# Algebraic Confluences

## Part I. Homology and Rewriting

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# String Rewriting and the Word Problem

# String Rewriting is 2-dimensional

• String rewriting system (Thue 1914) : **2-polygraph** (Street 1976, Burroni 1991) with only one 0-cell:

$$\begin{tikzcd}
\Sigma_0^* & \Sigma_1^* \arrow[l, shift right, "t_0"'] \arrow[l, shift left, "s_0"'] & \\
\Sigma_0 \arrow[u, equal] & \Sigma_1 \arrow[u, hook] \arrow[ul, shift right, "t_0"'] \arrow[ul, shift left, "s_0"] & \Sigma_2 \arrow[ul, shift right, "t_1"'] \arrow[ul, shift left, "s_1"]
\end{tikzcd}$$

- $\Sigma_0 = \{\cdot\}$
- $\Sigma_1$ set of **generators** : $s_0(x) \xrightarrow{x} t_0(x)$
- $\Sigma_1^*$ free monoid of **strings** : $\cdot \xrightarrow{x_1} \cdot \xrightarrow{x_2} \cdot \;\ldots\; \cdot \xrightarrow{x_k} \cdot$
- $\Sigma_2$ set of **rules** : $\varphi : u \Rightarrow v$

$$\begin{tikzcd}
 & \cdot \arrow[r, "x_2"] & \cdot \arrow[r, "x_3"] & \cdot & \cdots & \cdot \arrow[r, "x_{k-1}"] & \cdot \arrow[dr, "x_k"] & \\
\cdot \arrow[ur, "x_1"] \arrow[dr, "y_1"'] & & & \Downarrow \varphi & & & & \cdot \\
 & \cdot \arrow[r, "y_2"'] & \cdot \arrow[r, "y_3"'] & \cdot & \cdots & \cdot \arrow[r, "y_{l-1}"'] & \cdot \arrow[ur, "y_l"'] &
\end{tikzcd}$$

$$s_1(\varphi) = x_1 x_2 \ldots x_k, \quad t_1(\varphi) = y_1 y_2 \ldots y_l.$$

# 2-polygraphs

- Given a 2-polygraph $\Sigma$:

$$\begin{tikzcd}
\Sigma_0^* & & \Sigma_1^* \arrow[ll, shift right, "t_0"'] \arrow[ll, shift left, "s_0"] & & \\
\Sigma_0 \arrow[u, equal] & & \Sigma_1 \arrow[u, tail] \arrow[ull, shift right, "t_0"'] \arrow[ull, shift left, "s_0"] & & \Sigma_2 \arrow[ull, shift right, "t_1"'] \arrow[ull, shift left, "s_1"]
\end{tikzcd}$$

  - A **rewriting step** of $\Sigma$ is a 2-cell with shape

$$\begin{tikzcd}
\cdot \arrow[r, "w"] & \cdot \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\varphi"] & \cdot \arrow[r, "w'"] & \cdot
\end{tikzcd}$$

where $\varphi : u \Rightarrow v$ is a 2-cell of $\Sigma_2$ and $w$ and $w'$ are 1-cells of $\Sigma_1^*$.
  - A **rewriting sequence** of $\Sigma$ is a finite or infinite sequence

$$u_1 \overset{f_1}{\Longrightarrow} u_2 \overset{f_2}{\Longrightarrow} \cdots \overset{f_{n-1}}{\Longrightarrow} u_n \overset{f_n}{\Longrightarrow} \cdots$$

of rewriting steps.
- Rewriting sequences form a 2-category $\Sigma_2^*$.

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\end{tikzcd}$$

- $\Sigma$ is **convergent** if it is
  - **terminating**, *i.e.*, it does not generate any infinite reduction sequence

$$u_1 \Longrightarrow u_2 \Longrightarrow \cdots \Longrightarrow u_n \Longrightarrow \cdots$$

  - **confluent**, *i.e.*, all of its branchings are confluent:

$$\begin{tikzcd}
& w \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
w_1 \arrow[dr, Rightarrow] & & w_2 \arrow[dl, Rightarrow] \\
& w' &
\end{tikzcd}$$

# 2-polygraphs and the Word Problem

- **Finite convergent presentations** give a method for solving the word problem algorithmically.

  - Given a 2-polygraph $\Sigma$.

