Coherent presentations of Artin groups

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Joint work with Stéphane Gaussent and Yves Guiraud

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Motivation

▶ A **Coxeter system** $(\mathbf{W}, S)$ is a data made of a group $\mathbf{W}$ with a presentation by a (finite) set S of involutions, $s^2 = 1$, satisfying **braid relations**

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- Forgetting the involutive character of generators, one gets the **Artin's presentation**

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- ▷ Push further Artin's presentation and study the **relations among the braid relations**. (Brieskorn-Saito, 1972, Deligne, 1972, Deligne, 1997, Tits, 1981, Michel, 1999).
- ▷ We introduce a rewriting method to compute generators of relations among relations.

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- Set $\mathbf{W} = \mathbf{S}_4$ the group of permutations of $\{1, 2, 3, 4\}$, with $S = \{r, s, t\}$ where

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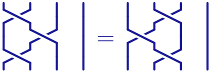
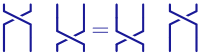
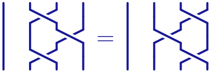
- The associated Artin group $\mathbf{B}(\mathbf{S}_4)$ is the group of braids on 4 strands:

$$\mathrm{Art}_2(\mathbf{S}_4) = \langle\, r, s, t \mid rsr = srs,\;\; rt = tr,\;\; tst = sts \,\rangle$$







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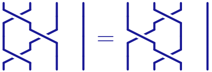

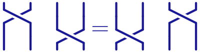

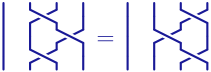

- The relations among the braid relations on 4 strands are generated by the **Zamolodchikov relation** (Deligne, 1997).

$$\begin{tikzcd}
 & strsrt \arrow[r, equal] & srtstr \arrow[r, equal] & srstsr \arrow[dr, equal] & \\
stsrst \arrow[ur, equal] \arrow[d, equal] & & & & rsrtsr \arrow[d, equal] \\
tstrst \arrow[d, equal] & & Z_{r,s,t} & & rstrsr \arrow[d, equal] \\
tsrtst \arrow[dr, equal] & & & & rstsrs \arrow[dl, equal] \\
 & tsrsts \arrow[r, equal] & trsrts \arrow[r, equal] & rtstrs &
\end{tikzcd}$$

Plan

I. Coherent presentations of categories

- Polygraphs as higher-dimensional rewriting systems
- Coherent presentations as cofibrant approximations

II. Homotopical completion-reduction procedure

- Tietze transformations
- Rewriting properties of 2-polygraphs
- The homotopical completion-procedure

III. Applications to Artin monoids

- Garside's coherent presentation
- Artin's coherent presentation

References

- S. Gaussent, Y. Guiraud, P.M., Coherent presentations of Artin monoids, 2013.
- Y. Guiraud, P.M., Higher-dimensional normalisation strategies for acyclicity, 2012.
- Y. Guiraud, P.M., A polygraphic survey on finiteness conditions for rewriting systems, 2013.

Part I. Coherent presentations of categories

Polygraphs

Polygraphs

- A **1-polygraph** is an oriented graph (Σ_0, Σ_1)

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\begin{tikzcd}
\Sigma_0 & \Sigma_1 \arrow[l, shift right, "s_0"'] \arrow[l, shift left, "t_0"]
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  - (Σ_0, Σ_1) is a 1-polygraph,
  - Σ_2 is a globular extension of the free category Σ_1^*.

$$\begin{tikzcd} \Sigma_0 & \Sigma_1^* \arrow[l, shift right, "s_0"'] \arrow[l, shift left, "t_0"] & \Sigma_2 \arrow[l, shift right, "s_1"'] \arrow[l, shift left, "t_1"] \end{tikzcd}$$

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where $u \overset{\alpha}{\Longrightarrow} v$ is a 2-cell of Σ_2 and w, w' are 1-cells of Σ_1^*.

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- A $(3,1)$-**polygraph** is a pair $\Sigma = (\Sigma_2, \Sigma_3)$ made of
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Let $\mathbf{C}$ be a category.

- A **presentation** of $\mathbf{C}$ is a 2-polygraph Σ such that

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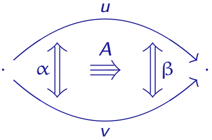

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- An **extended presentation** of $\mathbf{C}$ is a $(3, 1)$-polygraph Σ such that

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Coherent presentations of categories

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Theorem. [Gaussent-Guiraud-M., 2013]

Let Σ be an extended presentation of a category $\mathbf{C}$. Consider the Lack's model structure for 2-categories.

The following assertions are equivalent:

i) *The $(3,1)$-polygraph Σ is a coherent presentation of $\mathbf{C}$.*

ii) *The $(2,1)$-category $\Sigma_2^\top/\Sigma_3$ is a cofibrant 2-category weakly equivalent to $\mathbf{C}$, that is a **cofibrant approximation** of $\mathbf{C}$.*

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  - A homotopy basis can be made with only one 3-cell

$$\left\langle\; r, s, t \;\middle|\; sr \overset{\gamma_{rs}}{\Longrightarrow} rs,\; ts \overset{\gamma_{st}}{\Longrightarrow} st,\; tr \overset{\gamma_{rt}}{\Longrightarrow} rt \;\middle|\; Z_{r,s,t} \;\right\rangle$$

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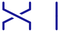
where the 3-cell $Z_{r,s,t}$ is the **permutohedron**

$$\begin{tikzcd}
& str \arrow[r, Rightarrow, "s\gamma_{rt}"] \arrow[ddr, phantom, "\Downarrow Z_{r,s,t}"] & srt \arrow[dr, Rightarrow, bend left, "\gamma_{rs}t"] & \\
tsr \arrow[ur, Rightarrow, bend left, "\gamma_{st}r"] \arrow[dr, Rightarrow, bend right, "t\gamma_{rs}"'] & & & rst \\
& trs \arrow[r, Rightarrow, "\gamma_{rt}s"'] & rts \arrow[ur, Rightarrow, bend right, "r\gamma_{st}"'] &
\end{tikzcd}$$

Examples

▶ **Artin's coherent presentation** of the monoid $\mathbf{B}^+(\mathbf{S}_3)$

$s =$  $t =$ 

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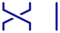
$s =$  $t =$ 

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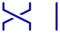
$s =$  $\quad t =$ 

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- **Artin's coherent presentation** of the monoid $\mathbf{B}^+(\mathbf{S}_4)$

$$\mathrm{Art}_3(\mathbf{S}_4) = \langle\, r, s, t \mid rsr \overset{\gamma_{sr}}{\Longrightarrow} srs,\ \ rt \overset{\gamma_{tr}}{\Longrightarrow} tr,\ \ tst \overset{\gamma_{st}}{\Longrightarrow} sts \mid Z_{r,s,t} \,\rangle$$

$$\begin{tikzcd}
 & strsrt \arrow[r, Rightarrow, "s\gamma_{rt}s\gamma^{-}_{rt}"] & srtstr \arrow[r, Rightarrow, "sr\gamma_{st}r"] & srstsr \arrow[dr, Rightarrow, "\gamma_{rs}tsr"] & \\
stsrst \arrow[ur, Rightarrow, "st\gamma_{rs}t"] & & & & rsrtsr \\
tstrst \arrow[u, Rightarrow, "\gamma_{st}rst"] \arrow[d, Rightarrow, "ts\gamma_{rt}st"'] & & \Downarrow Z_{r,s,t} & & rstrsr \arrow[u, Rightarrow, "rs\gamma_{rt}sr"'] \\
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\end{tikzcd}$$

Coherent presentations

Problems.

1. How to compute a coherent presentation ?
2. How to transform a coherent presentation ?

Part II. Homotopical completion-reduction procedure

Tietze transformations

Tietze transformations

▶ Two $(3,1)$-polygraphs Σ and Υ are **Tietze-equivalent** if there is an equivalence of 2-categories

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- Two $(3,1)$-polygraphs Σ and Υ are **Tietze-equivalent** if there is an equivalence of 2-categories

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inducing an isomorphism on presented categories: $\Sigma_1^* \simeq \Upsilon_1^*$.

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- **add a generator**: for $u \in \Sigma_1^*$, add a generating 1-cell x and add a generating 2-cell

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- **remove a generator**: for a generating 2-cell α in Σ_2 with $x \in \Sigma_1$,

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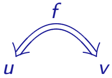

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$$\begin{tikzcd} u \arrow[rr, Leftrightarrow, bend left=40, "f"] \arrow[rr, Leftrightarrow, bend right=40, "g"'] \arrow[rr, phantom, "B \Updownarrow"] & & v \end{tikzcd}$$

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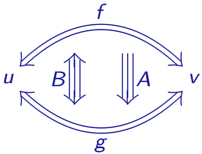

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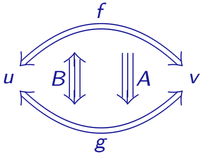

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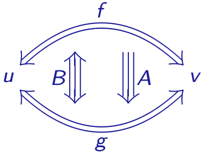

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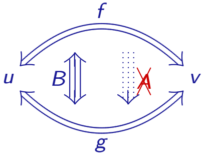

Tietze transformations

Theorem. [Gaussent-Guiraud-M., 2013]

Two (finite) $(3,1)$-polygraphs Σ and Υ are Tietze equivalent if, and only if, there exists a (finite) Tietze transformation

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Consequence.

