A Homotopical Completion Procedure with Applications to Coherence of Monoids

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Rewriting Techniques and Applications

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Plan

I. Motivation

II. Coherent presentations

III. Homotopical completion and reduction procedure

IV. Applications

Plan

Motivation

- Computation of homological and homotopical invariants:
  - Homological and homotopical finiteness conditions for convergence (Squier, '87, '94),
  - Higher-dimensional categories with finite derivation type (G.-M., '09).

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- Classification of normalisation strategies:
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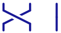
- Coherence theorems for algebraic and categorical structures:
  - Monoidal categories (MacLane, '63, G.-M., '12),
  - A homotopical completion procedure (G.-M.-M., '13),
  - Coherent presentations of Artin groups (Deligne, '72, Tits '81, Gaussent-G.-M., '13, arXiv :1203.5358v2).

Motivation

- The **positive braids** on 3 strands are generated by

 and 

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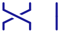 and 

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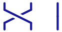 and 

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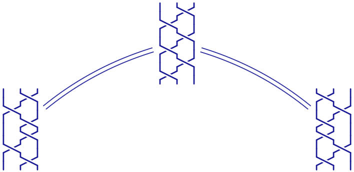

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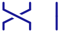 and 

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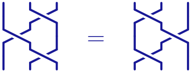

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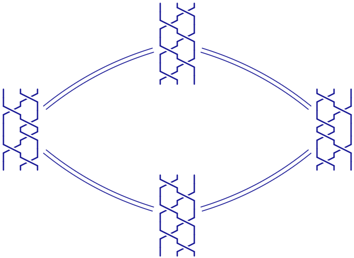

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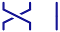

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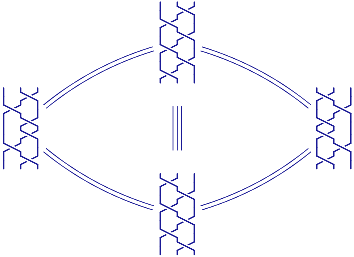

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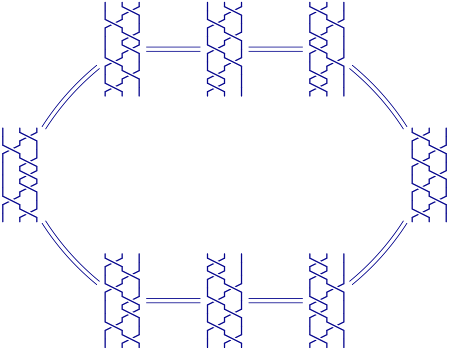

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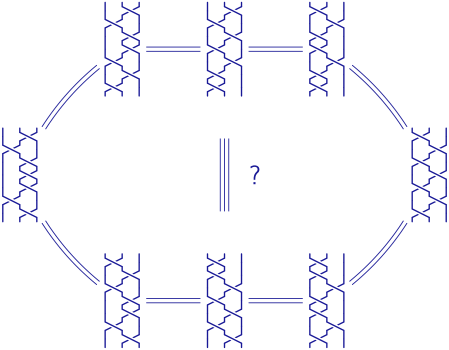

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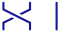
- How to compare proofs of equalities of two braids ?

Motivation

Method.

1. Consider **Artin's presentation** of the monoid $\mathbf{B}_3^+$ of positive braids on 3 strands

$$\Sigma_{\mathrm{Artin}} = \langle\ s, t \mid tst \overset{\rho}{\Rightarrow} sts\ \rangle$$

where $s =$  and $t =$ 

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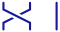
where $s =$  and $t =$ 

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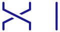
where $s =$  and $t =$ 

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Proposition.
For the presentation Σ_{Artin} of $\mathbf{B}_3^+$ any two proofs of the same equality are equal.

