Squier's theory for monoids and algebras

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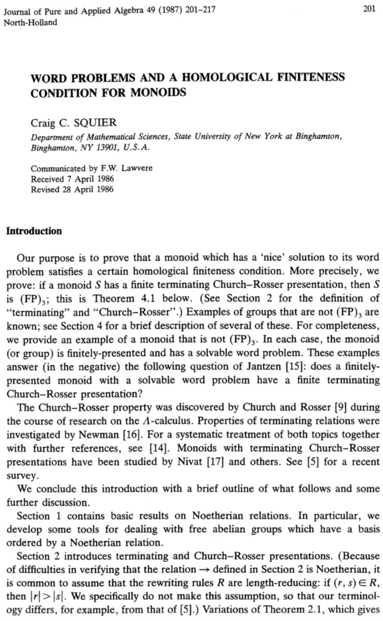
From the word problem to homology of monoids



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WORD PROBLEMS AND A HOMOLOGICAL FINITENESS CONDITION FOR MONOIDS

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Introduction

Our purpose is to prove that a monoid which has a 'nice' solution to its word problem satisfies a certain homological finiteness condition. More precisely, we prove: if a monoid S has a finite terminating Church–Rosser presentation, then S is $(\mathrm{FP})_3$; this is Theorem 4.1 below. (See Section 2 for the definition of "terminating" and "Church–Rosser".) Examples of groups that are not $(\mathrm{FP})_3$ are known; see Section 4 for a brief description of several of these. For completeness, we provide an example of a monoid that is not $(\mathrm{FP})_3$. In each case, the monoid (or group) is finitely-presented and has a solvable word problem. These examples answer (in the negative) the following question of Jantzen [15]: does a finitely-presented monoid with a solvable word problem have a finite terminating Church–Rosser presentation?

The Church–Rosser property was discovered by Church and Rosser [9] during the course of research on the Λ-calculus. Properties of terminating relations were investigated by Newman [16]. For a systematic treatment of both topics together with further references, see [14]. Monoids with terminating Church–Rosser presentations have been studied by Nivat [17] and others. See [5] for a recent survey.

We conclude this introduction with a brief outline of what follows and some further discussion.

Section 1 contains basic results on Noetherian relations. In particular, we develop some tools for dealing with free abelian groups which have a basis ordered by a Noetherian relation.

Section 2 introduces terminating and Church–Rosser presentations. (Because of difficulties in verifying that the relation $\rightarrow$ defined in Section 2 is Noetherian, it is common to assume that the rewriting rules R are length-reducing: if $(r,s)\in R$, then $|r|>|s|$. We specifically do not make this assumption, so that our terminology differs, for example, from that of [5].) Variations of Theorem 2.1, which gives

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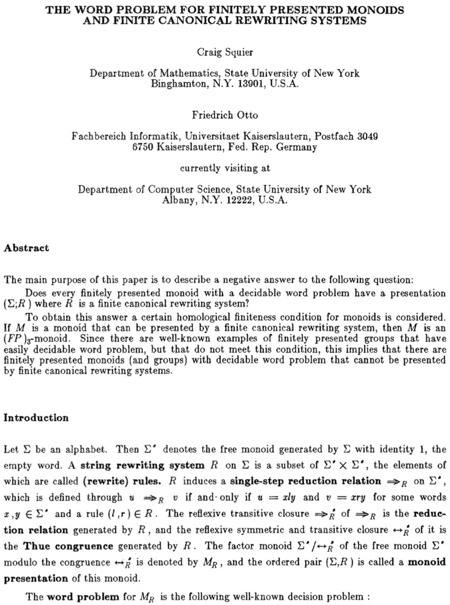

THE WORD PROBLEM FOR FINITELY PRESENTED MONOIDS AND FINITE CANONICAL REWRITING SYSTEMS

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Abstract

The main purpose of this paper is to describe a negative answer to the following question:

Does every finitely presented monoid with a decidable word problem have a presentation $(\Sigma;R)$ where R is a finite canonical rewriting system?

To obtain this answer a certain homological finiteness condition for monoids is considered. If M is a monoid that can be presented by a finite canonical rewriting system, then M is an $(FP)_3$-monoid. Since there are well-known examples of finitely presented groups that have easily decidable word problem, but that do not meet this condition, this implies that there are finitely presented monoids (and groups) with decidable word problem that cannot be presented by finite canonical rewriting systems.

Introduction

Let Σ be an alphabet. Then Σ^* denotes the free monoid generated by Σ with identity 1, the empty word. A **string rewriting system** R on Σ is a subset of $\Sigma^* \times \Sigma^*$, the elements of which are called **(rewrite) rules.** R induces a **single-step reduction relation** $\Rightarrow_R$ on Σ^*, which is defined through $u \Rightarrow_R v$ if and only if $u = xly$ and $v = xry$ for some words $x,y \in \Sigma^*$ and a rule $(l,r) \in R$. The reflexive transitive closure $\Rightarrow_R^*$ of $\Rightarrow_R$ is the **reduction relation** generated by R, and the reflexive symmetric and transitive closure $\leftrightarrow_R^*$ of it is the **Thue congruence** generated by R. The factor monoid $\Sigma^*/\leftrightarrow_R^*$ of the free monoid Σ^* modulo the congruence $\leftrightarrow_R^*$ is denoted by M_R, and the ordered pair (Σ,R) is called a **monoid presentation** of this monoid.

The **word problem** for M_R is the following well-known decision problem :

Squier's theory for monoids and algebras

I. Introduction: from the word problem to homology of monoids.

II. Low-dimensional coherence from convergence.

- Presentations of monoids and Syzygies.
- Coherence and three-dimensional presentations.

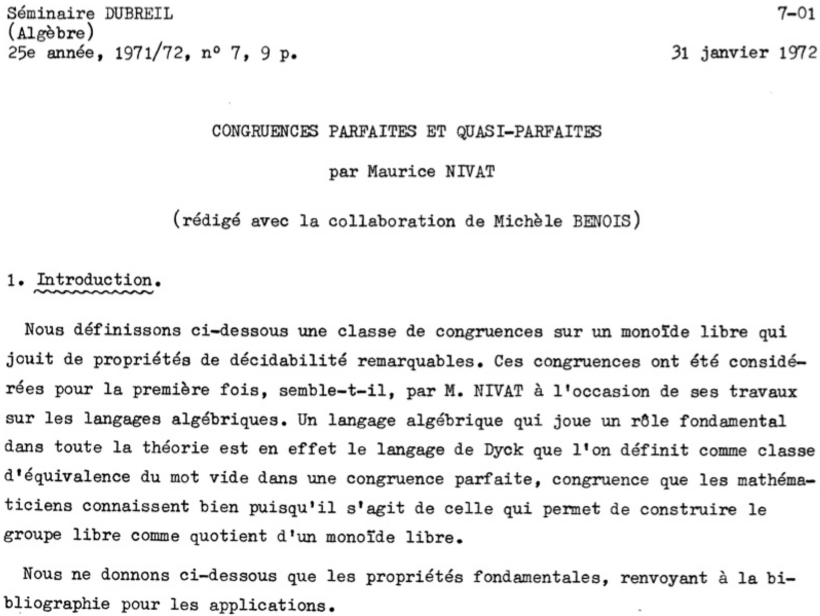
III. Homological syzygies from convergence.

- Homological finiteness conditions.
- Polygraphic resolutions from convergence.

IV. Linear rewriting.

Part I. Introduction: from the word problem to homology of monoids

Critical pair Lemma



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31 janvier 1972

CONGRUENCES PARFAITES ET QUASI-PARFAITES

par Maurice NIVAT

(rédigé avec la collaboration de Michèle BENOIS)

1. Introduction.

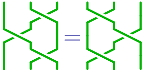
Nous définissons ci-dessous une classe de congruences sur un monoïde libre qui jouit de propriétés de décidabilité remarquables. Ces congruences ont été considérées pour la première fois, semble-t-il, par M. NIVAT à l'occasion de ses travaux sur les langages algébriques. Un langage algébrique qui joue un rôle fondamental dans toute la théorie est en effet le langage de Dyck que l'on définit comme classe d'équivalence du mot vide dans une congruence parfaite, congruence que les mathématiciens connaissent bien puisqu'il s'agit de celle qui permet de construire le groupe libre comme quotient d'un monoïde libre.

Nous ne donnons ci-dessous que les propriétés fondamentales, renvoyant à la bibliographie pour les applications.

Examples: confluence of 2-polygraphs

- Monoid of positive braids on three strands:

$s =$  $t =$ 



Examples: confluence of 2-polygraphs

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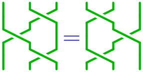

$$\mathbf{B}_3^+ = \langle\, s, t \mid sts = tst \,\rangle$$

Examples: confluence of 2-polygraphs

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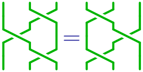

$$\mathbf{B}_3^+ = \langle\ s, t \mid sts = tst\ \rangle$$

- The 2-polygraph

$$\langle\ s, t \quad \mid \quad tst \overset{\gamma_{st}}{\Longrightarrow} sts\ \rangle$$

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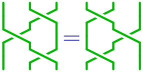

$$\mathbf{B}_3^+ = \langle\, s, t \mid sts = tst \,\rangle$$

- The 2-polygraph

$$\langle\ s, t \quad \Big| \quad tst \overset{\Upsilon_{st}}{\Longrightarrow} sts\ \rangle$$

has only one critical branching:

$$\begin{tikzcd} & \mathbf{sts}st \\ \mathbf{tst}st \arrow[ur, Rightarrow, bend left, "\Upsilon_{\mathbf{st}}st"] & \end{tikzcd}$$

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- The 2-polygraph

$$\langle\ s, t \quad\Big|\quad tst \overset{\gamma_{st}}{\Longrightarrow} sts \ \rangle$$

has only one critical branching:

$$\begin{tikzcd}
 & stsst \\
ts\mathbf{tst} \arrow[ur, Rightarrow, bend left, "\gamma_{st}st"] \arrow[dr, Rightarrow, bend right, "ts\gamma_{\mathbf{st}}"'] & \\
 & ts\mathbf{sts}
\end{tikzcd}$$

Examples: confluence of 2-polygraphs

- The 2-polygraph

$$\left\langle\; r, s, t \;\;\middle|\;\; sr \overset{\gamma_{rs}}{\Longrightarrow} rs,\; ts \overset{\gamma_{st}}{\Longrightarrow} st,\; tr \overset{\gamma_{rt}}{\Longrightarrow} rt \;\right\rangle$$

Examples: confluence of 2-polygraphs

- The 2-polygraph

$$\langle\, r, s, t \;\mid\; sr \overset{\gamma_{rs}}{\Longrightarrow} rs,\; ts \overset{\gamma_{st}}{\Longrightarrow} st,\; tr \overset{\gamma_{rt}}{\Longrightarrow} rt \,\rangle$$

  - It has only one critical branching

$$tsr$$

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  - It has only one critical branching

$$\begin{tikzcd}
 & \mathbf{st}r \\
\mathbf{ts}r \arrow[ur, Rightarrow, bend left, "\gamma_{\mathbf{st}}r"] &
\end{tikzcd}$$

Examples: confluence of 2-polygraphs

- The 2-polygraph

$$\langle\ r, s, t \ \mid\ sr \overset{\gamma_{rs}}{\Longrightarrow} rs,\ ts \overset{\gamma_{st}}{\Longrightarrow} st,\ tr \overset{\gamma_{rt}}{\Longrightarrow} rt \ \rangle$$

  - It has only one critical branching

$$\begin{tikzcd}
 & str \\
t\mathbf{sr} \arrow[ur, Rightarrow, bend left, "\gamma_{st} r"] \arrow[dr, Rightarrow, bend right, "t\gamma_{\mathbf{rs}}"'] & \\
 & t\mathbf{rs}
\end{tikzcd}$$

Examples: confluence of 2-polygraphs

- The 2-polygraph

$$\left\langle\, r, s, t \;\middle|\; sr \overset{\gamma_{rs}}{\Longrightarrow} rs,\ ts \overset{\gamma_{st}}{\Longrightarrow} st,\ tr \overset{\gamma_{rt}}{\Longrightarrow} rt \,\right\rangle$$

  - It has only one critical branching

$$\begin{tikzcd}
& s\mathbf{tr} \arrow[r, Rightarrow, "s\gamma_{\mathbf{rt}}"] & s\mathbf{rt} \\
tsr \arrow[ur, Rightarrow, bend left, "\gamma_{st} r"] \arrow[dr, Rightarrow, bend right, "t\gamma_{rs}"'] & & \\
& \mathbf{tr}s \arrow[r, Rightarrow, "\gamma_{\mathbf{rt}} s"'] & \mathbf{rt}s
\end{tikzcd}$$

Examples: confluence of 2-polygraphs

- The 2-polygraph

$$\left\langle\, r, s, t \;\middle|\; sr \overset{\gamma_{rs}}{\Longrightarrow} rs,\ ts \overset{\gamma_{st}}{\Longrightarrow} st,\ tr \overset{\gamma_{rt}}{\Longrightarrow} rt \,\right\rangle$$

  - It has only one critical branching

$$\begin{tikzcd}
 & str \arrow[r, Rightarrow, "s\gamma_{rt}"] & \mathbf{sr}t \arrow[dr, Rightarrow, bend left, "\boldsymbol{\gamma}_{\mathbf{rs}}t"] & \\
tsr \arrow[ur, Rightarrow, bend left, "\gamma_{st}r"] \arrow[dr, Rightarrow, bend right, "t\gamma_{rs}"'] & & & \mathbf{rst} \\
 & trs \arrow[r, Rightarrow, "\gamma_{rt}s"'] & r\mathbf{ts} \arrow[ur, Rightarrow, bend right, "r\boldsymbol{\gamma}_{\mathbf{st}}"'] &
\end{tikzcd}$$

Knuth-Bendix's completion procedure

```
Input: $\Sigma$ be a terminating $2$-polygraph with a total termination order $\prec$.
$\mathcal{KB}(\Sigma) := \Sigma$
$\mathcal{C}b := \{$ critical branchings of $\Sigma \}$
while $\mathcal{C}b \neq \emptyset$ do
    Picks a branching in $\mathcal{C}b$:
```

$$\begin{tikzcd}
& v \\
u \arrow[ur, Rightarrow, bend left, "f"] \arrow[dr, Rightarrow, bend right, "g"'] & \\
& w
\end{tikzcd}$$

```
    $\mathcal{C}b := \mathcal{C}b \setminus \{(f,g)\}$
    Reduce $v$ (resp. $w$) to a normal form $\widehat{v}$ (resp. $\widehat{w}$) with respect to $\mathcal{KB}(\Sigma)_2$
```

$$\begin{tikzcd}
& v \arrow[r, Rightarrow] & \widehat{v} \\
u \arrow[ur, Rightarrow, bend left, "f"] \arrow[dr, Rightarrow, bend right, "g"'] & & \\
& w \arrow[r, Rightarrow] & \widehat{w}
\end{tikzcd}$$

```
    if $\widehat{v} \neq \widehat{w}$ then
        if $\widehat{v} > \widehat{w}$ then
            $\mathcal{KB}(\Sigma)_2 := \mathcal{KB}(\Sigma)_2 \cup \{ \alpha : \widehat{v} \Rightarrow \widehat{w} \}$:
```

$$\begin{tikzcd}
& v \arrow[r, Rightarrow] & \widehat{v} \arrow[dd, Rightarrow, dotted, "\alpha"] \\
u \arrow[ur, Rightarrow, bend left, "f"] \arrow[dr, Rightarrow, bend right, "g"'] & & \\
& w \arrow[r, Rightarrow] & \widehat{w}
\end{tikzcd}$$

```
        end
        if $\widehat{w} > \widehat{v}$ then
            $\mathcal{KB}(\Sigma)_2 := \mathcal{KB}(\Sigma)_2 \cup \{ \alpha : \widehat{w} \Rightarrow \widehat{v} \}$:
```

$$\begin{tikzcd}
& v \arrow[r, Rightarrow] & \widehat{v} \\
u \arrow[ur, Rightarrow, bend left, "f"] \arrow[dr, Rightarrow, bend right, "g"'] & & \\
& w \arrow[r, Rightarrow] & \widehat{w} \arrow[uu, Rightarrow, dotted, "\alpha"']
\end{tikzcd}$$

```
        end
    end
    $\mathcal{C}b := \mathcal{C}b \cup \{$ critical branching created by $\alpha \}$
end
```

Knuth-Bendix's completion procedure

- If the procedure stops, it returns the 2-polygraph $\mathcal{KB}(\Sigma)$.