  - Consider the monoid $\mathbf{M}$ **presented** by $\Sigma$, i.e., the quotient of the free monoid $\Sigma_1^*$ by the congruence generated by $\Sigma_2$:

$$\mathbf{M} = \Sigma_1^* / \Sigma_2.$$

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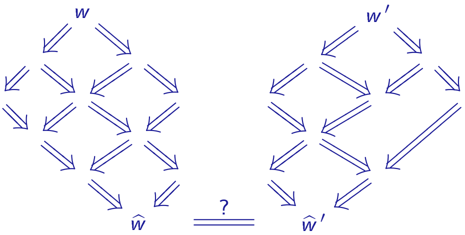
  - **Normal form algorithm** for finite and convergent 2-polygraphs:




**Fact.** Monoids having a finite convergent presentation are decidable.

# Finite Convergent Presentations

- Finitely presented monoids with a finite convergent presentation (Nivat, 1972, Book, Otto, Diekert, Jantzen, ... in eighties).

- Knuth-Bendix **completion algorithm**, 1970.
  - Input : a 2-polygraph $\Sigma$ a well-founded ordering $>$ on $\Sigma_1^*$
  - Algorithm will try to compute a set of rules $\Sigma_2^\infty$ such that

**i)** $u > v$ holds for each $u \Rightarrow v$ in $\Sigma_2^\infty$,

**ii)** $\Sigma_2^\infty$ is confluent,

**iii)** $\Sigma_2^\infty$ and $\Sigma_2$ are Tietze equivalent.

- The algorithm will terminate if and only if there exists a finite set $\Sigma_2'$ such **i)**, **ii)**, **iii)** hold.
  - Else it will run forever ... generating an infinite set of rules satisfying **i)**, **ii)**, **iii)**,
  - or it will fail when it encounters an unorientable 'equation'.

## String Rewriting and the Word Problem

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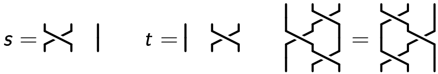
**Example.** (Kapur-Narendran, 1985)
Artin's presentation of monoid of **positive braids on 3 strands**:

$$\mathbf{B}_3^+ = \langle\, s, t \mid sts = tst \,\rangle$$




1. $\mathbf{B}_3^+$ has a decidable word problem.
2. There does not exist finite convergent presentation of $\mathbf{B}_3^+$ with two generators.
3. But with three generators by adding a generator $a$ standing for the product $st$.

$$\Sigma = \langle\, s, t \quad \mid tst \Rightarrow sts$$

$$\Sigma = \langle\, s, t, a \mid tst \Rightarrow sts$$

$$\Sigma = \langle\, s, t, a \mid tst \Rightarrow sts,\ st \overset{\beta}{\Rightarrow} a$$

$$\Sigma = \langle\ s, t, a \mid\ ta \overset{\alpha}{\Rightarrow} as\ , st \overset{\beta}{\Rightarrow} a$$

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$$\begin{tikzcd}
 & aa \\
sta \arrow[ur, Rightarrow, bend left, "\beta a"] \arrow[dr, Rightarrow, bend right, "s\alpha"'] & \\
 & sas
\end{tikzcd}$$

$$\Sigma = \langle\, s, t, a \mid ta \overset{\alpha}{\Rightarrow} as\,, st \overset{\beta}{\Rightarrow} a, sas \overset{\gamma}{\Rightarrow} aa$$

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 & aa \\
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$$\Sigma = \langle\, s, t, a \mid\ ta \overset{\alpha}{\Rightarrow} as\ , st \overset{\beta}{\Rightarrow} a, sas \overset{\gamma}{\Rightarrow} aa, saa \overset{\delta}{\Rightarrow} aat$$

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 & aa \\
sta \arrow[ur, Rightarrow, bend left, "\beta a"] \arrow[dr, Rightarrow, bend right, "s\alpha"'] & \\
 & sas \arrow[uu, Rightarrow, bend right, "\gamma"']
\end{tikzcd}$$

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\end{tikzcd}
\qquad
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 & aaas & \\
sasas \arrow[ur, Rightarrow, bend left, "\gamma as"] \arrow[dr, Rightarrow, bend right, "sa\gamma"'] & & aata \arrow[ul, Rightarrow, bend right, "aa\alpha"'] \\
 & saaa \arrow[ur, Rightarrow, bend right, "\delta a"'] &
\end{tikzcd}$$

$$\Sigma = \left\langle\, s, t, a \;\middle|\; ta \overset{\alpha}{\Rightarrow} as,\ st \overset{\beta}{\Rightarrow} a,\ sas \overset{\gamma}{\Rightarrow} aa,\ saa \overset{\delta}{\Rightarrow} aat \,\right\rangle$$