If Σ is a coherent presentation of a category $\mathbf{C}$ and if there exists a Tietze transformation

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then Υ is a coherent presentation of $\mathbf{C}$.

Rewriting properties of 2-polygraphs

Let $\Sigma = (\Sigma_0, \Sigma_1, \Sigma_2)$ be a 2-polygraph (**string rewriting system**).

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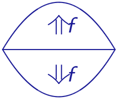
  - ▷ **aspherical** branchings have shape 

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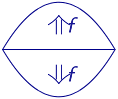

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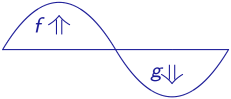
$$
\begin{tikzcd}
 & vu' \arrow[dr, Rightarrow, bend left, "u'g"] & \\
uu' \arrow[ur, Rightarrow, bend left, "fv"] \arrow[dr, Rightarrow, bend right, "ug"'] & \equiv & vv' \\
 & uv' \arrow[ur, Rightarrow, bend right, "fv'"'] &
\end{tikzcd}
\qquad fv \star_1 u'g = ug \star_1 fv'
$$



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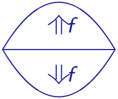

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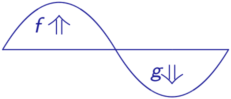
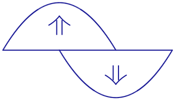


  ▷ **critical branchings** are all the other cases



Rewriting properties of 2-polygraphs

Example.

Consider the 2-polygraph

$$\langle\, s, t \mid tst \overset{\gamma_{st}}{\Longrightarrow} sts \,\rangle$$

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$$\begin{tikzcd}
 & s\mathbf{tr} \arrow[r, Rightarrow, "s\gamma_{\mathbf{rt}}"] & s\mathbf{rt} \\
tsr \arrow[ur, Rightarrow, bend left, "\gamma_{st}r"] \arrow[dr, Rightarrow, bend right, "t\gamma_{rs}"'] & & \\
 & \mathbf{tr}s \arrow[r, Rightarrow, "\gamma_{\mathbf{rt}}s"'] & \mathbf{rt}s
\end{tikzcd}$$

Rewriting properties of 2-polygraphs

Example.

Consider the 2-polygraph

$$\langle\, r, s, t \mid sr \overset{\gamma_{rs}}{\Longrightarrow} rs,\ ts \overset{\gamma_{st}}{\Longrightarrow} st,\ tr \overset{\gamma_{rt}}{\Longrightarrow} rt \,\rangle$$

▷ It has only one critical branching

$$\begin{tikzcd}
 & str \arrow[r, Rightarrow, "s\gamma_{rt}"] & \mathbf{sr}t \arrow[dr, Rightarrow, bend left, "\gamma_{\mathbf{rs}}t"] & \\
tsr \arrow[ur, Rightarrow, bend left, "\gamma_{st}r"] \arrow[dr, Rightarrow, bend right, "t\gamma_{rs}"'] & & & \mathbf{rst} \\
 & trs \arrow[r, Rightarrow, "\gamma_{rt}s"'] & r\mathbf{ts} \arrow[ur, Rightarrow, bend right, "r\gamma_{\mathbf{st}}"'] &
\end{tikzcd}$$

Rewriting properties of 2-polygraphs

Theorem. [Newman's diammond lemma, 1942]

For terminating 2-polygraphs, local confluence and confluence are equivalent properties.

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For terminating 2-polygraphs, local confluence and confluence are equivalent properties.

▶ The **Knuth-Bendix procedure** computes a convergent presentation from a terminating presentation (Knuth-Bendix, 1970).

Theorem. [Squier, 1994]

For a convergent presentation Σ of a category $\mathbf{C}$, the $(3,1)$-polygraph obtained from Σ by adjunction of a generating confluence

$$\begin{tikzcd} & v \arrow[dr, Rightarrow, "f'"] \arrow[d, phantom, "\Downarrow A_{f,g}"] & \\ u \arrow[ur, Rightarrow, "f"] \arrow[dr, Rightarrow, "g"'] & {} & t \\ & w \arrow[ur, Rightarrow, "g'"'] & \end{tikzcd}$$

for every critical branching (f,g) is a coherent presentation of $\mathbf{C}$.

Homotopical completion procedure

Let Σ be a terminating 2-polygraph (with a total termination order).

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▷ for every critical pair

$$\begin{tikzcd} & v \\ u \arrow[ur, Rightarrow, bend left, "f"] \arrow[dr, Rightarrow, bend right, "g"'] & \\ & w \end{tikzcd}$$

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$$\begin{tikzcd}[row sep=small] & v \arrow[r, Rightarrow, "f'"] & \widehat{v} \\ u \arrow[ur, Rightarrow, bend left, "f"] \arrow[dr, Rightarrow, bend right, "g"'] & & \\ & w \arrow[r, Rightarrow, "g'"'] & \widehat{w} \end{tikzcd}$$

compute f' and g' reducing to some normal forms.

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▷ if $\widehat{v} = \widehat{w}$, add a 3-cell $A_{f,g}$

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▷ if $\widehat{v} < \widehat{w}$, add the 2-cell χ and the 3-cell $A_{f,g}$

$$\begin{tikzcd} & v \arrow[r, Rightarrow, "f'"] \arrow[d, phantom, "\Downarrow A_{f,g}"] & \widehat{v} \\ u \arrow[ur, Rightarrow, bend left, "f"] \arrow[dr, Rightarrow, bend right, "g"'] & {} & \\ & w \arrow[r, Rightarrow, "g'"'] & \widehat{w} \arrow[uu, Rightarrow, dotted, "\chi"'] \end{tikzcd}$$

Homotopical completion procedure

- Potential adjunction of additional 2-cells χ can create new critical branchings,
  - whose confluence must also be examined,
  - possibly generating the adjunction of additional 2-cells and 3-cells
  - ...

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$$\langle \Sigma \mid \emptyset \rangle = \Sigma^0 \subseteq \Sigma^1 \subseteq \cdots \subseteq \Sigma^n \subseteq \Sigma^{n+1} \subseteq \cdots$$

- The **homotopical completion** of Σ is the $(3,1)$-polygraph

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Theorem. [Gaussent-Guiraud-M., 2013]

For every terminating presentation Σ of a category $\mathbf{C}$, the homotopical completion $\mathcal{S}(\Sigma)$ of Σ is a coherent convergent presentation of $\mathbf{C}$.

Homotopical completion procedure

Example. The **Kapur-Narendran's presentation** of $\mathbf{B}^+(\mathbf{S}_3)$, obtained from Artin's presentation by coherent adjunction of the Coxeter element st

$$\Sigma_2^{\mathrm{KN}} = \langle\ s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a \rangle$$

