\mathrm{ru}} \to \Lambda_1^{\mathrm{irr}}$ defines a linear map.

2) Given monomials w_f, w_g and w_h in the support of the 1-cells f, g and h respectively, such that the product $w_f w_g w_h$ is in Λ_1^{ru}. Show that for any finite composition of reductions r, then $fr(g)h$ is in Λ_1^{ru} and that $r_S(fr(g)h) = r_S(fgh)$ holds.

6.2.4. Ambiguities. A 5-tuple (σ, τ, u, v, w) with σ, τ in S and u, v, w monomials in Λ_1^*, such that $w_\sigma = uv$ and $w_\tau = vw$ (resp. $\sigma \neq \tau$, $w_\sigma = v$ and $w_\tau = uvw$) is an *overlap ambiguity* (resp. *inclusion ambiguity*) of S. Such an ambiguity is *resolvable* if there exist compositions of reductions r and r' that satisfy the *confluence condition*:

$$r(f_\sigma w) = r'(uf_\tau) \quad \big(\text{resp.} \quad r(uf_\sigma w) = r'(f_\tau)\,\big).$$

6.2.5. Reduction system compatible with a monomial order. The diamond lemma obtained by Bergman concern reduction systems compatible with a monomial order. A reduction system S is *compatible* with a monomial order $\prec$, if for any $\sigma = (w_\sigma, f_\sigma)$ in S, we have $w \prec w_\sigma$ for any monomial w in the support of f_σ.

Given a reduction system compatible with a monomial order $\prec$. For a monomial w in Σ_1^*, we denote by $I_{\prec w}$ the subspace of Λ_1^ℓ defined by

$$I_{\prec w} = \mathrm{Span}_{\mathbb{K}}\big(u(w_\sigma - f_\sigma)v \mid (w_\sigma, f_\sigma) \in S \text{ and } uw_\sigma v \prec w \big).$$

An overlap ambiguity (resp. inclusion ambiguity) (σ, τ, u, v, w) is *resolvable relative to* $\prec$ if

$$f_\sigma w - u f_\tau \in I_{\prec uvw}, \quad \big(\text{resp. } u f_\sigma w - f_\tau \in I_{\prec uvw}\,\big).$$

Let $\mathcal{G}$ be a subset of 1-cells of Λ_1^ℓ and let $\prec$ be a monomial order on Λ_1^ℓ. We denote by $S(\mathcal{G}, \prec)$ the *reduction system generated by* $\mathcal{G}$ with respect to $\prec$ defined by

$$S(\mathcal{G}, \prec) = \{\, (\mathrm{lm}(f), \mathrm{lm}(f) - \frac{1}{\mathrm{lc}(f)} f) \mid f \in \mathcal{G}\,\}.$$

6.2.6. Theorem (The Diamond Lemma, [Ber78, Theorem 1.2]). *Let* S *be a reduction system compatible with a monomial order* $\prec$*. The following conditions are equivalent.*

i) *All the ambiguities of* S *are resolvable.*

ii) *All the ambiguities of* S *are resolvable relative to* $\prec$*.*

iii) S *is reduction-unique.*

A fourth equivalent condition is given in [Ber78, Theorem 1.2] as follows. Consider the algebra **A** given as the quotient of the free algebra Λ_1^ℓ by the two-side ideal

$$I(S) = \{\, w_\sigma - f_\sigma \mid \sigma \in S\,\}.$$

If the reduction system S is compatible with a monomial order $\prec$, the confluence conditions **i)** - **iii)** above hold if and only if the set Λ_1^{irm} of irreducible monomial under S is a linear basis of the algebra **A**. In this case, the $\mathbb{K}$-algebra **A** is isomorphic to the $\mathbb{K}$-algebra Λ_1^{nf}, whose product is given by $f \cdot g = r_S(fg)$, for any 1-cells f and g in Λ_1^{nf}.

6.3. Noncommutative Gröbner bases

6.3.1. Noncommutative Gröbner bases. Let Λ_1^ℓ be a free algebra over a set Λ_1 and let $\prec$ be a monomial order on Λ_1^ℓ. A *(noncommutative) Gröbner basis* of an ideal I of Λ_1^ℓ with respect to the monomial order $\prec$ is a subset $\mathcal{G}$ of I such that the ideal generated by the leading monomials of the 1-cells of I coincides with the ideal generated by the leading monomials of the 1-cells of $\mathcal{G}$:

$$\langle\, \mathrm{lm}(I)\,\rangle = \langle\, \mathrm{lm}(\mathcal{G})\,\rangle.$$

Equivalently, for every 1-cell f in I, there exists g in $\mathcal{G}$ with $\mathrm{lm}(f) = u\,\mathrm{lm}(g)v$, where u and v are monomials of Λ_1^ℓ.

The two following results show that the notion of noncommutative Gröbner basis corresponds to the notion of left-monomial convergent linear 2-polygraph compatible with a monomial order.

6.3.2. Proposition. *Let Λ be a convergent left-monomial linear 2-polygraph, compatible with a monomial order $\prec$ on Λ_1^ℓ. The set of 1-cells $\{s_1(\alpha) - t_1(\alpha) \mid \alpha \in \Lambda_2\}$ is a Gröbner basis of the ideal $I(\Lambda)$ for the monomial order $\prec$.*

6.3.3. Exercise. Prove Proposition 6.3.2.

6.3.4. Proposition. *Let I be an ideal of a free 1-algebra Λ_1^ℓ. Let $\mathcal{G}$ be a Gröbner basis for I with respect to a monomial order $\prec$. Then the leading 2-polygraph $\Lambda(\mathcal{G}, \prec)$ is convergent and $I(\Lambda(\mathcal{G}, \prec)) = I$ holds.*

Proof. Suppose that $\mathcal{G}$ is a Gröbner basis of the ideal I with respect to $\prec$. By definition, the ideal $I(\Lambda(\mathcal{G}, \prec))$ is equal to the ideal I generated by $\mathcal{G}$. Prove that the linear 2-polygraph $\Lambda(\mathcal{G}, \prec)$ is convergent. Its termination is a consequence of its compatibility with the monomial order $\prec$. The monomials in Λ_1^* reduced with respect to $\Lambda(\mathcal{G}, \prec)$ are the monomials that cannot be decomposed as $u\,\mathrm{lm}(g)v$ with g in $\mathcal{G}$ and u and v monomials in Λ_1^*. As a consequence, if a reduced 1-cell f of Λ_1^ℓ is contained in the ideal I, its leading monomial must be 0, because $\mathcal{G}$ is a Gröbner basis of I. By Proposition 5.5.4, we deduce that the linear 2-polygraph $\Lambda(\mathcal{G}, \prec)$ is confluent. □

As a conclusion to this chapter, the following result summarizes all the characterizations of the confluence property of linear rewriting systems. Note that some equivalences are tautological.

6.3.5. Theorem. *Let Λ_1^ℓ be a free algebra over a set Λ_1. Let $\prec$ be a monomial order on Λ_1^ℓ. Given an ideal I of Λ_1^ℓ and a subset $\mathcal{G}$ of I, we denote by Λ the leading polygraph $\Lambda(\mathcal{G}, \prec)$ and by S the reduction system $S(\mathcal{G}, \prec)$. The following conditions are equivalent.*

i) *$\mathcal{G}$ is a Gröbner basis with respect to $\prec$.*

ii) *Λ is convergent.*

iii) *Λ is confluent.*

iv) *Λ is locally confluent.*

v) *All the critical branchings of Λ are confluent.*

vi) *Every composition $(f, g)_w$ is reduced to 0 with respect to the division by $\mathcal{G}$.*

vii) *All the ambiguities of S are resolvable.*

viii) *All the ambiguities of S are resolvable relative to $\prec$.*

ix) *S is reduction-unique.*

x) $\Lambda_1^\ell = \Lambda_1^{nf} \oplus I$.

xi) *Every 1-cell of I admits 0 as a normal form with respect to Λ_2.*

xii) *For any f in I, there exists a decomposition $\mathrm{lm}(f) = u\,\mathrm{lm}(g)v$ for some u, v in Λ_1^* and g in $\mathcal{G}$.*

xiii) *The set of $\mathcal{G}$-reduced monomials forms a linear basis of the algebra* $\mathbf{A}$.