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  - the **parallel composition** of rewriting steps

$$\begin{tikzcd} u \arrow[d, Rightarrow, "f"] \arrow[r, phantom, "\star_0"] & u' \arrow[d, Rightarrow, "g"] \\ v & v' \end{tikzcd}$$

is associative, unitary and **compatible with the sequential composition**

$$\begin{tikzcd} & vu' \arrow[dr, Rightarrow, bend left, "u'g"] \arrow[dd, phantom, "\equiv"] & \\ uu' \arrow[ur, Rightarrow, bend left, "fv"] \arrow[dr, Rightarrow, bend right, "ug"'] & & vv' \\ & uv' \arrow[ur, Rightarrow, bend right, "fv'"'] & \end{tikzcd}$$

$$fv \star_1 u'g = ug \star_1 fv'$$

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- An **extended presentation** consists of a presentation $\langle\, \Sigma_1 \mid \Sigma_2 \,\rangle$, together with a set Σ_3 of **coherence generators** with shape

$$\begin{tikzcd}[column sep=large] u \arrow[r, bend left, Leftrightarrow] \arrow[r, bend right, Leftrightarrow] \arrow[r, phantom, "\Downarrow A"] & v \end{tikzcd}$$

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$$\begin{tikzcd} u \arrow[rr, Leftrightarrow, bend left=40] \arrow[rr, Leftrightarrow, bend right=40] \arrow[rr, phantom, "\Downarrow A"] & & v \end{tikzcd}$$

• A **coherent presentation** is an extended presentation $\langle\, \Sigma_1 \mid \Sigma_2 \mid \Sigma_3 \,\rangle$ such that the congruence generated by Σ_3 , that is

- the equivalence relation $\equiv$ on parallel 2-cells in $\Sigma_2^\top$,
- closed by context: $f \equiv g$ implies $ufv \equiv ugv$,
- closed by composition: $f \equiv g$ implies $k \star_1 f \star_1 h \equiv k \star_1 g \star_1 h$.

contains every pair of parallel 2-cells.

Coherent presentations

Problems.

1. How to compute a coherent presentation ?
2. How to transform a coherent presentation ?

Computing coherent presentations

- A **family of generating confluences** of $\langle\, \Sigma_1 \mid \Sigma_2 \,\rangle$ is a set of coherence generators of shape

$$\begin{tikzcd}
 & v \arrow[dr, Rightarrow, bend left, "f'"] \arrow[dd, phantom, "\Downarrow A_{f,g}"] & \\
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Theorem. *(Squier, 1994) Let Σ be a convergent presentation of a monoid $\mathbf{M}$.*
The extended presentation defined by a chosen family of generating confluences is a coherent and convergent presentation of $\mathbf{M}$.

Tietze transformations of coherent presentations

- **add a generator**: for a word u,

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$$\begin{tikzcd} u \arrow[r, Rightarrow, dotted, "\cancel{\alpha}"] & \cancel{x} \end{tikzcd}$$

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Theorem.

If Σ is a coherent presentation of a monoid $\mathbf{M}$, then for any Tietze transformation $\mathcal{T}$, the presentation $\mathcal{T}(\Sigma)$ is a coherent presentation of $\mathbf{M}$.

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- if $\widehat{v} < \widehat{w}$, the 2-cell χ and the 3-cell $A_{f,g}$ are added by Tietze transformation

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However. The extended presentation $\mathcal{HC}(\Sigma_{\mathrm{KN}})$ obtained is bigger than necessary.

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- Let Σ be a convergent and coherent presentation.

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\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & v \arrow[dr, Rightarrow, "f_2"] & \\
u \arrow[ur, Rightarrow, bend left, "f"] \arrow[dr, Rightarrow, bend right, "h"'] \arrow[rr, phantom, "A_{f,h}\Downarrow"] & & w' \\
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\end{tikzcd}
\qquad
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 & v \arrow[rr, Rightarrow, bend left, "f_1"] \arrow[dr, Rightarrow, "f_2"] & & x' \arrow[dr, Rightarrow, bend left, "h"] \arrow[d, phantom, "C\Downarrow"] & \\
u \arrow[ur, Rightarrow, bend left, "f"] \arrow[dr, Rightarrow, bend right, "h"'] \arrow[r, phantom, "A_{f,h}\Downarrow"] & & w' \arrow[rr, Rightarrow, "g_3"] & & \widehat{u} \\
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$$