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$$\begin{tikzcd}
 & aa \\
sta \arrow[ur, Rightarrow, bend left, "\beta a"] \arrow[dr, Rightarrow, bend right, "s\alpha"'] & \\
 & sas
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$$\begin{tikzcd}
 & aa & \\
sta \arrow[ur, Rightarrow, bend left, "\beta a"] \arrow[dr, Rightarrow, bend right, "s\alpha"'] & \arrow[phantom, "\Downarrow A"] & \\
 & sas \arrow[uu, Rightarrow, bend right, "\gamma"'] &
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 & aa & & & aat \\
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$$\begin{tikzcd}
 & aa & & & aat \\
sta \arrow[ur, Rightarrow, bend left, "\beta a"] \arrow[dr, Rightarrow, bend right, "s\alpha"'] \arrow[r, phantom, "\Downarrow A"] & {} & & sast \arrow[ur, Rightarrow, bend left, "\gamma t"] \arrow[dr, Rightarrow, bend right, "sa\beta"'] \arrow[r, phantom, "\Downarrow B"] & {} \\
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$$\begin{tikzcd}
 & aa & \\
sta \arrow[ur, Rightarrow, bend left, "\beta a"] \arrow[dr, Rightarrow, bend right, "s\alpha"'] & \Downarrow A & \\
 & sas \arrow[uu, Rightarrow, bend right, "\gamma"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & aat & \\
sast \arrow[ur, Rightarrow, bend left, "\gamma t"] \arrow[dr, Rightarrow, bend right, "sa\beta"'] & \Downarrow B & \\
 & saa \arrow[uu, Rightarrow, bend right, "\delta"'] &
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$$\begin{tikzcd}
 & aa & \\
sta \arrow[ur, Rightarrow, bend left, "\beta a"] \arrow[dr, Rightarrow, bend right, "s\alpha"'] & \Downarrow A & \\
 & sas \arrow[uu, Rightarrow, bend right, "\gamma"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & aat & \\
sast \arrow[ur, Rightarrow, bend left, "\gamma t"] \arrow[dr, Rightarrow, bend right, "sa\beta"'] & \Downarrow B & \\
 & saa \arrow[uu, Rightarrow, bend right, "\delta"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & aaas & \\
sasas \arrow[ur, Rightarrow, bend left, "\gamma as"] \arrow[dr, Rightarrow, bend right, "sa\gamma"'] & \Downarrow C & aata \arrow[ul, Rightarrow, bend right, "aa\alpha"'] \\
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$$\begin{tikzcd}
 & aa & \\
sta \arrow[ur, Rightarrow, bend left, "\beta a"] \arrow[dr, Rightarrow, bend right, "s\alpha"'] & \Downarrow A & \\
 & sas \arrow[uu, Rightarrow, bend right, "\gamma"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & aat & \\
sast \arrow[ur, Rightarrow, bend left, "\gamma t"] \arrow[dr, Rightarrow, bend right, "sa\beta"'] & \Downarrow B & \\
 & saa \arrow[uu, Rightarrow, bend right, "\delta"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & aaas & \\
sasas \arrow[ur, Rightarrow, bend left, "\gamma as"] \arrow[dr, Rightarrow, bend right, "sa\gamma"'] & \Downarrow C & aata \arrow[ul, Rightarrow, bend right, "aa\alpha"'] \\
 & saaa \arrow[ur, Rightarrow, bend right, "\delta a"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & aaaa \\
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 & saaat
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$$\begin{tikzcd}
& aa \\
sta \arrow[ur, Rightarrow, bend left, "\beta a"] \arrow[dr, Rightarrow, bend right, "s\alpha"'] \arrow[r, phantom, "\Downarrow A"] & \\
& sas \arrow[u, Rightarrow, bend right, "\gamma"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& aat \\
sast \arrow[ur, Rightarrow, bend left, "\gamma t"] \arrow[dr, Rightarrow, bend right, "sa\beta"'] \arrow[r, phantom, "\Downarrow B"] & \\
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\end{tikzcd}
\qquad
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& saaa \arrow[ur, Rightarrow, bend right, "\delta a"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& aaaa & aaast \arrow[l, Rightarrow, "aaa\beta"'] \\
sasaa \arrow[ur, Rightarrow, bend left, "\gamma aa"] \arrow[dr, Rightarrow, bend right, "sa\delta"'] \arrow[rr, phantom, "\Downarrow D"] & & \\
& saaat \arrow[r, Rightarrow, "\delta at"'] & aatat \arrow[uu, Rightarrow, bend right, "aa\alpha t"']
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$$\begin{tikzcd}
& aa \\
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\end{tikzcd}
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& saaa \arrow[ur, Rightarrow, "\delta a"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& aaaa & aaast \arrow[l, Rightarrow, "aaa\beta"'] \\
sasaa \arrow[ur, Rightarrow, "\gamma aa"] \arrow[dr, Rightarrow, "sa\delta"'] \arrow[r, phantom, "\Downarrow D"] & {} & \\
& saaat \arrow[r, Rightarrow, "\delta at"'] & aatat \arrow[uu, Rightarrow, "aa\alpha t"']
\end{tikzcd}$$

However. The extended presentation $\mathcal{S}(\Sigma_2^{\mathrm{KN}})$ obtained is bigger than necessary.

Homotopical completion-reduction procedure

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- ▷ 3-spheres induced by some of the **generating triple confluences** of $\mathcal{S}(\Sigma)$,

$$\begin{tikzcd} & v \\ u \arrow[ur, Rightarrow, bend left, "f"] \arrow[r, Rightarrow, "g"] \arrow[dr, Rightarrow, bend right, "h"'] & w \\ & x \end{tikzcd}$$

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$$
\begin{tikzcd}
 & v \arrow[rr, Rightarrow, bend left, "f_1"] \arrow[dr, phantom, "\Downarrow A_{f,g}"] & & x' \\
u \arrow[ur, Rightarrow, bend left, "f"] \arrow[rr, Rightarrow, "g"] \arrow[dr, Rightarrow, bend right, "h"'] & & w \arrow[ur, Rightarrow, "g_1"] & \\
 & x & &
\end{tikzcd}
$$

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 & x \arrow[rr, Rightarrow, bend right, "h_2"'] \arrow[u, phantom, "\Downarrow A_{g,h}"] & & v'
\end{tikzcd}
$$

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$$
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 & x \arrow[r, Rightarrow, bend right, "h_2"'] & v'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
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u \arrow[ur, Rightarrow, bend left, "f"] \arrow[rr, Rightarrow, "g"] \arrow[dr, Rightarrow, bend right, "h"'] & {} \arrow[d, phantom, "\Downarrow A_{g,h}"] & w \arrow[ur, Rightarrow, "g_1"] \arrow[dr, Rightarrow, "g_2"'] & \\
& x \arrow[rr, Rightarrow, bend right, "h_2"'] & & v'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& v \arrow[dr, Rightarrow, "f_2"] \arrow[dd, phantom, "A_{f,h}\Downarrow"] & \\
u \arrow[ur, Rightarrow, bend left, "f"] \arrow[dr, Rightarrow, bend right, "h"'] & & w' \\
& x \arrow[ur, Rightarrow, "h_1"'] &
\end{tikzcd}
$$

Homotopical completion-reduction procedure

INPUT: A terminating 2-polygraph Σ.

Step 1. Compute the homotopical completion $\mathcal{S}(\Sigma)$ (convergent and coherent).

Step 2. Apply the homotopical reduction to $\mathcal{S}(\Sigma)$ with a **collapsible part** Γ made of

- ▷ 3-spheres induced by some of the **generating triple confluences** of $\mathcal{S}(\Sigma)$,

$$
\begin{tikzcd}
 & v \arrow[rr, Rightarrow, bend left, "f_1"] \arrow[d, phantom, "\Downarrow A_{f,g}"] & & x' \\
u \arrow[ur, Rightarrow, bend left, "f"] \arrow[rr, Rightarrow, "g"] \arrow[dr, Rightarrow, bend right, "h"'] & & w \arrow[ur, Rightarrow, "g_1"] \arrow[dr, Rightarrow, "g_2"'] & \\
 & x \arrow[rr, Rightarrow, bend right, "h_2"'] \arrow[u, phantom, "\Downarrow A_{g,h}"] & & v'
\end{tikzcd}
\qquad
\begin{tikzcd}
 & v \arrow[rr, Rightarrow, bend left, "f_1"] \arrow[dr, Rightarrow, "f_2"] & & x' \arrow[dr, Rightarrow, bend left, "h"] \arrow[d, phantom, "C\Downarrow"] & \\
u \arrow[ur, Rightarrow, bend left, "f"] \arrow[dr, Rightarrow, bend right, "h"'] \arrow[rr, phantom, "A_{f,h}\Downarrow"] & & w' \arrow[rr, Rightarrow, "g_3"] & & \widehat{u} \\
 & x \arrow[ur, Rightarrow, "h_1"'] & & &
\end{tikzcd}
$$

Homotopical completion-reduction procedure

INPUT: A terminating 2-polygraph Σ.

Step 1. Compute the homotopical completion $\mathcal{S}(\Sigma)$ (convergent and coherent).

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$$
\begin{tikzcd}
 & v \arrow[rr, Rightarrow, bend left, "f_1"] \arrow[d, phantom, "\Downarrow A_{f,g}"] & & x' \\
u \arrow[ur, Rightarrow, bend left, "f"] \arrow[rr, Rightarrow, "g"] \arrow[dr, Rightarrow, bend right, "h"'] & {} & w \arrow[ur, Rightarrow, "g_1"] \arrow[dr, Rightarrow, "g_2"] & \\
 & x \arrow[rr, Rightarrow, bend right, "h_2"'] \arrow[u, phantom, "\Downarrow A_{g,h}"] & & v'
\end{tikzcd}
\qquad
\begin{tikzcd}
 & v \arrow[rr, Rightarrow, bend left, "f_1"] \arrow[dr, Rightarrow, "f_2"] \arrow[dd, phantom, "A_{f,h}\Downarrow"] & & x' \arrow[dr, Rightarrow, bend left, "h"] \arrow[d, phantom, "C\Downarrow"] & \\
u \arrow[ur, Rightarrow, bend left, "f"] \arrow[dr, Rightarrow, bend right, "h"'] & & w' \arrow[rr, Rightarrow, "g_3"] & {} & \widehat{u} \\
 & x \arrow[ur, Rightarrow, "h_1"] \arrow[rr, Rightarrow, bend right, "h_2"'] & & v' \arrow[ur, Rightarrow, bend right, "k"'] \arrow[u, phantom, "D\Downarrow"] &
\end{tikzcd}
$$

Homotopical completion-reduction procedure

INPUT: A terminating 2-polygraph Σ.

Step 1. Compute the homotopical completion $\mathcal{S}(\Sigma)$ (convergent and coherent).