6.3.6. Exercise. Prove the equivalences of Theorem 6.3.5.

6.3.7. Example. Consider the linear 2-polygraph Λ given in Example 5.1.7. For the deglex order $\prec_{\text{deglex}}$ induced by the alphabetic order $x \prec y \prec z$, the leading monomial of $f = z^3 + y^3 + x^3 - xyz$ is z^3, so that

$$\Lambda(\{f\}, \prec_{\text{deglex}}) = \langle\, x, y, z \mid z^3 \overset{\alpha_f}{\Rightarrow} xyz - x^3 - y^3 \,\rangle$$

The left-monomial linear 2-polygraph $\Lambda(\{f\}, \prec_{\text{deglex}})$ is compatible with the monomial order $\prec_{\text{deglex}}$, hence it is terminating. It is not confluent, because neither of its two critical branchings is confluent:

$$z^4 \overset{\alpha_f z}{\Rightarrow} xyz^2 - x^3z - y^3z \qquad z^4 \overset{z\alpha_f}{\Rightarrow} zxyz - zx^3 - zy^3$$

$$z^5 \overset{\alpha_f z^2}{\Rightarrow} xyz^3 - x^3z^2 - y^3z^2 \xRightarrow{xy\alpha_f - x^3z^2 - y^3z^2} xyxyz - xy^4 - xyx^3 - x^3z^2 - y^3z^2$$

$$z^5 \overset{z^2\alpha_f}{\Rightarrow} z^2xyz - z^2x^3 - z^2y^3$$

In particular, $\{f\}$ does not form a Gröbner basis of the ideal $I(\Lambda)$ We add to the polygraph $\Lambda(\{f\}, \prec_{\text{deglex}})$ the following 2-cell

$$\beta : zy^3 \Rightarrow zxyz - zx^3 + y^3z + x^3z - xyz^2.$$

This new rule makes the two previous critical branchings confluent and create a new critical branching

$$z^3y^3 \overset{z^2\beta}{\Rightarrow} z^3xyz - z^3x^3 + z^2y^3z + z^2x^3z - z^2xyz^2 \qquad z^3y^3 \overset{\alpha y^3}{\Rightarrow} xyzy^3 - x^3y^3 - y^6$$

which is also confluent. Finally, the convergent linear 2-polygraph $\langle\, x, y, z \mid \alpha_f, \beta \,\rangle$ is Tietze equivalent to the initial linear 2-polygraph $\Lambda(\{f\}, \prec_{\text{deglex}})$. In particular, the set of 1-cells $\{f, s_1(\beta) - t_1(\beta)\}$ forms a Gröbner basis of the ideal $I(\Lambda)$ with respect to the order $\prec_{\text{deglex}}$.

6.3.8. Example. The algebra presented by the following linear 2-polygraph

$$\langle\, x, y, z \mid x^2 = 0,\ xy = zx \,\rangle$$

does not have a finite Gröbner bases on 3-generators x, y and z. Indeed, the first relation is oriented as $x^2 \Rightarrow 0$ and the orientation $xy \Rightarrow zx$ induce the addition of the 2-cells $xz^nx \Rightarrow 0$, for all integer $n \geqslant 1$. Another way is to orient the relation as $zx \Rightarrow xy$. But in this case, we need to add the 2-cells $xy^nx \Rightarrow 0$, for all integer $n \geqslant 1$.

6.3.9. Exercise. Show that we can compute a Gröbner bases for the algebra given in Example 6.3.8 with four generators. [Hint. Add a generator t and the relations $xy \Rightarrow t$ and $zx \Rightarrow t$.]

CHAPTER

7

Anick's resolution

Contents

In two seminal papers, Anick introduced a method to compute a free resolution for an algebra starting with a Gröbner basis of its ideal of relations. First he gave the construction for monomial algebras in [Ani85] then for associative augmented algebras in [Ani86]. For an algebra presented by a Gröbner basis, the nth chains of its Anick resolution are generated by the n-fold overlaps of the leading terms of the Gröbner basis, and the differentials are constructed by Noetherian induction. The chains defined by Anick are recall in Subsection 7.2. The construction of the resolution is given in Subsection 7.3.

Resolutions for path algebras using the same method were obtained by Anick and Green in [AG87]. For a deeper discussion on the theory of Gröbner bases for path algebras and how to apply this theory to the construction of free resolutions for path algebras, we refer the reader to [Gre99]. Let us mention that the Anick resolution has been achieved by other methods. In particular, the Anick resolution for a homogeneous algebra can be constructed by a deformation of the resolution computed on the associated monomial algebra, see [DK09, Sec. 2.4.] for details, see also the Backelin construction in [Bac78]. The Anick resolution can be also obtained using algebraic Morse theory with a Morse matching on the bar resolution, see [Skö06, Sec. 3.2.] for details. Morse theory allows to construct, starting from a chain complex, a new chain complex such that the homology of the two complexes coincides. This method was also applied to the computation of minimal resolutions starting from the Anick resolution, [JW09].

Note also that others constructions of free resolutions using convergent rewriting systems were obtained by several authors, [Bro92, Kob90, Gro90, Kob05, GM12b]. Finally, let us mention that noncommutative Gröbner bases where developed by Dotsenko and Khoroshkin for shuffle operads in [DK10], giving operadic versions of Newman's lemma and Buchberger's algorithm. The Anick resolution for shuffle operads was constructed by Dotsenko and Khoroshkin in [DK09, DK12]. Using this construction, they prove that a shuffle operad with a quadratic Gröbner basis is Koszul, [DK12].

7.1. Homology of an algebra

In this section, we briefly recall the definition of homology of associative algebras with coefficients in left modules. For a deeper discussion on basic notions of homological algebra we refer the reader to [HS97, Rot09].

7.1.1. Functor Tor. Let us recall the definition of the derived functor Tor^R of the tensor product of modules over a fixed ring R. Let M be a left R-module and N be a right R-module. Given a projective resolution $\mathcal{P}$ of the right R-module N:

$$\mathcal{P}\ :\quad \cdots \longrightarrow P_n \xrightarrow{d_{n-1}} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{d_0} P_0 \xrightarrow{\varepsilon} N \longrightarrow 0$$

we associate the *deleted complex*:

$$\mathcal{P}_N\ :\quad \cdots \longrightarrow P_n \xrightarrow{d_{n-1}} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{d_0} P_0 \longrightarrow 0$$

obtained by suppressing the module N. Note that, we have not lost any information in the complex $\mathcal{P}_N$, as $N = \mathrm{coker}(d_0)$ by exactness of complex $\mathcal{P}$. Then, applying the functor $-\otimes_R M$, we form a complex of $\mathbb{Z}$-modules, denoted by $\mathcal{P}_N \otimes_R M$:

$$\mathcal{P}_N \otimes_R M\ :\quad \cdots \longrightarrow P_n \otimes_R M \xrightarrow{\overline{d}_{n-1}} P_{n-1} \otimes_R M \longrightarrow \cdots \longrightarrow P_1 \otimes_R M \xrightarrow{\overline{d}_0} P_0 \otimes_R M \longrightarrow 0$$

where $\overline{d}_{n-1}$ denotes the map $d_{n-1} \otimes \mathrm{Id}_M$.

For a natural number $n \geqslant 0$, we defined the $\mathbb{Z}$-module $\mathrm{Tor}_n^R(M, N)$ as the nth homology group of this complex:

$$\mathrm{Tor}_n^R(N, M) = H_n(\mathcal{P}_N \otimes_R M) = \mathrm{Ker}\, \overline{d}_{n-1} / \mathrm{Im}\, \overline{d}_n.$$

This definition is functorial in each variables, giving a bifunctor Tor_n^R from R-modules with values in the category of $\mathbb{Z}$-modules.