- Otherwise, it builds an increasing sequence of 2-polygraphs, whose limit is denoted by $\mathcal{KB}(\Sigma)$.

- If the starting 2-polygraph Σ is already convergent, then $\mathcal{KB}(\Sigma) = \Sigma$.

Theorem. (Knuth-Bendix, 1970)

▹ A Knuth-Bendix's completion $\mathcal{KB}(\Sigma)$ of a 2-polygraph Σ is a convergent presentation of the category $\overline{\Sigma}$.

▹ Moreover, the 2-polygraph $\mathcal{KB}(\Sigma)$ is finite if, and only if, the 2-polygraph Σ is finite and if the Knuth-Bendix's completion procedure halts.

Existence of finite convergent presentations

- The normal form procedure proves that, if a monoid admits a finite convergent presentation, then it has a decidable word problem.

- The converse implication was still an open problem in the middle of the eighties.

Question. (Jantzen, 1982, see also Bauer, Book, Otto and Diekert)

Does every finitely presented monoid with a decidable word problem admit a finite convergent presentation ?

Knuth-Bendix's completion: example

Example. (Bauer-Otto, 1984) Knuth-Bendix completion of the 2-polygraph

$$\Sigma^{\mathrm{KN}} = \langle\, s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a \rangle$$

Knuth-Bendix's completion: example

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$$\mathcal{KB}(\Sigma^{\mathrm{KN}}) \ =$$

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$$\mathcal{KB}(\Sigma^{\mathrm{KN}}) = \langle\, s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a$$

$$\begin{tikzcd}
 & aa \\
sta \arrow[ur, Rightarrow, bend left, "\beta a"] \arrow[dr, Rightarrow, bend right, "s\alpha"'] & \\
 & sas
\end{tikzcd}$$

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$$\mathcal{KB}(\Sigma^{\mathrm{KN}}) = \langle\ s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a,\ sas \overset{\gamma}{\Longrightarrow} aa$$

$$\begin{tikzcd}
 & aa \\
sta \arrow[ur, Rightarrow, bend left, "\beta a"] \arrow[dr, Rightarrow, bend right, "s\alpha"'] & \\
 & sas \arrow[uu, Rightarrow, bend right, "\gamma"']
\end{tikzcd}$$

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$$\begin{tikzcd}
 & aa \\
sta \arrow[ur, Rightarrow, bend left, "\beta a"] \arrow[dr, Rightarrow, bend right, "s\alpha"'] & \\
 & sas \arrow[uu, Rightarrow, bend right, "\gamma"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & aat \\
sast \arrow[ur, Rightarrow, bend left, "\gamma t"] \arrow[dr, Rightarrow, bend right, "sa\beta"'] & \\
 & saa
\end{tikzcd}$$

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$$\begin{tikzcd}
& aa & & & aat \\
sta \arrow[ur, Rightarrow, bend left, "\beta a"] \arrow[dr, Rightarrow, bend right, "s\alpha"'] & & & sast \arrow[ur, Rightarrow, bend left, "\gamma t"] \arrow[dr, Rightarrow, bend right, "sa\beta"'] & \\
& sas \arrow[uu, Rightarrow, bend right, "\gamma"'] & & & saa \arrow[uu, Rightarrow, bend right, "\delta"']
\end{tikzcd}$$

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$$\begin{tikzcd}
& aa \\
sta \arrow[ur, Rightarrow, bend left, "\beta a"] \arrow[dr, Rightarrow, bend right, "s\alpha"'] & \\
& sas \arrow[uu, Rightarrow, bend right, "\gamma"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& aat \\
sast \arrow[ur, Rightarrow, bend left, "\gamma t"] \arrow[dr, Rightarrow, bend right, "sa\beta"'] & \\
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$$\mathcal{K}\mathcal{B}(\Sigma^{\mathrm{KN}}) = \langle\, s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\; st \overset{\beta}{\Longrightarrow} a,\; sas \overset{\gamma}{\Longrightarrow} aa,\; saa \overset{\delta}{\Longrightarrow} aat$$

$$\begin{tikzcd}
& aa \\
sta \arrow[ur, Rightarrow, bend left, "\beta a"] \arrow[dr, Rightarrow, bend right, "s\alpha"'] & \\
& sas \arrow[uu, Rightarrow, bend right, "\gamma"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& aat \\
sast \arrow[ur, Rightarrow, bend left, "\gamma t"] \arrow[dr, Rightarrow, bend right, "sa\beta"'] & \\
& saa \arrow[uu, Rightarrow, bend right, "\delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& aaas & \\
sasas \arrow[ur, Rightarrow, bend left, "\gamma as"] \arrow[dr, Rightarrow, bend right, "sa\gamma"'] & & aata \arrow[ul, Rightarrow, bend right, "aa\alpha"'] \\
& saaa \arrow[ur, Rightarrow, bend right, "\delta a"'] &
\end{tikzcd}$$

Knuth-Bendix's completion: example

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$$\begin{tikzcd}
 & aa \\
sta \arrow[ur, Rightarrow, bend left, "\beta a"] \arrow[dr, Rightarrow, bend right, "s\alpha"'] & \\
 & sas \arrow[uu, Rightarrow, bend right, "\gamma"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & aat \\
sast \arrow[ur, Rightarrow, bend left, "\gamma t"] \arrow[dr, Rightarrow, bend right, "sa\beta"'] & \\
 & saa \arrow[uu, Rightarrow, bend right, "\delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & aaas & \\
sasas \arrow[ur, Rightarrow, bend left, "\gamma as"] \arrow[dr, Rightarrow, bend right, "sa\gamma"'] & & aata \arrow[ul, Rightarrow, bend right, "aa\alpha"'] \\
 & saaa \arrow[ur, Rightarrow, bend right, "\delta a"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & aaaa \\
sasaa \arrow[ur, Rightarrow, bend left, "\gamma aa"] \arrow[dr, Rightarrow, bend right, "sa\delta"'] & \\
 & saaat
\end{tikzcd}$$

Knuth-Bendix's completion: example

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$$\Sigma^{\mathrm{KN}} = \langle\, s, t, a \mid ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a \,\rangle$$

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$$\begin{tikzcd}
& aa \\
sta \arrow[ur, Rightarrow, bend left, "\beta a"] \arrow[dr, Rightarrow, bend right, "s\alpha"'] & \\
& sas \arrow[uu, Rightarrow, bend right, "\gamma"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& aat \\
sast \arrow[ur, Rightarrow, bend left, "\gamma t"] \arrow[dr, Rightarrow, bend right, "sa\beta"'] & \\
& saa \arrow[uu, Rightarrow, bend right, "\delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& aaas & \\
sasas \arrow[ur, Rightarrow, bend left, "\gamma as"] \arrow[dr, Rightarrow, bend right, "sa\gamma"'] & & aata \arrow[ul, Rightarrow, bend right, "aa\alpha"'] \\
& saaa \arrow[ur, Rightarrow, bend right, "\delta a"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& aaaa & aaast \arrow[l, Rightarrow, "aaa\beta"'] \\
sasaa \arrow[ur, Rightarrow, bend left, "\gamma aa"] \arrow[dr, Rightarrow, bend right, "sa\delta"'] & & \\
& saaat \arrow[r, Rightarrow, "\delta at"'] & aatat \arrow[uu, Rightarrow, bend right, "aa\alpha t"']
\end{tikzcd}$$

Questions.

▷ Which condition a monoid need to satisfy to admit a presentation by a finite convergent rewriting system ?

▷ How can we caracterize the class of finitely presented monoids that have finite convergent presentations ?

Example

Example. Stallings, 1963, constructed a finitely presented group $\mathbf{G}$ whose $\mathrm{H}_3(\mathbf{G}, \mathbb{Z})$ is not finitely generated and thus it does not have homological type FP_3.

Example

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- ▶ Similar example (group of matrices) by Abels, 1979.

Squier's example

Example. Consider, for every $k \geqslant 1$, the monoid $\mathbf{S}_k$ presented by

$$\left\langle\ a, b, t, x_1, \ldots, x_k, y_1, \ldots, y_k \ \middle|\ (\alpha_n)_{n\in\mathbb{N}},\ (\beta_i)_{1\leqslant i\leqslant k},\ (\gamma_i)_{1\leqslant i\leqslant k},\ (\delta_i)_{1\leqslant i\leqslant k},\ (\varepsilon_i)_{1\leqslant i\leqslant k}\ \right\rangle$$

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Theorem. (Squier, 1987)

- ▷ For $k \geqslant 1$, $\mathbf{S}_k$ is finitely presented.
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Theorem. (Squier, 1994)

- ▷ $\mathbf{S}_1$ does not have of finite derivation type.
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Extensions of Squier's finiteness conditions

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▶ How to describe in the higher-categorical framework the constructions by Anick, Kobayashi, Groves, Brown ?

Question. (Lafont-Metayer, 2009)

Is it true that a monoid presented by a finite convergent rewriting system always has a finite cofibrant approximation in the folk model structure on ∞-categories ?

Part II. Low-dimensional coherence from convergence (proof of Theorem B).

- Presentations of monoids and Syzygies.
- Coherence and three-dimensional presentations.

Homotopical completion procedure

Example. The **Kapur-Narendran's presentation** of $\mathbf{B}^+(\mathbf{S}_3)$, obtained from Artin's presentation by coherent adjunction of the Coxeter element st

$$\Sigma_2^{\mathrm{KN}} = \langle\, s, t, a \;\mid\; ta \overset{\alpha}{\Longrightarrow} as,\; st \overset{\beta}{\Longrightarrow} a \,\rangle$$

The deglex order generated by $t > s > a$ proves the termination of Σ_2^{KN}.

$$\mathcal{S}(\Sigma_2^{\mathrm{KN}}) = \langle\, s, t, a \;\mid\; ta \overset{\alpha}{\Longrightarrow} as,\; st \overset{\beta}{\Longrightarrow} a,\; sas \overset{\gamma}{\Longrightarrow} aa,\; saa \overset{\delta}{\Longrightarrow} aat \;\mid\; A,\; B,\; C,\; D \,\rangle$$

$$\begin{tikzcd}
 & aa \\
sta \arrow[ur, Rightarrow, bend left, "\beta a"] \arrow[dr, Rightarrow, bend right, "s\alpha"'] & \Downarrow A \\
 & sas \arrow[uu, Rightarrow, bend right, "\gamma"']
\end{tikzcd}
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- We will see that this coherent presentation is bigger than necessary.

A variant of Squier's example

Example. (Lafont-Prouté, 1991) Consider the monoid $\mathbf{M}$ presented by the 2-polygraph:

$$\Sigma \;=\; \left\langle\, a, b, c, d, d' \;\middle|\; ab \overset{\alpha_0}{\Longrightarrow} a,\; da \overset{\beta}{\Longrightarrow} ac,\; d'a \overset{\beta'}{\Longrightarrow} ac \,\right\rangle.$$

- The monoid $\mathbf{M}$ admits a finite presentation, it has a decidable word problem, yet it is not of finite derivation type.
- Infinite Knuth-Bendix completion of Σ:

$$\mathcal{KB}(\Sigma) \;=\; \left\langle\, a, b, c, d, d' \;\middle|\; (ac^n b \overset{\alpha_n}{\Longrightarrow} ac^n)_{n\in\mathbb{N}},\; da \overset{\beta}{\Longrightarrow} ac,\; d'a \overset{\beta'}{\Longrightarrow} ac \,\right\rangle.$$

- Squier's completion of $\mathcal{KB}(\Sigma^{\mathrm{LP}})$ has two infinite families of 3-cells:

$$\begin{tikzcd}
 & ac^{n+1}b \arrow[dr, Rightarrow, bend left, "\alpha_{n+1}"] & \\
dac^n b \arrow[ur, Rightarrow, bend left, "\beta c^n b"] \arrow[dr, Rightarrow, bend right, "d\alpha_n"'] & \Downarrow A_n & ac^{n+1} \\
 & dac^n \arrow[ur, Rightarrow, bend right, "\beta c^n"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & ac^{n+1}b \arrow[dr, Rightarrow, bend left, "\alpha_{n+1}"] & \\
d'ac^n b \arrow[ur, Rightarrow, bend left, "\beta' c^n b"] \arrow[dr, Rightarrow, bend right, "d'\alpha_n"'] & \Downarrow B_n & ac^{n+1} \\
 & d'ac^n \arrow[ur, Rightarrow, bend right, "\beta' c^n"'] &
\end{tikzcd}$$

- The monoid $\mathbf{M}$ is not of finite derivation type:
  - $\mathcal{KB}(\Sigma)$ has no triple critical branching.
  - The 3-cells B_n induce a projection $\pi : \mathcal{KB}(\Sigma)^\top \to (\Sigma)^\top$, so that the family $(\pi(A_n))_{n\in\mathbb{N}}$ is an infinite homotopy basis of $\Sigma^\top$.
  - No finite subfamily of $(\pi(A_n))_{n\in\mathbb{N}}$ can be a homotopy basis of $(\Sigma)^\top$.

Application of Squier's completion: coherence for monoids

Homotopical completion procedure

Let Σ be a terminating 2-polygraph (with a total termination order).

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  ▷ for every critical branching

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- ▷ if $\widehat{v} = \widehat{w}$, add a 3-cell $A_{f,g}$

$$\begin{tikzcd}
 & v \arrow[dr, Rightarrow, bend left, "f'"] \arrow[d, phantom, "\Downarrow A_{f,g}"] & \\
u \arrow[ur, Rightarrow, bend left, "f"] \arrow[dr, Rightarrow, bend right, "g"'] & {} & \widehat{v} = \widehat{w} \\
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& w \arrow[ur, Rightarrow, bend right, "g'"'] &
\end{tikzcd}$$

- if $\widehat{v} < \widehat{w}$, add the 2-cell $\alpha_{f,g}$ and the 3-cell $A_{f,g}$

$$\begin{tikzcd}
& v \arrow[r, Rightarrow, "f'"] \arrow[dd, Rrightarrow, dotted, "A_{f,g}"] & \widehat{v} \\
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- Potential adjunction of additional 2-cells $\alpha_{f,g}$ can create new critical branchings,
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- This defines an increasing sequence of $(3,1)$-polygraphs

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For a terminating presentation Σ of a category $\mathbf{C}$, the homotopical completion $\mathcal{S}(\Sigma)$ is a coherent convergent presentation of $\mathbf{C}$.

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Proof.

- $\mathcal{S}(\Sigma)$ obtained from Σ by successive application of Knuth-Bendix's procedure.
- Squier's coherence theorem.

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However. The coherent presentation $\mathcal{S}(\Sigma_2^{\mathrm{KN}})$ is bigger than necessary.