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$$
\begin{tikzcd}
& v \arrow[rr, Rightarrow, bend left, "f_1"] \arrow[d, phantom, "\Downarrow A_{f,g}"] & & x' \arrow[dr, Rightarrow, bend left, "h"] & \\
u \arrow[ur, Rightarrow, bend left, "f"] \arrow[rr, Rightarrow, "g"] \arrow[dr, Rightarrow, bend right, "h"'] & & w \arrow[ur, Rightarrow, "g_1"] \arrow[dr, Rightarrow, "g_2"'] \arrow[rr, phantom, "\Downarrow B"] & & \widehat{u} \\
& x \arrow[rr, Rightarrow, bend right, "h_2"'] \arrow[u, phantom, "\Downarrow A_{g,h}"] & & v' \arrow[ur, Rightarrow, bend right, "k"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& v \arrow[rr, Rightarrow, bend left, "f_1"] \arrow[dr, Rightarrow, "f_2"] & & x' \arrow[dr, Rightarrow, bend left, "h"] \arrow[d, phantom, "C \Downarrow"] & \\
u \arrow[ur, Rightarrow, bend left, "f"] \arrow[dr, Rightarrow, bend right, "h"'] \arrow[r, phantom, "A_{f,h} \Downarrow"] & {} & w' \arrow[rr, Rightarrow, "g_3"] & {} & \widehat{u} \\
& x \arrow[ur, Rightarrow, "h_1"] \arrow[rr, Rightarrow, bend right, "h_2"'] & & v' \arrow[ur, Rightarrow, bend right, "k"'] \arrow[u, phantom, "D \Downarrow"] &
\end{tikzcd}
$$

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INPUT: A terminating 2-polygraph Σ.

Step 1. Compute the homotopical completion $\mathcal{S}(\Sigma)$ (convergent and coherent).

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$$
\begin{tikzcd}
 & v \arrow[rr, Rightarrow, bend left, "f_1"] \arrow[d, phantom, "\Downarrow A_{f,g}"] & & x' \arrow[dr, Rightarrow, bend left, "h"] \arrow[d, phantom, "\Downarrow B"] & \\
u \arrow[ur, Rightarrow, bend left, "f"] \arrow[rr, Rightarrow, "g"] \arrow[dr, Rightarrow, bend right, "h"'] & & w \arrow[ur, Rightarrow, "g_1"] \arrow[dr, Rightarrow, "g_2"'] & & \widehat{u} \\
 & x \arrow[rr, Rightarrow, bend right, "h_2"'] \arrow[u, phantom, "\Downarrow A_{g,h}"] & & v' \arrow[ur, Rightarrow, bend right, "k"'] &
\end{tikzcd}
\quad \Rrightarrow \quad
\begin{tikzcd}
 & v \arrow[rr, Rightarrow, bend left, "f_1"] \arrow[dr, Rightarrow, "f_2"] & & x' \arrow[dr, Rightarrow, bend left, "h"] \arrow[d, phantom, "C\Downarrow"] & \\
u \arrow[ur, Rightarrow, bend left, "f"] \arrow[dr, Rightarrow, bend right, "h"'] \arrow[r, phantom, "A_{f,h}\Downarrow"] & & w' \arrow[rr, Rightarrow, "g_3"] & & \widehat{u} \\
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\end{tikzcd}
$$

Homotopical completion-reduction procedure

INPUT: A terminating 2-polygraph Σ.

Step 1. Compute the homotopical completion $\mathcal{S}(\Sigma)$ (convergent and coherent).

Step 2. Apply the homotopical reduction to $\mathcal{S}(\Sigma)$ with a **collapsible part** Γ made of

- ▷ 3-spheres induced by some of the **generating triple confluences** of $\mathcal{S}(\Sigma)$,
- ▷ the 3-cells adjoined with a 2-cell by homotopical completion to reach confluence,

$$
\begin{tikzcd}
 & v \arrow[r, Rightarrow, "f'"] \arrow[dd, phantom, "\Downarrow A_{f,g}"] & \widehat{v} \\
u \arrow[ur, Rightarrow, bend left, "f"] \arrow[dr, Rightarrow, bend right, "g"'] & & \\
 & w \arrow[r, Rightarrow, "g'"'] & \widehat{w} \arrow[uu, Rrightarrow, dotted, "\chi"']
\end{tikzcd}
$$

Homotopical completion-reduction procedure

INPUT: A terminating 2-polygraph Σ.

Step 1. Compute the homotopical completion $\mathcal{S}(\Sigma)$ (convergent and coherent).

Step 2. Apply the homotopical reduction to $\mathcal{S}(\Sigma)$ with a **collapsible part** Γ made of

- ▷ 3-spheres induced by some of the **generating triple confluences** of $\mathcal{S}(\Sigma)$,
- ▷ the 3-cells adjoined with a 2-cell by homotopical completion to reach confluence,
- ▷ some collapsible 2-cells or 3-cells already present in the initial presentation Σ.

Homotopical completion-reduction procedure

INPUT: A terminating 2-polygraph Σ.

Step 1. Compute the homotopical completion $\mathcal{S}(\Sigma)$ (convergent and coherent).

Step 2. Apply the homotopical reduction to $\mathcal{S}(\Sigma)$ with a **collapsible part** Γ made of

- ▷ 3-spheres induced by some of the **generating triple confluences** of $\mathcal{S}(\Sigma)$,
- ▷ the 3-cells adjoined with a 2-cell by homotopical completion to reach confluence,
- ▷ some collapsible 2-cells or 3-cells already present in the initial presentation Σ.

The **homotopical completion-reduction** of terminating 2-polygraph Σ is the $(3,1)$-polygraph

$$\mathcal{R}(\Sigma) = \pi_{\Gamma}(\mathcal{S}(\Sigma))$$

Theorem. [Gaussent-Guiraud-M., 2013]

For every terminating presentation Σ of a category $\mathbf{C}$*, the homotopical completion-reduction $\mathcal{R}(\Sigma)$ of Σ is a coherent convergent presentation of* $\mathbf{C}$*.*

The homotopical completion-reduction procedure

Example.

$$\Sigma_2^{\mathrm{KN}} = \langle\, s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a \,\rangle$$

The homotopical completion-reduction procedure

Example.

$$\Sigma_2^{\mathrm{KN}} = \langle\; s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\; st \overset{\beta}{\Longrightarrow} a \;\rangle$$

$$\mathcal{S}(\Sigma_2^{\mathrm{KN}}) = \langle\; s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\; st \overset{\beta}{\Longrightarrow} a, sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, C, D \;\rangle$$

The homotopical completion-reduction procedure

Example.

$$\Sigma_2^{\mathrm{KN}} = \langle\, s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a \,\rangle$$

$$\mathcal{S}(\Sigma_2^{\mathrm{KN}}) = \langle\, s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a, sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, C, D \,\rangle$$

$$\langle\, s, t, a \mid\ ta \overset{\alpha}{\Longrightarrow} as\ , st \overset{\beta}{\Longrightarrow} a, sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, C, D \,\rangle$$

The homotopical completion-reduction procedure

Example.

$$\Sigma_2^{\mathrm{KN}} = \langle\, s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a \,\rangle$$

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$$\langle\, s, t, a \mid\ ta \overset{\alpha}{\Longrightarrow} as\ , st \overset{\beta}{\Longrightarrow} a, sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, C, D \,\rangle$$

- There are four critical triple branchings, overlapping on

$$sasta,\ sasast,\ sasasas,\ sasasaa.$$

The homotopical completion-reduction procedure

Example.

$$\Sigma_2^{\mathrm{KN}} = \langle\ s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a\ \rangle$$

$$\mathcal{S}(\Sigma_2^{\mathrm{KN}}) = \langle\ s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a, sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, C, D\ \rangle$$

$$\langle\ s, t, a \mid\ ta \overset{\alpha}{\Longrightarrow} as\ , st \overset{\beta}{\Longrightarrow} a, sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, \cancel{C}, D\ \rangle$$

▶ There are four critical triple branchings, overlapping on

$$sasta,\ sasast,\ sasasas,\ sasasaa.$$

▷ Critical triple branching on $sasta$ proves that C is redundant:

$$\begin{tikzcd}
 & aata \arrow[r, Rightarrow, "aa\alpha"] & aaas \\
sasta \arrow[ur, Rightarrow, bend left, "\gamma ta"] \arrow[r, Rightarrow, "sa\beta a"] \arrow[dr, Rightarrow, bend right, "sas\alpha"'] \arrow[r, phantom, shift left=4, "\Downarrow Ba"] \arrow[r, phantom, shift right=4, "\Downarrow saA"] & saaa \arrow[u, Rightarrow, "\delta a"'] & \\
 & sasas \arrow[u, Rightarrow, "sa\gamma"'] &
\end{tikzcd}
\quad \Rrightarrow \quad
\begin{tikzcd}
 & aata \arrow[dr, Rightarrow, "aa\alpha"] & & \\
sasta \arrow[ur, Rightarrow, "\gamma ta"] \arrow[dr, Rightarrow, "sas\alpha"'] \arrow[rr, phantom, "="] & & aaas \arrow[d, phantom, "\Downarrow C"] & \\
 & sasas \arrow[ur, Rightarrow, "\gamma as"] \arrow[dr, Rightarrow, "sa\gamma"'] & {} & aata \arrow[ul, Rightarrow, "aa\alpha"'] \\
 & & saaa \arrow[ur, Rightarrow, "\delta a"'] &
\end{tikzcd}$$

$$C = sas\alpha^{-1} \star_1 (Ba \star_1 aa\alpha) \star_2 (saA \star_1 \delta a \star_1 aa\alpha)$$