7.1.2. Following the definition, the functor $\mathrm{Tor}_0^R(N, -)$ is naturally equivalent to $N \otimes_R -$ and the functor $\mathrm{Tor}_n^R(-, M)$ is naturally equivalent to $- \otimes_R M$. Indeed, we have $\mathrm{Tor}_0^R(N, M) = \mathrm{coker}(\overline{d}_0)$. Furthermore, the functor $N \otimes_R -$ is right exact, hence

$$\mathrm{coker}(\overline{d}_0) = P_0 \otimes_R M / \mathrm{Im}\, (\overline{d}_0) = P_0 \otimes_R M / \ker(\varepsilon \otimes \mathrm{Id}_M) = N \otimes_R M.$$

This proves that

$$\mathrm{Tor}_0^R(N, M) = N \otimes_R M.$$

7.1.3. Contracting homotopy. Recall that a method to prove that a complex of R-modules

$$\cdots \longrightarrow M_{n+1} \xrightarrow{d_n} M_n \xrightarrow{d_{n-1}} M_{n-1} \longrightarrow \cdots \longrightarrow M_1 \xrightarrow{d_0} M_0 \xrightarrow{\varepsilon} N \longrightarrow 0$$

is acyclic is to construct a *contracting homotopy*, that is a sequence of morphisms of abelian groups

$$\cdots \longleftarrow M_{n+1} \xleftarrow{\iota_{n+1}} M_n \xleftarrow{\iota_n} M_{n-1} \longleftarrow \cdots \longleftarrow M_1 \xleftarrow{\iota_1} M_0 \xleftarrow{\iota_0} N$$

such that

$$\varepsilon\iota_0 = \mathrm{Id}_N, \qquad d_0\iota_1 + \iota_0\varepsilon = \mathrm{Id}_{M_0}, \qquad d_n\iota_{n+1} + \iota_n d_{n-1} \ = \ \mathrm{Id}_{M_n},$$

for every $n \geqslant 1$.

7.1.4. Homology of an algebra. Let $\mathbf{A}$ be an associative algebra over a field $\mathbb{K}$. For $n \geqslant 0$, the n-*th homology space* of the algebra $\mathbf{A}$ with coefficient in a left $\mathbf{A}$-module M is defined by

$$H_n(\mathbf{A}, M) = \mathrm{Tor}_n^{\mathbf{A}}(\mathbb{K}, M).$$

In practice, to compute the n-th homology spaces $H_n(\mathbf{A}, \mathbb{K})$, for all $n \geqslant 0$, we construct a free resolution of $\mathbb{K}$, seen as a trivial right-$\mathbf{A}$-module:

$$\mathcal{F}_{\mathbb{K}} \ : \cdots \longrightarrow F_n \xrightarrow{d_{n-1}} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_0} F_0 \xrightarrow{\varepsilon} \mathbb{K} \longrightarrow 0$$

and we compute the homology of the complex $\mathcal{F}_{\mathbb{K}} \otimes_{\mathbf{A}} \mathbb{K}$.

7.2. Anick's chains

In this subsection, Λ denotes a reduced left-monomial linear 2-polygraph. The set of sources of rules in Λ_2 we will be denoted by $s_1(\Lambda) = \{ s_1(\alpha) \in \Lambda_1^* \mid \alpha \in \Lambda_2 \}$. For a monomial u in Λ_1^*, we denote by $\deg_{s_1(\Lambda)}(u)$ the number of possible reductions on u with respect to Λ_2.

7.2.1. Anick's chains, [Ani86]. For an integer $n \geqslant -1$, the *Anick* n-*chains* of the linear 2-polygraph Λ and their *tails* are defined by induction as follows.

- The unique (-1)-*chain* is the empty monomial, denoted by 1, it is its own tail.
- The 0-*chains* are the 1-cells in Λ_1, and the *tail* of a 0-chain x in Λ_1 is x itself.
- For $n \geqslant 1$, suppose that $(n-1)$-chains and their tails constructed. An n-*chain* is a monomial u in Λ_1^* of the form

$$u = vt$$

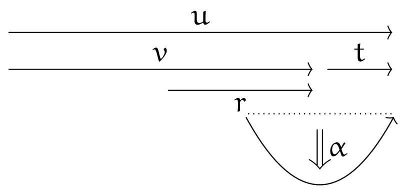
such that

i) v is $(n-1)$-chain,

ii) t is a reduced monomial with respect to Λ_2, called the *tail* of u,

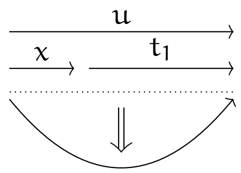
iii) if r is the tail of v, then $\deg_{s_1(\Lambda)}(rt) = 1$,

iv) the unique reduction on rt is rightmost, that is, given by a 2-cell α in Λ_2 reducing the ending of the monomial rt:

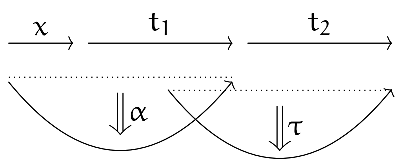


We will denote by $\Omega_n(\Lambda)$, or by Ω_n if there is no possible confusion, the set of n-chains of the linear 2-polygraph Λ.

7.2.2. Anick's chains and overlapping. The linear 2-polygraph Λ being reduced, we have the following description of Anick's chains. We have $\Omega_1(\Lambda) = s_1(\Lambda)$. Indeed, a 1-chain is a non reduced monomial u written $u = xt_1$, where x is a 1-cell in Λ_1 and t_1 is a reduced monomial:

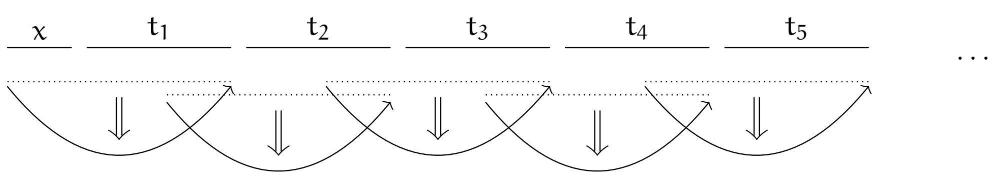


and such that there is only one 2-cell in Λ_2 that can be applied on the monomial u. A 2-chain u is the source of a critical branching. Indeed, $u = xt_1t_2$, where xt_1 is the source of a 2-cell α in Λ_2 and there is a rightmost reduction τ reducing t_1t_2, and thus overlapping α:



Moreover, u is not the source of a critical triple branching, as we have $\deg_{s_1(\Lambda)} u = 2$. In this way, there is a one-to-one correspondence between $\Omega_2(\Lambda)$ and the set of critical branchings of the 2-polygraph Λ.

For $n \geqslant 3$, a n-chain u corresponds to a n-fold overlapping composed by $(n-1)$ chained critical branchings. It is possible that $\deg_{s_1(\Lambda)} u > n$, see Example 7.2.5.



7.2.3. Proposition ([Ani86]). *Suppose* $n \geqslant 1$. *If* $u = x_{i_1} \ldots x_{i_t}$ *is an* n*-chain, then there is a unique* $s \leqslant t$ *such that* $x_{i_1} \ldots x_{i_s}$ *is an* $(n-1)$*-chain. Moreover,* $x_{i_{s+1}} \ldots x_{i_t}$ *is reduced.*

Indeed, suppose that there is two $(n-1)$-chains $x_{i_1} \ldots x_{i_s}$ and $x_{i_1} \ldots x_{i_{s'}}$ which factorise u. By uniqueness of the reduction on the tail, condition **iii)** in (7.2.1), necessarily we have $s = s'$

7.2.4. Notation. An n-chain u, whose $(n-1)$-chain is v and tail is t, will be denoted by $u = v|t$. Expanding this notation, any n-chain can be written $x|t_1|t_2| \ldots |t_n$, where $x \in s_1(\Lambda)$ and $x|t_1|\ldots|t_i$ is an i-chain for any $0 < i < n$.

