Abstract abstract coherence and acyclicity

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Joint work with **Cameron Calk**, **Eric Goubault**, **Georg Struth**

Plan

- **Motivations**
- **I. Confluence proofs in modal Kleene algebras**
- **II. Coherence by rewriting**
- **III. Coherent proofs in higher modal Kleene algebras**
- **IV. Work in progress**

Joint work with **Cameron Calk**, **Eric Goubault**, **Georg Struth**

Motivations:
syzygies, coherence and resolutions

From syzygies to resolutions

- **Syzygies** are relations between generators of a module.

- Let M be a finitely generated R-module, and a set of generators:

$$Y = \{\mathbf{y_1}, \ldots, \mathbf{y}_k\}$$

  - a **syzygy** of M is an element $(\lambda_1, ..., \lambda_k)$ in R^k connecting generators:

$$\lambda_1 \mathbf{y_1} + \ldots + \lambda_k \mathbf{y_k} = 0$$

- The set of all syzygies wrt Y forms a submodule of R^n: the **module of 1st syzygies**.

- For $n \geqslant 2$, the **nth syzygy module** is the module of all syzygies of the $(n-1)$th syzygy module.

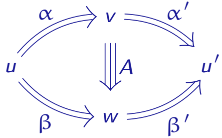
Theorem. (Hilbert's Syzygy Theorem, 1890)
If M is a finitely generated module over the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$, then the nth syzygy module of M is a always a free module.

- F.-O. Schreyer, 1980 : computation of syzygies by means of the **division algorithm**.
  - Buchberger's completion algorithm computes **Gröbner bases**.
  - The reduction to zero of a **S-polynomial** in a Gröbner basis gives a syzygy.

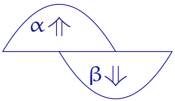
Syzygies by rewriting and coherence

- Squier's machinery (Squier, 1994)
  - ▷ Let $X = (X_1, X_2)$ be a convergent string rewriting systems.
  - ▷ Family of **generating confluences**

$\mathrm{cg}(X) := \{$

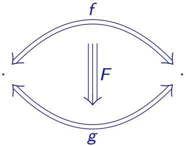


$|$ (α, β) critical branching of X



$\}$

Theorem. (Squier, 1994)
Any two parallel zig-zag rewriting sequences (f, g) can be filled by pasting of these generating confluences:



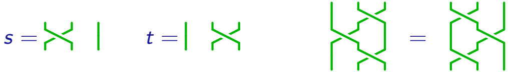
$\in \quad \mathrm{cg}(X)^\top$

- ▷ $\mathrm{cg}(X)$ is an **acyclic extension** of the free $(2,1)$-category $X_2^\top$.
- ▷ $(X_1, X_2, \mathrm{cg}(X))$ is a **coherent presentation** of the monoid X_1^*/X_2.

- **Homotopical completion-reduction procedure** to compute coherent presentations (Guiraud-M.-Mimram, 2013).

Coherent presentations: examples

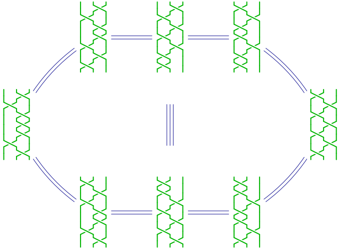
- The **Artin monoid** B_3^+ of braids on three strands.



- **Artin presentation** of B_3^+

$$\mathrm{Art}_2(B_3^+) = \langle\, s, t \quad \big| \quad tst = sts \rangle$$

- We prove that there is no syzygy between relations induced by $tst = sts$.



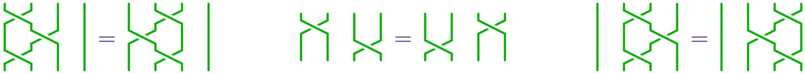
With presentation $\mathrm{Art}_2(B_3^+)$
two proofs of the same equality in B_3^+
are equal.