- The **homotopical reduction in dimension 3**, denoted by $\overline{\mathcal{HC}}$,

  - builds such a 4-cell, for each critical triple branching,

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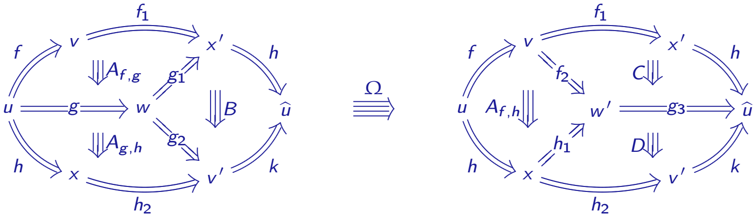

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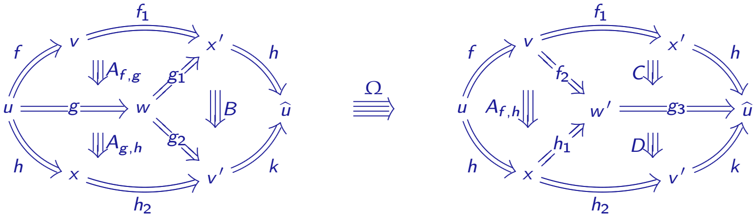

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The extended presentation $\overline{\mathcal{HC}}(\Sigma)$ is a reduced coherent and convergent presentation of $\mathbf{M}$.

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Example.

$$\mathcal{HC}(\Sigma_{\mathrm{KN}}) = \left\langle\, s, t, a \mid ta \overset{\alpha}{\Rightarrow} as,\; st \overset{\beta}{\Rightarrow} a, sas \overset{\gamma}{\Rightarrow} aa, saa \overset{\delta}{\Rightarrow} aat \mid A, B, C, D \,\right\rangle$$

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- There are four critical triple branchings, overlapping on

$$sasta,\; sasast,\; sasasas,\; sasasaa.$$

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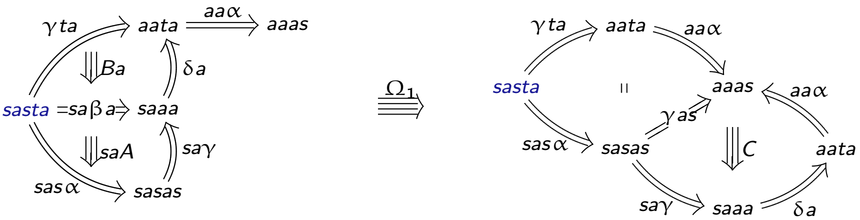
  - Critical triple branching on $sasta$



  - The 4-cell Ω_1 proves that C is superfluous in the coherent presentation.
  - The 3-cell C can be written as a composition of 3-cells A and B

$$C = sas\alpha^{-1} \star_1 (Ba \star_1 aa\alpha) \star_2 (saA \star_1 \delta a \star_1 aa\alpha)$$

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- Critical triple branching on $sasast$