7.2.5. Example, [Ani86]. Let Λ be a reduced left-monomial linear 2-polygraph with $\Lambda_1 = \{x\}$ and $s_1(\Lambda) = \{x^3\}$. The 1-cell x is the unique 0-chain. The monomial $x^3 = x|x^2$ is the unique 1-chain, xx is not a 1-chain because $\deg_{s_1(\Lambda)} x^2 = 0$. The monomial $x^4 = x^3|x$ is the unique 2-chain. Note that $x^5 = x^3|x^2$ is not a 2-chain. Indeed, $\deg_{s_1(\Lambda)} x^4 = 2$, and on the monomial x^5 there are three possible reductions, with the first one that intersects the last one, giving a critical triple branching:

$$\overline{x\,\underline{x\,x}}\,\overline{\underline{x}\,x}$$

The monomial $x^6 = x^4|x^2$ is the unique 3-chain. Note that $x^5 = x^4x$ is not a 3-chain because $\deg_{s_1(\Lambda)} xx = 0$. Note that there are four possible reductions on the 3-chain x^6:

$$\overline{x\,\underline{x\,\overline{x\,x}}\,\overline{x}\,x}$$

Thus we have

$$\Omega_0 = \Lambda_1, \quad \Omega_1 = s_1(\Lambda), \quad \Omega_2 = \{x^4\}, \quad \Omega_3 = \{x^6\}.$$

More generally, we show that for any integer $n \geqslant 0$, we have

$$\Omega_{2n-1} = \{x^{3n}\}, \quad \Omega_{2n} = \{x^{3n+1}\}.$$

7.2.6. Example, [Ani86]. Suppose that $\Lambda_1 = \{x, y\}$ and $s_1(\Lambda) = \{x^2yxy, xyxy^2\}$. Then we have

$$\Omega_0 = \{x, y\}, \quad \Omega_1 = \{x|xyxy, x|yxy^2\}, \quad \Omega_2 = \{x|xyxy|y, x|xyxy|xy^2\}, \quad \Omega_n = \emptyset, \ \text{ for } n \geqslant 3.$$

7.2.7. Exercise, [Ani85]. Let Λ be a linear 2-polygraph such that $\Lambda_1 = \{x, y, z\}$. Determine Anick's chains in the following situations

1) $s_1(\Lambda) = \{xyzx, zxy\}$,

2) $s_1(\Lambda) = \{xyzx, xxy\}$. In this case, show that the number of n-chains equals the $(n+2)$nd Fibonacci number when $n \geqslant 1$.

7.3. Anick's resolution

In this subsection, Λ denotes a convergent reduced left-monomial linear 2-polygraph, whose 2-cells are compatible with a monomial order $\prec$ defined on Λ_1^*. Let denote by **A** the algebra presented by Λ. We define a section $\iota : \mathbf{A} \longrightarrow \Lambda_1^\ell$ of the canonical projection $\pi : \Lambda_1^\ell \longrightarrow \mathbf{A}$, sending every 1-cell f of **A** to the normal form $\widehat{f}$ of any representative 1-cell of f in Λ_1^ℓ, as in (5.5.5).

7.3.1. Anick's resolution. Let $\mathbf{A}[\Omega_n(\Lambda)] = \mathbb{K}[\Omega_n(\Lambda)] \otimes_{\mathbb{K}} \mathbf{A}$ be the free right $\mathbf{A}$-module over the set of n-chains $\Omega_n(\Lambda)$. We will identify $\mathbf{A}[\Omega_0(\Lambda)]$ to $\mathbf{A}[\Lambda_1]$ and $\mathbf{A}[\Omega_{-1}(\Lambda)]$ to $\mathbf{A}$. Anick constructs in [Ani86] a free resolution of right $\mathbf{A}$-modules defined by the complex

$$\mathcal{A}(\Lambda)\ :\cdots \longrightarrow \mathbf{A}[\Omega_n(\Lambda)] \overset{d_n}{\to} \mathbf{A}[\Omega_{n-1}(\Lambda)] \to \cdots \longrightarrow \mathbf{A}[\Omega_1(\Lambda)] \overset{d_1}{\to} \mathbf{A}[\Lambda_1] \overset{d_0}{\to} \mathbf{A} \overset{\varepsilon}{\to} \mathbb{K} \longrightarrow 0,$$

whose differentials d_n are constructed inductively simultaneously with the contracting homotopy

$$\iota_n : \operatorname{Ker} d_{n-1} \longrightarrow \mathbf{A}[\Omega_n(\Lambda)].$$

The applications d_n are morphisms of right $\mathbf{A}$-modules and the applications ι_n are linear maps.

7.3.2. For the first steps of the resolution

$$\mathbf{A}[\Lambda_1] \underset{\iota_0}{\overset{d_0}{\rightleftarrows}} \mathbf{A} \underset{\iota_{-1}}{\overset{\varepsilon}{\rightleftarrows}} \mathbb{K} \longrightarrow 0, \tag{7.3.3}$$

we define $\iota_{-1} : \mathbb{K} \hookrightarrow \mathbf{A}$ as the embedding of $\mathbb{K}$ in $\mathbf{A}$, and we define the *augmentation map* $\varepsilon : \mathbf{A} \to \mathbb{K}$ by setting $\varepsilon(x) = 0$, for all $x \in \Lambda_1$. Hence, we have $\mathbf{A} = \mathbb{K} \oplus \operatorname{Ker} \varepsilon$, and $\varepsilon\iota_{-1} = \mathrm{Id}_{\mathbb{K}}$. Then, we set

$$d_0(x \otimes 1) = x,$$

for all x in Λ_1. By convergence hypothesis, any monomial in $\mathbf{A}$ admits a unique normal form in Λ_1^* with respect to Λ_2. For a monomial u in $\mathbf{A}$ such that the normal form is written $\widehat{u} = x_1x_2\ldots x_k$ in Λ_1^*, we define

$$\iota_0(1 \otimes u) = x_1 \otimes x_2 \ldots x_k. \tag{7.3.4}$$

Then, we extend ι_0 to any f in $\mathbf{A}$ by linearity. The map ι_0 is well defined by uniqueness of the normal form due to the convergence of the linear 2-polygraph Λ. The exactness of the sequence (7.3.3) in $\mathbf{A}$ is a consequence of the two equalities:

$$\varepsilon d_0(x \otimes 1) = 0 \ \text{ and } \ d_0\iota_0 = \mathrm{id}_{\operatorname{Ker}(\varepsilon)}.$$

7.3.5. For $n \geqslant 1$, we define the pair (d_n, ι_n) by induction on n:

$$\mathbf{A}[\Omega_n(\Lambda)] \underset{\iota_n}{\overset{d_n}{\rightleftarrows}} \mathbf{A}[\Omega_{n-1}(\Lambda)] \underset{\iota_{n-1}}{\overset{d_{n-1}}{\rightleftarrows}} \mathbf{A}[\Omega_{n-2}(\Lambda)] \underset{\iota_{n-2}}{\overset{d_{n-2}}{\rightleftarrows}} \quad \cdots$$

We suppose that the maps d_k and $\iota_k : \operatorname{Ker} d_{k-1} \longrightarrow \mathbf{A}[\Omega_k(\Lambda)]$ are constructed such that

$$d_{k-1}d_k = 0 \quad \text{and} \quad d_k\iota_k = \mathrm{Id}_{\operatorname{Ker} d_{k-1}},$$

for all $k \leqslant n-1$. We define inductively d_n on an n-chain $v|t$ with tail t by

$$d_n(v|t \otimes 1) = v \otimes t - \iota_{n-1}d_{n-1}(v \otimes t). \tag{7.3.6}$$

7.3.7. In the definition of $d_n(v|t \otimes 1)$, the term $v \otimes t$ will be the leading term with respect the well-founded order defined on $\mathbf{A}[\Omega_n(\Lambda)]$ as follows. We extend the monomial order $\prec$ on Λ_1^{ℓ} into a well-founded order on $\mathbf{A}[\Omega_n(\Lambda)]$ by setting

$$f_1 \otimes u_1 \prec f_2 \otimes u_2 \quad \text{if} \quad f_1\widehat{u}_1 \prec f_2\widehat{u}_2,$$

for all f_1, f_2 in $\mathbb{K}[\Omega_n(\Lambda)]$ and u_1, u_2 in $\mathbf{A}$.