Coherent presentations: examples

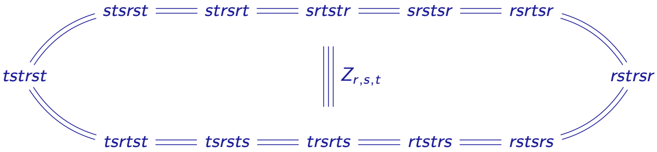
▶ The **Artin monoid** B_4^+ of braids on four strands.

▶ **Artin presentation** of B_4^+

$$\mathrm{Art}_2(B_4^+) = \langle\, r, s, t \;\mid\; rsr = srs,\;\; rt = tr,\;\; tst = sts \,\rangle$$



▶ The syzygies of the braid relations on four strands are generated by the **Zamolodchikov relation** (Deligne, 1997).



▶ The Artin monoid $B^+(W)$ on a Coxeter group W with Garside's presentation, (Gaussent-Guiraud-M., 2015)

Higher order syzygies

- **Higher order syzygies** are relations between relations between relations, and so on.

- Higher order syzygy problem for a 1-category $\mathbf{C}$ (or a p-category)

Problem.

- *Given a presentation of $\mathbf{C}$ by generators and relations.*
- *We would like to build a (small !) **cofibrant approximation** of $\mathbf{C}$ in the category of ω-categories (or $(\omega, 1)$-categories).*

- That is, a free ω-category homotopically equivalent to $\mathbf{C}$.

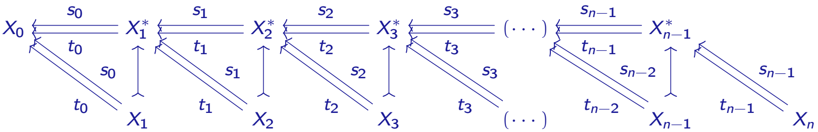
- Squier's machinery extends to higher dimensions.
  - Solution in $(\omega, 1)$-categories (Guiraud-M., 2012).
  - Still an open problem in ω-categories.

Polygraphic resolutions

- An n-**polygraph** is a sequence

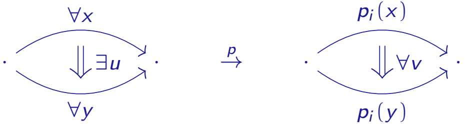
$$X = (X_0, X_1, \ldots, X_n)$$

constructed by induction



- An ω-functor $p : C \to D$ is an **acyclic fibration** if
  - $p_0 : C_0 \to D_0$ is onto,
  - p has the **lifting property**:

$$\forall x || y \in C_i, \;\; p_i(x) \xrightarrow{\forall\, v} p_i(y) \in D_{i+1}, \;\; x \xrightarrow{\exists\, u} y \in C_{i+1} \;\middle|\; p_{i+1}(u) = v$$



- A **polygraphic resolution** of an ω-category C is an acyclic fibration

$$p : X^* \to C$$

where X^* is a free ω-category on an ω-polygraph X.

Algebraic formulation problems on polygraphic resolutions

Problem.

How can polygraphic resolutions be algebraically formulated?
With a view to formalization in proof assistants?

Issues.

◀ 1 ▶ Algebraic formulation of the structure of polygraphs (higher dimensional rewriting):

- ▷ Abstraction of **shapes**: globular, cubical, simplicial...
- ▷ **Homotopical properties**: acyclicity, contracting homotopies, normalisation strategies...

◀ 2 ▶ Algebraic formulation of the calculation machinery of syzygies by rewriting.

- ▷ Abstraction of **diagrammatic reasoning**: confluence, termination...
- ▷ Church-Rosser, Newman, and Squier machineries...

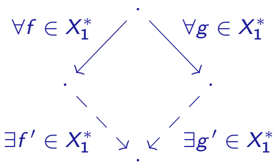
◀ 3 ▶ The formalisation in proof assistants.

