Confluence of linear rewriting and homology of algebras

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Joint work with Yves Guiraud and Eric Hoffbeck

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Motivation : homology of algebras

- Let $\mathbf{A}$ be an associative $\mathbb{K}$-algebra presented by generators X and relations R

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Applications.

- ▷ Non-commutative algebraic geometry (Artin-Schelter algebras, ...)
- ▷ Theoretical physics (Yang-Mills algebras, Calabi-Yau algebras, ...)
- ▷ Combinatorial algebras, ...

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  - We have

$$\mathrm{Im}(\delta_n \otimes_{\mathbf{A}} \mathrm{Id}_{\mathbb{K}}) \subset \mathrm{Ker}(\delta_{n-1} \otimes_{\mathbf{A}} \mathrm{Id}_{\mathbb{K}})$$

  - The ***n*th homology space** of $\mathbf{A}$ is defined by

$$\mathrm{H}_n(\mathbf{A}, \mathbb{K}) = \mathrm{Ker}(\delta_{n-1} \otimes_{\mathbf{A}} \mathrm{Id}_{\mathbb{K}}) / \mathrm{Im}(\delta_n \otimes_{\mathbf{A}} \mathrm{Id}_{\mathbb{K}})$$

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$\mathbf{A}_0 = \mathbb{K} \ni 1$, $\quad \mathbf{A}_1 = \mathbb{K}\langle X \rangle \ni x, y, x + y$, $\quad \mathbf{A}_2 \ni x^2, x^2 + y^2$, ...

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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
5	•	•	•	•	•	$\cdots$
4	•	•	•	•	•	$\cdots$
3	•	•	•	•	•	$\cdots$
2	•	•	•	•	•	$\cdots$
1	•	•	•	•	•	$\cdots$
0	•	•	•	•	•	$\cdots$
k	$\mathrm{H}_0(\mathbf{A}, \mathbb{K})$	$\mathrm{H}_1(\mathbf{A}, \mathbb{K})$	$\mathrm{H}_2(\mathbf{A}, \mathbb{K})$	$\mathrm{H}_3(\mathbf{A}, \mathbb{K})$	$\mathrm{H}_4(\mathbf{A}, \mathbb{K})$	$\cdots$

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Objective.

Describe the vector spaces $\mathrm{H}_n(\mathbf{A}, \mathbb{K})$ in term of ***n*-fold critical branching**.

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Plan

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I. Linear rewriting systems

II. Computation of polygraphic resolutions

III. Free resolutions of algebras

References.

- Y. Guiraud, E. Hoffbeck, P.M., Linear polygraphs and Koszulity of algebras, 2014, arXiv:1406.0815.

- Y. Guiraud, P.M., Higher-dimensional normalisation strategies for acyclicity, Advances in Mathematics, 2012, arXiv:1011.0558.

- Y. Guiraud, P.M., S. Mimram, A homotopical completion procedure with application to coherence of monoids, RTA 2013.

- Y. Guiraud, P.M. Polygraphs of finite derivation type, 2014, arXiv:1402.2587.

Part I. Linear rewriting systems

Linear rewriting and higher-dimensional rewriting

- **Gröbner bases**.
  - ▷ An algorithm for polynomial ideals.
    - — Solving word problems, computing normal forms, ...

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- **Linear polygraphs**
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- Consider an oriented graph, with $\Sigma_0 = \{\bullet\}$ (see G.-H.-M. '14 in the case of several objects)

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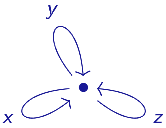
Example. $\Sigma_0 = \{\bullet\}$ and $\Sigma_1 = \{x, y, z\}$



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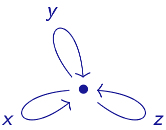

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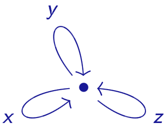

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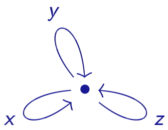

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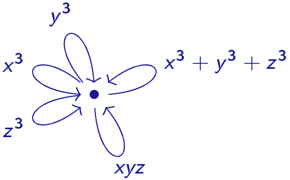

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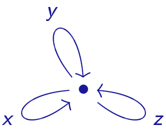

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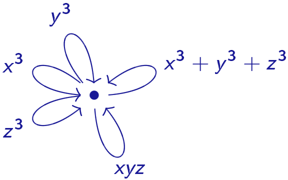
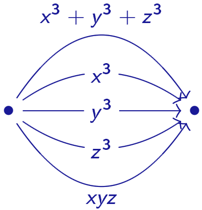
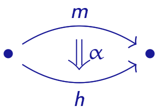




Linear rewriting systems

- A (monic) **rule** is a 2-cell of the form



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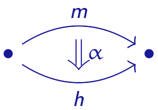

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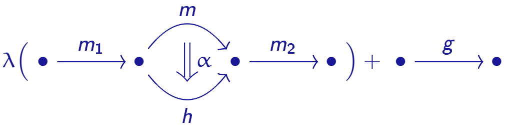

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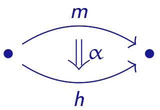

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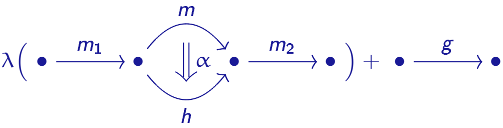