7.3.8. Let us define recursively the map

$$\iota_n : \operatorname{Ker} d_{n-1} \longrightarrow \mathbf{A}[\Omega_n(\Lambda)]$$

as follows. Given h in $\operatorname{Ker} d_{n-1} \subset \mathbf{A}[\Omega_{n-1}(\Lambda)]$, we denote by $u_{n-1} \otimes t$ the leading term of h, that is

$$h = \lambda u_{n-1} \otimes t + (\text{lower terms}),$$

where λ in $\mathbb{K}$ is non-zero. The $(n-1)$-chain u_{n-1} can be uniquely decomposed in

$$u_{n-1} = u_{n-2}|t',$$

where u_{n-2} is an $(n-2)$-chain and t' is the tail of u_{n-1}. By induction, we have

$$d_{n-1}(u_{n-1} \otimes 1) = u_{n-2} \otimes t' + (\text{lower terms}).$$

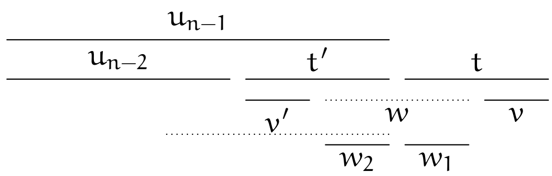
As d_{n-1} is a morphism of right $\mathbf{A}$-modules, we have

$$\begin{aligned} d_{n-1}(h) &= \lambda d_{n-1}(u_{n-1} \otimes t) + d_{n-1}(\text{lower terms}) \\ &= \lambda u_{n-2} \otimes t't + (\text{lower terms}). \end{aligned}$$

Suppose now that the monomial $t't$ is reduced, then $u_{n-2} \otimes t't$ remain the leading term of $d_{n-1}(h)$, hence h cannot be in $\operatorname{Ker} d_{n-1}$ thus contradicting the hypothesis. It follows that $t't$ can be reduced, and we set

$$t't = v'wv,$$

where w is the 1-source of the leftmost reduction with respect to Λ_2 that can be applied on $t't$:



(7.3.9)

Consider the factorization $w = w_2w_1$ and $t = w_1v$ as in the picture (7.3.9). It follows that $u_{n-2}v'w = u_{n-2}|t'|w_1$ forms an n-chain, and $u_{n-2}v'w \otimes v \in \mathbf{A}[\Omega_n(\Lambda)]$. We set

$$\begin{aligned} \iota_n(h) &= \iota_n(\lambda u_{n-1} \otimes t + \text{lower terms}) \\ &= \lambda u_{n-2}v'w \otimes v + \iota_n(h - \lambda d_n(u_{n-2}v'w \otimes v)). \end{aligned}$$

This is well defined, because $h - \lambda d_n(u_{n-2}v'w \otimes v) \prec h$ by construction. Indeed

$$\begin{aligned} d_n(u_{n-2}v'w \otimes v) = d_n(u_{n-2}v'w_2w_1 \otimes v) &= u_{n-2}v'w_2 \otimes w_1v + (\text{lower terms}) \\ &= u_{n-1} \otimes t + (\text{lower terms}). \end{aligned}$$

Moreover, $d_{n-1}(h - \lambda d_n(u_{n-2}v'w \otimes v)) = 0$.

From this construction, we deduce the following result:

7.3.10. Theorem ([Ani86, Theorem 1.4]). *Let* **A** *be an algebra presented by a convergent reduced left-monomial linear 2-polygraph* Λ*, compatible with a monomial order* $\prec$*. The complex of right* **A***-modules* $\mathcal{A}(\Lambda)$ *defined by*

$$\cdots \longrightarrow \mathbf{A}[\Omega_n(\Lambda)] \overset{d_n}{\to} \mathbf{A}[\Omega_{n-1}(\Lambda)] \to \cdots \longrightarrow \mathbf{A}[\Omega_1(\Lambda)] \overset{d_1}{\to} \mathbf{A}[\Lambda_1] \overset{d_0}{\to} \mathbf{A} \overset{\varepsilon}{\to} \mathbb{K} \longrightarrow 0$$

where, for any $n \geqslant 0$*, the morphism* d_n *is defined on a* n*-chain* $v|t$ *by*

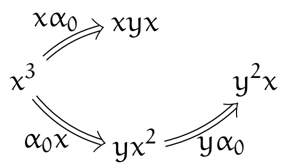
$$d_n(v|t \otimes 1) = v \otimes t + h,$$

where $\mathrm{lt}(h) \prec v|t \otimes 1$*, if* $h \neq 0$*, is a resolution of the trivial right* **A***-module* $\mathbb{K}$*.*

7.3.11. Example. Let consider the algebra **A** presented by the linear 2-polygraph

$$\Lambda = \langle\, x, y \mid x^2 \overset{\alpha_0}{\Rightarrow} yx \,\rangle,$$

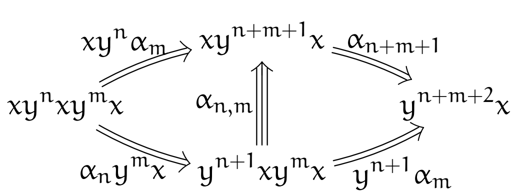
compatible with the deglex order $\prec_{\text{deglex}}$ induced by the alphabetic order $y \prec x$. It appears one critical branching



We complete the linear 2-polygraph Λ with the 2-cells

$$\alpha_n : xy^nx \Longrightarrow y^{n+1}x,$$

for all $n > 0$. We note that, for any integers $n, m \geqslant 0$, we have a critical branching



Then the linear 2-polygraph Λ', whose set of 1-cell is Λ_1 and $\Lambda'_2 = \{\alpha_n \mid n \geqslant 0\}$ is convergent, compatible with the monomial order $\prec$ and Tietze equivalent to Λ. Equivalently, the set $\{xy^nx - y^{n+1}x \mid n \geqslant 0\}$

forms a Gröbner basis for the ideal $I(\Lambda)$. Anick's 1-chains are of the form $x|y^n x$ with $n \geqslant 0$ and Anick's 2-chains are of the form $x|y^n x|y^m x$ with $n, m \geqslant 0$. More generally, for any $k \geqslant 2$, we have

$$\Omega_k = \{x|y^{n_1}x|y^{n_2}x|\dots|y^{n_k}x \;\text{ for }\; n_1, \dots, n_k \geqslant 0\},$$

Let us compute the boundary maps d_0, d_1, d_2 and d_3. We have $d_0(x \otimes 1) = x$, $d_0(y \otimes 1) = y$ and

$$\begin{aligned} d_1(x|y^n x \otimes 1) &= x \otimes y^n x - \iota_0 d_0(x \otimes y^n x),\\ &= x \otimes y^n x - \iota_0(1 \otimes x y^n x),\\ &= x \otimes y^n x - \iota_0(1 \otimes y^{n+1} x),\\ &= x \otimes y^n x - y \otimes y^n x. \end{aligned}$$

The last equality is consequence of the definition of the map ι_0 in (7.3.4).

$$\begin{aligned} d_2(x|y^n x|y^m x \otimes 1) &= x|y^n x \otimes y^m x - \iota_1 d_1(x|y^n x \otimes y^m x),\\ &= x|y^n x \otimes y^m x - \iota_1(x \otimes y^n x y^m x - y \otimes y^n x y^m x),\\ &= x|y^n x \otimes y^m x - \iota_1(x \otimes y^{n+m+1} x - y \otimes y^{n+m+1} x),\\ &= x|y^n x \otimes y^m x - x|y^{n+m+1} x \otimes 1 + \iota_1\big(x \otimes y^{n+m+1} x - y \otimes y^{n+m+1} x - d_1(x|y^{n+m+1} x \otimes 1)\big),\\ &= x|y^n x \otimes y^m x - x|y^{n+m+1} x \otimes 1\\ &\qquad + \iota_1\big(x \otimes y^{n+m+1} x - y \otimes y^{n+m+1} x - x \otimes y^{n+m+1} x + y \otimes y^{n+m+1} x)\big),\\ &= x|y^n x \otimes y^m x - x|y^{n+m+1} x \otimes 1. \end{aligned}$$

$$\begin{aligned} d_3(x|y^n x|y^m x|y^k x \otimes 1) &= x|y^n x|y^m x \otimes y^k x - \iota_2 d_2(x|y^n x|y^m x \otimes y^k x),\\ &= x|y^n x|y^m x \otimes y^k x - \iota_2\big(x|y^n x \otimes y^m x y^k x - x|y^{n+m+1} x \otimes y^k x\big),\\ &= x|y^n x|y^m x \otimes y^k x - \iota_2\big(x|y^n x \otimes y^{m+k+1} x - x|y^{n+m+1} x \otimes y^k x\big),\\ &= x|y^n x|y^m x \otimes y^k x - x|y^n x|y^{m+k+1} x \otimes 1\\ &\qquad - \iota_2\big(x|y^n x \otimes y^{m+k+1} x - x|y^{n+m+1} x \otimes y^k x - d_2(x|y^n x|y^{m+k+1} x \otimes 1)\big),\\ &= x|y^n x|y^m x \otimes y^k x - x|y^n x|y^{m+k+1} x \otimes 1\\ &\qquad - \iota_2\big(x|y^n x \otimes y^{m+k+1} x - x|y^{n+m+1} x \otimes y^k x - x|y^n x \otimes y^{m+k+1} x - x|y^{n+m+k+2} x \otimes 1\big)\\ &= x|y^n x|y^m x \otimes y^k x - x|y^n x|y^{m+k+1} x \otimes 1 - \iota_2\big(-x|y^{n+m+1} x \otimes y^k x - x|y^{n+m+k+1} x \otimes 1\big),\\ &= x|y^n x|y^m x \otimes y^k x - x|y^n x|y^{m+k+1} x \otimes 1 + x|y^{n+m+1} x|y^k x \otimes 1\\ &\qquad + \iota_2(x|y^{n+m+1} x \otimes y^k x - x|y^{n+m+k+1} x \otimes 1 - d_2(x|y^{n+m+1} x|y^k x \otimes 1)\big),\\ &= x|y^n x|y^m x \otimes y^k x - x|y^n x|y^{m+k+1} x \otimes 1 + x|y^{n+m+1} x|y^k x \otimes 1\\ &\qquad + \iota_2(x|y^{n+m+1} x \otimes y^k x - x|y^{n+m+k+1} x \otimes 1 - x|y^{n+m+1} x \otimes y^k x - x|y^{n+m+k+1} x \otimes 1)\big)\\ &= x|y^n x|y^m x \otimes y^k x - x|y^n x|y^{m+k+1} x \otimes 1 + x|y^{n+m+1} x|y^k x \otimes 1. \end{aligned}$$