- ▷ Isabelle... **see Georg Struth's talk tomorrow morning.**

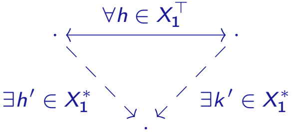
I. Calculating confluence proofs in modal Kleene algebras

Church-Rosser Theorem (diagrammatic formulation)

- An **abstract rewriting system** is a 1-polygraph (X_0, X_1)
  - It is **confluent** if



  - It has the **Church-Rosser property** if



Theorem. (Church-Rosser, 1936)

A 1-polygraph is confluent if and only if it is Church-Rosser.

Theorem. (Newman, 1942)

A terminating 1-polygraph is confluent if and only if it is locally confluent.

Church-Rosser Theorem (algebraic formulation)

- A **Kleene algebra** is a dioid (idempotent semiring) $(K, +, 0, \cdot, 1)$ equipped with a Kleene star operation $(-)^* : K \to K$ satisfying unfold and induction axioms.

- The **path Kleene algebra** on a 1-polygraph X is the structure

$$K(X) := (\mathcal{P}(X_1^*), \cup, \odot, \emptyset, \mathbb{1}, (-)^*).$$

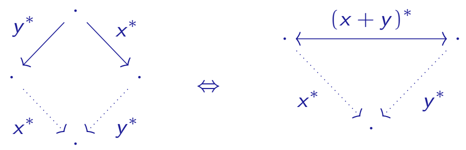
  - Composition of φ and ψ in $\mathcal{P}(X_1^*)$:

$$\varphi \odot \psi := \{\, u \star_0 v \mid u \in \varphi \wedge v \in \psi \wedge t_0(u) = s_0(v) \,\}.$$

  - $\mathbb{1}$ is the set of all identity arrows of X.
  - The operation $(-)^*$ is defined by $\varphi^* = \bigcup_{i \in \mathbb{N}} \varphi^i$, with $\varphi^0 = \mathbb{1}$ and $\varphi^{i+1} = \varphi \odot \varphi^i$.

Theorem. (Church-Rosser Theorem *à la* Struth, 2002)
For all x, y in a Kleene algebra

$$y^* x^* \leqslant x^* y^* \;\Leftrightarrow\; (x+y)^* \leqslant x^* y^*.$$



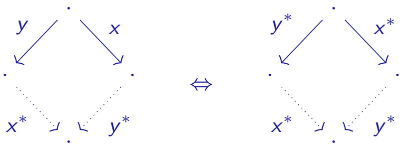
Newman's Theorem (algebraic formulation)

- Algebraic notion of termination (Desharnais-Möller-Struth, 2011).
  - **Modal Kleene algebra tests**.

Theorem. (Desharnais-Möller-Struth, 2004)

In a modal Kleene algebra K with complete test algebra, if $x + y$ is Noetherian, then

$$\langle y||x\rangle \leqslant |x^*\rangle\langle y^*| \;\Leftrightarrow\; \langle y^*||x^*\rangle \leqslant |x^*\rangle\langle y^*|.$$

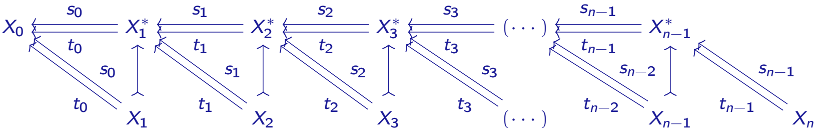


Objective. To give a coherent formulation of these two algebraic results for higher dimensional rewriting systems.

II. Coherence by rewriting

Polygraphs

- Consider an n-polygraph $X = (X_0, X_1, \dots, X_n)$



- It induces an ARS on the free $(n-1)$-category X^*_{n-1}, whose rules are

$$C[s_{n-1}(\alpha)] \xrightarrow{C[\alpha]} C[t_{n-1}]$$

with $s_{n-1} \overset{\alpha}{\to} t_{n-1}$ an n-generator in X_n and C a context:

$$C[\square] = f_n \star_{n-1} (f_{n-1} \star_{n-2} \cdots (f_1 \star_0 \square \star_0 g_1) \cdots \star_{n-2} g_{n-1}) \star_{n-1} g_n,$$

where f_k and g_k are identities k-cells.