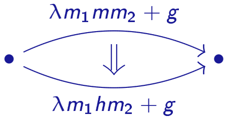

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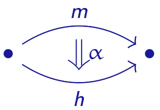

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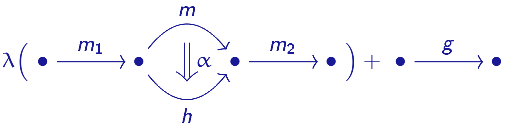

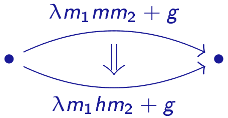

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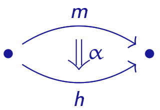

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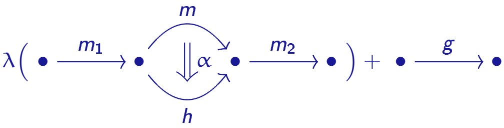

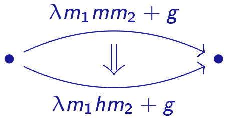

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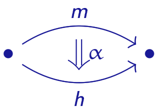

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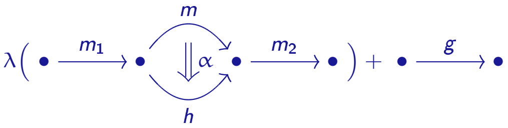

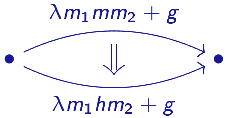

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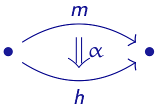

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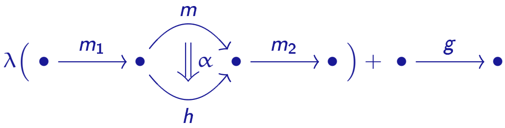

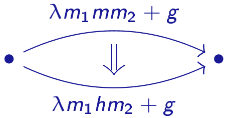

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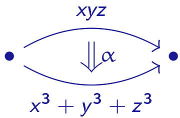

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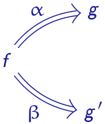

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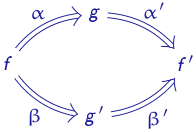

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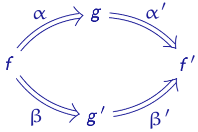

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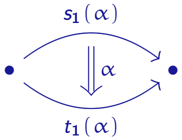

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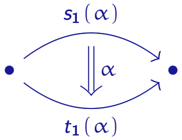

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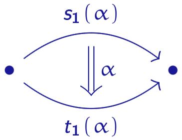

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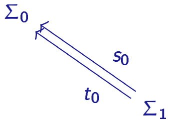

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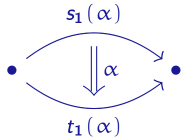

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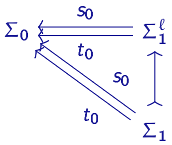

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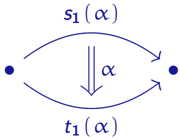
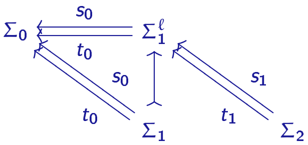


- We will denote a polygraph as a diagram:



The 2-category of reductions

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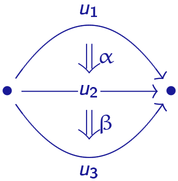

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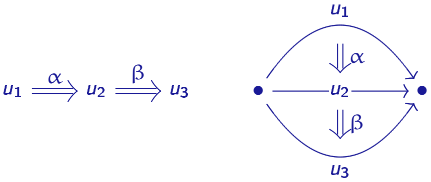
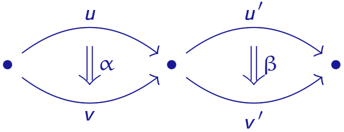


  - the **parallel composition** $\alpha \star_0 \beta$ of rewriting steps is associative and unitary



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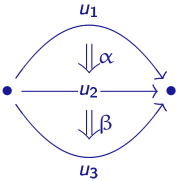

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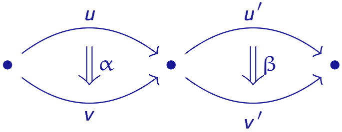
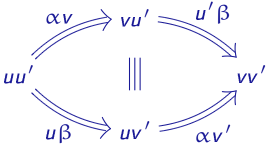


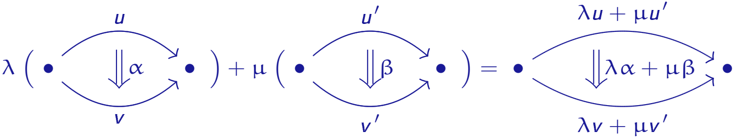
and compatible with the sequential composition (**exchange relations**)



$$\alpha v \star_1 u'\beta = u\beta \star_1 \alpha v'$$

The 2-algebroid of rewriting

- The linear structure on 1-cells induces a linear structure on 2-cells:



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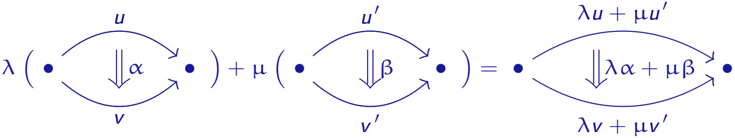