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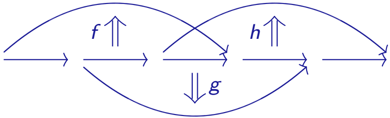

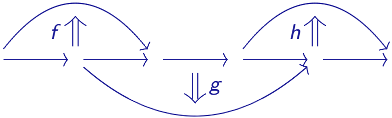

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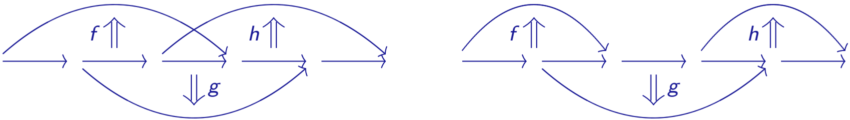

Step 3. Apply the homotopical reduction to $\mathcal{S}(\Sigma)$ with a **collapsible part** Γ made of

- ▷ 3-spheres induced by some of the **generating triple confluences** of $\mathcal{S}(\Sigma)$,

$$\begin{tikzcd}
 & v \\
u \arrow[ur, Rightarrow, bend left, "f"] \arrow[r, Rightarrow, "g"] \arrow[dr, Rightarrow, bend right, "h"'] & w \\
 & x
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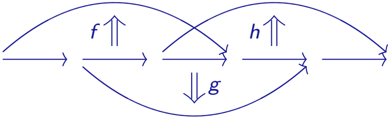
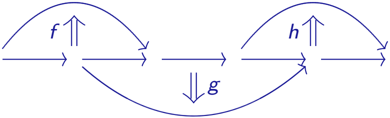
Step 2. Compute **critical triple branching**, that is overlappings of three rewriting steps:





Step 3. Apply the homotopical reduction to $\mathcal{S}(\Sigma)$ with a **collapsible part** Γ made of

- ▷ 3-spheres induced by some of the **generating triple confluences** of $\mathcal{S}(\Sigma)$,

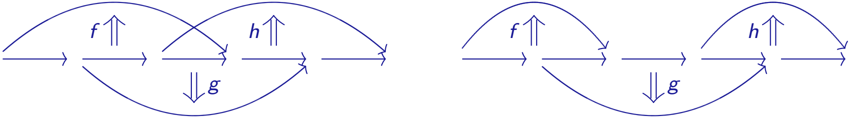
$$\begin{tikzcd}
 & v \arrow[r, Rightarrow, bend left, "f_1"] \arrow[d, phantom, "\Lleftarrow A_{f,g}"] & x' \\
u \arrow[ur, Rightarrow, bend left, "f"] \arrow[r, Rightarrow, "g"] \arrow[dr, Rightarrow, bend right, "h"'] & w \arrow[ur, Rightarrow, "g_1"] & \\
 & x &
\end{tikzcd}$$

Homotopical completion-reduction procedure

INPUT: A terminating 2-polygraph Σ.

Step 1. Compute an homotopical completion $\mathcal{S}(\Sigma)$ (convergent and coherent).

Step 2. Compute **critical triple branching**, that is overlappings of three rewriting steps:



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 & x \arrow[rr, Rightarrow, bend right, "h_2"'] & & v'
\end{tikzcd}$$

Homotopical completion-reduction procedure

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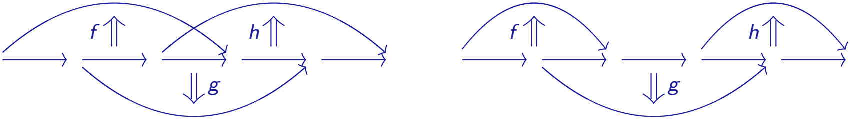

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u \arrow[ur, Rightarrow, bend left, "f"] \arrow[rr, Rightarrow, "g"] \arrow[dr, Rightarrow, bend right, "h"'] & & w \arrow[ur, Rightarrow, "g_1"] \arrow[dr, Rightarrow, "g_2"'] & \\
& x \arrow[rr, Rightarrow, bend right, "h_2"'] \arrow[u, phantom, "\Rrightarrow A_{g,h}"] & & v'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& v \\
u \arrow[ur, Rightarrow, bend left, "f"] \arrow[dr, Rightarrow, bend right, "h"'] & \\
& x
\end{tikzcd}$$

Homotopical completion-reduction procedure

INPUT: A terminating 2-polygraph Σ.

Step 1. Compute an homotopical completion $\mathcal{S}(\Sigma)$ (convergent and coherent).

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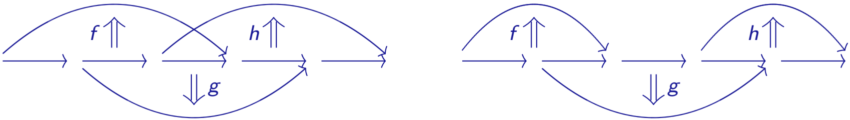

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& x \arrow[rr, Rightarrow, bend right, "h_2"'] \arrow[ur, phantom, "\Downarrow A_{g,h}"] & & v'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& v \arrow[dr, Rightarrow, "f_2"] & \\
u \arrow[ur, Rightarrow, bend left, "f"] \arrow[dr, Rightarrow, bend right, "h"'] \arrow[rr, phantom, "A_{f,h}\Downarrow"] & & w' \\
& x \arrow[ur, Rightarrow, "h_1"'] &
\end{tikzcd}$$

Homotopical completion-reduction procedure

INPUT: A terminating 2-polygraph Σ.

Step 1. Compute an homotopical completion $\mathcal{S}(\Sigma)$ (convergent and coherent).

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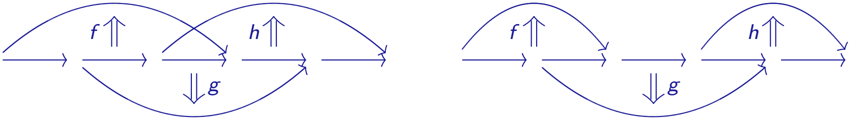

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 & x \arrow[rr, Rightarrow, bend right, "h_2"'] \arrow[u, phantom, "\Downarrow A_{g,h}"] & & v'
\end{tikzcd}
\qquad
\begin{tikzcd}
 & v \arrow[rr, Rightarrow, bend left, "f_1"] \arrow[dr, Rightarrow, "f_2"] & & x' \arrow[dr, Rightarrow, bend left, "h"] \arrow[d, phantom, "C \Downarrow"] & \\
u \arrow[ur, Rightarrow, bend left, "f"] \arrow[dr, Rightarrow, bend right, "h"'] \arrow[r, phantom, "A_{f,h} \Downarrow"] & & w' \arrow[rr, Rightarrow, "g_3"] & & \widehat{u} \\
 & x \arrow[ur, Rightarrow, "h_1"] & & &
\end{tikzcd}$$

Homotopical completion-reduction procedure

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Step 1. Compute an homotopical completion $\mathcal{S}(\Sigma)$ (convergent and coherent).

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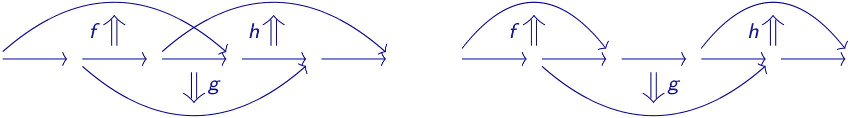

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 & x \arrow[rr, Rightarrow, bend right, "h_2"'] \arrow[ur, phantom, "\Downarrow A_{g,h}"] & & v'
\end{tikzcd}
\qquad
\begin{tikzcd}
 & v \arrow[rr, Rightarrow, bend left, "f_1"] \arrow[dr, Rightarrow, "f_2"] & & x' \arrow[dr, Rightarrow, bend left, "h"] \arrow[d, phantom, "C\Downarrow"] & \\
u \arrow[ur, Rightarrow, bend left, "f"] \arrow[dr, Rightarrow, bend right, "h"'] \arrow[r, phantom, "A_{f,h}\Downarrow"] & & w' \arrow[rr, Rightarrow, "g_3"] & & \widehat{u} \\
 & x \arrow[ur, Rightarrow, "h_1"] \arrow[rr, Rightarrow, bend right, "h_2"'] & & v' \arrow[ur, Rightarrow, bend right, "k"'] \arrow[u, phantom, "D\Downarrow"] &
\end{tikzcd}$$

Homotopical completion-reduction procedure

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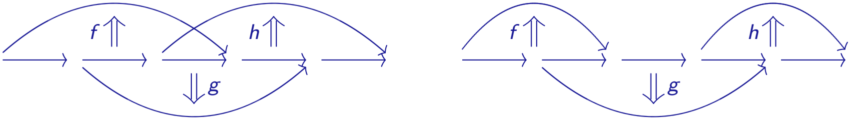

Step 3. Apply the homotopical reduction to $\mathcal{S}(\Sigma)$ with a **collapsible part** Γ made of

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$$\begin{tikzcd}
 & v \arrow[rr, Rightarrow, bend left, "f_1"] & & x' \arrow[ddr, Rightarrow, bend left, "h"] & \\
 & \Downarrow A_{f,g} & & & \\
u \arrow[uur, Rightarrow, bend left, "f"] \arrow[rr, Rightarrow, "g"] \arrow[ddr, Rightarrow, bend right, "h"'] & & w \arrow[uur, Rightarrow, "g_1"] \arrow[ddr, Rightarrow, "g_2"'] & \Downarrow B & \widehat{u} \\
 & \Downarrow A_{g,h} & & & \\
 & x \arrow[rr, Rightarrow, bend right, "h_2"'] & & v' \arrow[uur, Rightarrow, bend right, "k"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & v \arrow[rr, Rightarrow, bend left, "f_1"] \arrow[ddr, Rightarrow, "f_2"] & & x' \arrow[ddr, Rightarrow, bend left, "h"] & \\
 & & & \Downarrow C & \\
u \arrow[uur, Rightarrow, bend left, "f"] \arrow[ddr, Rightarrow, bend right, "h"'] & A_{f,h} \Downarrow & w' \arrow[rr, Rightarrow, "g_3"] & & \widehat{u} \\
 & & & \Downarrow D & \\
 & x \arrow[uur, Rightarrow, "h_1"] \arrow[rr, Rightarrow, bend right, "h_2"'] & & v' \arrow[uur, Rightarrow, bend right, "k"'] &
\end{tikzcd}$$

Homotopical completion-reduction procedure

INPUT: A terminating 2-polygraph Σ.

Step 1. Compute an homotopical completion $\mathcal{S}(\Sigma)$ (convergent and coherent).

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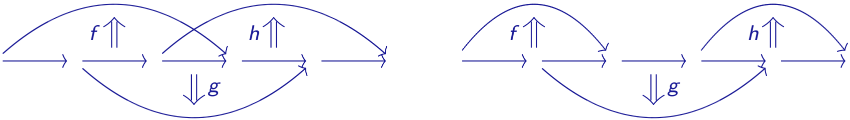

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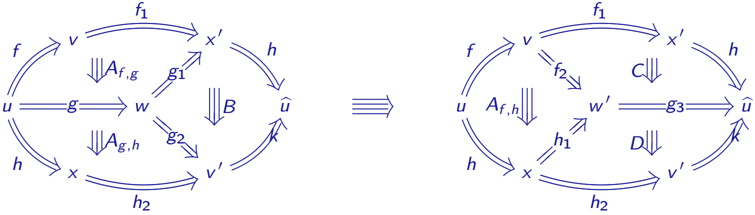

Homotopical completion-reduction procedure

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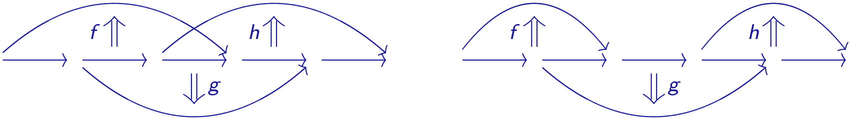


Step 3. Apply the homotopical reduction to $\mathcal{S}(\Sigma)$ with a **collapsible part** Γ made of

- ▷ 3-spheres induced by some of the **generating triple confluences** of $\mathcal{S}(\Sigma)$,
- ▷ the 3-cells adjoined with a 2-cell by homotopical completion to reach confluence:

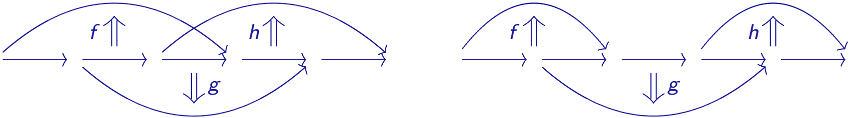
$$\begin{tikzcd}
 & v \arrow[r, Rightarrow, "f'"] \arrow[d, phantom, "\Downarrow A_{f,g}"] & \widehat{v} \\
u \arrow[ur, Rightarrow, bend left, "f"] \arrow[dr, Rightarrow, bend right, "g"'] & & \\
 & w \arrow[r, Rightarrow, "g'"'] & \widehat{w} \arrow[uu, Rightarrow, dotted, "\chi"']
\end{tikzcd}$$

Homotopical completion-reduction procedure

INPUT: A terminating 2-polygraph Σ.

Step 1. Compute an homotopical completion $\mathcal{S}(\Sigma)$ (convergent and coherent).

Step 2. Compute **critical triple branching**, that is overlappings of three rewriting steps:



Step 3. Apply the homotopical reduction to $\mathcal{S}(\Sigma)$ with a **collapsible part** Γ made of

- ▷ 3-spheres induced by some of the **generating triple confluences** of $\mathcal{S}(\Sigma)$,
- ▷ the 3-cells adjoined with a 2-cell by homotopical completion to reach confluence:
- ▷ some collapsible 2-cells or 3-cells already present in the initial presentation Σ.

Homotopical completion-reduction procedure

INPUT: A terminating 2-polygraph Σ.

Step 1. Compute an homotopical completion $\mathcal{S}(\Sigma)$ (convergent and coherent).

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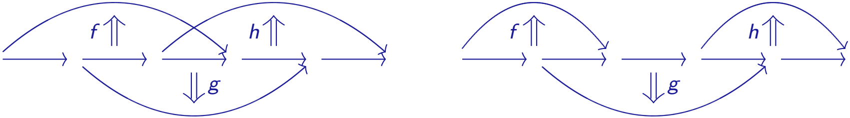

Step 3. Apply the homotopical reduction to $\mathcal{S}(\Sigma)$ with a **collapsible part** Γ made of

- 3-spheres induced by some of the **generating triple confluences** of $\mathcal{S}(\Sigma)$,
- the 3-cells adjoined with a 2-cell by homotopical completion to reach confluence:
- some collapsible 2-cells or 3-cells already present in the initial presentation Σ.

► The **homotopical completion-reduction** of the 2-polygraph Σ is the $(3, 1)$-polygraph

$$\mathcal{R}(\Sigma) = \pi_{\Gamma}(\mathcal{S}(\Sigma))$$

Theorem. [Gaussent-Guiraud-M., 2015]

For every terminating presentation Σ of a category $\mathbf{C}$, the homotopical completion-reduction $\mathcal{R}(\Sigma)$ is a coherent presentation of $\mathbf{C}$.