The homotopical completion-reduction procedure

Example.

$$\Sigma_2^{\mathrm{KN}} = \langle\, s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\; st \overset{\beta}{\Longrightarrow} a \,\rangle$$

$$\mathcal{S}(\Sigma_2^{\mathrm{KN}}) = \langle\, s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\; st \overset{\beta}{\Longrightarrow} a, sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, C, D \,\rangle$$

$$\langle\, s, t, a \mid\; ta \overset{\alpha}{\Longrightarrow} as\;, st \overset{\beta}{\Longrightarrow} a, sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, \cancel{C}, \cancel{D} \rangle$$

- There are four critical triple branchings, overlapping on

$$sasta,\; sasast,\; sasasas,\; sasasaa.$$

  - ▹ Critical triple branching on $sasast$ proves that D is redundant:

$$\begin{tikzcd}
& aaast \arrow[r, Rightarrow, "aaa\beta"] \arrow[d, phantom, "Ct\Downarrow"] & aaaa \\
sasast \arrow[ur, Rightarrow, bend left, "\gamma ast"] \arrow[r, Rightarrow, "sa\gamma t"] \arrow[dr, Rightarrow, bend right, "sasa\beta"'] & saaat \arrow[r, Rightarrow, "\delta at"'] \arrow[d, phantom, "\Downarrow saB"] & aatat \arrow[ul, Rightarrow, "aa\alpha t"'] \\
& sasaa \arrow[u, Rightarrow, bend right, "sa\delta"'] &
\end{tikzcd}
\quad \Rrightarrow \quad
\begin{tikzcd}
& aaast \arrow[dr, Rightarrow, bend left, "aaa\beta"] & & \\
sasast \arrow[ur, Rightarrow, bend left, "\gamma ast"] \arrow[dr, Rightarrow, bend right, "sasa\beta"'] \arrow[rr, phantom, "="] & & aaaa \arrow[d, phantom, "\Downarrow D"] & aaast \arrow[l, Rightarrow, "aaa\beta"'] \\
& sasaa \arrow[ur, Rightarrow, "\gamma aa"'] \arrow[dr, Rightarrow, bend right, "sa\delta"'] & & \\
& & saaat \arrow[r, Rightarrow, "\delta at"'] & aatat \arrow[uu, Rightarrow, "aa\alpha t"']
\end{tikzcd}$$

$$D = sasa\beta^{-1} \star_1 \big((Ct \star_1 aaa\beta) \star_2 (saB \star_1 \delta at \star_1 aa\alpha t \star_1 aaa\beta) \big)$$

The homotopical completion-reduction procedure

Example.

$$\Sigma_2^{\mathrm{KN}} = \langle\; s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\; st \overset{\beta}{\Longrightarrow} a \;\rangle$$

$$\mathcal{S}(\Sigma_2^{\mathrm{KN}}) = \langle\; s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\; st \overset{\beta}{\Longrightarrow} a, sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, C, D \;\rangle$$

$$\langle\; s, t, a \mid\; ta \overset{\alpha}{\Longrightarrow} as\,, st \overset{\beta}{\Longrightarrow} a, sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, \cancel{C}, \cancel{D} \rangle$$

▷ The 3-cells A and B are collapsible and the rules γ and δ are redundant.

$$\begin{tikzcd}
 & aa \\
sta \arrow[ur, Rightarrow, bend left, "\beta a"] \arrow[dr, Rightarrow, bend right, "s\alpha"'] \arrow[r, phantom, "\Downarrow A"] & {} \\
 & sas \arrow[u, Rightarrow, bend right, "\gamma"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & aat \\
sast \arrow[ur, Rightarrow, bend left, "\gamma t"] \arrow[dr, Rightarrow, bend right, "sa\beta"'] \arrow[r, phantom, "\Downarrow B"] & {} \\
 & saa \arrow[u, Rightarrow, bend right, "\delta"']
\end{tikzcd}$$

The homotopical completion-reduction procedure

Example.

$$\Sigma_2^{\mathrm{KN}} = \langle\; s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\; st \overset{\beta}{\Longrightarrow} a \;\rangle$$

$$\mathcal{S}(\Sigma_2^{\mathrm{KN}}) = \langle\; s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\; st \overset{\beta}{\Longrightarrow} a, sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, C, D \;\rangle$$

$$\langle\; s, t, a \mid\; ta \overset{\alpha}{\Longrightarrow} as\,, st \overset{\beta}{\Longrightarrow} a,\; \cancel{sas \overset{\gamma}{\Longrightarrow} aa}, \cancel{, saa \overset{\delta}{\Longrightarrow} aat} \mid \cancel{A}, \cancel{B}, \cancel{C}, \cancel{D} \rangle$$

▷ The 3-cells A and B are collapsible and the rules γ and δ are redundant.

$$\begin{tikzcd} & aa \\ sta \arrow[ur, Rightarrow, bend left, "\beta a"] \arrow[dr, Rightarrow, bend right, "s\alpha"'] & \\ & sas \end{tikzcd} \qquad \begin{tikzcd} & aat \\ sast \arrow[ur, Rightarrow, bend left, "\gamma t"] \arrow[dr, Rightarrow, bend right, "sa\beta"'] & \\ & saa \end{tikzcd}$$

The homotopical completion-reduction procedure

Example.

$$\Sigma_2^{\mathrm{KN}} = \langle\, s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a \,\rangle$$

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$$\langle\, s, t, \cancel{a} \mid ta \overset{\alpha}{\Longrightarrow} as\ ,\cancel{st \overset{\beta}{\Longrightarrow} a},\ \cancel{sas \overset{\gamma}{\Longrightarrow} aa},\ ,\cancel{saa \overset{\delta}{\Longrightarrow} aat} \mid \cancel{A}, \cancel{B}, \cancel{C}, \cancel{D} \,\rangle$$

▷ The rule $st \overset{\beta}{\Longrightarrow} a$ is collapsible and the generator a is redundant.

The homotopical completion-reduction procedure

Example.

$$\Sigma_2^{\mathrm{KN}} = \langle\ s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a\ \rangle$$

$$\mathcal{S}(\Sigma_2^{\mathrm{KN}}) = \langle\ s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a, sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, C, D\ \rangle$$

$$\langle\ s, t, \cancel{a} \mid tst \overset{\alpha}{\Longrightarrow} sts, \cancel{st \overset{\beta}{\Longrightarrow} a}, \cancel{sas \overset{\gamma}{\Longrightarrow} aa}, \cancel{, saa \overset{\delta}{\Longrightarrow} aat} \mid \cancel{A}, \cancel{B}, \cancel{C}, \cancel{D}\ \rangle$$

$$\mathcal{R}(\Sigma_2^{\mathrm{KN}}) = \langle\ s, t \mid tst \overset{\alpha}{\Longrightarrow} sts \mid \emptyset\ \rangle$$

$$= \mathrm{Art}_3(\mathbf{S}_3)$$

Part III. Applications to Artin monoids

Garside's presentation

- Let $\mathbf{W}$ be a Coxeter group

$$\mathbf{W} = \left\langle\ S \mid s^2 = 1,\ \ \langle ts \rangle^{m_{st}}\ =\ \langle st \rangle^{m_{st}}\ \right\rangle$$

where $\langle ts \rangle^{m_{st}}$ stands for the word $tsts\ldots$ with m_{st} letters.

Garside's presentation

- Let $\mathbf{W}$ be a Coxeter group

$$\mathbf{W} = \langle\, S \mid s^2 = 1,\ \ \langle ts\rangle^{m_{st}} \ = \ \langle st\rangle^{m_{st}} \,\rangle$$

where $\langle ts\rangle^{m_{st}}$ stands for the word $tsts\ldots$ with m_{st} letters.

- **Artin's presentation** of the Artin monoid $\mathbf{B}^+(\mathbf{W})$:

$$\mathrm{Art}_2(\mathbf{W}) = \langle\, S \mid \langle ts\rangle^{m_{st}} \ = \ \langle st\rangle^{m_{st}} \,\rangle$$

Garside's presentation

- Let $\mathbf{W}$ be a Coxeter group

$$\mathbf{W} = \langle\, S \mid s^2 = 1,\ \ \langle ts\rangle^{m_{st}} \ = \ \langle st\rangle^{m_{st}} \,\rangle$$

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- **Artin's presentation** of the Artin monoid $\mathbf{B}^+(\mathbf{W})$:

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- **Garside's presentation** of $\mathbf{B}^+(\mathbf{W})$

$$\mathrm{Gar}_2(\mathbf{W}) = \langle\, \mathbf{W}\setminus\{1\} \mid \ u|v \overset{\alpha_{u,v}}{\Longrightarrow} uv, \ \text{whenever } u \frown v \,\rangle$$

where

uv is the product in $\mathbf{W}$,

$u|v$ is the product in the free monoid over $\mathbf{W}$.