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sta \arrow[ur, Rightarrow, bend left, "\beta a"] \arrow[dr, Rightarrow, bend right, "s\alpha"'] & \\
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 & saaa \arrow[ur, Rightarrow, bend right, "\delta a"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & aaaa & aaast \arrow[l, Rightarrow, "aaa\beta"'] \\
sasaa \arrow[ur, Rightarrow, bend left, "\gamma aa"] \arrow[dr, Rightarrow, bend right, "sa\delta"'] & & \\
 & saaat \arrow[r, Rightarrow, "\delta at"'] & aatat \arrow[uu, Rightarrow, bend right, "aa\alpha t"']
\end{tikzcd}$$

# String Rewriting and the Word Problem

**Questions.** (Book, 1985, Kapur-Narendran, 1985, Jantzen, 1985, ...)

1. Does a finitely presented decidable monoid have a finite convergent presentation ?
2. Does rewriting is universal to decide the word problem in a finite presented monoid ?
3. What conditions a monoid must satisfy if it can be presented by a finite convergent rewriting system ?

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3. What conditions a monoid must satisfy if it can be presented by a finite convergent rewriting system ?

**Answears.** (Squier, 1987)

1. No in general.
2. No.
3. Homological finiteness condition left-$\mathrm{FP}_3$.

# Homological Squier Theorem

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**Theorem.** (Squier, 1987)

- **i)** A monoid having a finite convergent presentation is of **homological type left-$\mathrm{FP}_3$**.
- **ii)** There are finitely presented decidable monoids which are not of homological type left-$\mathrm{FP}_3$.

# Homological Squier Theorem

**Definition.**

- $\mathbb{Z}\mathbf{M}$ : ring of monoid $\mathbf{M}$:
  - free $\mathbb{Z}$-module on $\mathbf{M}$
  - bilinear product extending product in $\mathbf{M}$:

$$\Big(\sum_{u\in\mathbf{M}} \lambda_u u\Big)\Big(\sum_{v\in\mathbf{M}} \lambda_v v\Big) = \sum_{w\in\mathbf{M}} \sum_{uv=w} \lambda_u \lambda_v w$$

- A monoid $\mathbf{M}$ is of **homological type left-$\mathrm{FP}_3$** if there exists a projective resolution

$$\ldots \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

of $\mathbb{Z}\mathbf{M}$-modules such that each $P_i$ is a finitely generated $\mathbb{Z}\mathbf{M}$-module, for $0 \leqslant i \leqslant 3$.

**Objective.** Construct a resolution of free $\mathbb{Z}\mathbf{M}$-modules induced by a convergent polygraph $\Sigma$.

# Homological Squier Theorem

**Theorem.** (Squier, 1987)

**i)** A monoid having a finite convergent presentation is of homological type left-$\mathrm{FP}_3$.

**Proof.** (idea)

- $\Sigma$ be a 2-polygraph, let $\mathbf{M} = \Sigma_1^* / \Sigma_2$.
- We construct an **exact sequence** of $\mathbb{Z}\mathbf{M}$-modules

$$
\begin{array}{ccccccccc}
\mathbb{Z}\mathbf{M}[\Sigma_2] & \xrightarrow{\delta_2} & \mathbb{Z}\mathbf{M}[\Sigma_1] & \xrightarrow{\delta_1} & \mathbb{Z}\mathbf{M} & \xrightarrow{\varepsilon} & \mathbb{Z} & \longrightarrow & 0 \\
 & & & & u & \longmapsto & 1 & & \\
 & & u[x] & \longmapsto & ux - u & & & & \\
u[l \Rightarrow r] & \longmapsto & u[l] - u[r] & & & & & &
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with $[xy] = [x] + x[y]$, pour tous $x, y \in \Sigma_1^*$

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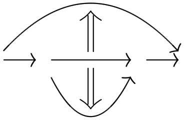
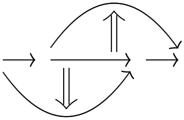
\end{array}
$$

with $[xy] = [x] + x[y]$, pour tous $x, y \in \Sigma_1^*$

1. $\varepsilon$ surjective,
2. $\mathrm{Im}\delta_1 = \ker \varepsilon$, $\mathrm{Im}\delta_2 = \ker \delta_1$
3. The $\mathbb{Z}\mathbf{M}$-module $\ker \delta_2$ of **homological 2-syzygies** is generated by **critical branchings**:





4. If $\Sigma_2$ is finite, then $\ker \delta_2$ is finitely generated.

# Homological Squier Theorem

• $\ker \delta_2$ is generated by critical branchings.