The homotopical completion-reduction procedure

Example.

$$\Sigma_2^{\mathrm{KN}} = \langle\ s, t, a \quad \big| \quad ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a\ \rangle$$

The homotopical completion-reduction procedure

Example. $$\Sigma_2^{\mathrm{KN}} = \langle\ s, t, a \quad \big| \quad ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a\ \rangle$$

$$\mathcal{S}(\Sigma_2^{\mathrm{KN}}) = \langle\ s, t, a \quad \big| \quad ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a, sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, C, D\ \rangle$$

The homotopical completion-reduction procedure

Example.

$$\Sigma_2^{\mathrm{KN}} = \langle\ s, t, a \ \ \mid \ \ ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a \ \rangle$$

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$$\langle\ s, t, a \ \ \mid \ \ ta \overset{\alpha}{\Longrightarrow} as\ , st \overset{\beta}{\Longrightarrow} a, sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, C, D \ \rangle$$

The homotopical completion-reduction procedure

Example.

$$\Sigma_2^{\mathrm{KN}} = \langle\; s, t, a \;\mid\; ta \overset{\alpha}{\Longrightarrow} as,\; st \overset{\beta}{\Longrightarrow} a \;\rangle$$

$$\mathcal{S}(\Sigma_2^{\mathrm{KN}}) = \langle\; s, t, a \;\mid\; ta \overset{\alpha}{\Longrightarrow} as,\; st \overset{\beta}{\Longrightarrow} a, sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, C, D \;\rangle$$

$$\langle\; s, t, a \;\mid\; ta \overset{\alpha}{\Longrightarrow} as\;, st \overset{\beta}{\Longrightarrow} a, sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, C, D \;\rangle$$

- There are four critical triple branchings, overlapping on

$$sasta,\; sasast,\; sasasas,\; sasasaa.$$

The homotopical completion-reduction procedure

Example.

$$\Sigma_2^{\mathrm{KN}} = \langle\, s, t, a \ \mid\ ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a \,\rangle$$

$$\mathcal{S}(\Sigma_2^{\mathrm{KN}}) = \langle\, s, t, a \ \mid\ ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a, sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, C, D \,\rangle$$

$$\langle\, s, t, a \ \mid\ ta \overset{\alpha}{\Longrightarrow} as\ , st \overset{\beta}{\Longrightarrow} a, sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, \cancel{C}, D \,\rangle$$

- There are four critical triple branchings, overlapping on

  $sasta,\ sasast,\ sasasas,\ sasasaa.$

  - Critical triple branching on $sasta$ proves that C is redundant:

$$\begin{tikzcd}
 & aata \arrow[r, Rightarrow, "aa\alpha"] & aaas \\
sasta \arrow[ur, Rightarrow, bend left, "\gamma ta"] \arrow[r, Rightarrow, "sa\beta a"] \arrow[dr, Rightarrow, bend right, "sas\alpha"'] \arrow[ur, phantom, shift right=6, "\Downarrow Ba"] \arrow[dr, phantom, shift left=6, "\Downarrow saA"] & saaa \arrow[u, Rightarrow, bend right, "\delta a"'] & \\
 & sasas \arrow[u, Rightarrow, bend right, "sa\gamma"'] &
\end{tikzcd}
\quad \Rrightarrow \quad
\begin{tikzcd}
 & aata \arrow[dr, Rightarrow, bend left, "aa\alpha"] & & \\
sasta \arrow[ur, Rightarrow, bend left, "\gamma ta"] \arrow[dr, Rightarrow, bend right, "sas\alpha"'] \arrow[rr, phantom, "="] & & aaas & \\
 & sasas \arrow[ur, Rightarrow, "\gamma as"] \arrow[dr, Rightarrow, bend right, "sa\gamma"'] \arrow[rr, phantom, "\Downarrow C"] & & aata \arrow[ul, Rightarrow, bend right, "aa\alpha"'] \\
 & & saaa \arrow[ur, Rightarrow, bend right, "\delta a"'] &
\end{tikzcd}$$

$$C = sas\alpha^{-1} \star_1 (Ba \star_1 aa\alpha) \star_2 (saA \star_1 \delta a \star_1 aa\alpha)$$

The homotopical completion-reduction procedure

Example.

$$\Sigma_2^{\mathrm{KN}} = \langle\, s, t, a \;\mid\; ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a \,\rangle$$

$$\mathcal{S}(\Sigma_2^{\mathrm{KN}}) = \langle\, s, t, a \;\mid\; ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a,\ sas \overset{\gamma}{\Longrightarrow} aa,\ saa \overset{\delta}{\Longrightarrow} aat \mid A, B, C, D \,\rangle$$

$$\langle\, s, t, a \;\mid\; ta \overset{\alpha}{\Longrightarrow} as\ ,\ st \overset{\beta}{\Longrightarrow} a,\ sas \overset{\gamma}{\Longrightarrow} aa,\ saa \overset{\delta}{\Longrightarrow} aat \mid A, B, \cancel{C}, \cancel{D} \,\rangle$$

- There are four critical triple branchings, overlapping on

$$sasta,\ sasast,\ sasasas,\ sasasaa.$$

  - ▷ Critical triple branching on $sasast$ proves that D is redundant:

$$\begin{tikzcd}
 & aaast \arrow[r, Rightarrow, "aaa\beta"] \arrow[d, phantom, "Ct\Rrightarrow"'] & aaaa \\
sasast \arrow[ur, Rightarrow, bend left, "\gamma ast"] \arrow[r, Rightarrow, "sa\gamma t"] \arrow[dr, Rightarrow, bend right, "sasa\beta"'] \arrow[dr, phantom, "\Rrightarrow saB", near start] & saaat \arrow[r, Rightarrow, "\delta at"'] & aatat \arrow[ul, Rightarrow, "aa\alpha t"'] \\
 & sasaa \arrow[u, Rightarrow, bend right, "sa\delta"'] &
\end{tikzcd}
\qquad \Rrightarrow \qquad
\begin{tikzcd}
 & aaast \arrow[dr, Rightarrow, bend left, "aaa\beta"] & & \\
sasast \arrow[ur, Rightarrow, bend left, "\gamma ast"] \arrow[dr, Rightarrow, bend right, "sasa\beta"'] & = & aaaa & aaast \arrow[l, Rightarrow, "aaa\beta"'] \\
 & sasaa \arrow[ur, Rightarrow, "\gamma aa"'] \arrow[dr, Rightarrow, bend right, "sa\delta"'] & \Rrightarrow D & \\
 & & saaat \arrow[r, Rightarrow, "\delta at"'] & aatat \arrow[uu, Rightarrow, "aa\alpha t"']
\end{tikzcd}$$

$$D = sasa\beta^{-1} \star_1 \big((Ct \star_1 aaa\beta) \star_2 (saB \star_1 \delta at \star_1 aa\alpha t \star_1 aaa\beta) \big)$$

The homotopical completion-reduction procedure

Example.

$$\Sigma_2^{\mathrm{KN}} = \langle\ s, t, a \ \mid\ ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a\ \rangle$$

$$\mathcal{S}(\Sigma_2^{\mathrm{KN}}) = \langle\ s, t, a \ \mid\ ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a, sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, C, D\ \rangle$$

$$\langle\ s, t, a \ \mid\ ta \overset{\alpha}{\Longrightarrow} as\ , st \overset{\beta}{\Longrightarrow} a, sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, \xcancel{C}, \xcancel{D}\rangle$$

- ▷ The 3-cells A and B are collapsible and the rules γ and δ are redundant.

$$\begin{tikzcd}
 & aa \\
sta \arrow[ur, Rightarrow, bend left, "\beta a"] \arrow[dr, Rightarrow, bend right, "s\alpha"'] & \Rrightarrow A \\
 & sas \arrow[uu, Rightarrow, bend right, "\gamma"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & aat \\
sast \arrow[ur, Rightarrow, bend left, "\gamma t"] \arrow[dr, Rightarrow, bend right, "sa\beta"'] & \Rrightarrow B \\
 & saa \arrow[uu, Rightarrow, bend right, "\delta"']
\end{tikzcd}$$

The homotopical completion-reduction procedure

Example.

$$\Sigma_2^{\mathrm{KN}} = \langle\, s, t, a \;\mid\; ta \xRightarrow{\alpha} as,\ st \xRightarrow{\beta} a \,\rangle$$

$$\mathcal{S}(\Sigma_2^{\mathrm{KN}}) = \langle\, s, t, a \;\mid\; ta \xRightarrow{\alpha} as,\ st \xRightarrow{\beta} a, sas \xRightarrow{\gamma} aa, saa \xRightarrow{\delta} aat \mid A, B, C, D \,\rangle$$

$$\langle\, s, t, a \;\mid\; ta \xRightarrow{\alpha} as\,, st \xRightarrow{\beta} a,\ \xcancel{sas \xRightarrow{\gamma} aa,}\ \xcancel{, saa \xRightarrow{\delta} aat} \mid \xcancel{A}, \xcancel{B}, \xcancel{C}, \xcancel{D} \,\rangle$$

▷ The 3-cells A and B are collapsible and the rules γ and δ are redundant.

$$\begin{tikzcd}
 & aa \\
sta \arrow[ur, Rightarrow, bend left, "\beta a"] \arrow[dr, Rightarrow, bend right, "s\alpha"'] & \\
 & sas
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & aat \\
sast \arrow[ur, Rightarrow, bend left, "\gamma t"] \arrow[dr, Rightarrow, bend right, "sa\beta"'] & \\
 & saa
\end{tikzcd}$$

The homotopical completion-reduction procedure

Example.

$$\Sigma_2^{\mathrm{KN}} = \langle\ s, t, a \ \ \big|\ \ ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a\ \rangle$$

$$\mathcal{S}(\Sigma_2^{\mathrm{KN}}) = \langle\ s, t, a \ \ \big|\ \ ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a, sas \overset{\gamma}{\Longrightarrow} aa, saa \overset{\delta}{\Longrightarrow} aat \mid A, B, C, D\ \rangle$$

$$\langle\ s, t, \cancel{a} \ \ \big|\ \ ta \overset{\alpha}{\Longrightarrow} as\ ,\ \cancel{st \overset{\beta}{\Longrightarrow} a},\ \cancel{sas \overset{\gamma}{\Longrightarrow} aa},\ ,\ \cancel{saa \overset{\delta}{\Longrightarrow} aat} \mid \cancel{A}, \cancel{B}, \cancel{C}, \cancel{D}\ \rangle$$

▷ The rule $st \overset{\beta}{\Longrightarrow} a$ is collapsible and the generator a is redundant.

The homotopical completion-reduction procedure

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$$\Sigma_2^{\mathrm{KN}} = \langle\ s, t, a \ \mid\ ta \xRightarrow{\alpha} as,\ st \xRightarrow{\beta} a\ \rangle$$

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$$\langle\ s, t, \xcancel{a} \ \mid\ tst \xRightarrow{\alpha} sts, \xcancel{st \xRightarrow{\beta} a,}\ \xcancel{sas \xRightarrow{\gamma} aa,}\ \xcancel{, saa \xRightarrow{\delta} aat} \mid \xcancel{A}, \xcancel{B}, \xcancel{C}, \xcancel{D} \rangle$$

$$\mathcal{R}(\Sigma_2^{\mathrm{KN}}) = \langle\ s, t \ \mid\ tst \xRightarrow{\alpha} sts \ \mid\ \emptyset\ \rangle$$

$$= \mathrm{Art}_3(\mathbf{S}_3)$$

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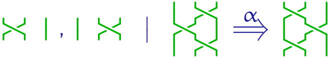
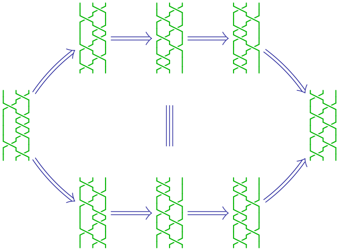
$= \langle$  $\mid\ \emptyset\ \rangle$



With presentation $\mathrm{Art}_2(\mathbf{S}_3)$ two proofs of the same equality in $\mathbf{B}_3^+$ are equal.

The homotopical completion-reduction procedure

Exemple.

$$\mathrm{Art}_2(\mathbf{S}_4) \;=\; \langle\; r, s, t \mid rsr = srs,\; sts = tst,\; rt = tr \;\rangle$$

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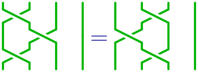
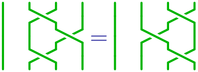
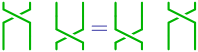
$r =$  $s =$  $t =$ 







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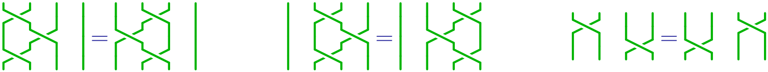

Proposition. (Deligne, 1997)

For presentation $\mathrm{Art}_2(\mathbf{S}_4)$ of $\mathbf{B}_4^+$ two proofs of the same equality are equal modulo **Zamolodchikov relation**:

$$\begin{tikzcd}
 & stsrst \arrow[r, Rightarrow] & strsrt \arrow[r, Rightarrow] & srtstr \arrow[r, Rightarrow] \arrow[dd, Rrightarrow] & srstsr \arrow[r, Rightarrow] & rsrtsr & \\
tstrst \arrow[ur, Rightarrow, bend left] \arrow[dr, Rightarrow, bend right] & & & & & & rstrsr \arrow[ul, Rightarrow, bend right] \\
 & tsrtst \arrow[r, Rightarrow] & tsrsts \arrow[r, Rightarrow] & trsrts \arrow[r, Rightarrow] & rtstrs \arrow[r, Rightarrow] & rstsrs \arrow[ur, Rightarrow, bend right] &
\end{tikzcd}$$

Artin monoids: Garside's presentation

- Let **W** be a **Coxeter group**

$$\mathbf{W} = \left\langle\ S \quad\middle|\quad s^2 = 1,\ \ \langle ts\rangle^{m_{st}} \ =\ \langle st\rangle^{m_{st}}\ \right\rangle$$

where $\langle ts\rangle^{m_{st}}$ stands for the word $tsts\ldots$ with m_{st} letters.

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- **Garside's extended presentation** of the Artin monoid $\mathbf{B}^+(\mathbf{W})$
  - 1-cells:

$$\mathsf{Gar}_1(\mathbf{W}) = \mathbf{W} \setminus \{1\}$$

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$$\mathrm{Gar}_2(\mathbf{W}) = \left\{ \; u|v \overset{\alpha_{u,v}}{\Longrightarrow} uv \; \text{ whenever } l(uv) = l(u) + l(v) \; \right\}$$

where uv is the product in $\mathbf{W}$ and $u|v$ is the product in the free monoid over $\mathbf{W}$.

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  - $\mathrm{Gar}_3(\mathbf{W})$ made of one 3-cell

$$\begin{tikzcd}
 & uv|w \arrow[dr, Rightarrow, bend left, "\alpha_{uv,w}"] \arrow[dd, Rightarrow, "A_{u,v,w}"] & \\
u|v|w \arrow[ur, Rightarrow, bend left, "\alpha_{u,v}|w"] \arrow[dr, Rightarrow, bend right, "u|\alpha_{v,w}"'] & & uvw \\
 & u|vw \arrow[ur, Rightarrow, bend right, "\alpha_{u,vw}"'] &
\end{tikzcd}$$

for every u, v, w in $\mathbf{W} \setminus \{1\}$ such that the lengths can be added.

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Theorem. [Gaussent-Guiraud-M., 2015]
$\mathrm{Gar}_3(\mathbf{W})$ is a coherent presentation the Artin monoid $\mathbf{B}^+(\mathbf{W})$

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 & u|vw \arrow[ur, Rightarrow, bend right, "\alpha_{u,vw}"'] &
\end{tikzcd}$$

for every u, v, w in $\mathbf{W} \setminus \{1\}$ such that the lengths can be added.

Theorem. [Gaussent-Guiraud-M., 2015]

$\mathrm{Gar}_3(\mathbf{W})$ is a coherent presentation the Artin monoid $\mathbf{B}^+(\mathbf{W})$

Proof.

By homotopical completion-reduction of the 2-polygraph $\mathrm{Gar}_2(\mathbf{W})$.

Artin monoids: Artin's coherent presentation

Theorem. [Tits, 1981, Gaussent-Guiraud-M., 2015]

The Artin monoid $\mathbf{B}^+(\mathbf{W})$ admits the coherent presentation $\mathrm{Art}_3(\mathbf{W})$ made of

- ▷ Artin's presentation

$$\mathrm{Art}_2(\mathbf{W}) = \langle\ S \quad \big| \quad \langle ts\rangle^{m_{st}} \ = \ \langle st\rangle^{m_{st}}\ \rangle$$

- ▷ one 3-cell $Z_{r,s,t}$ for every $t > s > r$ in S such that the subgroup $\mathbf{W}_{\{r,s,t\}}$ is finite.