$$\begin{tikzcd}
& aaast \arrow[r, Rightarrow, "aaa\beta"] \arrow[d, phantom, "Ct\Downarrow"] & aaaa \\
sasast \arrow[ur, Rightarrow, bend left, "\gamma ast"] \arrow[r, Rightarrow, "sa\gamma t"] \arrow[dr, Rightarrow, bend right, "sasa\beta"'] \arrow[dr, phantom, "\Downarrow saB"] & saaat \arrow[r, Rightarrow, "\delta at"'] & aatat \arrow[ul, Rightarrow, "aa\alpha t"'] \\
& sasaa \arrow[u, Rightarrow, bend right, "sa\delta"'] &
\end{tikzcd}
\quad \overset{\Omega_2}{\Rrightarrow} \quad
\begin{tikzcd}
& aaast \arrow[dr, Rightarrow, bend left, "aaa\beta"] & & \\
sasast \arrow[ur, Rightarrow, bend left, "\gamma ast"] \arrow[dr, Rightarrow, bend right, "sasa\beta"'] & = & aaaa \arrow[d, phantom, "\Downarrow D"] & aaast \arrow[l, Rightarrow, "aaa\beta"'] \\
& sasaa \arrow[ur, Rightarrow, "\gamma aa"'] \arrow[dr, Rightarrow, bend right, "sa\delta"'] & & \\
& & saaat \arrow[r, Rightarrow, "\delta at"'] & aatat \arrow[uu, Rightarrow, "aa\alpha t"']
\end{tikzcd}$$

- This 4-cell Ω_2 proves that D is superfluous in the coherent presentation.
- The 3-cell D can be written as a composition of 3-cells A and B

$$D = sasa\beta^{-1} \star_1 \big((Ct \star_1 aaa\beta) \star_2 (saB \star_1 \delta at \star_1 aa\alpha t \star_1 aaa\beta) \big)$$

The homotopical completion-reduction procedure: step 1

Example.

$$\mathcal{HC}(\Sigma_{\mathrm{KN}}) = \langle\, s, t, a \mid ta \overset{\alpha}{\Rightarrow} as,\ st \overset{\beta}{\Rightarrow} a, sas \overset{\gamma}{\Rightarrow} aa, saa \overset{\delta}{\Rightarrow} aat \mid A, B, \cancel{C}, \cancel{D} \,\rangle$$

- There are four critical triple branchings, overlapping on

$$sasta,\ sasast,\ sasasas,\ sasasaa.$$

Conclusion.

$$\overline{\mathcal{HC}}(\Sigma_{\mathrm{KN}}) = \langle\, s, t, a \mid ta \overset{\alpha}{\Rightarrow} as,\ st \overset{\beta}{\Rightarrow} a, sas \overset{\gamma}{\Rightarrow} aa, saa \overset{\delta}{\Rightarrow} aat \mid A, B \,\rangle$$

The homotopical completion-reduction procedure: step 2

• Let Σ be a coherent presentation.

• The **homotopical reduction in dimension 2** eliminates the rules added during the homotopical completion process.

$$\begin{tikzcd}
 & v \arrow[r, Rightarrow, "f'"] \arrow[d, phantom, "\Downarrow A_{f,g}"] & \widehat{v} \\
u \arrow[ur, Rightarrow, bend left, "f"] \arrow[dr, Rightarrow, bend right, "g"'] & & \\
 & w \arrow[r, Rightarrow, "g'"'] & \widehat{w} \arrow[uu, dotted, "\chi"']
\end{tikzcd}$$

- both χ and A are removed by a Tietze transformation.

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$$

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\end{tikzcd}
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- both χ and A are removed by a Tietze transformation.

• The homotopical reduction in dimension 2 applied on $\overline{\mathcal{HC}}(\Sigma)$ construct the extended presentation $\mathcal{HCR}(\Sigma)$.

Theorem. *Let Σ be a terminating presentation of a monoid $\mathbf{M}$.*
The extended presentation $\mathcal{HCR}(\Sigma)$ is a coherent presentation of $\mathbf{M}$, whose underlying presentation is Σ.

The homotopical completion-reduction procedure: step 2

Example.

$$\overline{\mathcal{HC}}(\Sigma_{\mathrm{KN}}) = \left\langle\ s, t, a \mid ta \overset{\alpha}{\Rightarrow} as,\ st \overset{\beta}{\Rightarrow} a, sas \overset{\gamma}{\Rightarrow} aa, saa \overset{\delta}{\Rightarrow} aat \mid A, B\ \right\rangle$$