7.3.12. Example. Let consider the algebra **A** given in 7.3.11 with the following presentation

$$\langle\, x, y \mid yx \overset{\beta}{\Rightarrow} x^2 \,\rangle,$$

compatible with the deglex order induced by the alphabetic order $x \prec y$. This polygraph does not have critical branching, thus the sets of Anick's n-chains are empty for $n \geqslant 2$. It follows that the associated Anick resolution is

$$\cdots \longrightarrow 0 \longrightarrow \mathbf{A}[y|x] \overset{d_1}{\to} \mathbf{A}[x,y] \overset{d_0}{\to} \mathbf{A} \overset{\varepsilon}{\to} \mathbb{K} \longrightarrow 0$$

with $d_0(x \otimes 1) = x$, $d_0(y \otimes 1) = y$ and

$$\begin{aligned} d_1(y|x \otimes 1) &= y \otimes x - \iota_0(1 \otimes yx), \\ &= x \otimes y - \iota_0(1 \otimes x^2), \\ &= x \otimes y - x \otimes x. \end{aligned}$$

7.3.13. Example. Consider Example 5.1.7 with the algebra **A** presented by

$$\langle\, x, y, z \mid xyz = x^3 + y^3 + z^3 \,\rangle.$$

With the Gröbner basis computed in 6.3.7:

$$z^3 \overset{\alpha_f}{\Rightarrow} xyz - x^3 - y^3 \qquad zy^3 \overset{\beta}{\Rightarrow} zxyz - zx^3 + y^3z + x^3z - xyz^2$$

Anick's chains are of the form z^n and $z^n y^3$, for $n \geqslant 0$, so that the Anick resolution, defined in [Ani86], is infinite.

7.3.14. Exercise, [Ani86, Section 3]. Compute the Anick resolution for the algebra presented by the linear 2-polygraph $\langle\, x, y \mid xyxyx \Rightarrow xyx \,\rangle$.

7.3.15. Anick's resolution for a monomial algebra. We construct the Anick resolution in the case of a monomial algebra **A**. Recall from 5.1.19, that such an algebra can be presented by a monomial linear 2-polygraph Λ, that is, left-monomial and $t_1(\alpha) = 0$ for all α in Λ_2. Obviously, such a presentation is always convergent. Suppose that the polygraph Λ is reduced. The sets of chains for Λ are $\Omega_0(\Lambda) = \Lambda_1$, $\Omega_1(\Lambda) = s_1(\Lambda)$ and for any $n \geqslant 2$, $\Omega_n(\Lambda)$ is the set of n-overlapping $x|t_1|\ldots|t_{n-1}|t_n$ of branchings of Λ with $x \in \Lambda_1$, and $t_1, \ldots, t_n \in \Lambda_1^*$, such that xt_1, $t_i t_{i+1}$ in $s_1(\Lambda)$ for any $1 \leqslant i \leqslant n-1$. We have

$$\widehat{xt_1} = 0 \qquad \text{and} \qquad \widehat{t_{i-1}t_i} = 0, \text{for all } 1 \leqslant i \leqslant n. \tag{7.3.16}$$

Consider the boundary map

$$d_n : \mathbf{A}[\Omega_n(\Lambda)] \longrightarrow \mathbf{A}[\Omega_{n-1}(\Lambda)]$$

defined by

$$d_n(x|t_1|\ldots|t_{n-1}|t_n \otimes 1) = x|t_1|\ldots|t_{n-1} \otimes t_n - \iota_{n-1} d_{n-1}(x|t_1|\ldots|t_{n-1} \otimes t_n).$$

By definition of d_{n-1}, we have

$$d_{n-1}(x|t_1|\ldots|t_{n-1} \otimes t_n) = x|t_1|\ldots|t_{n-2} \otimes t_{n-1}t_n - \iota_{n-2} d_{n-2}(x|t_1|\ldots|t_{n-2} \otimes t_{n-1}t_n)$$

Using relation in (7.3.16), we have $d_{n-1}(x|t_1|\ldots|t_{n-1} \otimes t_n) = 0$, hence

$$d_n(x|t_1|\ldots|t_{n-1}|t_n \otimes 1) = x|t_1|\ldots|t_{n-1} \otimes t_n.$$

7.4. Computing homology with Anick's resolution

7.4.1. Computing homology. Given an algebra $\mathbf{A}$ and a left $\mathbf{A}$-module M. When the algebra is presented by a convergent reduced left-monomial linear 2-polygraph Λ, compatible with a monomial order, the Anick resolution $\mathcal{A}(\Lambda)$ gives a method to compute the homology groups of $\mathbf{A}$ with coefficient in M. In this section, we give several examples of computations of homology groups with coefficients in $\mathbb{K}$. From the resolution $\mathcal{A}(\Lambda)$, we compute the complex $\mathcal{A}(\Lambda) \otimes_{\mathbf{A}} \mathbb{K}$ given by

$$\cdots \longrightarrow \mathbb{K}[\Omega_n(\Lambda)] \xrightarrow{\overline{d}_n} \mathbb{K}[\Omega_{n-1}(\Lambda)] \to \cdots \longrightarrow \mathbb{K}[\Omega_1(\Lambda)] \xrightarrow{\overline{d}_1} \mathbb{K}[\Lambda_1] \xrightarrow{\overline{d}_0} \mathbb{K} \longrightarrow 0$$

where $\mathbb{K}[\Omega_n(\Lambda)]$ denotes the free vector space on $\Omega_n(\Lambda)$ and $\overline{d}_n$ denotes the map $d_n \otimes \mathrm{Id}_{\mathbb{K}}$. These maps satisfy $\overline{d}_n\overline{d}_{n+1} = 0$, for all $n \geqslant 0$, and we have

$$H_0(\mathbf{A}, \mathbb{K}) = \mathbb{K}, \quad \text{and} \quad H_n(\mathbf{A}, \mathbb{K}) = \mathrm{Ker}\, \overline{d}_{n-1}/\mathrm{Im}\, \overline{d}_n.$$

As a first application, we have the following finiteness properties.

7.4.2. Proposition. *Let $\mathbf{A}$ be an algebra presented by a finite convergent left-monomial linear 2-polygraph. The following statements hold.*

i) *$\mathbf{A}$ is of homological type right-FP_∞, that is, there exists an infinite length free finitely generated resolution of the trivial right $\mathbf{A}$-module $\mathbb{K}$.*

ii) *For any $n \geqslant 0$, the vector space $H_n(\mathbf{A}, \mathbb{K})$ is finitely generated.*

iii) *[Ani86, Lemma 3.1] The algebra $\mathbf{A}$ has a Poincaré series*

$$P_{\mathbf{A}}(t) = \sum_{n=0}^{\infty} \dim_{\mathbb{K}}(H_n(\mathbf{A}, \mathbb{K}))t^n,$$

with exponential or slower growth, that is, there are constants $c_1, c_2 > 0$, such that

$$0 \leqslant \dim_{\mathbb{K}}(H_n(\mathbf{A}, \mathbb{K})) \leqslant c_2(c_1)^n.$$

Note that the finiteness conditions **i)** and **ii)** were obtained by Kobayashi for monoids. A monoid $\mathbf{M}$ is of *homological type right-*FP_∞ *over* $\mathbb{K}$ if the monoid algebra $\mathbb{K}\mathbf{M}$ is of homological type right-FP_∞. In [Kob90], by constructing a resolution similar to the Anick resolution, Kobayashi shows that a monoid $\mathbf{M}$ having a presentation by a finite convergent rewriting system is of homological type FP_∞. Similar constructions of resolutions of monoids presented by convergent rewriting systems were also obtained by Brown [Bro92] and by Groves [Gro90]. The diferent constructions are based on distinct ways to describe the n-fold critical branchings of a convergent rewriting system.