Artin monoids: Zamolodchikov $Z_{r,s,t}$ according to Coxeter type

Type A_3

$$\begin{tikzcd}
 & strsrt \arrow[r, Rightarrow, "s\gamma_{rt}s\gamma^{-}_{rt}"] & srtstr \arrow[r, Rightarrow, "sr\gamma_{st}r"] & srstsr \arrow[dr, Rightarrow, bend left, "\gamma_{rs}tsr"] & \\
stsrst \arrow[ur, Rightarrow, bend left, "st\gamma_{rs}t"] & & & & rsrtsr \\
tstrst \arrow[u, Rightarrow, "\gamma_{st}rst"] \arrow[d, Rightarrow, "ts\gamma_{rt}st"'] & & \Downarrow Z_{r,s,t} & & rstrsr \arrow[u, Rightarrow, "rs\gamma_{rt}sr"'] \\
tsrtst \arrow[dr, Rightarrow, bend right, "tsr\gamma_{st}"'] & & & & rstsrs \arrow[u, Rightarrow, "rst\gamma_{rs}"'] \\
 & tsrsts \arrow[r, Rightarrow, "t\gamma_{rs}ts"'] & trsrts \arrow[r, Rightarrow, "\gamma_{rt}s\gamma^{-}_{rt}s"'] & rtstrs \arrow[ur, Rightarrow, bend right, "r\gamma_{st}rs"'] &
\end{tikzcd}$$

Type B_3

$$\begin{tikzcd}
 & srtsrtstr \arrow[r, Rightarrow, "srts\gamma^{-}_{rt}str"] & srtstrstr \arrow[r, Rightarrow, "sr\gamma_{st}rs\gamma_{rt}"] & srstsrsrt \arrow[r, Rightarrow, "srst\gamma_{rs}t"] & srstrsrst \arrow[r, Rightarrow, "srs\gamma_{rt}srst"] & srsrtsrst \arrow[dr, Rightarrow, bend left, "\gamma_{rs}tsrst"] & \\
strsrstsr \arrow[ur, Rightarrow, bend left, "s\gamma_{rt}sr\gamma^{-}_{st}r"] & & & & & & rsrstsrst \\
stsrsrtsr \arrow[u, Rightarrow, "st\gamma_{rs}tsr"] & & & & & & rsrtstrst \arrow[u, Rightarrow, "rsr\gamma_{st}rst"'] \\
tstrsrtsr \arrow[u, Rightarrow, "\gamma_{st}rsrtsr"] \arrow[d, Rightarrow, "ts\gamma_{rt}s\gamma^{-}_{rt}sr"'] & & & \Downarrow Z_{r,s,t} & & & rsrtsrtst \arrow[u, Rightarrow, "rsrts\gamma^{-}_{rt}st"'] \\
tsrtstrsr \arrow[d, Rightarrow, "tsr\gamma_{st}rsr"'] & & & & & & rstrsrsts \arrow[u, Rightarrow, "rs\gamma_{rt}sr\gamma^{-}_{st}"'] \\
tsrstsrsr \arrow[dr, Rightarrow, bend right, "tsrst\gamma_{rs}"'] & & & & & & rstsrsrts \arrow[u, Rightarrow, "rst\gamma_{rs}ts"'] \\
 & tsrstrsrs \arrow[r, Rightarrow, "tsrs\gamma_{rt}srs"'] & tsrsrtsrs \arrow[r, Rightarrow, "t\gamma_{rs}tsrs"'] & trsrstsrs \arrow[r, Rightarrow, "\gamma_{rt}sr\gamma_{st}rs"'] & rtsrtstrs \arrow[r, Rightarrow, "rts\gamma^{-}_{rt}strs"'] & rtstrstrs \arrow[ur, Rightarrow, bend right, "r\gamma_{st}rs\gamma_{rt}s"'] &
\end{tikzcd}$$

Type H_3

$$\begin{tikzcd}
 & srstrsrsrtsrsrt \arrow[r, Rightarrow] & srsrtsrstrsrsrt \arrow[r, Rightarrow] & srsrtsrstsrsrst \arrow[r, Rightarrow] & srsrtsrtstrsrst \arrow[dr, Rightarrow, bend left] & \\
srstsrsrstsrsrt \arrow[ur, Rightarrow, bend left] & & & & & srsrtstrsrtsrst \arrow[d, Rightarrow] \\
srtstrsrtstrsrt \arrow[u, Rightarrow] & & & & & srsrstsrsrtsrst \arrow[d, Rightarrow] \\
srtsrtstrsrtstr \arrow[u, Rightarrow] & & & & & rsrsrtsrsrtsrst \\
srtsrstsrsrstsr \arrow[u, Rightarrow] & & & & & rsrstrsrsrtsrst \arrow[u, Rightarrow] \\
srtsrstrsrsrtsr \arrow[u, Rightarrow] & & & & & rsrstsrsrstsrst \arrow[u, Rightarrow] \\
strsrsrtsrsrtsr \arrow[u, Rightarrow] & & & & & rsrtstrsrtstrst \arrow[u, Rightarrow] \\
stsrsrstsrsrtsr \arrow[u, Rightarrow] & & & & & rsrtsrtstrsrtst \arrow[u, Rightarrow] \\
tstrsrstsrsrtsr \arrow[u, Rightarrow] \arrow[d, Rightarrow] & & \Downarrow Z_{r,s,t} & & & rsrtsrstsrsrsts \arrow[u, Rightarrow] \\
tsrtsrstsrstrsr \arrow[d, Rightarrow] & & & & & rsrtsrstrsrsrts \arrow[u, Rightarrow] \\
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tsrtstrsrtstrsr \arrow[d, Rightarrow] & & & & & rstsrsrstsrsrts \arrow[u, Rightarrow] \\
tsrstsrsrstsrsr \arrow[d, Rightarrow] & & & & & rtstrsrtstrsrts \arrow[u, Rightarrow] \\
tsrstrsrsrtsrsr \arrow[d, Rightarrow] & & & & & rtsrtstrsrtstrs \arrow[u, Rightarrow] \\
tsrsrtsrstrsrsr \arrow[d, Rightarrow] & & & & & rtsrstsrsrstsrs \arrow[u, Rightarrow] \\
tsrsrtsrstsrsrs \arrow[dr, Rightarrow, bend right] & & & & & rtsrstrsrsrtsrs \arrow[u, Rightarrow] \\
 & tsrsrtsrtstrsrs \arrow[r, Rightarrow] & tsrsrtstrsrtsrs \arrow[r, Rightarrow] & tsrsrstsrsrtsrs \arrow[r, Rightarrow] & trsrsrtsrsrtsrs \arrow[ur, Rightarrow, bend right] &
\end{tikzcd}$$

Type $A_1 \times A_1 \times A_1$

$$\begin{tikzcd}
 & str \arrow[r, Rightarrow, "s\gamma_{rt}"] & srt \arrow[dr, Rightarrow, bend left, "\gamma_{rs}t"] & \\
tsr \arrow[ur, Rightarrow, bend left, "\gamma_{st}r"] \arrow[dr, Rightarrow, bend right, "t\gamma_{rs}"'] & \Downarrow Z_{r,s,t} & & rst \\
 & trs \arrow[r, Rightarrow, "\gamma_{rt}s"'] & rts \arrow[ur, Rightarrow, bend right, "r\gamma_{st}"'] &
\end{tikzcd}$$

Type $I_2(p) \times A_1$, $p \geqslant 3$,

$$\begin{tikzcd}
 & st\langle rs\rangle^{p-1} \arrow[r, Rightarrow, "s\gamma_{rt}\langle rs\rangle^{p-2}"] & (\cdots) \arrow[r, Rightarrow] & \langle sr\rangle^{p}t \arrow[dr, Rightarrow, bend left, "\gamma_{rs}t"] & \\
t\langle sr\rangle^{p} \arrow[ur, Rightarrow, bend left, "\gamma_{st}\langle rs\rangle^{p-1}"] \arrow[dr, Rightarrow, bend right, "t\gamma_{rs}"'] & & \Downarrow Z_{r,s,t} & & \langle rs\rangle^{p}t \\
 & t\langle rs\rangle^{p} \arrow[r, Rightarrow, "\gamma_{rt}\langle sr\rangle^{p-1}"'] & rt\langle sr\rangle^{p-1} \arrow[r, Rightarrow, "r\gamma_{st}\langle sr\rangle^{p-2}"'] & (\cdots) \arrow[ur, Rightarrow, bend right] &
\end{tikzcd}$$

Plactic monoids

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$$\mathrm{Knuth}_2(n) \;=\; \left\{ \begin{array}{ll} zxy = xzy & \text{for all } 1 \leqslant x \leqslant y < z \leqslant n \\ yzx = yxz & \text{for all } 1 \leqslant x < y \leqslant z \leqslant n \end{array} \right\}$$

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1	1	1	2	2	3	4
2	2	3	3	4	6	
4	5	6	6			
6	7					

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- **Column presentation** (Cain-Gray-Malheiro, 2015)
  - add **columns** as generators:

$$c_u = x_p \ldots x_2 x_1 \in \mathrm{Knuth}_1^*(n) \quad \text{such that} \quad x_p > \ldots > x_2 > x_1.$$

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  - 2-cells: $\mathrm{Col}_2(n)$ is the set of 2-cells

$$c_u c_v \overset{\alpha_{u,v}}{\Longrightarrow} c_w c_{w'}$$

such that

- u and v are columns,
- the planar representation of $P(uv)$ is not the juxtaposition of columns u and v,
- w and w' are respectively the left and right columns of $P(uv)$.

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  - the planar representation of $P(uv)$ is not the juxtaposition of columns u and v,
  - w and w' are respectively the left and right columns of $P(uv)$.

  - 3-cells:

$$\begin{tikzcd}
 & c_e c_{e'} c_t \arrow[r, Rightarrow, "c_e \alpha_{e',t}"] & c_e c_b c_{b'} \arrow[dr, Rightarrow, "\alpha_{e,b} c_{b'}"] & \\
c_x c_v c_t \arrow[ur, Rightarrow, "\alpha_{x,v} c_t"] \arrow[dr, Rightarrow, "c_u \alpha_{v,t}"'] & \arrow[r, phantom, "\Downarrow \mathcal{X}_{x,v,t}"] & & c_a c_d c_{b'} \\
 & c_x c_w c_{w'} \arrow[r, Rightarrow, "\alpha_{x,w} c_{w'}"'] & c_a c_{a'} c_{w'} \arrow[ur, Rightarrow, "c_a \alpha_{a',w'}"'] &
\end{tikzcd}$$

with x in $\mathrm{Knuth}_1(n)$ and v, t are columns.

Theorem. [Hage-M., 2015]

For $n \geqslant 2$, $\mathrm{Col}_3(n)$ is a finite coherent presentation of the plactic monoid $\mathbf{P}_n$.

Plactic monoids: column presentation

- **Column extended presentation** of the plactic monoid $\mathbf{P}_n$
  - 1-cells:

$$\mathrm{Col}_1(n) = \{\ c_u \mid u \text{ is a column}\ \}$$

  - 2-cells: $\mathrm{Col}_2(n)$ is the set of 2-cells

$$c_u c_v \overset{\alpha_{u,v}}{\Longrightarrow} c_w c_{w'}$$

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c_x c_v c_t \arrow[ur, Rightarrow, "\alpha_{x,v} c_t"] \arrow[dr, Rightarrow, "c_u \alpha_{v,t}"'] & & & c_a c_d c_{b'} \\
 & c_x c_w c_{w'} \arrow[r, Rightarrow, "\alpha_{x,w} c_{w'}"', ""{name=D}] & c_a c_{a'} c_{w'} \arrow[ur, Rightarrow, "c_a \alpha_{a',w'}"'] &
\arrow[from=U, to=D, Rrightarrow, "\mathcal{X}_{x,v,t}"]
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For $n \geqslant 2$, $\mathrm{Col}_3(n)$ is a finite coherent presentation of the plactic monoid $\mathbf{P}_n$.

Proof. By homotopical completion-reduction of the 2-polygraph $\mathrm{Col}_2(n)$.

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Higher-dimensional categories with finite derivation type

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$$(\Sigma_0, \Sigma_1, \Sigma_2, \ldots, \Sigma_n)$$

constructed by induction

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 & \Sigma_1 \arrow[u, tail] \arrow[ul, shift right, "s_0"'] \arrow[ul, shift left, "t_0"] & \Sigma_2 \arrow[u, tail] \arrow[ul, shift right, "s_1"'] \arrow[ul, shift left, "t_1"] & \Sigma_3 \arrow[u, tail] \arrow[ul, shift right, "s_2"'] \arrow[ul, shift left, "t_2"] & (\cdots) \arrow[ul, shift right, "s_3"'] \arrow[ul, shift left, "t_3"] & \Sigma_{n-1} \arrow[u, tail] \arrow[ul, shift right, "s_{n-2}"'] \arrow[ul, shift left, "t_{n-2}"]
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- An n-polygraph Σ induces an abstract rewriting system on Σ_{n-1}^*.

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- An n-polygraph Σ induces an abstract rewriting system on Σ_{n-1}^*.

- We extend the (abstract) rewriting properties:

termination / **confluence** / **locally confluence** / **convergence**.

Squier's completion

- Let Σ be a convergent n-polygraph.

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- A **family of generating confluences** of Σ is a cellular extension of the $(n, n-1)$-category $\Sigma_n^\top$ that contains exactly one $(n+1)$-cell

$$\begin{tikzcd}
 & v \arrow[dr, bend left, "\alpha'"] \arrow[dd, "A"] & \\
u \arrow[ur, bend left, "\alpha"] \arrow[dr, bend right, "\beta"'] & & u' \\
 & w \arrow[ur, bend right, "\beta'"'] &
\end{tikzcd}$$

for every critical branching (α, β) of Σ.

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Squier's completion and finite derivation type

Proposition.

If Σ is a convergent presentation of an $(n-1)$-category $\mathbf{C}$, that is $\mathbf{C} \simeq \Sigma_{n-1}^{*}/\Sigma_{n}$, then a Squier's completion $\mathcal{S}(\Sigma) = (\Sigma, \Gamma)$ is a coherent presentation of $\mathbf{C}$, that is $\Sigma_{n}^{\top}/\Gamma$ is aspherical.

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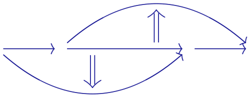
Regular critical branchings



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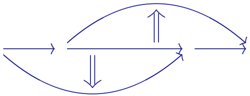
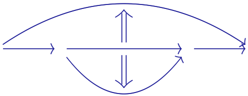


Inclusion critical branchings



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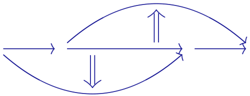

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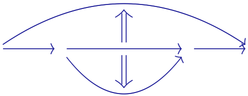
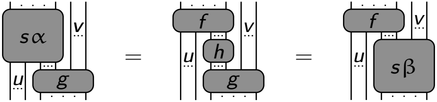


▶ For $n \geqslant 3$, there exist finite convergent n-polygraphs which does not have finite derivation type.

Critical branchings in 3-polygraphs

- **Regular** critical branchings:



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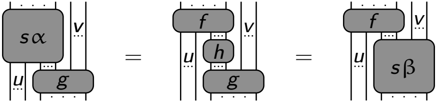

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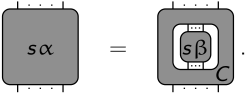

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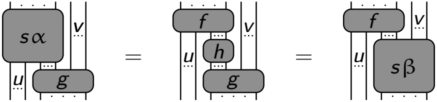

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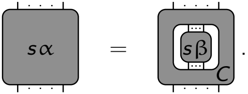
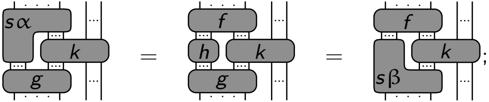


- **Right-indexed** critical branchings:



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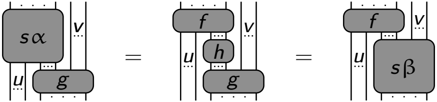
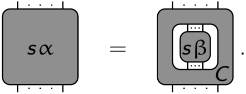
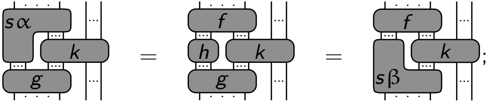


- **Inclusion** critical branchings:



- **Right-indexed** critical branchings:



- **Left-indexed** critical branchings, **multi-indexed** critical branchings.