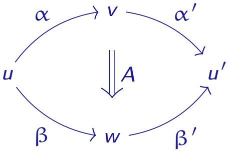
- We extend the (abstract) rewriting properties on X:

termination / **confluence** / **locally confluence** / **convergence**.

Squier's completion

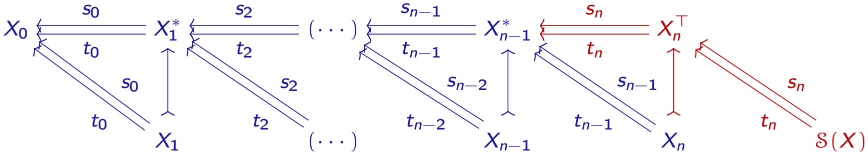
- Let X be a convergent n-polygraph.

- A **family of generating confluences** of X is a cellular extension of the $(n, n-1)$-category $X_n^\top$ that contains exactly one $(n+1)$-cell



for every critical branching (α, β) of X.

- A **Squier's completion** of X is a $(n+1, n-1)$-polygraph

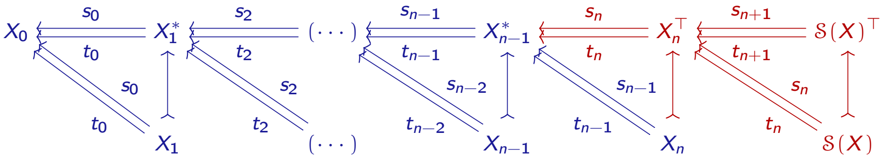


where $\mathcal{S}(X)$ is a chosen family of generating confluences of X

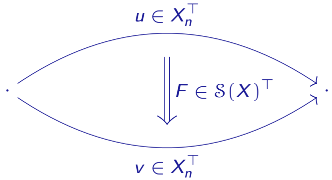
Squier's completion and finite derivation type

Theorem. (Guiraud-M. 2009)

If X is convergent, then the Squier completion $\mathcal{S}(X)$ is acyclic.

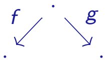


$$\forall u || v \in X_n^\top, \quad \exists u \overset{F}{\Rightarrow} v \in \mathcal{S}(X)^\top \quad | \quad s_n(F) = u \text{ and } t_n(F) = v$$



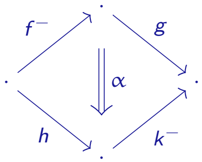
- The proof relies on the following two coherent confluent results:
  - Coherent Newman's lemma.
  - Coherent Church-Rosser theorem.

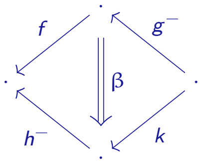
Coherent confluence

- Let Γ be a cellular extension of $X_n^\top$.
- Consider the free $(n+1)$-category $X_n^\top[\Gamma]$ with invertible k-cells only for $k = n$.
- Γ is a **confluence filler of a branching** 

of X

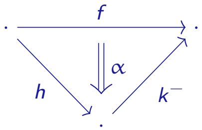
if there exist n-cells h, k in X_n^*, and $(n+1)$-cells α, β in $X_n^\top[\Gamma]$ with shapes:





- Γ is a **confluence filler** of an n-cell f in $X_n^\top$

if there exist n-cells h, k in X_n^* and an $(n+1)$-cell α in $X_n^\top[\Gamma]$ of the shape:



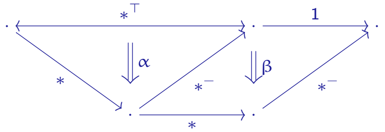
Coherent confluence

Theorem. (Coherent Church-Rosser filler lemma)

Let X be an n-polygraph, and Γ a cellular extension of $X_n^\top$. Then

$$\Big(\ \Gamma \text{ is a confluence filler of } X\ \Big) \quad \Leftrightarrow \quad \Big(\ \Gamma \text{ is a Church-Rosser filler of } X\ \Big)$$

Proof.