7.4.3. Exercise. Prove the conditions **i)** and **ii)** in Proposition 7.4.2.

7.4.4. Low-dimensional homology. In the first dimensions, we have the following complex

$$\mathbb{K}[\Omega_2(\Lambda)] \xrightarrow{\overline{d}_2} \mathbb{K}[\Omega_1(\Lambda)] \xrightarrow{\overline{d}_1} \mathbb{K}[\Lambda_1] \xrightarrow{\overline{d}_0} \mathbb{K} \longrightarrow 0$$

The map $\overline{d}_0$ is zero, hence

$$H_1(\mathbf{A}, \mathbb{K}) = \mathbb{K}[\Lambda_1]/\mathrm{Im}\,\overline{d}_1.$$

A 1-cell x of Λ_1 in $\mathrm{Im}\,\overline{d}_1$ comes from a relation with source or target x. It follows that x is a redundant generator in the presentation. Indeed, a term $x \otimes 1$, with x in Λ_1 appears in $\mathrm{Im}\,d_1$ if and only if x is the source or the target of a 2-cell in Λ_2. Let $\alpha : x \Rightarrow y_1 \ldots y_k$ be a 2-cell in Λ_2, where by hypothesis $y_1 \ldots y_k$ is reduced. Thus we have

$$\begin{aligned} d_1(1|x \otimes 1) &= x \otimes 1 - \iota_0 d_0(x \otimes 1) \\ &= x \otimes 1 - \iota_0(1 \otimes y_1 \ldots y_k) \\ &= x \otimes 1 - y_1 \otimes y_2 \ldots y_k. \end{aligned}$$

Hence $\overline{d}_1(x) = x$. Suppose now that $x_1 \ldots x_k \overset{\alpha}{\Rightarrow} y$ is a 2-cell in Λ_2. We have

$$\begin{aligned} d_1(x_1 \ldots x_k \otimes 1) &= x_1 \otimes x_2 \ldots x_k - \iota_0 d_0(x_1 \otimes x_2 \ldots x_k) \\ &= x_1 \otimes x_2 \ldots x_k - \iota_0(1 \otimes y) \\ &= x_1 \otimes x_2 \ldots x_k - y \otimes 1 \end{aligned}$$

Hence $\overline{d}_1(x_1 \ldots x_k) = -y$. Thus, we have $\overline{d}_1 = 0$ if and only if the number of generators is minimal. In this way, $\dim_{\mathbb{K}} H_1(\mathbf{A}, \mathbb{K})$ is equal to the minimal number of generators for a presentation of the algebra $\mathbf{A}$. For analogous reasons, we show that $\dim_{\mathbb{K}} H_2(\mathbf{A}, \mathbb{K})$ is the minimal required number of the defining relations.

7.4.5. Example. Consider the algebra $\mathbf{A}$ presented by the linear 2-polygraph $\langle\, x, y \mid yx \Rightarrow x^2 \,\rangle$. From the Anick resolution computed in 7.3.12, we deduce the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{K}[y|x] \xrightarrow{\overline{d}_1} \mathbb{K}[x, y] \xrightarrow{\overline{d}_0} \mathbb{K} \longrightarrow 0$$

whose boundary maps $\overline{d}_0$ and $\overline{d}_1$ are zero. We deduce

$$H_n(\mathbf{A}, \mathbb{K}) = \begin{cases} \mathbb{K} & \text{if } n = 0, 2, \\ \mathbb{K}^2 & \text{if } n = 1, \\ 0 & \text{if } n \geqslant 3. \end{cases}$$

7.4.6. Exercise [Ani86, Theorem 3.2]. Let $\mathbf{A}$ be an algebra admitting a presentation by a left-monomial reduced linear 2-polygraph compatible with a monomial order and having no critical branching. Show that $H_n(\mathbf{A}, \mathbb{K}) = 0$, for any $n \geqslant 3$. A presentation without critical branching is called *combinatorially free* in [Ani86].

7.5. MINIMALITY OF ANICK'S RESOLUTION

7.5.1. Minimal complex. A complex of free right **A**-modules

$$\cdots \longrightarrow F_{n+1} \xrightarrow{d_n} F_n \xrightarrow{d_{n-1}} F_{n-1} \longrightarrow \cdots$$

is *minimal* if all induced maps $\overline{d}_n = d_n \otimes \mathrm{Id}_{\mathbb{K}} : F_{n+1} \otimes_{\mathbf{A}} \mathbb{K} \longrightarrow F_n \otimes_{\mathbf{A}} \mathbb{K}$ are zero. A resolution is *minimal* if the associated complex is minimal. Note that a minimal free resolution is one in which each free module has the minimal number of generators as illustrated in the following example.

7.5.2. Example. Let consider the algebra **A** presented by the linear 2-polygraph $\langle x, y \mid x \Rightarrow y \rangle$, which is compatible with the deglex order induced by $y \prec x$. The Anick resolution is

$$0 \longrightarrow \mathbf{A}[x|1] \xrightarrow{d_1} \mathbf{A}[x, y] \xrightarrow{d_0} \mathbf{A} \xrightarrow{\varepsilon} \mathbb{K} \longrightarrow 0$$

with

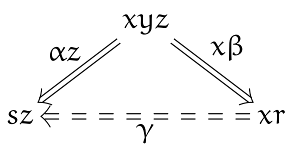
$$d_0(x \otimes 1) = x, \quad d_0(y \otimes 1) = y, \quad d_1(x|1 \otimes 1) = x \otimes 1 - 1 \otimes y.$$

This resolution is not minimal because $\overline{d}_1 \neq 0$. A minimal resolution for the algebra **A** can be constructed from the polygraph $\langle\, x \mid \emptyset \,\rangle$ with no 2-cell.

7.5.3. Example. Let consider the algebra **A** presented by the linear 2-polygraph

$$\langle\, x, y, z, r, s \mid xy \overset{\alpha}{\Rightarrow} s,\ yz \overset{\beta}{\Rightarrow} r \,\rangle$$

compatible with the deglex order induced by the alphabetic order $s \prec r \prec z \prec y \prec x$. There is a critical branching:



which is confluent by adding the rule $xr \overset{\gamma}{\Rightarrow} sz$. The linear 2-polygraph $\Lambda' = \langle\, \Lambda_1 \mid \alpha, \beta, \gamma \,\rangle$ is compatible with the deglex order considered above, convergent and Tietze equivalent to Λ. The induced the Anick resolution $\mathcal{A}(\Lambda')$ is

$$\cdots \longrightarrow 0 \longrightarrow \mathbf{A}[xy|z] \xrightarrow{d_2} \mathbf{A}[x|y, x|r, y|z] \xrightarrow{d_1} \mathbf{A}[x, y, z, r, s] \xrightarrow{d_0} \mathbf{A} \xrightarrow{\varepsilon} \mathbb{K} \longrightarrow 0$$

with

$$d_1(x|y \otimes 1) = x \otimes y - s \otimes 1, \quad d_1(x|r \otimes 1) = x \otimes r - s \otimes z, \quad d_1(y|z \otimes 1) = y \otimes z - r \otimes 1,$$

and

$$d_2(x|y|z \otimes 1) = xy \otimes z - xr \otimes 1.$$

This resolution is not minimal, because the maps $\overline{d}_1$ and $\overline{d}_2$ are non zero. Note that

$$H_n(\mathbf{A}, \mathbb{K}) = \begin{cases} \mathbb{K} & \text{if } n = 0, \\ \mathbb{K}^3 & \text{if } n = 1, \\ 0 & \text{if } n \geqslant 2. \end{cases}$$

and a minimal resolution for the algebra $\mathbf{A}$ can be constructed from the linear 2-polygraph $\langle\, x, y, z \mid \emptyset \,\rangle$ which produces the following resolution

$$\cdots \longrightarrow 0 \longrightarrow \mathbf{A}[x, y, z] \overset{d_0}{\to} \mathbf{A} \overset{\varepsilon}{\to} \mathbb{K} \longrightarrow 0$$

7.5.4. Exercise. Consider the linear 2-polygraph

$$\Lambda = \langle\, x, y, z, r, s \mid xy \overset{\alpha}{\Rightarrow} ss,\ yz \overset{\beta}{\Rightarrow} sr \,\rangle.$$

1) Complete the polygraph Λ into a convergent polygraph Λ'.