Critical branchings in 3-polygraphs

Proposition.

Let Σ be a finite, convergent 3-polygraph.

▷ If Σ does not have indexed critical branchings, then Σ has finite derivation type.

▷ If Σ has indexed critical branchings, but each of them has a finite number of normal instances, then Σ has finite derivation type.

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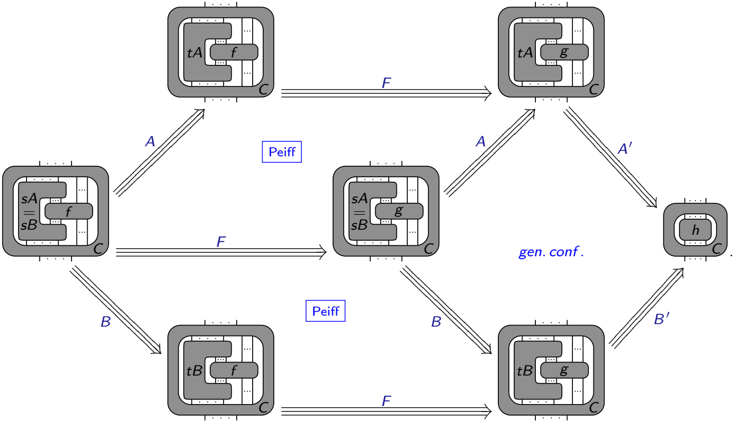

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For every $n \geqslant 2$, there exists an n-category which does not have finite derivation type and admits a presentation by a finite convergent $(n+1)$-polygraph.

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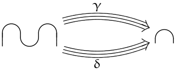

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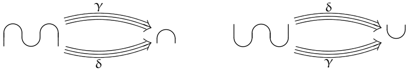

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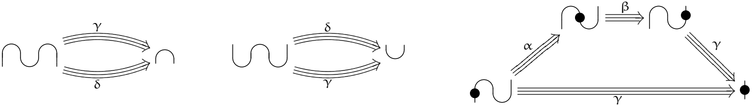

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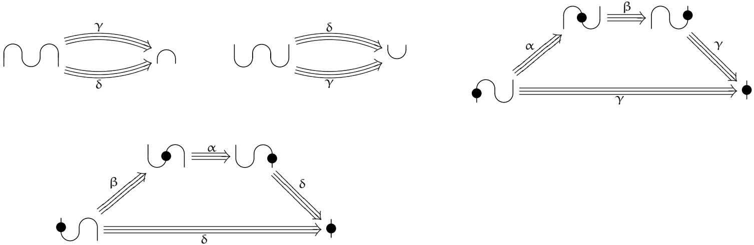

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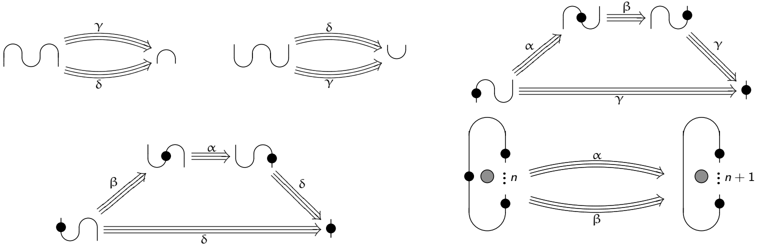

Part III. Homological syzygies from convergence.

- **Proof of Theorem A.**
- **Polygraphic resolutions from convergence.**

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Theorem. (Anick 1986, Kobayashi 1990, Groves 1990, Brown 1992)

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▶ A construction with $(\infty, 1)$-polygraphs.

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- **Higher-dimensional normalisation strategies for acyclicity.**

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- (Guiraud-M., 2012) Method to compute polygraphic resolutions for 1-categories from convergence.

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- Let $\Sigma = (\Sigma_0, \Sigma_1, \ldots, \Sigma_n)$ be an $(n,1)$-polygraph and $\mathbf{C}$ be the 1-category presented by Σ.

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$$\begin{tikzcd}
u \arrow[rr, Rightarrow, bend left, "f"] \arrow[dr, Rightarrow, "\sigma_u"'] & {} \arrow[d, Rrightarrow, shorten >=12pt, "\sigma_f"] & v \\
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  - For a 3-cell A of $\Sigma_3^\top$,

$$\begin{tikzcd} u \arrow[r, Rightarrow, bend left=50, "f", ""{name=F, below}] \arrow[r, Rightarrow, bend right=50, "g"', ""{name=G, above}] & v \arrow[from=F, to=G, phantom, "\Downarrow A"] \end{tikzcd}$$

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$$\begin{tikzcd} u \arrow[rr, Rightarrow, bend left=60, "f"] \arrow[rr, Rightarrow, bend right=60, "g"'] \arrow[r, Rightarrow, "\sigma_u"] & \widehat{u} \arrow[r, Rightarrow, "\sigma_v^-"] & v \end{tikzcd}$$

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$$\begin{tikzcd} & \Downarrow \sigma_f & \\ u \arrow[r, Rightarrow, "\sigma_u"] \arrow[rr, Rightarrow, bend left=70, "f"] \arrow[rr, Rightarrow, bend right=70, "g"'] & \widehat{u} \arrow[r, Rightarrow, "\sigma_v^-"] & v \end{tikzcd}$$

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$$\begin{tikzcd}
u \arrow[rr, Rightarrow, bend left=20, "f"] \arrow[dr, Rightarrow, "\sigma_u"'] & {} \arrow[d, phantom, "\Downarrow \sigma_f"] & v \\
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\end{tikzcd}$$

  - For a 3-cell A of $\Sigma_3^\top$,

$$\begin{tikzcd}
u \arrow[r, Rightarrow, bend left=50, "f", ""{name=F, below}] \arrow[r, Rightarrow, bend right=50, "g"', ""{name=G, above}] & v \arrow[from=F, to=G, phantom, "\Downarrow A"]
\end{tikzcd}$$

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$$\begin{tikzcd}
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\end{tikzcd}$$

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$$\begin{tikzcd}
u \arrow[r, Rightarrow, bend left=50, "f", ""{name=F, below}] \arrow[r, Rightarrow, bend right=50, "g"', ""{name=G, above}] & v \arrow[phantom, from=F, to=G, "\Downarrow A"]
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\quad \overset{\sigma_A}{\Rrightarrow} \quad
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\end{tikzcd}$$

Normalisation strategy

- A **normalisation strategy** for Σ is a mapping σ of every k-cell f of $\Sigma_k^\top$ to a $(k+1)$-cell of $\Sigma_{k+\mathbf{1}}^\top$

$$\begin{tikzcd}
s_{k-\mathbf{1}}(f) \arrow[rr, Rightarrow, bend left, "f"] \arrow[dr, Rightarrow, "\sigma_{s_{k-\mathbf{1}}(f)}"'] & \arrow[d, Rrightarrow, shorten >=6pt, "\sigma_f"] & t_{k-\mathbf{1}}(f) \\
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Theorem. (Guiraud-M., 2012)
An $(n, 1)$-polygraph is acyclic if and only if it admits a normalisation strategy.

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$$F_{\mathbf{C}}[\Sigma_n] \xrightarrow{d_n} F_{\mathbf{C}}[\Sigma_{n-1}] \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} F_{\mathbf{C}}[\Sigma_1] \xrightarrow{d_1} F_{\mathbf{C}}[\Sigma_0] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

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If Σ is a polygraphic resolution of a 1-category $\mathbf{C}$, then the complex $F_{\mathbf{C}}[\Sigma]$ is a free resolution of the trivial natural system $\mathbb{Z}$.

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Consequence.

▹ If $\mathbf{C}$ has a finite convergent presentation, then $\mathbf{C}$ is of homological type FP_∞.

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- Suppose that Σ is a reduced convergent 2-polygraph.

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- Denote ρ_u the **rightmost** rewriting step on a reducible 1-cell u:

$$\begin{tikzcd}[column sep=large] {} \arrow[rr, "u"] & {} \arrow[r, bend right=60, ""{name=L}] \arrow[r, phantom, "\Downarrow\rho_u"] & {} \end{tikzcd}$$

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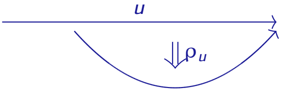

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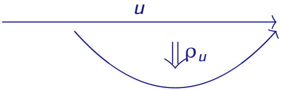

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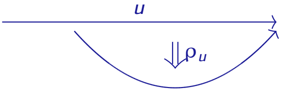

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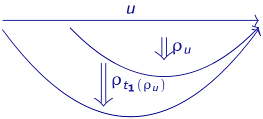

$$\rho_u = \rho_u \star_1 \rho_{t_1(\rho_u)}$$

Basis of generating confluences

- Critical branchings of Σ are of the following shape:

$$\begin{tikzcd}
{} \arrow[r] \arrow[rr, bend left=60] \arrow[rr, phantom, bend left=30, "\varphi\Uparrow"] & {} \arrow[r] \arrow[rr, bend right=60] \arrow[rr, phantom, bend right=30, "\Downarrow\psi"] & {} \arrow[r] & {}
\end{tikzcd}$$

with φ and ψ in Σ_2.

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with φ and ψ in Σ_2.

- Suppose that Σ is equipped with its rightmost normalisation strategy ρ.

Basis of generating confluences

- Critical branchings of Σ are of the following shape:

$$\begin{tikzcd}
{} \arrow[r] \arrow[rr, bend left=60] \arrow[r, phantom, shift left=4ex, "\varphi\Uparrow"] & {} \arrow[r] \arrow[rr, bend right=60] \arrow[r, phantom, shift right=4ex, "\Downarrow\psi"] & {} \arrow[r] & {}
\end{tikzcd}$$

with φ and ψ in Σ_2.

- Suppose that Σ is equipped with its rightmost normalisation strategy ρ.
  - ▷ Any critical branching has the shape $(\varphi\widehat{u}, \rho_{u_1\widehat{u}})$:

$$\begin{tikzcd}
{} \arrow[r, bend left=60, "\widehat{u}_1"] \arrow[r, "u_1"] \arrow[r, phantom, shift left=3ex, "\varphi\Uparrow"] \arrow[rr, bend right=50] \arrow[rr, phantom, shift right=4ex, "\Downarrow\rho_{u_1\widehat{u}}"] & {} \arrow[r, "\widehat{u}"] & {}
\end{tikzcd}$$

with φ in Σ_2.

Basis of generating confluences

- Any critical branching has the shape $(\varphi\widehat{u},\, \rho_{u_1\widehat{u}})$:

$$\begin{tikzcd} {} \arrow[r, bend left=70, "\widehat{u}_1"{name=T}] \arrow[r, "u_1"'{name=M}] \arrow[rr, bend right=50, ""{name=B}] & {} \arrow[r, "\widehat{u}"] & {} \arrow[from=M, to=T, phantom, "\varphi\Uparrow"] \arrow[from=M, to=B, phantom, "\Downarrow\rho_{u_1\widehat{u}}"] \end{tikzcd}$$

with φ in Σ_2.

- The **basis of generating confluences** of Σ is the cellular extension $\mathrm{Cg}_3(\Sigma)$ of $\Sigma_2^\top$ made of one 3-cell

$$\begin{tikzcd} & \widehat{u}_1\widehat{u} \arrow[dr, Rightarrow, bend left, "\rho_{\widehat{u}_1\widehat{u}}"] & \\ u_1\widehat{u} \arrow[ur, Rightarrow, bend left, "\varphi\widehat{u}"] \arrow[rr, Rightarrow, bend right, "\rho_{u_1\widehat{u}}"'] & \Downarrow\!\!\Downarrow\!\!\Downarrow\,\omega_b & \widehat{u_1\widehat{u}} \end{tikzcd}$$

for every critical branching $b = (\varphi\widehat{u},\, \rho_{u_1\widehat{u}})$.

Basis of generating confluences

- Any critical branching has the shape $(\varphi\,\widehat{u},\ \rho_{u_1\,\widehat{u}})$:

$$\begin{tikzcd}
\cdot \arrow[r, bend left=60, "\widehat{u}_1"{name=T}] \arrow[r, "u_1"'{name=M}] \arrow[rr, bend right=50, ""{name=B}] & \cdot \arrow[r, "\widehat{u}"] & \cdot
\arrow[from=M, to=T, phantom, "\varphi\Uparrow"]
\arrow[from=M, to=B, phantom, "\Downarrow\rho_{u_1\,\widehat{u}}"]
\end{tikzcd}$$

with φ in Σ_2.

- The **basis of generating confluences** of Σ is the cellular extension $\mathrm{Cg}_3(\Sigma)$ of $\Sigma_2^\top$ made of one 3-cell

$$\begin{tikzcd}
 & \widehat{u}_1\,\widehat{u} \arrow[dr, Rightarrow, "\rho_{\widehat{u}_1\,\widehat{u}}"] & \\
u_1\,\widehat{u} \arrow[ur, Rightarrow, "\varphi\,\widehat{u}"] \arrow[rr, Rightarrow, bend right=30, "\rho_{u_1\,\widehat{u}}"'] \arrow[rr, phantom, "\Downarrow\!\!\!\Downarrow\!\!\!\Downarrow\omega_b"] & & \widehat{u_1\,\widehat{u}}
\end{tikzcd}$$

for every critical branching $b = (\varphi\,\widehat{u},\ \rho_{u_1\,\widehat{u}})$.

Theorem B. (Squier's Theorem)

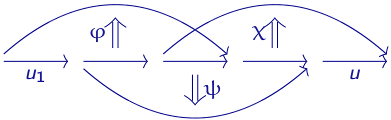
The $(3,1)$-polygraph

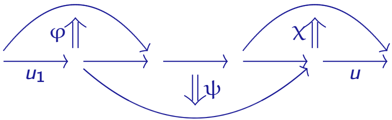
$$\begin{tikzcd}
\Sigma_0 & \Sigma_1^* \arrow[l, shift left] \arrow[l, shift right] & \Sigma_2^\top \arrow[l, shift left] \arrow[l, shift right] & \\
 & \Sigma_1 \arrow[ul, shift left, "s_0"'] \arrow[ul, shift right, "t_0"] \arrow[u, tail] & \Sigma_2 \arrow[ul, shift left, "s_1"'] \arrow[ul, shift right, "t_1"] \arrow[u, tail] & \mathrm{Cg}_3(\Sigma) \arrow[ul, shift left, "s_2"'] \arrow[ul, shift right, "t_2"]
\end{tikzcd}$$

is acyclic.

The basis of generating triple confluences

- A **critical** triple branching is an overlapping of three rewriting steps:





  - ▷ For both shapes, the corresponding critical triple branching can be written

$$b = \left(\, c\widehat{u},\ \rho_{u'\widehat{u}} \,\right) = \left(\, f\widehat{u},\ \rho_{u'}\widehat{u},\ \rho_{u'\widehat{u}} \,\right)$$

where $c = (f, \rho_{u'})$ is a critical branching and $\rho_{u'} = u_1\psi$.