The homotopical completion-reduction procedure: step 2

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- The 3-cells A and B correspond to the adjunction of the rules γ and δ during the $\mathcal{HC}$ procedure

$$\begin{tikzcd}
 & aa \\
sta \arrow[ur, Rightarrow, bend left, "\beta a"] \arrow[dr, Rightarrow, bend right, "s\alpha"'] \arrow[r, phantom, "\Downarrow A"] & \\
 & sas \arrow[uu, Rightarrow, bend right, "\gamma"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & aat \\
sast \arrow[ur, Rightarrow, bend left, "\gamma t"] \arrow[dr, Rightarrow, bend right, "sa\beta"'] \arrow[r, phantom, "\Downarrow B"] & \\
 & saa \arrow[uu, Rightarrow, bend right, "\delta"']
\end{tikzcd}$$

The homotopical completion-reduction procedure: step 2

Example.

$$\overline{\mathcal{HC}}(\Sigma_{\mathrm{KN}}) = \langle\; s, t, a \mid ta \overset{\alpha}{\Rightarrow} as,\; st \overset{\beta}{\Rightarrow} a, sas \overset{\gamma}{\Rightarrow} aa, saa \overset{\delta}{\Rightarrow} aat \mid A, B \;\rangle$$

- The 3-cells A and B correspond to the adjunction of the rules γ and δ during the $\mathcal{HC}$ procedure

$$\begin{tikzcd}
 & aa \\
sta \arrow[ur, Rightarrow, bend left, "\beta a"] \arrow[dr, Rightarrow, bend right, "s\alpha"'] \arrow[r, phantom, "\Rrightarrow A"] & {} \\
 & sas \arrow[uu, Rightarrow, bend right, "\gamma"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & aat \\
sast \arrow[ur, Rightarrow, bend left, "\gamma t"] \arrow[dr, Rightarrow, bend right, "sa\beta"'] \arrow[r, phantom, "\Rrightarrow B"] & {} \\
 & saa \arrow[uu, Rightarrow, bend right, "\delta"']
\end{tikzcd}$$

- They are removed by the $\mathcal{HCR}$ procedure:

The homotopical completion-reduction procedure: step 2

Example.

$$\overline{\mathcal{HC}}(\Sigma_{\mathrm{KN}}) = \langle\, s, t, a \mid ta \overset{\alpha}{\Rightarrow} as,\ st \overset{\beta}{\Rightarrow} a,\ \cancel{sas \overset{\gamma}{\Rightarrow} aa},\ \cancel{saa \overset{\delta}{\Rightarrow} aat} \mid \cancel{A}, \cancel{B} \,\rangle$$

- The 3-cells A and B correspond to the adjunction of the rules γ and δ during the $\mathcal{HC}$ procedure

$$\begin{tikzcd}[row sep=small] & aa \\ sta \arrow[ur, Rightarrow, bend left, "\beta a"] \arrow[dr, Rightarrow, bend right, "s\alpha"'] & \\ & sas \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] & aat \\ sast \arrow[ur, Rightarrow, bend left, "\gamma t"] \arrow[dr, Rightarrow, bend right, "sa\beta"'] & \\ & saa \end{tikzcd}$$

- They are removed by the $\mathcal{HCR}$ procedure:

$$\mathcal{HCR}(\Sigma_{\mathrm{KN}}) = \langle\, s, t, a \mid ta \overset{\alpha}{\Rightarrow} as,\ st \overset{\beta}{\Rightarrow} a \mid \emptyset \,\rangle$$

The homotopical completion-reduction procedure: step 3

- The generators added before the homotopical completion can be removed at the end.

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Proposition.
For the presentation Σ_{Artin} of $\mathbf{B}_3^+$ any two proofs of the same equality are equal.