2) Show that the Anick resolution of Λ' is not minimal.

3) Compute the homology of the algebra $\mathbf{A}$ presented by Λ.

4) Compute a minimal Anick's resolution of the algebra $\mathbf{A}$.

7.5.5. Example. Let consider the algebra

$$\mathbf{A}\langle\, a, b, c, d, e \mid ab = ee,\ bc = ed \,\rangle.$$

The alphabetic order $e \prec d \prec c \prec b \prec a$ induces the following orientation:

$$ab \overset{\alpha}{\Rightarrow} ee, \quad bc \overset{\beta}{\Rightarrow} ed.$$

There is only one critical branching:

$$\begin{array}{ccc} & abc & \\ {\scriptstyle \alpha c} \swarrow & & \searrow {\scriptstyle a\beta} \\ eec & \overset{\gamma}{\Leftarrow = = = = = =} & aed \end{array}$$

completed by adding the rule $aed \overset{\gamma}{\Rightarrow} eec$. The rewriting system $\{\alpha, \beta, \gamma\}$ is convergent. Anick's chains are

$$\Omega_{-1} = \{1\}, \quad \Omega_0 = \{a, b, c, d, e\}, \quad \Omega_1 = \{a|b, a|ed, b|c\}, \quad \Omega_2 = \{ab|c\}, \quad \Omega_n = \emptyset, \text{ for } n \geqslant 3.$$

The Anick resolution with this oriented presentation is

$$0 \to \mathbb{K}\{ab|c\} \otimes \mathbf{A} \overset{d_2}{\to} \mathbb{K}\{a|b, a|ed, b|c\} \otimes \mathbf{A} \overset{d_1}{\to} \mathbb{K}\{a, b, c, d, e\} \otimes \mathbf{A} \overset{d_0}{\to} \mathbf{A} \overset{\varepsilon}{\to} \mathbb{K} \longrightarrow 0 \qquad (7.5.6)$$

with $d_0(x \otimes 1) = x$, for any $x \in \Omega_0$,

$$d_1(a|b \otimes 1) = a \otimes b - e \otimes e, \quad d_1(a|ed \otimes 1) = a \otimes ed - e \otimes ec, \quad d_1(b|c \otimes 1) = b \otimes c - e \otimes d,$$

and

$$d_2(ab|c \otimes 1) = ab \otimes c - aed \otimes 1.$$

It follows that

$$H_n(\mathbf{A}, \mathbb{K}) = \begin{cases} \mathbb{K} & \text{if } n = 0, \\ \mathbb{K}^5 & \text{if } n = 1, \\ \mathbb{K}^2 & \text{if } n = 2, \\ 0 & \text{if } n \geqslant 3. \end{cases}$$

Hence the Anick resolution with these presentation is not minimal. A minimal Anick's resolution for the same algebra **A** can be constructed with the following orientation, induced by the alphabetic order with $b \prec e \prec a$:

$$ab \overset{\alpha}{\Rightarrow} ee, \quad ed \overset{\beta'}{\Rightarrow} bc$$

which produces the following chains:

$$\Omega_{-1} = \{1\}, \quad \Omega_0 = \{a, b, c, d, e\}, \quad \Omega_1 = \{a|b, e|d\}, \quad \Omega_2 = \emptyset, \text{ for } n \geqslant 1.$$

The Anick resolution with this oriention is

$$0 \to \mathbb{K}\{a|b, e|d\} \otimes \mathbf{A} \overset{d_1}{\to} \mathbb{K}\{a, b, c, d, e\} \otimes \mathbf{A} \overset{d_0}{\to} \mathbf{A} \overset{\varepsilon}{\to} \mathbb{K} \longrightarrow 0 \qquad (7.5.7)$$

with $d_0(x \otimes 1) = x$, for any $x \in \Omega_0$ and

$$d_1(a|b \otimes 1) = a \otimes b - e \otimes e,$$
$$d_1(e|d \otimes 1) = e \otimes d - b \otimes c.$$

This resolution is minimal.

7.5.8. Exercise. Let consider the algebra presented by

$$\langle x, y, z, r, s \mid xy = ss, \; yz = rr \rangle.$$

Show that there is no orientation of rules of this presentation giving a convergent linear 2-polygraph, and thus there is no minimal Anick's resolution for this algebra.

7.5.9. Proposition. *Let Λ be a monomial linear 2-polygraph. Let* **A** *be the monomial algebra presented by Λ. The following statements hold.*

i) *The Anick resolution $\mathcal{A}(\Lambda)$ defined in (7.3.1) is a minimal resolution.*

ii) *There is an isomorphism $\mathrm{Tor}_n^{\mathbf{A}}(\mathbb{K}, \mathbb{K}) \simeq \mathbb{K}\Omega_{n-1}$, for all $n \geqslant 0$.*

Let us mention another consequence for quadratic algebras. Given a monomial linear 2-polygraph Λ which is quadratic, that is its 2-cells are of the form $x_i x_j \Rightarrow 0$, with x_i, x_j in Λ_1. Then the Anick resolution $\mathcal{A}(\Lambda)$ is concentrated in the diagonal in the following sense. The set of 0-chains is Λ_1 and they are of degree 1. The set of 1-chains is $s_1(\Lambda)$ and they are of degree 2. More generally, an n-chains $x|t_1 \ldots |t_{n-1}|t_n$ is of degree $n+1$. As a consequence, we have

7.5.10. Theorem. *A quadratic monomial algebra is Koszul.*

7.5.11. Proposition. *Let* **A** *be an algebra and let* Λ *be a left-monomial reduced convergent linear* 2*-polygraph compatible with a monomial order that presents* **A**. *If the Anick resolution* $\mathcal{A}(\Lambda)$ *is minimal, then, for any* $n \geqslant 0$*, we have a isomorphism of spaces*

$$H_n(\mathbf{A}, \mathbb{K}) \simeq \mathbb{K}[\Omega_{n-1}(\Lambda)].$$

7.5.12. Exercise. Prove Proposition 7.5.11.

7.5.13. When Anick's resolution is minimal. We have seen in Proposition 7.5.9 that the Anick resolution $\mathcal{A}(\Lambda)$ is minimal when the presentation is monomial. Following exercise gives an other situation for which the Anick resolution is minimal.

7.5.14. Exercise. Let Λ be a left-monomial reduced linear 2-polygraph compatible with a monomial order. Suppose that Λ is convergent and quadratic, that is, any 2-cell in Λ_2 is of the form $x_{i_1}x_{i_2} \Rightarrow y_{i_1}y_{i_2}$ with $x_{i_1}, x_{i_2}, y_{i_1}, y_{i_2}$ in Λ_1. Show that the Anick resolution $\mathcal{A}(\Lambda)$ is minimal.

7.5.15. Exercise. A linear 2-polygraph is *cubical* if its 2-cells are of the form $x_{i_1}x_{i_2}x_{i_3} \Rightarrow y_{i_1}y_{i_2}y_{i_3}$. Is the result of Exercise 7.5.14 can be extended to cubical convergent linear 2-polygraphs ?

7.5.16. Exercises. Compute homology spaces of the algebras presented by the following linear 2-polygraphs

1) $\langle\, x, y \mid xy \Rightarrow yx \,\rangle$.	**2)** $\langle\, x, y \mid x^2 \Rightarrow 0 \,\rangle$.	**3)** $\langle\, x, y \mid x^2 \Rightarrow y^2 \,\rangle$.
4) $\langle\, x, y \mid x^2 \Rightarrow xy \,\rangle$.	**5)** $\langle\, x, y \mid x^2 \Rightarrow xy - y^2 \,\rangle$.	**6)** $\langle\, x, y \mid xyx \Rightarrow yxy \,\rangle$.

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