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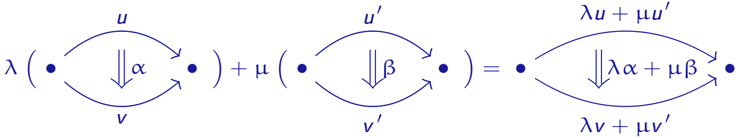

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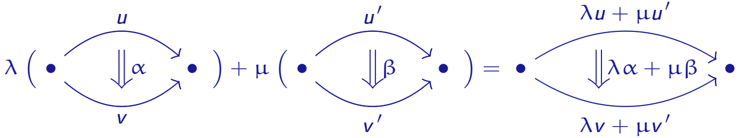

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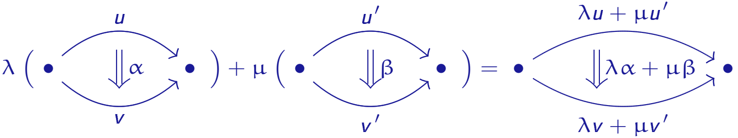

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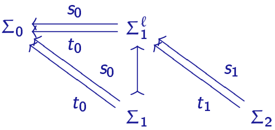

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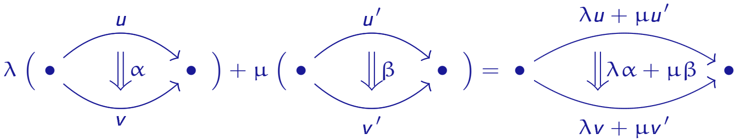

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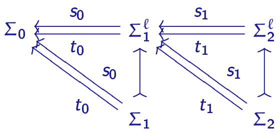

Part II. Computation of polygraphic resolutions

Polygraphic resolutions

Aim.

▷ Starting with a presentation of an algebra $\mathbf{A}$ by a linear 2-polygraph, we would like to compute a **small categorical globular model** for $\mathbf{A}$.

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▷ Apply homotopical reduction in order to obtain a smaller model.

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$$\big(\ \Sigma_0, \Sigma_1, \Sigma_2, \Sigma_3,\ \dots\ , \Sigma_n, \dots\ \big)$$

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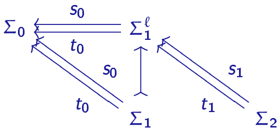

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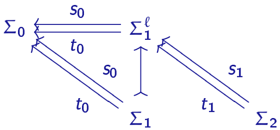

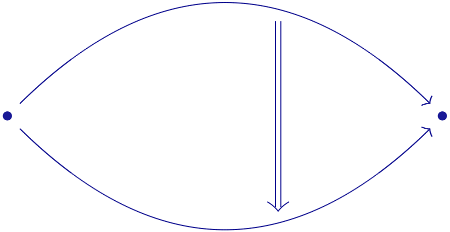

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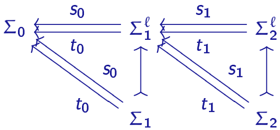

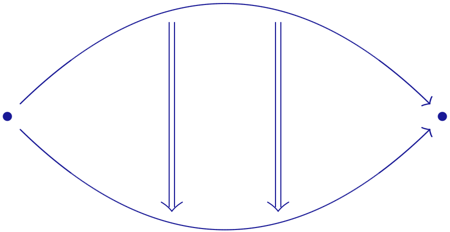

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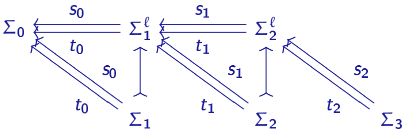

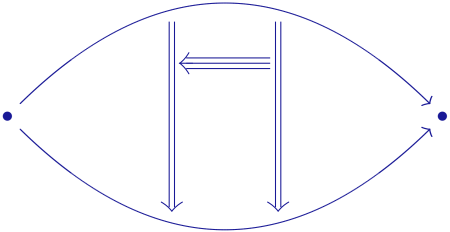

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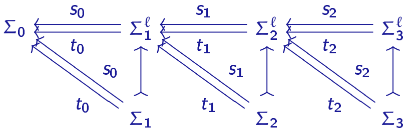

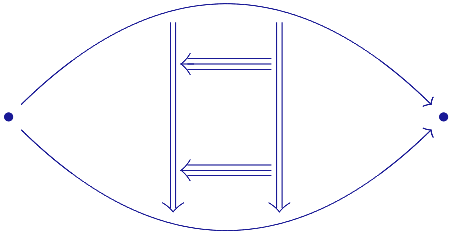

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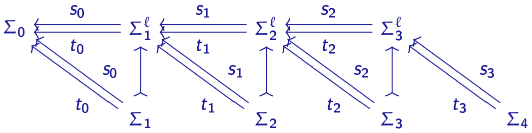

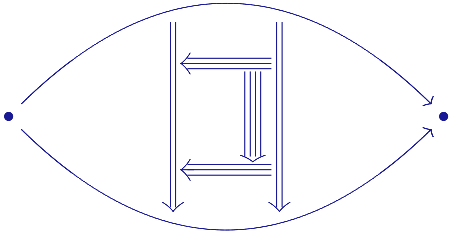

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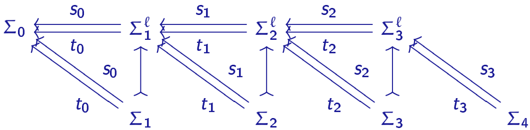