Garside's presentation

- Let $\mathbf{W}$ be a Coxeter group

$$\mathbf{W} = \langle\ S \mid s^2 = 1,\ \ \langle ts \rangle^{m_{st}}\ = \langle st \rangle^{m_{st}}\ \rangle$$

where $\langle ts \rangle^{m_{st}}$ stands for the word $tsts\ldots$ with m_{st} letters.

- **Artin's presentation** of the Artin monoid $\mathbf{B}^+(\mathbf{W})$:

$$\mathrm{Art}_2(\mathbf{W}) = \langle\ S \mid \langle ts \rangle^{m_{st}}\ = \langle st \rangle^{m_{st}}\ \rangle$$

- **Garside's presentation** of $\mathbf{B}^+(\mathbf{W})$

$$\mathrm{Gar}_2(\mathbf{W}) = \langle\ \mathbf{W} \setminus \{1\} \ \mid\ u|v \overset{\alpha_{u,v}}{\Longrightarrow} uv,\ \text{whenever } u \frown v\ \rangle$$

where

uv is the product in $\mathbf{W}$,

$u|v$ is the product in the free monoid over $\mathbf{W}$.

- Notations :
  - $u \frown v$ whenever $l(uv) = l(u) + l(v)$.
  - $u \overset{\times}{\ } v$ whenever $l(uv) < l(u) + l(v)$.

Garside's coherent presentation

▶ The **Garside's coherent presentation** of $\mathbf{B}^+(\mathbf{W})$ is the extended presentation $\mathrm{Gar}_3(\mathbf{W})$ obtained from $\mathrm{Gar}_2(\mathbf{W})$ by adjunction of one 3-cell

$$
\begin{tikzcd}
& uv|w \arrow[dr, Rightarrow, "\alpha_{uv,w}"] \arrow[dd, phantom, "\Rrightarrow A_{u,v,w}"] & \\
u|v|w \arrow[ur, Rightarrow, "\alpha_{u,v}|w"] \arrow[dr, Rightarrow, "u|\alpha_{v,w}"'] & & uvw \\
& u|vw \arrow[ur, Rightarrow, "\alpha_{u,vw}"'] &
\end{tikzcd}
$$

for every u, v, w in $\mathbf{W} \setminus \{1\}$ with $u \quad v \quad w$.

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▶ The **Garside's coherent presentation** of $\mathbf{B}^+(\mathbf{W})$ is the extended presentation $\mathrm{Gar}_3(\mathbf{W})$ obtained from $\mathrm{Gar}_2(\mathbf{W})$ by adjunction of one 3-cell

$$\begin{tikzcd}
& uv|w \arrow[dr, Rightarrow, bend left, "\alpha_{uv,w}"] \arrow[dd, phantom, "\Downarrow A_{u,v,w}"] & \\
u|v|w \arrow[ur, Rightarrow, bend left, "\alpha_{u,v}|w"] \arrow[dr, Rightarrow, bend right, "u|\alpha_{v,w}"'] & & uvw \\
& u|vw \arrow[ur, Rightarrow, bend right, "\alpha_{u,vw}"'] &
\end{tikzcd}$$

for every u, v, w in $\mathbf{W} \setminus \{1\}$ with $u \quad v \quad w$.

Theorem. [Gaussent-Guiraud-M., 2013]

For every Coxeter group $\mathbf{W}$, the Artin monoid $\mathbf{B}^+(\mathbf{W})$ admits $\mathrm{Gar}_3(\mathbf{W})$ as a coherent presentation.

Proof. By homotopical completion-reduction of the 2-polygraph $\mathrm{Gar}_2(\mathbf{W})$.

Garside's coherent presentation

Step 1. We compute the coherent convergent presentation $\mathcal{S}(\mathsf{Gar}_2(\mathbf{W}))$

Garside's coherent presentation

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▷ The 2-polygraph $\mathrm{Gar}_2(\mathbf{W})$ has one critical branching for every u, v, w in $\mathbf{W} \setminus \{1\}$ when

$$u \qquad v \qquad w$$

Garside's coherent presentation

Step 1. We compute the coherent convergent presentation $\mathcal{S}(\mathrm{Gar}_2(\mathbf{W}))$

▷ The 2-polygraph $\mathrm{Gar}_2(\mathbf{W})$ has one critical branching for every u, v, w in $\mathbf{W} \setminus \{1\}$ when

$u|v|w$

Garside's coherent presentation

Step 1. We compute the coherent convergent presentation $\mathcal{S}(\mathrm{Gar}_2(\mathbf{W}))$

▷ The 2-polygraph $\mathrm{Gar}_2(\mathbf{W})$ has one critical branching for every u, v, w in $\mathbf{W} \setminus \{1\}$ when

$$u \quad v \quad w \qquad\qquad u|v|w \xRightarrow{\alpha_{u,v}|w} uv|w$$

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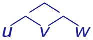
▷ There are two possibilities.

if 

$$\begin{tikzcd}
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 & u|vw
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$$\begin{tikzcd}
& uv|w \arrow[dr, Rightarrow, bend left, "\alpha_{uv,w}"] \arrow[dd, phantom, "\Downarrow A_{u,v,w}"] & \\
u|v|w \arrow[ur, Rightarrow, bend left, "\alpha_{u,v}|w"] \arrow[dr, Rightarrow, bend right, "u|\alpha_{v,w}"'] & & uvw \\
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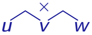
otherwise 

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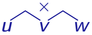
otherwise 

$$\begin{tikzcd} u|v|w \arrow[rr, Rightarrow, bend left, "\alpha_{u,v}|w"] \arrow[dr, Rightarrow, "u|\alpha_{v,w}"'] & & uv|w \\ & u|vw \arrow[ur, Rightarrow, "\beta_{u,v,w}"'] & \end{tikzcd}$$

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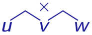
otherwise 

$$\begin{tikzcd} u|v|w \arrow[rr, Rightarrow, bend left, "\alpha_{u,v}|w"] \arrow[dr, Rightarrow, "u|\alpha_{v,w}"'] & \arrow[d, phantom, "\Downarrow B_{u,v,w}"] & uv|w \\ & u|vw \arrow[ur, Rightarrow, "\beta_{u,v,w}"'] & \end{tikzcd}$$

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$$\begin{tikzcd}
& uv|w \arrow[dr, Rightarrow, "\alpha_{uv,w}"] \arrow[dd, phantom, "\Downarrow A_{u,v,w}"] & \\
u|v|w \arrow[ur, Rightarrow, "\alpha_{u,v}|w"] \arrow[dr, Rightarrow, "u|\alpha_{v,w}"'] & & uvw \\
& u|vw \arrow[ur, Rightarrow, "\alpha_{u,vw}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
u|v|w \arrow[rr, Rightarrow, bend left, "\alpha_{u,v}|w"] \arrow[dr, Rightarrow, "u|\alpha_{v,w}"'] & \arrow[d, phantom, "\Downarrow B_{u,v,w}"] & uv|w \\
& u|vw \arrow[ur, Rightarrow, "\beta_{u,v,w}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& uv|wx \arrow[dr, Rightarrow, "\beta_{uv,w,x}"] \arrow[dd, phantom, "\Downarrow C_{u,v,w,x}"] & \\
u|v|wx \arrow[ur, Rightarrow, "\alpha_{u,v}|wx"] \arrow[dr, Rightarrow, "u|\beta_{v,w,x}"'] & & uvw|x \\
& u|vw|x \arrow[ur, Rightarrow, "\alpha_{u,vw}|x"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
u|v|wx \arrow[rrr, Rightarrow, bend left, "\alpha_{u,v}|wx"] \arrow[dr, Rightarrow, "u|\beta_{v,w,x}"'] & & & uv|wx \\
& u|vw|x \arrow[r, Rightarrow, "\beta_{u,v,w}|x"'] \arrow[r, phantom, shift left=6, "\Downarrow D_{u,v,w,x}"] & uv|w|x \arrow[ur, Rightarrow, "uv|\alpha_{w,x}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& uv|w|xy \arrow[rr, Rightarrow, "uv|\alpha_{w,xy}"] & & uv|wxy \\
u|vw|xy \arrow[ur, Rightarrow, "\beta_{u,v,w}|xy"] \arrow[dr, Rightarrow, "u|\beta_{vw,x,y}"'] & & \arrow[phantom, "\Downarrow F_{u,v,w,x,y}"] & \\
& u|vwx|y \arrow[r, Rightarrow, "\beta_{u,v,wx}|y"'] & uv|wx|y \arrow[uur, Rightarrow, "uv|\alpha_{vw,x}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& uv|w|x \arrow[dr, Rightarrow, "uv|\alpha_{w,x}"] \arrow[dd, phantom, "\Downarrow E_{u,v,w,x}"] & \\
u|vw|x \arrow[ur, Rightarrow, "\beta_{u,v,w}|x"] \arrow[dr, Rightarrow, "u|\alpha_{vw,x}"'] & & uv|wx \\
& u|vwx \arrow[ur, Rightarrow, "\beta_{u,v,wx}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& uv|w|xy \arrow[dr, Rightarrow, "uv|\beta_{w,x,y}"] \arrow[dd, phantom, "\Downarrow G_{u,v,w,x,y}"] & \\
u|vw|xy \arrow[ur, Rightarrow, "\beta_{u,v,w}|xy"] \arrow[dr, Rightarrow, "u|\beta_{vw,x,y}"'] & & uv|wx|y \\
& u|vwx|y \arrow[ur, Rightarrow, "\beta_{u,v,wx}|y"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& uv|xy \arrow[dr, Rightarrow, "\beta_{uv,x,y}"] \arrow[d, phantom, "\Downarrow H_{u,v,x,y}"] & \\
u|vxy \arrow[ur, Rightarrow, "\beta_{u,v,xy}"] \arrow[rr, Rightarrow, bend right, "\beta_{u,vx,y}"'] & & uvx|y
\end{tikzcd}$$