**Proof.** (idea)

- Consider the free $\mathbb{Z}\mathbf{M}$-module $\mathbb{Z}\mathbf{M}[\Sigma_3]$ generated by the set of critical branchings $\Sigma_3$

$$\begin{array}{ccccccccccc}
\mathbb{Z}\mathbf{M}[\Sigma_3] & \xrightarrow{\delta_3} & \mathbb{Z}\mathbf{M}[\Sigma_2] & \xrightarrow{\delta_2} & \mathbb{Z}\mathbf{M}[\Sigma_1] & \xrightarrow{\delta_1} & \mathbb{Z}\mathbf{M} & \xrightarrow{\varepsilon} & \mathbb{Z} & \longrightarrow & 0 \\
 & & & & & & u & \longmapsto & 1 & & \\
 & & & & u[x] & \longmapsto & ux - u & & & & \\
 & & u[l \Rightarrow r] & \longmapsto & u[l] - u[r] & & & & & &
\end{array}$$

- Define the boundary map

$$\delta_3\left(\begin{tikzcd} & uvw \arrow[dl, Rightarrow, "\varphi w"'] \arrow[dr, Rightarrow, "u\psi"] & \\ u'w \arrow[dr, Rightarrow, "f"'] \arrow[rr, phantom, "\Rrightarrow"] & & uw' \arrow[dl, Rightarrow, "g"] \\ & t & \end{tikzcd}\right) = [\varphi w \star_1 f] - [u\psi \star_1 g] = [\varphi]w + [f] - u[\psi] - [g].$$

where $uv \overset{\varphi}{\Rightarrow} u'$ and $vw \overset{\psi}{\Rightarrow} w'$ are rules in $\Sigma_2$.

- Show that $\mathrm{Im}\ \delta_3 = \ker \delta_2$.

## Homological Squier Theorem: examples

**Theorem.** (Squier, 1987)

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$$\Sigma_0 = \{\cdot\} \qquad \Sigma_1 = \{a, b, t, x_1, \dots, x_k, y_1, \dots, y_k\}, \qquad k \geqslant 2,$$

$$\Sigma_2 = \{\ at^n b \Rightarrow 1,\ n \in \mathbb{N},\ \ x_i a \Rightarrow atx_i,\ x_i t \Rightarrow tx_i,\ x_i b \Rightarrow bx_i,\ x_i y_i \Rightarrow 1\ \}$$

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**Example.** (Stallings, 1963) A finitely presented decidable group that is not left-$\mathrm{FP}_3$, (Bieri, 1976).

**Example.** (Abels, 1979) A finitely presented decidable group that is not left-$\mathrm{FP}_3$, (Bieri, 1980) :

$$\begin{bmatrix} 1 & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & 0 & \star & \star \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathcal{M}_4\big(\mathbb{Z}[1/p]\big), \quad p \text{ a prime.}$$

# Homological Squier Theorem in practice

- Let $\mathbf{M}$ be a monoid.

**First step:** - Compute a convergent presentation $\Sigma = (\Sigma_1, \Sigma_2)$ of $\mathbf{M}$.

- Consider the set of **critical pairs** $\Sigma_3$, of **critical triples** $\Sigma_4$.

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**Second step:** from Squier's resolution

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\mathbb{Z}\mathbf{M}[\Sigma_4] & \xrightarrow{\delta_4} & \mathbb{Z}\mathbf{M}[\Sigma_3] & \xrightarrow{\delta_3} & \mathbb{Z}\mathbf{M}[\Sigma_2] & \xrightarrow{\delta_2} & \mathbb{Z}\mathbf{M}[\Sigma_1] & \xrightarrow{\delta_1} & \mathbb{Z}\mathbf{M} & \xrightarrow{\varepsilon} & \mathbb{Z} \\
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\end{array}$$

construct a complex $\mathbb{Z}\mathbf{M}[\Sigma_*] \otimes_{\mathbb{Z}\mathbf{M}} \mathbb{Z}$ of free abelian groups:

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 & & & & & & [x] & \longmapsto & x \\
 & & & & [l \Rightarrow r] & \longmapsto & [l] - [r] & &
\end{array}$$

with

$$\delta_3 \left( \begin{array}{ccc} \varphi w \swarrow & & \searrow u\psi \\ f \searrow & & \swarrow g \end{array} \right) = [\varphi] + [f] - [g] - [\psi].$$

# Homological Squier Theorem in practice

**Third step:** Compute homology groups.

$$\mathbb{Z}[\Sigma_4] \xrightarrow{\overline{\delta}_4} \mathbb{Z}[\Sigma_3] \xrightarrow{\overline{\delta}_3} \mathbb{Z}[\Sigma_2] \xrightarrow{\overline{\delta}_2} \mathbb{Z}[\Sigma_1] \xrightarrow{\overline{\delta}_1} \mathbb{Z}$$

$$H_0(\mathbf{M}, \mathbb{Z}) = \mathbb{Z}$$
$$H_1(\mathbf{M}, \mathbb{Z}) = \ker \overline{\delta}_1 / \mathrm{Im} \overline{\delta}_2$$
$$H_2(\mathbf{M}, \mathbb{Z}) = \ker \overline{\delta}_2 / \mathrm{Im} \overline{\delta}_3$$
$$H_3(\mathbf{M}, \mathbb{Z}) = \ker \overline{\delta}_3 / \mathrm{Im} \overline{\delta}_4$$

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**Third step:** Compute homology groups.

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$$\begin{aligned}
H_0(\mathbf{M}, \mathbb{Z}) &= \mathbb{Z} \\
H_1(\mathbf{M}, \mathbb{Z}) &= \ker \overline{\delta}_1 / \mathrm{Im} \overline{\delta}_2 \\
H_2(\mathbf{M}, \mathbb{Z}) &= \ker \overline{\delta}_2 / \mathrm{Im} \overline{\delta}_3 \\
H_3(\mathbf{M}, \mathbb{Z}) &= \ker \overline{\delta}_3 / \mathrm{Im} \overline{\delta}_4
\end{aligned}$$

**Conclusion:** If $H_3(\mathbf{M}, \mathbb{Z})$ is not finitely generated, then $\mathbf{M}$ does not have a finite convergent presentation.

# Homological Squier Theorem in practice

**Example** (Lafont-Prouté, 1991) Consider monoid

$$\mathbf{M} = \langle\, a, b, c, d, d' \mid ab = a,\ da = ac,\ d'a = ac \,\rangle$$

# Homological Squier Theorem in practice

**Example** (Lafont-Prouté, 1991) Consider monoid

$$\mathbf{M} = \langle\, a, b, c, d, d' \mid ab = a,\ da = ac,\ d'a = ac \,\rangle$$

**First step:** Compute a convergent presentation (Knuth-Bendix)

$\Sigma_1 = \{a, b, c, d, d'\}$

$\Sigma_2 = \{\ ac^n b \overset{\alpha_n}{\Rightarrow} ac^n,\ n \in \mathbb{N},\ \ da \overset{\beta}{\Rightarrow} ac,\ d'a \overset{\beta'}{\Rightarrow} ac\ \}$

$\Sigma_3 = \{A_n,\ A'_n,\ n \in \mathbb{N}\}$ with

$$\begin{tikzcd}
 & dac^n b \arrow[dl, Rightarrow, "\beta c^n b"'] \arrow[dr, Rightarrow, "d\alpha_n"] & \\
ac^{n+1} b \arrow[dr, Rightarrow, "\alpha_{n+1}"'] & A_n & dac^n \arrow[dl, Rightarrow, "\beta c^n"] \\
 & ac^{n+1} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & d'ac^n b \arrow[dl, Rightarrow, "\beta' c^n b"'] \arrow[dr, Rightarrow, "d\alpha_n"] & \\
ac^{n+1} b \arrow[dr, Rightarrow, "\alpha_{n+1}"'] & A'_n & d'ac^n \arrow[dl, Rightarrow, "\beta' c^n"] \\
 & ac^{n+1} &
\end{tikzcd}$$

$\Sigma_4 = \emptyset$ (no critical triple).