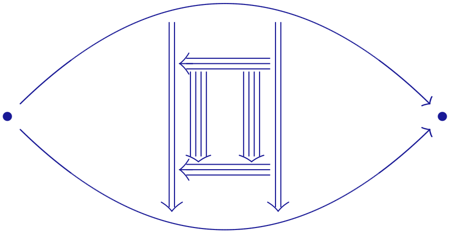

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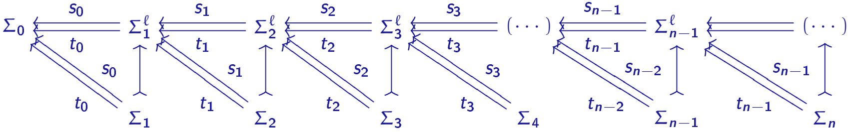
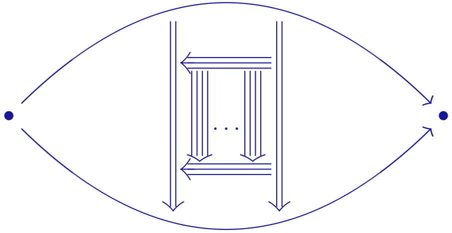
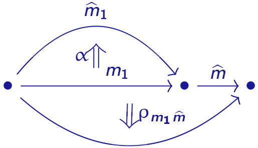




Computation of resolution in dimension 3

▶ Suppose that $(\Sigma_0, \Sigma_1, \Sigma_2)$ is a reduced convergent 2-polygraph. Consider the rightmost reduction strategy ρ.

▷ Any **critical branching** has the following shape:



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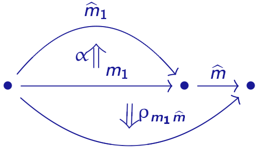

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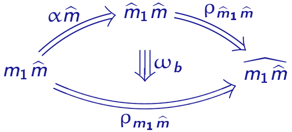

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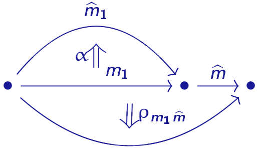

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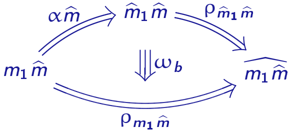

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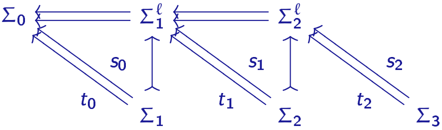
Proposition. The following linear 3-polygraph is acyclic



Computation of resolution in dimension 3 : example

Example.

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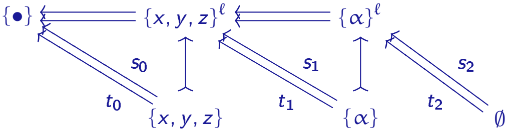

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$$yzy \overset{\alpha y}{\Longrightarrow} -x^2y \qquad\qquad zyz \overset{z\alpha}{\Longrightarrow} -zx^2$$

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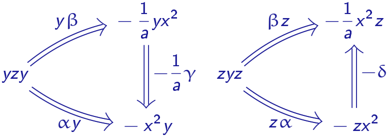

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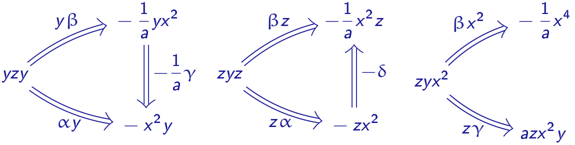

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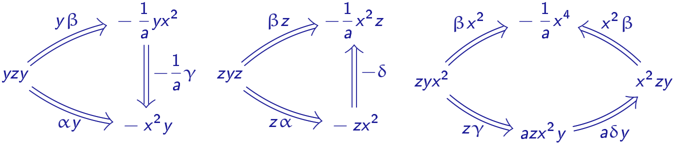

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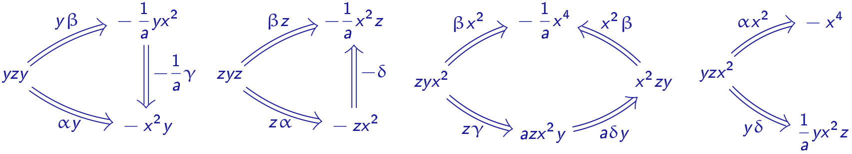

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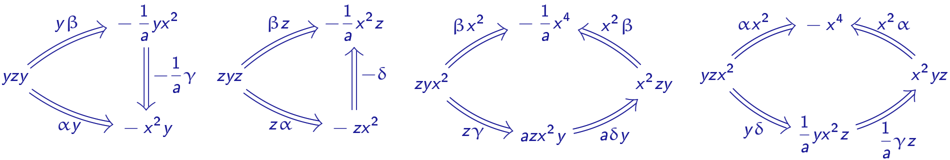