$$\begin{tikzcd}
& uv|w \;=\; uv|xy \arrow[dr, Rightarrow, "\beta_{uv,x,y}"] \arrow[dd, phantom, "\Downarrow I_{u,v,w,v',w'}"] & \\
\begin{array}{c} u|vw \\ = \\ u|v'w' \end{array} \arrow[ur, Rightarrow, "\beta_{u,v,w}"] \arrow[dr, Rightarrow, "\beta_{u,v',w'}"'] & & \begin{array}{c} uvx|y \\ = \\ uv'x'|y \end{array} \\
& uv'|w' \;=\; uv'|x'y \arrow[ur, Rightarrow, "\beta_{uv',x',y}"'] &
\end{tikzcd}$$

Garside's coherent presentation

Proposition.

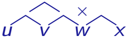
For every Coxeter group $\mathbf{W}$, the Artin monoid $\mathbf{B}^+(\mathbf{W})$ admits, as a coherent convergent presentation, the $(3,1)$-polygraph $\mathcal{S}(\mathrm{Gar}_2(\mathbf{W}))$ where

- ▷ the 1-cells are the elements of $\mathbf{W} \setminus \{1\}$,
- ▷ there is a 2-cell $u|v \xRightarrow{\alpha_{u,v}} uv$ for every u, v in $\mathbf{W} \setminus \{1\}$ with $u \frown v$,
- ▷ the 2-cells $u|vw \xRightarrow{\beta_{u,v,w}} uv|w$, for every u, v, w in $\mathbf{W} \setminus \{1\}$ with $u \frown \overset{\times}{v} \frown w$,
- ▷ the nine families of 3-cells A, B, C, D, E, F, G, H, I.

Garside's coherent presentation

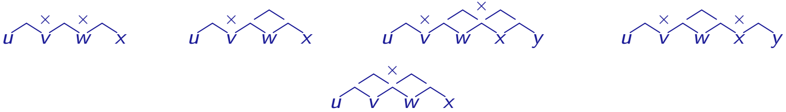
Step 2. Homotopical reduction of $\mathcal{S}(\mathrm{Gar}_2(\mathbf{W}))$.

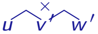
▷ We consider some triple critical branchings of $\mathcal{S}(\mathrm{Gar}_2(\mathbf{W}))$

In the case 

$$
\begin{tikzcd}
 & uv|w|x \arrow[rr, Rightarrow, bend left, "\alpha_{uv,w}|x"] & & uvw|x \\
u|v|w|x \arrow[ur, Rightarrow, "\alpha_{u,v}|w|x"] \arrow[rr, Rightarrow, "u|\alpha_{v,w}|x"] \arrow[dr, Rightarrow, "u|v|\alpha_{w,x}"'] & \arrow[r, phantom, "A_{u,v,w}|x"] & u|vw|x \arrow[ur, Rightarrow, "\alpha_{u,vw}|x"'] & \\
 & u|v|wx \arrow[ur, Rightarrow, "u|\beta_{v,w,x}"'] \arrow[u, phantom, "u|B_{v,w,x}"] & &
\end{tikzcd}
\quad \Rrightarrow \quad
\begin{tikzcd}
 & uv|w|x \arrow[dr, Rightarrow, "uv|\alpha_{w,x}"'] \arrow[drrr, Rightarrow, bend left, "\alpha_{uv,w}|x"] & \arrow[r, phantom, "B_{uv,w,x}"] & & \\
u|v|w|x \arrow[ur, Rightarrow, "\alpha_{u,v}|w|x"] \arrow[dr, Rightarrow, "u|v|\alpha_{w,x}"'] & = & uv|wx \arrow[rr, Rightarrow, "\beta_{uv,w,x}"] & & uvw|x \\
 & u|v|wx \arrow[ur, Rightarrow, "\alpha_{u,v}|wx"] \arrow[rr, Rightarrow, bend right, "u|\beta_{v,w,x}"'] & \arrow[r, phantom, "C_{u,v,w,x}"] & u|vw|x \arrow[ur, Rightarrow, "\alpha_{u,vw}|x"'] &
\end{tikzcd}
$$

and similar 3-spheres for the following cases



and 

with $vw = v'w'$

Artin's coherent presentation

Theorem. [Gaussent-Guiraud-M., 2013]

For every Coxeter group $\mathbf{W}$*, the Artin monoid* $\mathbf{B}^+(\mathbf{W})$ *admits the coherent presentation* $\mathrm{Art}_3(\mathbf{W})$ *made of*

▷ *Artin's presentation* $\mathrm{Art}_2(\mathbf{W}) = \langle\ S \mid \langle ts\rangle^{m_{st}} \ = \ \langle st\rangle^{m_{st}}\ \rangle$

▷ *one 3-cell* $Z_{r,s,t}$ *for every elements* $t > s > r$ *of* S *such that the subgroup* $\mathbf{W}_{\{r,s,t\}}$ *is finite.*

Artin's coherent presentation

- The 3-cells $Z_{r,s,t}$ for Coxeter types A_3

$$
\begin{tikzcd}
 & strsrt \arrow[r, Rightarrow, "s\gamma_{rt}s\gamma_{rt}^{-}"] & srtstr \arrow[r, Rightarrow, "sr\gamma_{st}r"] & srstsr \arrow[dr, Rightarrow, bend left, "\gamma_{rs}tsr"] & \\
stsrst \arrow[ur, Rightarrow, bend left, "st\gamma_{rs}t"] & & & & rsrtsr \\
tstrst \arrow[u, Rightarrow, "\gamma_{st}rst"] \arrow[d, Rightarrow, "ts\gamma_{rt}st"'] & & {} \arrow[r, phantom, "\Rrightarrow Z_{r,s,t}"] & & rstrsr \arrow[u, Rightarrow, "rs\gamma_{rt}sr"'] \\
tsrtst \arrow[dr, Rightarrow, bend right, "tsr\gamma_{st}"'] & & & & rstsrs \arrow[u, Rightarrow, "rst\gamma_{rs}"'] \\
 & tsrsts \arrow[r, Rightarrow, "t\gamma_{rs}ts"'] & trsrts \arrow[r, Rightarrow, "\gamma_{rt}s\gamma_{rt}^{-}s"'] & rtstrs \arrow[ur, Rightarrow, bend right, "r\gamma_{st}rs"'] &
\end{tikzcd}
$$

Artin's coherent presentation

- The 3-cells $Z_{r,s,t}$ for Coxeter types B_3

$$
\begin{tikzcd}[column sep=small]
 & srtsrtstr \arrow[r, Rightarrow, "srts\gamma^{-}_{rt}str"] & srtstrstr \arrow[r, Rightarrow, "sr\gamma_{st}rs\gamma_{rt}"] & srstsrsrt \arrow[r, Rightarrow, "srst\gamma_{rs}t"] & srstrsrst \arrow[r, Rightarrow, "srs\gamma_{rt}srst"] & srsrtsrst \arrow[dr, Rightarrow, bend left, "\gamma_{rs}tsrst"] & \\
strsrstsr \arrow[ur, Rightarrow, bend left, "s\gamma_{rt}sr\gamma^{-}_{st}r"] & & & & & & rsrstsrst \\
stsrsrtsr \arrow[u, Rightarrow, "st\gamma_{rs}tsr"] & & & \arrow[dd, phantom, "\Downarrow Z_{r,s,t}"] & & & rsrtstrst \arrow[u, Rightarrow, "rsr\gamma_{st}rst"'] \\
tstrsrtsr \arrow[u, Rightarrow, "\gamma_{st}rsrtsr"] \arrow[d, Rightarrow, "ts\gamma_{rt}s\gamma^{-}_{rt}sr"'] & & & & & & rsrtsrtst \arrow[u, Rightarrow, "rsrts\gamma^{-}_{rt}st"'] \\
tsrtstrsr \arrow[d, Rightarrow, "tsr\gamma_{st}rsr"'] & & & & & & rstrsrsts \arrow[u, Rightarrow, "rs\gamma_{rt}sr\gamma^{-}_{st}"'] \\
tsrstsrsr \arrow[dr, Rightarrow, bend right, "tsrst\gamma_{rs}"'] & & & & & & rstsrsrts \arrow[u, Rightarrow, "rst\gamma_{rs}ts"'] \\
 & tsrstrsrs \arrow[r, Rightarrow, "tsrs\gamma_{rt}srs"'] & tsrsrtsrs \arrow[r, Rightarrow, "t\gamma_{rs}tsrs"'] & trsrstsrs \arrow[r, Rightarrow, "\gamma_{rt}sr\gamma^{-}_{st}rs"'] & rtsrtstrs \arrow[r, Rightarrow, "rts\gamma^{-}_{rt}strs"'] & rtstrstrs \arrow[ur, Rightarrow, bend right, "r\gamma_{st}rs\gamma_{rt}s"'] &
\end{tikzcd}
$$