The basis of generating triple confluences

- The **basis of generating triple confluences** is the cellular extension $\mathrm{Cg}_4(\Lambda)$ of $\mathrm{Cg}_3(\Lambda)^\top$ made of one 4-cell

$$\begin{tikzcd}
& \widehat{u}'\widehat{u} \arrow[dr, Rightarrow, "\rho_{\widehat{u}'\widehat{u}}"] \arrow[d, phantom, "\Downarrow \rho_A"] & & & & \widehat{u}'\widehat{u} \arrow[dr, Rightarrow, "\rho_{\widehat{u}'\widehat{u}}"] \arrow[d, phantom, "\Downarrow \rho_B"] & \\
u'\widehat{u} \arrow[ur, Rightarrow, bend left=40, "f^*\widehat{u}"] \arrow[ur, Rightarrow, "\rho_{u'}\widehat{u}"'] \arrow[ur, phantom, bend left=20, "\Downarrow \omega_c\widehat{u}"] \arrow[rr, Rightarrow, bend right=10, "\rho_{u'\widehat{u}}"'] & & \widehat{u'u} \arrow[r, phantom, "\overset{\omega_b}{\Rrightarrow}"] & {} & u'\widehat{u} \arrow[ur, Rightarrow, "f^*\widehat{u}"] \arrow[rr, Rightarrow, bend right=10, "\rho_{u'\widehat{u}}"'] & & \widehat{u'u}
\end{tikzcd}$$

for every critical triple branching

$$b = (f\widehat{u}, \rho_{u'}\widehat{u}, \rho_{u'\widehat{u}})$$

where $c = (f, \rho_{u'})$ is a critical branching.

The basis of generating triple confluences

- The **basis of generating triple confluences** is the cellular extension $\mathrm{Cg}_4(\Lambda)$ of $\mathrm{Cg}_3(\Lambda)^{\top}$ made of one 4-cell

$$\begin{tikzcd}
& \widehat{u}'\widehat{u} \arrow[dr, Rightarrow, bend left, "\rho_{\widehat{u}'\widehat{u}}"] \arrow[d, phantom, "\Downarrow \rho_A"] & & & & \widehat{u}'\widehat{u} \arrow[dr, Rightarrow, "\rho_{\widehat{u}'\widehat{u}}"] \arrow[d, phantom, "\Downarrow \rho_B"] & \\
u'\widehat{u} \arrow[ur, Rightarrow, bend left=60, "f^*\widehat{u}"] \arrow[ur, Rightarrow, "\rho_{u'}\widehat{u}"'] \arrow[ur, phantom, bend left=30, "\Rrightarrow \omega_c\widehat{u}"] \arrow[rr, Rightarrow, bend right, "\rho_{u'\widehat{u}}"'] & & \widehat{u'u} & \overset{\omega_b}{\Rrightarrow} & u'\widehat{u} \arrow[ur, Rightarrow, "f^*\widehat{u}"] \arrow[rr, Rightarrow, bend right, "\rho_{u'\widehat{u}}"'] & & \widehat{u'u}
\end{tikzcd}$$

for every critical triple branching

$$b = (f\widehat{u}, \rho_{u'}\widehat{u}, \rho_{u'\widehat{u}})$$

where $c = (f, \rho_{u'})$ is a critical branching.

Proposition.

The $(4,1)$-polygraph

$$\begin{tikzcd}
\Sigma_0 & \Sigma_1^{\ell} \arrow[l, shift left] \arrow[l, shift right] & \Lambda_2^{\top} \arrow[l, shift left] \arrow[l, shift right] & \mathrm{Cg}_3(\Lambda)^{\top} \arrow[l, shift left] \arrow[l, shift right] & \\
& \Sigma_1 \arrow[ul, shift left, "s_0"] \arrow[ul, shift right, "t_0"'] \arrow[u, tail] & \Lambda_2 \arrow[ul, shift left, "s_1"] \arrow[ul, shift right, "t_1"'] \arrow[u, tail] & \mathrm{Cg}_3(\Lambda) \arrow[ul, shift left, "s_2"] \arrow[ul, shift right, "t_2"'] \arrow[u, tail] & \mathrm{Cg}_4(\Lambda) \arrow[ul, shift left, "s_2"] \arrow[ul, shift right, "t_2"']
\end{tikzcd}$$

is acyclic.

Basis of generating n-fold confluences

- An **n-critical branching** of Σ has the shape

$$b \;=\; \left(\, c\widehat{u},\; \rho_{u'\widehat{u}} \,\right)$$

where c is a critical $(n-1)$-fold branching with source u'.

- The **basis of generating n-fold confluences** is the cellular extension $\mathrm{Cg}_{n+1}(\Sigma)$ of $\mathrm{Cg}_n(\Sigma)^\top$ made of one $(n+1)$-cell

$$\omega_b \;:\; (\omega_c \widehat{u})^* \;\longrightarrow\; \widehat{\omega_c u}^*$$

for every critical n-fold branching $b = (c\widehat{u}, \rho_{u'\widehat{u}})$.

Theorem. (Guiraud-M., 2012)

Any convergent 2-polygraph Σ extends to a Tietze-equivalent polygraphic resolution $\mathrm{Cg}_\infty(\Sigma)$

$$\begin{tikzcd}
\Sigma_0 & \Sigma_1^* \arrow[l, shift left] \arrow[l, shift right] & \Sigma_2^\top \arrow[l, shift left] \arrow[l, shift right] & \mathrm{Cg}_3(\Sigma)^\top \arrow[l, shift left] \arrow[l, shift right] & \cdots \arrow[l, shift left] \arrow[l, shift right] & \mathrm{Cg}_n(\Sigma)^\top \arrow[l, shift left] \arrow[l, shift right] & \cdots \arrow[l, shift left] \arrow[l, shift right] \\
& \Sigma_1 \arrow[ul, shift left, "s_0"] \arrow[ul, shift right, "t_0"'] \arrow[u, tail] & \Sigma_2 \arrow[ul, shift left, "s_1"] \arrow[ul, shift right, "t_1"'] \arrow[u, tail] & \mathrm{Cg}_3(\Sigma) \arrow[ul, shift left, "s_2"] \arrow[ul, shift right, "t_2"'] \arrow[u, tail] & \cdots \arrow[ul, shift left] \arrow[ul, shift right] & \mathrm{Cg}_n(\Sigma) \arrow[ul, shift left, "s_{n-1}"] \arrow[ul, shift right, "t_{n-1}"'] \arrow[u, tail] & \cdots \arrow[ul, shift left] \arrow[ul, shift right]
\end{tikzcd}$$

whose n-cells, for $n \geqslant 3$, are indexed by the critical $(n-1)$-fold branchings.

Part IV. Linear rewriting

- **Linear 2-polygraphs.**
- **Linear polygraphic resolutions and Koszulity.**

Motivation

- Consider a homogeneous algebras **A** (eg. quadratic algebras, $xy = x^2 + zy$, ...)

Motivation

- Consider a homogeneous algebras $\mathbf{A}$ (eg. quadratic algebras, $xy = x^2 + zy$, ...)
  - The algebra $\mathbf{A}$ is naturally graded:

$$\mathbf{A} = \mathbf{A}_0 \oplus \mathbf{A}_1 \oplus \mathbf{A}_2 \oplus \mathbf{A}_3 \oplus \mathbf{A}_4 \oplus \ \cdots \oplus \mathbf{A}_k \oplus \mathbf{A}_{k+1} \oplus \cdots$$

$\mathbf{A}_0 = \mathbb{K} \ni 1, \quad \mathbf{A}_1 = \mathbb{K}\langle X \rangle \ni x, y, x+y, \quad \mathbf{A}_2 \ni x^2, x^2 + y^2$, ...

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$\mathbf{A}_0 = \mathbb{K} \ni 1, \quad \mathbf{A}_1 = \mathbb{K}\langle X \rangle \ni x, y, x + y, \quad \mathbf{A}_2 \ni x^2, x^2 + y^2$, ...

  - This induces a graduation on the vectors spaces $\mathrm{Tor}^{\mathbf{A}}_{k,(i)}(\mathbb{K}, \mathbb{K})$,
    - k refers to the homological degree and (i) refers to the weight grading.

⋮	⋮	⋮	⋮	⋮		
5	●	●	●	●	●	⋯
4	●	●	●	●	●	⋯
3	●	●	●	●	0	⋯
2	●	●	●	0	0	⋯
1	●	●	0	0	0	⋯
0	●	0	0	0	0	⋯
k	$\mathrm{Tor}^{\mathbf{A}}_{0}(\mathbb{K}, \mathbb{K})$	$\mathrm{Tor}^{\mathbf{A}}_{1}(\mathbb{K}, \mathbb{K})$	$\mathrm{Tor}^{\mathbf{A}}_{2}(\mathbb{K}, \mathbb{K})$	$\mathrm{Tor}^{\mathbf{A}}_{3}(\mathbb{K}, \mathbb{K})$	$\mathrm{Tor}^{\mathbf{A}}_{4}(\mathbb{K}, \mathbb{K})$	⋯

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$\mathbf{A}_0 = \mathbb{K} \ni 1$, $\quad \mathbf{A}_1 = \mathbb{K}\langle X \rangle \ni x, y, x + y$, $\quad \mathbf{A}_2 \ni x^2, x^2 + y^2$, ...

- This induces a graduation on the vectors spaces $\mathrm{Tor}^{\mathbf{A}}_{k,(i)}(\mathbb{K}, \mathbb{K})$,
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⋮	⋮	⋮	⋮	⋮		
5	0	0	0	0	0	⋯
4	0	0	0	0	●	⋯
3	0	0	0	●	0	⋯
2	0	0	●	0	0	⋯
1	0	●	0	0	0	⋯
0	●	0	0	0	0	⋯
k	$\mathrm{Tor}^{\mathbf{A}}_{0}(\mathbb{K}, \mathbb{K})$	$\mathrm{Tor}^{\mathbf{A}}_{1}(\mathbb{K}, \mathbb{K})$	$\mathrm{Tor}^{\mathbf{A}}_{2}(\mathbb{K}, \mathbb{K})$	$\mathrm{Tor}^{\mathbf{A}}_{3}(\mathbb{K}, \mathbb{K})$	$\mathrm{Tor}^{\mathbf{A}}_{4}(\mathbb{K}, \mathbb{K})$	⋯

Definition. A graded algebra $\mathbf{A}$ is **Koszul** if the $\mathrm{Tor}^{\mathbf{A}}_{k,(i)}(\mathbb{K}, \mathbb{K})$ are **"*concentrated on the diagonal*"**:

$$\mathrm{Tor}^{\mathbf{A}}_{k,(i)}(\mathbb{K}, \mathbb{K}) = 0, \quad \text{for } k \neq i.$$

Motivation

Theorem. (Priddy, 1970)

An algebra admitting a Poincaré-Birkhoff-Witt basis is Koszul.

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An algebra having a presentation by a quadratic Gröbner basis is Koszul.

Proofs:

▷ Anick, 1986, Green, 1999. Computation of free resolutions using non-commutative Gröbner bases.

— Hilbert series, Poincaré-Betti series, Betti numbers, ...

Description of the vector spaces $\mathrm{Tor}^{\mathbf{A}}_{k,(i)}(\mathbb{K},\mathbb{K})$ in term of **k-fold critical branching**.

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▷ Berger, 1998: Confluence and Koszulity (X-**confluence**).

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Description of the vector spaces $\mathrm{Tor}^{\mathbf{A}}_{k,(i)}(\mathbb{K},\mathbb{K})$ in term of ***k*-fold critical branching**.

▷ Berger, 1998: Confluence and Koszulity (***X*-confluence**).

Definition. (Berger, 2001)

An N-homogeneous algebra $\mathbf{A}$ is **Koszul** if

$$\mathrm{Tor}^{\mathbf{A}}_{k,(i)}(\mathbb{K},\mathbb{K}) = 0, \quad \text{for } i \neq \ell_N(k), \quad \text{where} \quad \ell_N(k) = \begin{cases} lN & \text{if } k = 2l \\ lN+1 & \text{if } k = 2l+1 \end{cases}$$

Motivation

- Computation of free resolutions using Gröbner bases (Anick, 1986, Green, 1999, ...)

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  - Given a Gröbner basis for $\langle R\rangle$ wrt a monomial order, that is a subset G of R such that
    - $\langle G\rangle = \langle R\rangle$,
    - $\langle \mathrm{lt}(G)\rangle = \langle \mathrm{lt}(R)\rangle$.

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  - Given a Gröbner basis for $\langle R\rangle$ wrt a monomial order, that is a subset G of R such that
    - $\langle G\rangle = \langle R\rangle$,
    - $\langle \mathrm{lt}(G)\rangle = \langle \mathrm{lt}(R)\rangle$.

- Anick's resolution:

$$0 \longleftarrow \mathbb{K} \xleftarrow{\delta_{-1}} \mathbf{A} \xleftarrow{\delta_0} \mathbf{A}[X] \xleftarrow{\delta_1} \mathbf{A}[R] \xleftarrow{\delta_2} \mathbf{A}[\mathcal{O}_2] \longleftarrow \ \ldots \ \longleftarrow \mathbf{A}[\mathcal{O}_{n-1}] \xleftarrow{\delta_n} \mathbf{A}[\mathcal{O}_n] \longleftarrow \cdots$$

where

- $\mathbf{A}[\mathcal{O}_n]$ is the free $\mathbf{A}$-module generated by minimal ***n*-fold overlapping**,
- the map δ_n decomposes n-fold overlappings into $(n-1)$-fold overlappings of G.

Motivation

Example.

$$\mathbf{A}\langle\, x, y, z \;\mid\; xyz = x^3 + y^3 + z^3 \,\rangle$$

Motivation

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$$\mathbf{A}\langle\, x, y, z \;\mid\; xyz = x^3 + y^3 + z^3 \,\rangle$$

- Gröbner basis wrt lexicographic order with $x < y < z$

$$G = \left\{\; z^3 - xyz + x^3 + y^3 \;,\; zy^3 - zxyz + zx^3 + xyz^2 - x^3z - y^3z \;\right\}$$

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$$0 \longleftarrow \mathbb{K} \xleftarrow{\delta_{-1}} \mathbf{A} \xleftarrow{\delta_0} \mathbf{A}[x, y, z] \xleftarrow{\delta_1} \mathbf{A}[z^3, zy^3] \xleftarrow{\delta_2} \mathbf{A}[z^4, z^3y^3] \xleftarrow{\delta_3} \mathbf{A}[z^5, z^4y^3] \longleftarrow \ \ldots$$

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  - giving an infinite free resolution,

Motivation

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  - giving an infinite free resolution,
  - that computes $\mathrm{Tor}^{\mathbf{A}}_{k,(i)}(\mathbb{K}, \mathbb{K})$:

k	$\mathrm{Tor}^{\mathbf{A}}_0(\mathbb{K},\mathbb{K})$	$\mathrm{Tor}^{\mathbf{A}}_1(\mathbb{K},\mathbb{K})$	$\mathrm{Tor}^{\mathbf{A}}_2(\mathbb{K},\mathbb{K})$	$\mathrm{Tor}^{\mathbf{A}}_3(\mathbb{K},\mathbb{K})$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
4	0	0	0	0	$\cdots$
3	0	0	$\mathbb{K}$	0	$\cdots$
2	0	0	0	0	$\cdots$
1	0	$\mathbb{K}^3$	0	0	$\cdots$
0	$\mathbb{K}$	0	0	0	$\cdots$

Motivation

Example.

$$\mathbf{A}\langle\, x, y, z \mid xyz = x^3 + y^3 + z^3 \,\rangle$$

- Gröbner basis wrt lexicographic order with $x < y < z$

$$G = \left\{\, z^3 - xyz + x^3 + y^3 \,,\; zy^3 - zxyz + zx^3 + xyz^2 - x^3z - y^3z \,\right\}$$

$$0 \longleftarrow \mathbb{K} \xleftarrow{\delta_{-1}} \mathbf{A} \xleftarrow{\delta_0} \mathbf{A}[x, y, z] \xleftarrow{\delta_1} \mathbf{A}[z^3, zy^3] \xleftarrow{\delta_2} \mathbf{A}[z^4, z^3y^3] \xleftarrow{\delta_3} \mathbf{A}[z^5, z^4y^3] \longleftarrow \;\ldots$$

  - giving an infinite free resolution,
  - that computes $\mathrm{Tor}^{\mathbf{A}}_{k,(i)}(\mathbb{K}, \mathbb{K})$:

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
4	0	0	0	0	$\cdots$
3	0	0	$\mathbb{K}$	0	$\cdots$
2	0	0	0	0	$\cdots$
1	0	$\mathbb{K}^3$	0	0	$\cdots$
0	$\mathbb{K}$	0	0	0	$\cdots$
k	$\mathrm{Tor}^{\mathbf{A}}_0(\mathbb{K}, \mathbb{K})$	$\mathrm{Tor}^{\mathbf{A}}_1(\mathbb{K}, \mathbb{K})$	$\mathrm{Tor}^{\mathbf{A}}_2(\mathbb{K}, \mathbb{K})$	$\mathrm{Tor}^{\mathbf{A}}_3(\mathbb{K}, \mathbb{K})$	$\cdots$

  - It follows that the algebra $\mathbf{A}$ is Koszul.