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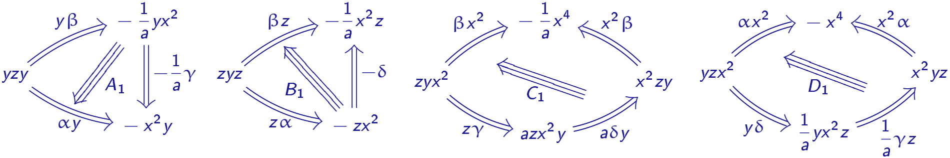

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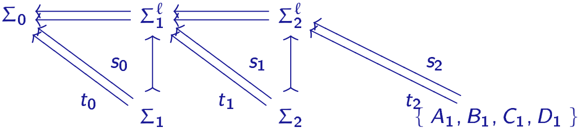
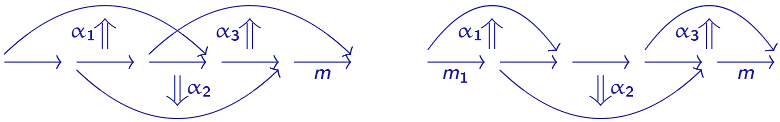


Higher-dimensional branchings

- A 3-fold branching is an overlapping of three rewriting steps:



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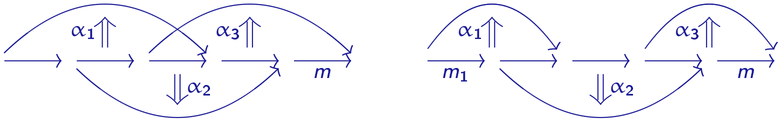
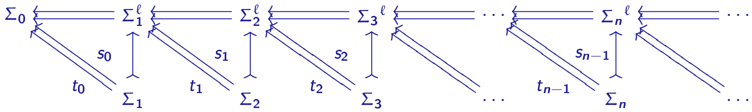


Theorem. (G.-H.-M., 2014)

Any convergent linear 2-polygraph $(\Sigma_0, \Sigma_1, \Sigma_2)$ extends to an acyclic linear ∞-polygraph Σ_∞



whose n-cells are indexed by the critical $(n-1)$-fold branchings.

Part III. Free resolutions of algebras

A free modules resolution

Theorem. (G.-H.-M., 2014)

- ▷ Let $\mathbf{A}$ be an algebra.
- ▷ Let Σ be a polygraphic resolution of $\mathbf{A}$.

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The complex of $\mathbf{A}$-modules

$$0 \longleftarrow \mathbf{A} \xleftarrow{\mu} \mathbf{A}[\Sigma_0] \xleftarrow{\delta_0} \mathbf{A}[\Sigma_1] \longleftarrow \dots \longleftarrow \mathbf{A}[\Sigma_k] \xleftarrow{\delta_k} \mathbf{A}[\Sigma_{k+1}] \longleftarrow \dots$$

- ▷ where $\mathbf{A}[\Sigma_k]$ is the free $\mathbf{A}$-module on Σ_k,
- ▷ the maps δ_k are defined by

$$\delta_1(u \otimes v) = uv, \qquad \delta_k[f] = [s_k(f)] - [t_k(f)].$$

is exact, that is

$$\mathrm{Im}\ \delta_n = \mathrm{Ker}\ \delta_{n-1}, \qquad \text{for all} \quad n \geqslant 0.$$

Koszul algebras

Definition.

A quadratic algebra **A** is **Koszul** if the vector spaces $\mathrm{H}_n(\mathbf{A}, \mathbb{K})$ are "concentrated on the diagonal"

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
5	0	0	0	0	0	$\cdots$
4	0	0	0	0	•	$\cdots$
3	0	0	0	•	0	$\cdots$
2	0	0	•	0	0	$\cdots$
1	0	•	0	0	0	$\cdots$
0	•	0	0	0	0	$\cdots$
k	$\mathrm{H}_0(\mathbf{A}, \mathbb{K})$	$\mathrm{H}_1(\mathbf{A}, \mathbb{K})$	$\mathrm{H}_2(\mathbf{A}, \mathbb{K})$	$\mathrm{H}_3(\mathbf{A}, \mathbb{K})$	$\mathrm{H}_4(\mathbf{A}, \mathbb{K})$	$\cdots$

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Theorem. (Anick, 1986, Green 1999)

An algebra having a presentation by a quadratic Gröbner basis is Koszul.

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An algebra having a presentation by a quadratic convergent linear 2-polygraph is Koszul.

Koszul algebras

Definition. (Berger 2001) A N-homogeneous algebra, with $N > 1$ is **Koszul** if

$$\mathrm{H}_n^{(i)}(\mathbf{A}, \mathbb{K}) = 0, \qquad \text{for } i \neq \ell_N(n),$$

where

- n refers to the homological degree and (i) refers to the length grading,
- ℓ_N is the weight function defined by:

$$\ell_N(n) = \begin{cases} lN & \text{if } n = 2l, \\ lN + 1 & \text{if } n = 2l + 1. \end{cases}$$

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Convergence and Koszulity

Example. The algebra

$$\mathbf{A}\langle\, x, y, z \mid xyz = x^3 + y^3 + z^3 \,\rangle$$

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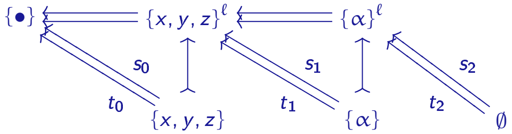