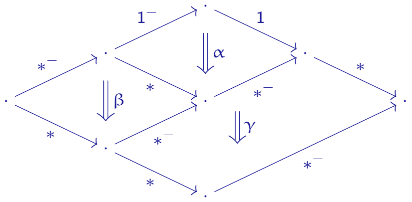


Theorem. (Coherent Newman filler lemma)

Let X be a terminating n-polygraph, and Γ a cellular extension of $X_n^\top$. Then

$$\Big(\ \Gamma \text{ is a local confluence filler of } X\ \Big) \quad \Leftrightarrow \quad \Big(\ \Gamma \text{ is a confluence filler of } X\ \Big)$$

Proof.



III. Calculating coherent proofs in higher modal Kleene algebras

Higher dioids

- The domain algebra of the path Kleene algebra $K(X)$ on a 1-polygraph X forms a bounded distributive lattice.

- A 0-**dioid** is a bounded distributive lattice:

- A 1-**dioid** is a dioid $(S, +, 0, \odot, 1)$.

- For $n \geqslant 1$, an n-**dioid** is a structure $(S, +, 0, \odot_i, 1_i)_{0 \leqslant i < n}$ such that
  - $(S, +, 0, \odot_i, 1_i)$ is a dioid for $0 \leqslant i < n$,
  - The **lax interchange laws** hold, for all $0 \leqslant i < j < n$,

$$(x \odot_j x') \odot_i (y \odot_j y') \leqslant (x \odot_i y) \odot_j (x' \odot_i y'),$$

  - Higher units are idempotents of lower multiplications, for all $0 \leqslant i < j < n$,

$$1_j \odot_i 1_j = 1_j.$$

Higher modal semirings

- An **antidomain 0-semiring** is a 0-dioid.
- For $n \geqslant 1$, an **antidomain n-semiring** is an n-dioid $(S, +, 0, \odot_i, 1_i)_i$ equipped with antidomain maps $(ad_i : S \to S)_{0 \leqslant i < n}$ such that
  - $(S, +, 0, \odot_i, 1_i, ad_i)$ is an antidomain semiring:
  $$ad_i(x)x = 0, \quad ad_i(xy) \leqslant ad_i(x\, ad^2(y)), \quad ad_i^2(x) + ad_i(x) = 1.$$
  - $ad_{i+1} \circ ad_i = ad_i$.
- An **anticodomain n-semiring** is an n-dioid S such that $(S_i^{op})_{0 \leqslant i < n}$ is a antidomain n-semiring. Denote $(ar_i : S \to S)_{0 \leqslant i < n}$ the codomain operators.
- A **Boolean modal n-semiring** is an antidomain n-semiring that is also an anticodomain n-semiring for $n \geqslant 1$, and a Boolean algebra for $n = 0$.
- Note that, setting $d_i := ad_i^2$ and $r_i := ar_i^2$, we recover a domain and codomain n-semirings.

Modal n-Kleene algebra

- An **n-Kleene algebra** (n-KA) is an n-dioid K equipped with operations $(-)^{*_i} : K \to K$ such that
  - $(K, +, 0, \odot_i, 1_i, (-)^{*_i})$ is a KA for $0 \leqslant i < n$,
  - For $0 \leqslant i < j < n$, the operation $(-)^{*_j}$ is a lax morphism wrt i-whiskering of j-dimensional elements:

$$\varphi \odot_i A^{*_j} \leqslant (\varphi \odot_i A)^{*_j} \qquad A^{*_j} \odot_i \varphi \leqslant (A \odot_i \varphi)^{*_j}$$

for all $A \in K$, $\varphi \in K_j$.

- A **modal n-Kleene algebra** (n-MKA) is an n-KA that is a modal n-semiring (domain and codomain semiring).
- A **Boolean n-MKA** is an n-KA that is a Boolean modal n-semiring.