# Homological Squier Theorem in practice

**Example** (Lafont-Prouté, 1991) Consider monoid

$$\mathbf{M} = \langle\, a, b, c, d, d' \mid ab = a,\ da = ac,\ d'a = ac \,\rangle$$

**Second step:** Compute complex

$$0 \xrightarrow{\overline{\delta}_4} \mathbb{Z}[\Sigma_3] \xrightarrow{\overline{\delta}_3} \mathbb{Z}[\Sigma_2] \xrightarrow{\overline{\delta}_2} \mathbb{Z}[\Sigma_1] \xrightarrow{\overline{\delta}_1} \mathbb{Z}$$

$$\overline{\delta}_1(a) = \overline{\delta}_1(b) = \overline{\delta}_1(c) = \overline{\delta}_1(d) = \overline{\delta}_1(d') = 0.$$

$$\overline{\delta}_2(ac^n b \overset{\alpha_n}{\Rightarrow} ac^n) = [ac^n b] - [ac^n] = [a] + n[c] + [b] - [a] - n[c] = [b],$$

$$\overline{\delta}_2(da \overset{\beta}{\Rightarrow} ac) = [da] - [ac] = [d] + [a] - [a] - [c] = [d] - [c],$$

$$\overline{\delta}_2(d'a \overset{\beta'}{\Rightarrow} ac) = [d'a] - [ac] = [d'] + [a] - [a] - [c] = [d'] - [c].$$

$$\overline{\delta}_3\left(\begin{tikzcd} & dac^n b \arrow[dl, Rightarrow, "\beta c^n b"'] \arrow[dr, Rightarrow, "d\alpha_n"] & \\ ac^{n+1} b \arrow[dr, Rightarrow, "\alpha_{n+1}"'] \arrow[rr, phantom, "A_n"] & & dac^n \arrow[dl, Rightarrow, "\beta c^n"] \\ & ac^{n+1} & \end{tikzcd}\right) = [\beta] + [\alpha_{n+1}] - [\beta] - [\alpha_n] = [\alpha_{n+1}] - [\alpha_n],$$

# Homological Squier Theorem in practice

**Example** (Lafont-Prouté, 1991) Consider monoid

$$\mathbf{M} = \langle\, a, b, c, d, d' \mid ab = a,\ da = ac,\ d'a = ac \,\rangle$$

**Third step:** Compute homology groups:

$H_0(\mathbf{M}, \mathbb{Z}) = \mathbb{Z}$.
$H_1(\mathbf{M}, \mathbb{Z}) = \ker \overline{\delta}_1 / \mathrm{Im}\overline{\delta}_2 = \mathbb{Z}[a, b, c, d, d'] / \big([b], [d] - [c], [d'] - [c]\big) = \mathbb{Z}[a, c] \simeq \mathbb{Z}^2$.
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**Conclusion:**

- $H_3(\mathbf{M}, \mathbb{Z})$ is not finitely generated,
- hence $\mathbf{M}$ is not of type left-$\mathrm{FP}_3$,
- by Squier's theorem, $\mathbf{M}$ does not have a finite convergent presentation.

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**Conclusion:** we cannot conclude ...

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$H_3(\mathbf{M}, \mathbb{Z}) = \ker \overline{\delta}_3 / \mathrm{Im}\overline{\delta}_4 = \ker \overline{\delta}_3 / 0 = 0$.

**Conclusion:** we cannot conclude ...

Monoid $\mathbf{M}$ has the following convergent presentation:

$$\Sigma_1 = \{\, a, b, c, d \,\}, \quad \Sigma_2 = \{\, ab \Rightarrow a,\ ac \Rightarrow da \,\}.$$

# Homological Squier Theorem, other proofs and generalisations

- Anick, 1986 : infinite resolution for associative $k$-algebras presented by a finite Gröbner basis.
- Kobayashi, 1990, Groves, 1991, Farkas 1992, Brown 1992, Cohen 1993.
- Guiraud-Malbos, 2012.

# Toward another Finiteness Conditions

**Fact.** The homological finiteness condition left-$\mathrm{FP}_3$ is not sufficient for a finitely presented decidable monoid to admit a finite convergent presentation.

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**Example.** (Squier, 1994) The monoid

$$\mathbf{S_1} = \langle\, a, b, t, x, y \mid at^n b \Rightarrow 1,\ xa \Rightarrow atx,\ xt \Rightarrow tx,\ xb \Rightarrow bx,\ xy \Rightarrow 1 \,\rangle.$$

- has a decidable word problem,
- is of homological type left-$\mathrm{FP}_3$,
- does not have a finite convergent presentation,
- does not have **finite derivation type**.

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# References


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