  - The homology of **A**

⋮	⋮	⋮	⋮	⋮	
3	0	0	0	0	⋯
2	0	0	$\mathbb{K}$	0	⋯
1	0	$\mathbb{K}^3$	0	0	⋯
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Example. (Backelin 1991, Polishchuk-Positselski 2005)

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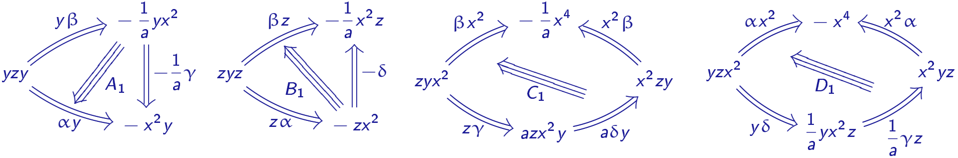

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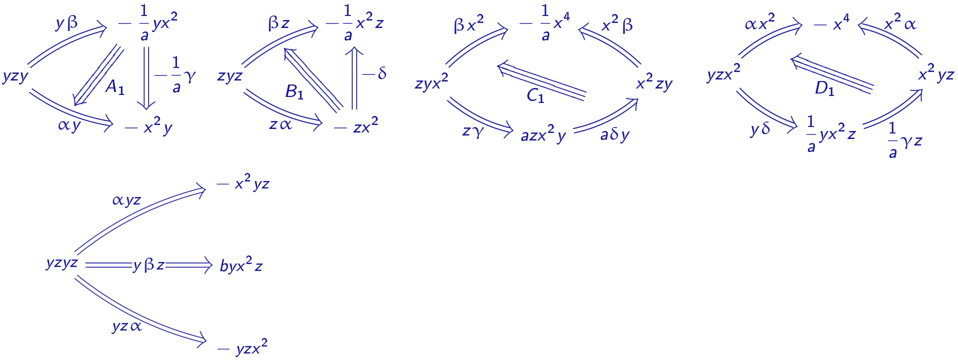

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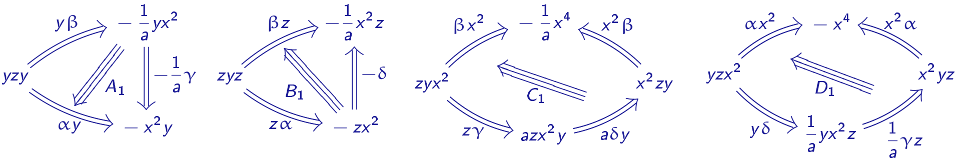

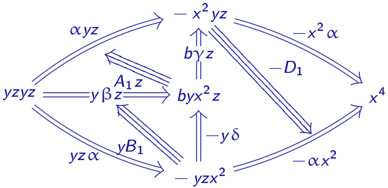

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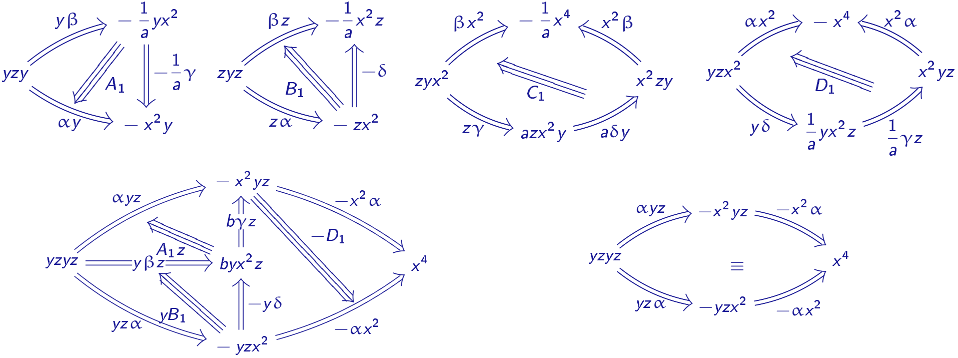

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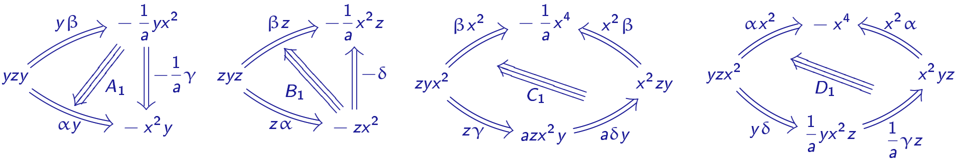