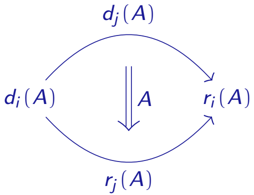
Globular Kleene algebras

- An n-MKA K is **globular** (n-gMKA) if the **globular relations** hold for $0 \leqslant i < j < n$:

$$d_i \circ d_j = d_i, \qquad d_i \circ r_j = d_i, \qquad r_i \circ d_j = r_i, \qquad r_i \circ r_j = r_i,$$

$$d_j(A \odot_i B) = d_j(A) \odot_i d_j(B), \qquad r_j(A \odot_i B) = r_j(A) \odot_i r_j(B).$$

- An element A in K is a collection of cells, and for $i < j$:



  - ▷ $d_k(A)$ is the set of k-cells that are k-sources of some cells belonging to A.
  - ▷ $r_k(A)$ is the set of k-cells that are k-targets of some cells belonging to A.

- The **right** and **left** **i-whiskering** of $A \in K$ by $\varphi \in K_j$ is

$$A \odot_i \varphi \qquad \text{and} \qquad \varphi \odot_i A$$

Confluence fillers

- Let K be an n-gMKA and $0 \leqslant j < n$.
- Define the **forward j-diamond** operators, for all $A \in K$ and $\varphi \in K_j$,

$$|A\rangle_j(\varphi) := d_j(A \odot_j \varphi).$$

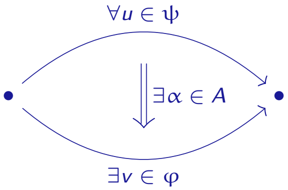
- Thus, for $A \in K$ and $\varphi, \psi \in K_j$, we have

$$|A\rangle_j(\varphi) \geqslant \psi \quad \text{iff} \quad d_j(A \odot_j \varphi) \geqslant \psi.$$

  - In the polygraphic model:

$$\forall u \in \psi,\ \exists v \in \varphi \text{ and } \exists \alpha \in A \text{ such that } s_j(\alpha) = u \text{ and } t_j(\alpha) = v.$$

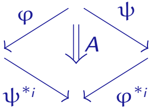
$$|A\rangle_j(\varphi) \geqslant \psi$$



Confluence fillers

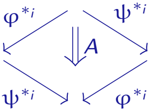
- Let $0 \leqslant i < j < n$, and φ, ψ in K_j. An element A in K is a
  - **local i-confluence filler** for (φ, ψ) if

$$|A\rangle_j(\psi^{*_i} \odot_i \varphi^{*_i}) \geqslant \varphi \odot_i \psi$$



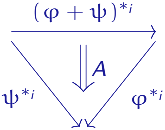
  - **i-confluence filler** for (φ, ψ) if

$$|A\rangle_j(\psi^{*_i} \odot_i \varphi^{*_i}) \geqslant \varphi^{*_i} \odot_i \psi^{*_i}$$



  - **i-Church-Rosser filler** for (φ, ψ) if

$$|A\rangle_j(\psi^{*_i} \odot_i \varphi^{*_i}) \geqslant (\psi + \varphi)^{*_i}$$



- Note that $(\psi + \varphi)^{*_i} \geqslant \varphi^{*_i} \odot_i \psi^{*_i} \geqslant \varphi \odot_i \psi$.

Completion fillers

- Coherent proofs are obtained using **completion by fillers**.
- Completion of an i-confluence filler A of a pair (φ, ψ) in K_j:
  - The **j-dimensional i-whiskering of A**

$$(\varphi + \psi)^{*_i} \odot_i A \odot_i (\varphi + \psi)^{*_i} \in K$$

  - The **i-whiskered j-completion of A**, denoted by $\hat{A}^{*_j}$, is

$$\big((\varphi + \psi)^{*_i} \odot_i A \odot_i (\varphi + \psi)^{*_i}\big)^{*_j} \in K$$

Coherent Church-Rosser and Newman in globular MKA

Theorem A. (Calk-Goubault-M.-Struth, 2023)

Let K be an n-gMKA and $0 \leqslant i < j < n$.
Let $\varphi, \psi \in K_j$.
If A is an i-confluence filler of (φ, ψ), then

$$|\hat{A}^{*_j}\rangle_j(\psi^{*_i} \odot_i \varphi^{*_i}) \geqslant (\varphi + \psi)^{*_i},$$

that is, the completion $\hat{A}^{*_j}$ is an i-Church-Rosser filler for (φ, ψ).