Plan

Braids on 4 strands

- Artin presentation of the monoid $\mathbf{B}_4^+$ of **braids on 4 strands**:

$$\Sigma_{\mathrm{Artin}} = \langle\, r, s, t \mid rsr = srs,\ sts = tst,\ rt = tr \,\rangle$$

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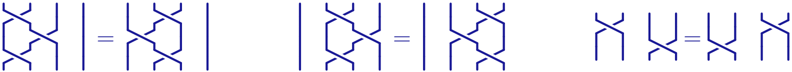
$r =$  $s =$  $t =$ 



Braids on 4 strands

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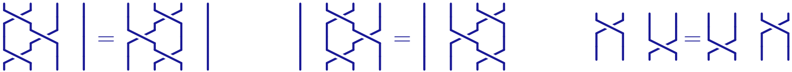

Proposition.
For the presentation Σ_{Artin} of $\mathbf{B}_4^+$ any two proofs of the same equality are equal modulo the ***Zamolodchikov relation***

$$\begin{tikzcd}
& stsrst \arrow[r, Rightarrow] & strsrt \arrow[r, Rightarrow] & srtstr \arrow[r, Rightarrow] & srstsr \arrow[r, Rightarrow] & rsrtsr & \\
tstrst \arrow[ur, Rightarrow, bend left] \arrow[dr, Rightarrow, bend right] & & & \equiv & & & rstrsr \arrow[ul, Rightarrow, bend right] \\
& tsrtst \arrow[r, Rightarrow] & tsrsts \arrow[r, Rightarrow] & trsrts \arrow[r, Rightarrow] & rtstrs \arrow[r, Rightarrow] & rstsrs \arrow[ur, Rightarrow, bend right] &
\end{tikzcd}$$

Computations

- The monoid of **braids on n strands**

$$\mathbf{B}_n^+ = \left\langle\, s_1, \ldots, s_{n-1} \;\middle|\; \begin{array}{ll} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \text{for } 1 \leqslant i < n-1 \\ s_i s_j = s_j s_i & \text{for } |i-j| \geqslant 2 \end{array} \,\right\rangle$$

- Computations with Artin, Kapur-Narendran and Brieskorn-Saito presentations.

- more generally, for the **generalised Artin monoid $\mathbf{B}^+(\mathbf{W})$** on a Coxeter group $\mathbf{W}$, using Garside presentation, see [Gaussent-Guiraud-Malbos, 2013].

- The **plactic monoid**

$$\mathbf{P}_n = \left\langle\, x_1, \ldots, x_n \;\middle|\; \begin{array}{l} x_j x_i x_k = x_j x_k x_i \text{ for } i < j \leqslant k \\ x_i x_k x_j = x_k x_i x_j \text{ for } i \leqslant j < k \end{array} \,\right\rangle$$

- Computations with Knuth and Column presentations.

- The **Chinese monoid**

$$\mathbf{Ch}_n = \left\langle\, x_1, \ldots, x_n \;\middle|\; x_j x_k x_i = x_k x_i x_j = x_k x_j x_i \text{ for } i \leqslant j \leqslant k \,\right\rangle$$

Results of experiments

http://www.pps.univ-paris-diderot.fr/~smimram/rewr

Coherent presentations						
Monoid	Presentation	Gen.	Rel.	Rel. comp.	Hom. gen.	Hom. gen. red.
$\mathbf{B}_3^+$	Artin	2	1	∞	∞	0
	Kapur-Narendran	**3**	**2**	**4**	**4**	**2**
	Brieskorn-Saito	3	2	4	6	2
	Garside	5	4	12	24	8
$\mathbf{B}_4^+$	Artin	3	3	∞	∞	1
	Kapur-Narendran	7	7	47	356	31
	Brieskorn-Saito	7	7	46	378	35
$\mathbf{B}_5^+$	Artin	4	6	∞	∞	4
	Kapur-Narendran	15	17	692	48260	?
	Brieskorn-Saito	15	17	598	28384	?
$\mathbf{P}_2 = \mathbf{Ch}_2$	Knuth	2	2	2	1	1
	Column	3	3	3	1	1
$\mathbf{P}_3$	Knuth	3	8	11	27	23
	Column	7	12	22	42	30
$\mathbf{P}_4$	Knuth	4	20	∞	∞	?
	Column	15	31	115	621	212
$\mathbf{P}_5$	Column	31	66	531	6893	?