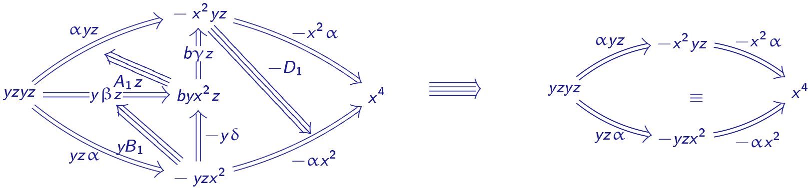

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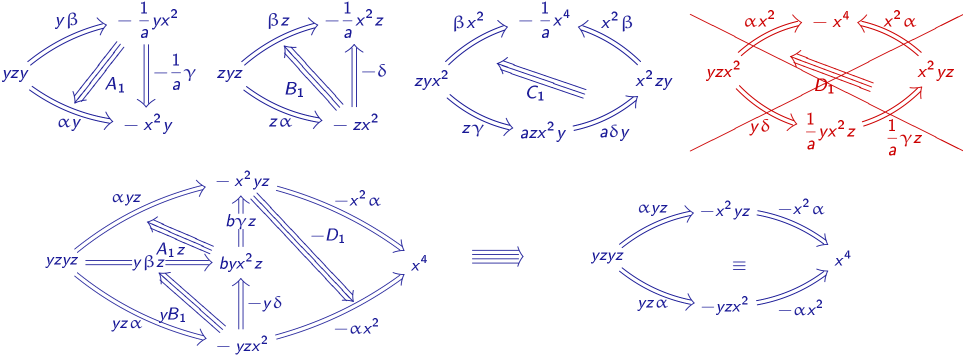

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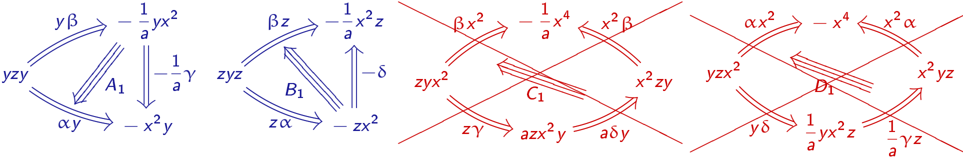

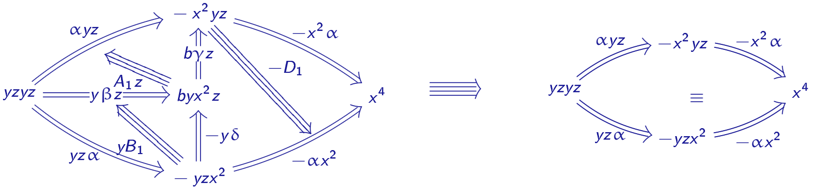

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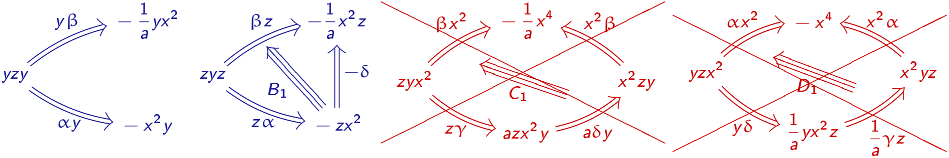

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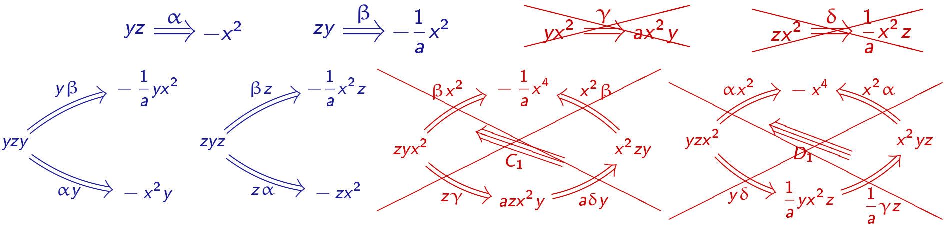

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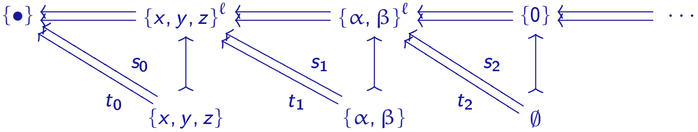
▶ From the polygraphic resolution



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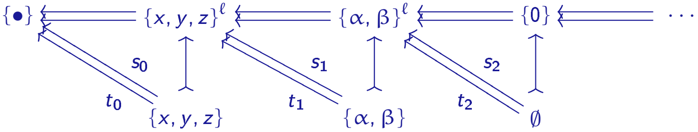

▷ we deduce that the algebra **A** is Koszul,

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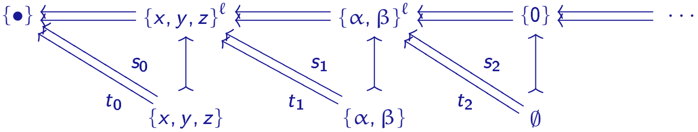
$$\mathbf{A}\langle\, x, y, z \mid x^2 + yz = 0,\ x^2 + azy = 0 \,\rangle \qquad \text{with } a \neq 0, 1$$

▶ From the polygraphic resolution



▷ we deduce that the algebra **A** is Koszul,

▷ the homology of **A**

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
4	0	0	0	0	$\cdots$
3	0	0	0	0	$\cdots$
2	0	0	$\mathbb{K}^2$	0	$\cdots$
1	0	$\mathbb{K}^3$	0	0	$\cdots$
0	$\mathbb{K}$	0	0	0	$\cdots$
k	$\mathrm{H}_0(\mathbf{A}, \mathbb{K})$	$\mathrm{H}_1(\mathbf{A}, \mathbb{K})$	$\mathrm{H}_2(\mathbf{A}, \mathbb{K})$	$\mathrm{H}_3(\mathbf{A}, \mathbb{K})$	$\cdots$