Artin's coherent presentation

- The 3-cells $Z_{r,s,t}$ for Coxeter types $A_1 \times A_1 \times A_1$

$$
\begin{tikzcd}
 & str \arrow[r, Rightarrow, "s\gamma_{rt}"] & srt \arrow[dr, Rightarrow, "\gamma_{rs}t"] & \\
tsr \arrow[ur, Rightarrow, "\gamma_{st}r"] \arrow[dr, Rightarrow, "t\gamma_{rs}"'] \arrow[rrr, phantom, "\Downarrow Z_{r,s,t}"] & & & rst \\
 & trs \arrow[r, Rightarrow, "\gamma_{rt}s"'] & rts \arrow[ur, Rightarrow, "r\gamma_{st}"'] &
\end{tikzcd}
$$

Artin's coherent presentation

- The 3-cells $Z_{r,s,t}$ for Coxeter type H_3

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & srstrsrsrtsrsrt \arrow[r, Rightarrow] & srsrtsrstrsrsrt \arrow[r, Rightarrow] & srsrtsrstsrsrst \arrow[r, Rightarrow] & srsrtsrtstrsrst \arrow[dr, Rightarrow, bend left] & \\
srstsrsrstsrsrt \arrow[ur, Rightarrow, bend left] & & & & & srsrtstrsrtsrst \arrow[d, Rightarrow] \\
srtstrsrtstrsrt \arrow[u, Rightarrow] & & & & & srsrstsrsrtsrst \arrow[d, Rightarrow] \\
srtsrtstrsrtstr \arrow[u, Rightarrow] & & & & & rsrsrtsrsrtsrst \\
srtsrstsrsrstsr \arrow[u, Rightarrow] & & & & & rsrstrsrsrtsrst \arrow[u, Rightarrow] \\
srtsrstrsrsrtsr \arrow[u, Rightarrow] & & & & & rsrstsrsrstsrst \arrow[u, Rightarrow] \\
strsrsrtsrsrtsr \arrow[u, Rightarrow] & & & & & rsrtstrsrtstrst \arrow[u, Rightarrow] \\
stsrsrstsrsrtsr \arrow[u, Rightarrow] & & & & & rsrtsrtstrsrtst \arrow[u, Rightarrow] \\
tstrsrstsrsrtsr \arrow[u, Rightarrow] \arrow[d, Rightarrow] & & & \arrow[d, phantom, "\Downarrow Z_{r,s,t}"] & & rsrtsrstsrsrsts \arrow[u, Rightarrow] \\
tsrtsrstsrstrsr \arrow[d, Rightarrow] & & & \, & & rsrtsrstrsrsrts \arrow[u, Rightarrow] \\
tsrtsrtstrstrsr \arrow[d, Rightarrow] & & & & & rstrsrsrtsrsrts \arrow[u, Rightarrow] \\
tsrtstrsrtstrsr \arrow[d, Rightarrow] & & & & & rstsrsrstsrsrts \arrow[u, Rightarrow] \\
tsrstsrsrstsrsr \arrow[d, Rightarrow] & & & & & rtstrsrtstrsrts \arrow[u, Rightarrow] \\
tsrstrsrsrtsrsr \arrow[d, Rightarrow] & & & & & rtsrtstrsrtstrs \arrow[u, Rightarrow] \\
tsrsrtsrstrsrsr \arrow[d, Rightarrow] & & & & & rtsrstsrsrstsrs \arrow[u, Rightarrow] \\
tsrsrtsrstsrsrs \arrow[dr, Rightarrow, bend right] & & & & & rtsrstrsrsrtsrs \arrow[u, Rightarrow] \\
 & tsrsrtsrtstrsrs \arrow[r, Rightarrow] & tsrsrtstrsrtsrs \arrow[r, Rightarrow] & tsrsrstsrsrtsrs \arrow[r, Rightarrow] & trsrsrtsrsrtsrs \arrow[ur, Rightarrow, bend right] &
\end{tikzcd}
$$

Artin's coherent presentation

- The 3-cells $Z_{r,s,t}$ for Coxeter type $I_2(p) \times A_1$, $p \geqslant 3$

$$
\begin{tikzcd}
 & st\langle rs\rangle^{p-1} \arrow[r, Rightarrow, "s\gamma_{rt}\langle rs\rangle^{p-2}"] & (\cdots) \arrow[r, Rightarrow] & \langle sr\rangle^{p} t \arrow[dr, Rightarrow, bend left, "\gamma_{rs} t"] & \\
t\langle sr\rangle^{p} \arrow[ur, Rightarrow, bend left, "\gamma_{st}\langle rs\rangle^{p-1}"] \arrow[dr, Rightarrow, bend right, "t\gamma_{rs}"'] & & \arrow[d, phantom, "\Downarrow Z_{r,s,t}"] & & \langle rs\rangle^{p} t \\
 & t\langle rs\rangle^{p} \arrow[r, Rightarrow, "\gamma_{rt}\langle sr\rangle^{p-1}"'] & rt\langle sr\rangle^{p-1} \arrow[r, Rightarrow, "r\gamma_{st}\langle sr\rangle^{p-2}"'] & (\cdots) \arrow[ur, Rightarrow, bend right] &
\end{tikzcd}
$$

Coherent presentations and actions on categories

Definition. (Deligne, 1997)

An action T of a monoid $\mathbf{M}$ on categories is specified by

- a category $\mathbf{C} = T(\bullet)$
- an endofunctor $T(u) : \mathbf{C} \to \mathbf{C}$, for every element u of $\mathbf{M}$,
- natural isomorphisms $T_{u,v} : T(u)T(v) \Rightarrow T(uv)$ and $T_\bullet : 1_{\mathbf{C}} \Rightarrow T(1)$

satisfying the following coherence conditions:

$$
\begin{tikzcd}
 & T(uv)T(w) \arrow[dr, Rightarrow, "T_{uv,w}"] & \\
T(u)T(v)T(w) \arrow[ur, Rightarrow, "T_{u,v}T(w)"] \arrow[dr, Rightarrow, "T(u)T_{v,w}"'] \arrow[rr, phantom, "="] & & T(uvw) \\
 & T(u)T(vw) \arrow[ur, Rightarrow, "T_{u,vw}"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & T(1)T(u) \arrow[dr, Rightarrow, "T_{1,u}"] \arrow[d, phantom, "="] & \\
T(u) \arrow[ur, Rightarrow, "T_\bullet T(u)"] \arrow[rr, equal, bend right] & {} & T(u)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & T(u)T(1) \arrow[dr, Rightarrow, "T_{u,1}"] \arrow[d, phantom, "="] & \\
T(u) \arrow[ur, Rightarrow, "T(u)T_\bullet"] \arrow[rr, equal, bend right] & {} & T(u)
\end{tikzcd}
$$

Coherent presentations and actions on categories

Theorem. [Gaussent-Guiraud-M., 2013]

Let $\mathbf{M}$ *be a monoid and let* Σ *be a coherent presentation of* $\mathbf{M}$.
There is an equivalence of categories

$$\mathrm{Act}(\mathbf{M}) \approx 2\mathrm{Cat}(\Sigma_2^\top/\Sigma_3, \mathbf{Cat})$$

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▶ Such equivalence was known for the Garside's presentation of spherical Artin monoids (Deligne, 1997)

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There is an equivalence of categories

$$\mathrm{Act}(\mathbf{M}) \approx 2\mathrm{Cat}(\Sigma_2^\top/\Sigma_3, \mathbf{Cat})$$

▶ Such equivalence was known for the Garside's presentation of spherical Artin monoids (Deligne, 1997)

Consequence.

To determine an action of an Artin monoid $\mathbf{B}^+(\mathbf{W})$ on a category $\mathbf{C}$, it suffices to attach

▷ to any generating 1-cell $s \in S$ an endofunctor $T(s) : \mathbf{C} \to \mathbf{C}$,

▷ to any generating 2-cell an isomorphism of functors such that these satisfy coherence Zamolochikov relations.

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Conjecture.

Higher Artin's coherent presentation of $\mathbf{B}(\mathbf{W}, S)$ has exactly on k-cell, $k \geqslant 0$, for every subset I of S of rank k such that the subgroup $\mathbf{W}_I$ is finite.