Theorem B. (Calk-Goubault-M.-Struth, 2023)

Let K be a Boolean n-gMKA, and $0 \leqslant i < j < n$, such that

- $(K_i, +, 0, \odot_i, 1_i, \neg_i)$ *is a complete Boolean algebra,*
- K_j *is i-continuous.*

Let $\psi \in K_j$ be i-Noetherian and $\varphi \in K_j$ i-well-founded.
If A is a local i-confluence filler for (φ, ψ), then

$$|\hat{A}^{*_j}\rangle_j(\psi^{*_i} \odot_i \varphi^{*_i}) \geqslant \varphi^{*_i} \odot_i \psi^{*_i},$$

that is, the completion $\hat{A}^{*_j}$ is a confluence filler for (φ, ψ).

Polygraphic model of higher Kleene algebras

- Let (X, Γ) be an $(n+1, n-1)$-polygraph.
- Define $K(X, \Gamma)$ the **full path** $(n+1)$**-MKA**:

$$K(X) := \mathcal{P}\big(X^*_{n-1}(X_n)[\Gamma]\big),$$

  - Composition of A and B in $K(X)$:

$$A \odot_i B := \big\{\alpha \star_i \beta \mid \alpha \in A \wedge \ \beta \in B \wedge t_i(\alpha) = s_i(\beta)\big\}.$$

  - Unit for $\odot_i$

$$\mathbb{1}_i \ = \ \big\{\iota_i^{n+1}(u) \mid u \in X^*_{n-1}(X_n)[\Gamma]_i\big\}.$$

  - Addition is the set union $\cup$, and the ordering is the set inclusion.
  - i-domain and i-codomain maps:

$$d_i(A) := \big\{\iota_i^{n+1}(s_i(\alpha)) \mid \alpha \in A\big\}, \qquad r_i(A) := \big\{\iota_i^{n+1}(t_i(\alpha)) \mid \alpha \in A\big\}.$$

  - i-antidomain and i-anticodomain maps:

$$ad_i(A) := \mathbb{1}_i \setminus \big\{\iota_i^{n+1}(s_i(\alpha)) \mid \alpha \in A\big\}, \qquad ar_i(A) := \mathbb{1}_i \setminus \big\{\iota_i^{n+1}(t_i(\alpha)) \mid \alpha \in A\big\}.$$

  - The i-star is $A^{*_i} = \bigcup_{k\in\mathbb{N}} A^{k_i}$, with $A^{0_i} := \mathbb{1}_i$ and $A^{k_i} := A \odot_i A^{(k-1)_i}$.

Polygraphic model of higher Kleene algebras

Proposition.

$K(X, \Gamma)$ is a Boolean $(n+1)$-gMKA.

Theorem. (Calk-Goubault-M.-Struth, 2023)

Theorems A & B in the polygraphic model give polygraphic coherent Church-Rosser and Newman filler results.

Conclusion:
IV. Work in progress

Three lines of research

◀ 1 ▶ Algebraic formulation of normalisation strategies.

- ▷ Normalisation strategies give constructive proofs of acyclicity in polygraphs (Guiraud-M., 2012).
- ▷ In low dimension, Squier's theorem for ARS using normalisation strategies in MKA (Calk-Goubault-M., 2021).
- ▷ Higher normalisation strategies in ω-quantales (M.-Struth, work in progress).

◀ 2 ▶ Algebraic formulation of cubical polygraphic resolutions (M.-Massacrier-Struth, work in progress).

- ▷ Cubical description of confluence properties.
- ▷ Functional definition of cubical categories and normalisation strategies.

◀ 3 ▶ Polygraphic resolutions for algebraic polygraphs (cartesian, linear, algebraic over an operad...), (Dabrowski-M.-Ren, work in progress).

- ▷ Formalisation of the coherent critical branching lemma (strings, terms, terms modulo).
- ▷ (Algebraically enriched) n-gMKA.

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