Example

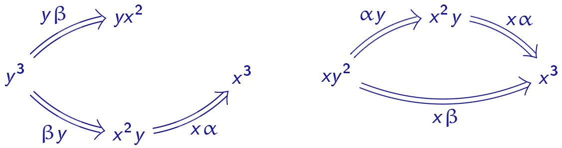
Example.

$$\mathbf{A}\langle\, x, y \mid x^2 = y^2 = xy \,\rangle,$$

▷ Consider the presentation by the 2-polygraph

$$xy \overset{\alpha}{\Longrightarrow} x^2\ , \qquad y^2 \overset{\beta}{\Longrightarrow} x^2$$

▷ There are two critical pairs

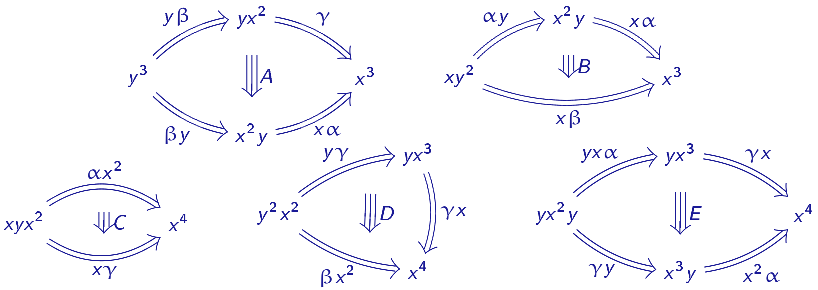


▷ We obtain a convergent 2-polygraph by adding the rule

$$yx^2 \overset{\gamma}{\Longrightarrow} x^3$$

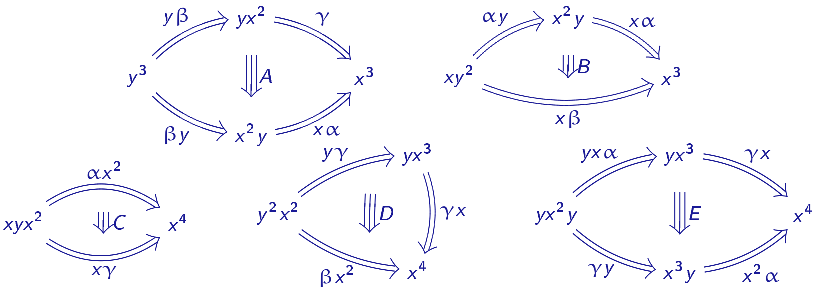
Example

- The following 3-cells form an homotopy basis:



Example

- The following 3-cells form an homotopy basis:



- There are seven critical triples on the following 1-cells:

$$xyx^2y, \quad xy^2x^2, \quad xy^3, \quad yx^2yy, \quad y^2x^2y, \quad y^3x^2, \quad y^4.$$

Example

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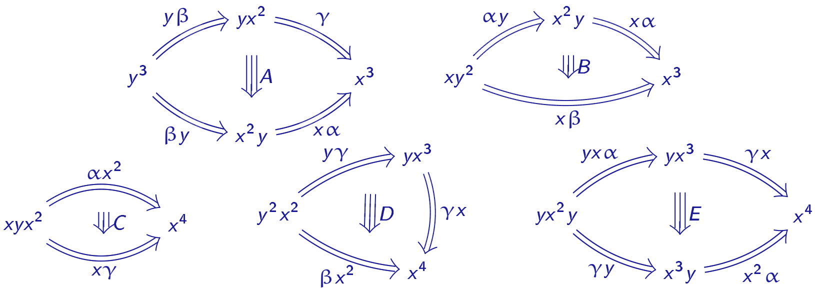

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▷ Only the 4-cell on 3th or 7th 1-cells could relate the 3-cells C, D and E without whisker in term of A and B.

▷ It is not enough to eliminate the 3-cells C, D and E, hence **A** is not Koszul.

Example

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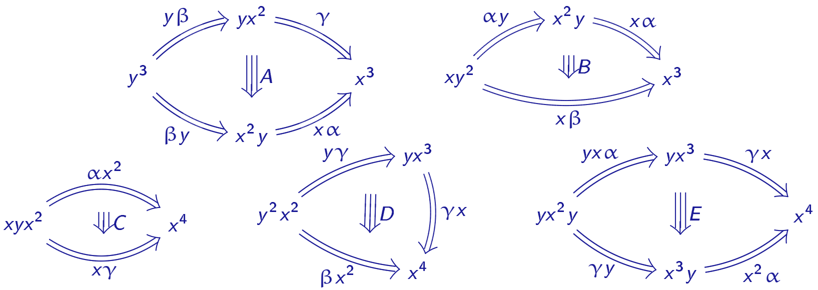

- Note that

k	$\mathrm{H}_0(\mathbf{A}, \mathbb{K})$	$\mathrm{H}_1(\mathbf{A}, \mathbb{K})$	$\mathrm{H}_2(\mathbf{A}, \mathbb{K})$	$\mathrm{H}_3(\mathbf{A}, \mathbb{K})$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
4	0	0	0	$\mathbb{K}$	$\cdots$
3	0	0	0	$\mathbb{K}$	$\cdots$
2	0	0	$\mathbb{K}^2$	0	$\cdots$
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0	$\mathbb{K}$	0	0	0	$\cdots$