Motivation

Example. (Backelin 1991, Polishchuk-Positselski 2005)

- ▷ A Koszul algebra that has no Poincaré-Birkhoff-Witt basis:

$$\mathbf{A}\langle\, x, y, z \;\mid\; x^2 + yz = 0,\; x^2 + azy = 0 \,\rangle \qquad \text{with } a \neq 0, 1$$

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- ▷ The algebra $\mathbf{A}$ is presented by the **linear rewriting system**

$$yz \overset{\alpha}{\Longrightarrow} -x^2 \qquad\qquad zy \overset{\beta}{\Longrightarrow} -\frac{1}{a}x^2$$

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- ▷ The algebra **A** is presented by the **linear rewriting system**

$$yz \xRightarrow{\alpha} -x^2 \qquad\qquad zy \xRightarrow{\beta} -\frac{1}{a}x^2$$

$$\begin{tikzcd}
 & -\frac{1}{a}yx^2 & & -\frac{1}{a}x^2z \\
yzy \arrow[ur, Rightarrow, bend left, "y\beta"] \arrow[dr, Rightarrow, bend right, "\alpha y"'] & & zyz \arrow[ur, Rightarrow, bend left, "\beta z"] \arrow[dr, Rightarrow, bend right, "z\alpha"'] & \\
 & -x^2y & & -zx^2
\end{tikzcd}$$

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- The algebra **A** is presented by the **linear rewriting system**

$$yz \xRightarrow{\alpha} -x^2 \qquad zy \xRightarrow{\beta} -\frac{1}{a}x^2 \qquad yx^2 \xRightarrow{\gamma} ax^2y \qquad zx^2 \xRightarrow{\delta} \frac{1}{a}x^2z$$

$$\begin{tikzcd}
 & -\frac{1}{a}yx^2 \arrow[dd, Rightarrow, "-\frac{1}{a}\gamma"] \\
yzy \arrow[ur, Rightarrow, bend left, "y\beta"] \arrow[dr, Rightarrow, bend right, "\alpha y"'] & \\
 & -x^2y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & -\frac{1}{a}x^2z \\
zyz \arrow[ur, Rightarrow, bend left, "\beta z"] \arrow[dr, Rightarrow, bend right, "z\alpha"'] & \\
 & -zx^2 \arrow[uu, Rightarrow, "-\delta"']
\end{tikzcd}$$

Motivation

Example. (Backelin 1991, Polishchuk-Positselski 2005)

- ▷ A Koszul algebra that has no Poincaré-Birkhoff-Witt basis:

$$\mathbf{A}\langle\, x, y, z \;\mid\; x^2 + yz = 0,\; x^2 + azy = 0 \,\rangle \qquad \text{with } a \neq 0, 1$$

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\end{tikzcd}
\qquad
\begin{tikzcd}
 & -\frac{1}{a}x^4 \\
zyx^2 \arrow[ur, Rightarrow, bend left, "\beta x^2"] \arrow[dr, Rightarrow, bend right, "z\gamma"'] & \\
 & azx^2y
\end{tikzcd}$$

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\qquad
\begin{tikzcd} & -\frac{1}{a}x^2z \\ zyz \arrow[ur, Rightarrow, bend left, "\beta z"] \arrow[dr, Rightarrow, bend right, "z\alpha"'] & \\ & -zx^2 \arrow[uu, Rightarrow, "-\delta"'] \end{tikzcd}
\qquad
\begin{tikzcd} & -\frac{1}{a}x^4 & \\ zyx^2 \arrow[ur, Rightarrow, bend left, "\beta x^2"] \arrow[dr, Rightarrow, bend right, "z\gamma"'] & & x^2zy \arrow[ul, Rightarrow, bend right, "x^2\beta"'] \\ & azx^2y \arrow[ur, Rightarrow, bend right, "a\delta y"'] & \end{tikzcd}$$

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\end{tikzcd}
\qquad
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\end{tikzcd}
\qquad
\begin{tikzcd}
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$$\begin{tikzcd}
 & -\frac{1}{a}yx^2 \arrow[dd, Rightarrow, "-\frac{1}{a}\gamma"] \arrow[ddl, Rightarrow, shorten=12pt, "A_1"] \\
yzy \arrow[ur, Rightarrow, bend left, "y\beta"] \arrow[dr, Rightarrow, bend right, "\alpha y"'] & \\
 & -x^2y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & -\frac{1}{a}x^2z \\
zyz \arrow[ur, Rightarrow, bend left, "\beta z"] \arrow[dr, Rightarrow, bend right, "z\alpha"'] & \\
 & -zx^2 \arrow[uu, Rightarrow, "-\delta"'] \arrow[uul, Rightarrow, shorten=12pt, "B_1"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & -\frac{1}{a}x^4 & \\
zyx^2 \arrow[ur, Rightarrow, bend left, "\beta x^2"] \arrow[dr, Rightarrow, bend right, "z\gamma"'] & & x^2zy \arrow[ul, Rightarrow, bend right, "x^2\beta"'] \arrow[ll, Rightarrow, shorten=20pt, "C_1"] \\
 & azx^2y \arrow[ur, Rightarrow, bend right, "a\delta y"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & -x^4 & \\
yzx^2 \arrow[ur, Rightarrow, bend left, "\alpha x^2"] \arrow[dr, Rightarrow, bend right, "y\delta"'] & & x^2yz \arrow[ul, Rightarrow, bend right, "x^2\alpha"'] \arrow[ll, Rightarrow, shorten=20pt, "D_1"] \\
 & \frac{1}{a}yx^2z \arrow[ur, Rightarrow, bend right, "\frac{1}{a}\gamma z"'] &
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yzy \arrow[ur, Rightarrow, bend left, "y\beta"] \arrow[r, Rightarrow, bend right, "\alpha y"'] & -x^2y
\end{tikzcd}
\qquad
\begin{tikzcd}
& -\frac{1}{a}x^2z \\
zyz \arrow[ur, Rightarrow, bend left, "\beta z"] \arrow[r, Rightarrow, bend right, "z\alpha"'] \arrow[r, phantom, "B_1"] & -zx^2 \arrow[u, Rightarrow, "-\delta"']
\end{tikzcd}$$

$$\begin{tikzcd}
& -\frac{1}{a}x^4 & \\
zyx^2 \arrow[ur, Rightarrow, bend left, "\beta x^2"] \arrow[dr, Rightarrow, bend right, "z\gamma"'] & \arrow[l, phantom, "C_1"] & x^2zy \arrow[ul, Rightarrow, bend right, "x^2\beta"'] \\
& azx^2y \arrow[ur, Rightarrow, bend right, "a\delta y"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& -x^4 & \\
yzx^2 \arrow[ur, Rightarrow, bend left, "\alpha x^2"] \arrow[dr, Rightarrow, bend right, "y\delta"'] & \arrow[l, phantom, "D_1"] & x^2yz \arrow[ul, Rightarrow, bend right, "x^2\alpha"'] \\
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\end{tikzcd}$$

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& -x^2yz \\
yzyz \arrow[ur, Rightarrow, bend left, "\alpha yz"] \arrow[r, Rightarrow, "y\beta z"] \arrow[dr, Rightarrow, bend right, "yz\alpha"'] & byx^2z \\
& -yzx^2
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$$\begin{tikzcd}
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yzy \arrow[urr, Rightarrow, bend left, "y\beta"] \arrow[drr, Rightarrow, bend right, "\alpha y"'] & & \\
 & & -x^2y
\arrow[from=1-3, to=3-2, Rrightarrow, "A_1"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & -\frac{1}{a}x^2z \\
zyz \arrow[urr, Rightarrow, bend left, "\beta z"] \arrow[drr, Rightarrow, bend right, "z\alpha"'] & & \\
 & & -zx^2 \arrow[uu, Rightarrow, "-\delta"']
\arrow[from=3-3, to=1-2, Rrightarrow, "B_1"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & -\frac{1}{a}x^4 & \\
zyx^2 \arrow[ur, Rightarrow, bend left, "\beta x^2"] \arrow[dr, Rightarrow, bend right, "z\gamma"'] & & x^2zy \arrow[ul, Rightarrow, bend right, "x^2\beta"'] \arrow[ll, Rrightarrow, phantom, "C_1"] \\
 & azx^2y \arrow[ur, Rightarrow, bend right, "a\delta y"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & -x^4 & \\
yzx^2 \arrow[ur, Rightarrow, bend left, "\alpha x^2"] \arrow[dr, Rightarrow, bend right, "y\delta"'] & & x^2yz \arrow[ul, Rightarrow, bend right, "x^2\alpha"'] \arrow[ll, Rrightarrow, phantom, "D_1"] \\
 & \frac{1}{a}yx^2z \arrow[ur, Rightarrow, bend right, "\frac{1}{a}\gamma z"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & -x^2yz \arrow[drr, Rightarrow, bend left, "-x^2\alpha"] & & \\
yzyz \arrow[urr, Rightarrow, bend left, "\alpha yz"] \arrow[rr, Rightarrow, "y\beta z" description] \arrow[drr, Rightarrow, bend right, "yz\alpha"'] & & byx^2z \arrow[u, Rightarrow, "b\gamma z" description] & & x^4 \\
 & & -yzx^2 \arrow[u, Rightarrow, "-y\delta"'] \arrow[urr, Rightarrow, bend right, "-\alpha x^2"'] & &
\arrow[from=2-3, to=1-2, Rrightarrow, "A_1 z"]
\arrow[from=3-3, to=2-2, Rrightarrow, "yB_1"]
\arrow[from=1-3, to=3-4, Rrightarrow, "-D_1"]
\end{tikzcd}$$

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$$\begin{tikzcd}
 & -\frac{1}{a}yx^2 \arrow[d, Rightarrow, "-\frac{1}{a}\gamma"] \arrow[ld, Rrightarrow, shorten=4mm, "A_1"] \\
yzy \arrow[ru, Rightarrow, bend left, "y\beta"] \arrow[r, Rightarrow, bend right, "\alpha y"'] & -x^2y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & -\frac{1}{a}x^2z \\
zyz \arrow[ru, Rightarrow, bend left, "\beta z"] \arrow[r, Rightarrow, bend right, "z\alpha"'] & -zx^2 \arrow[u, Rightarrow, "-\delta"'] \arrow[lu, Rrightarrow, shorten=4mm, "B_1"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & -\frac{1}{a}x^4 & \\
zyx^2 \arrow[ru, Rightarrow, bend left, "\beta x^2"] \arrow[rd, Rightarrow, bend right, "z\gamma"'] & & x^2zy \arrow[lu, Rightarrow, bend right, "x^2\beta"'] \arrow[ll, Rrightarrow, shorten=8mm, "C_1"] \\
 & azx^2y \arrow[ru, Rightarrow, bend right, "a\delta y"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & -x^4 & \\
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 & \frac{1}{a}yx^2z \arrow[ru, Rightarrow, bend right, "\frac{1}{a}\gamma z"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & -x^2yz \arrow[rd, Rightarrow, bend left, "-x^2\alpha"] \arrow[rd, Rrightarrow, shift left=8mm, shorten=6mm, "-D_1"] & \\
yzyz \arrow[rru, Rightarrow, bend left, "\alpha yz"] \arrow[rr, Rightarrow, "y\beta z" description] \arrow[rrd, Rightarrow, bend right, "yz\alpha"'] & & byx^2z \arrow[u, Rightarrow, "b\gamma z"] \arrow[lu, Rrightarrow, shorten=3mm, "A_1z"'] & x^4 \\
 & & -yzx^2 \arrow[u, Rightarrow, "-y\delta"'] \arrow[ru, Rightarrow, bend right, "-\alpha x^2"'] \arrow[lu, Rrightarrow, shorten=3mm, "yB_1"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & -x^2yz \arrow[rd, Rightarrow, bend left, "-x^2\alpha"] & \\
yzyz \arrow[ru, Rightarrow, bend left, "\alpha yz"] \arrow[rd, Rightarrow, bend right, "yz\alpha"'] \arrow[rr, phantom, "\equiv"] & & x^4 \\
 & -yzx^2 \arrow[ru, Rightarrow, bend right, "-\alpha x^2"'] &
\end{tikzcd}$$

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$$\begin{tikzcd}
 & -\frac{1}{a}yx^2 \arrow[dd, Rightarrow, "-\frac{1}{a}\gamma", ""{name=V}] \\
yzy \arrow[ur, Rightarrow, bend left, "y\beta"] \arrow[dr, Rightarrow, bend right, "\alpha y"', ""{name=L}] & \\
 & -x^2y
\arrow[Rightarrow, from=V, to=L, "A_1"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & -\frac{1}{a}x^2z \\
zyz \arrow[ur, Rightarrow, bend left, "\beta z", ""{name=U}] \arrow[dr, Rightarrow, bend right, "z\alpha"'] & \\
 & -zx^2 \arrow[uu, Rightarrow, "-\delta"', ""{name=V}]
\arrow[Rightarrow, from=V, to=U, "B_1"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & -\frac{1}{a}x^4 & \\
zyx^2 \arrow[ur, Rightarrow, bend left, "\beta x^2", ""{name=U}] \arrow[dr, Rightarrow, bend right, "z\gamma"'] & & x^2zy \arrow[ul, Rightarrow, bend right, "x^2\beta"'] \\
 & azx^2y \arrow[ur, Rightarrow, bend right, "a\delta y"', ""{name=D}] &
\arrow[Rightarrow, from=D, to=U, "C_1"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & -x^4 & \\
yzx^2 \arrow[ur, Rightarrow, bend left, "\alpha x^2", ""{name=U}] \arrow[dr, Rightarrow, bend right, "y\delta"'] & & x^2yz \arrow[ul, Rightarrow, bend right, "x^2\alpha"'] \\
 & \frac{1}{a}yx^2z \arrow[ur, Rightarrow, bend right, "\frac{1}{a}\gamma z"', ""{name=D}] &
\arrow[Rightarrow, from=D, to=U, "D_1"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & & -x^2yz \arrow[drr, Rightarrow, bend left, "-x^2\alpha"] & & \\
yzyz \arrow[urr, Rightarrow, bend left, "\alpha yz", ""{name=T}] \arrow[rr, Rightarrow, "y\beta z", ""{name=M}] \arrow[drr, Rightarrow, bend right, "yz\alpha"'] & & byx^2z \arrow[u, Rightarrow, "b\gamma z"] & & x^4 \\
 & & -yzx^2 \arrow[u, Rightarrow, "-y\delta"'] \arrow[urr, Rightarrow, bend right, "-\alpha x^2"', ""{name=B}] & &
\arrow[Rightarrow, from=2-3, to=T, "A_1z"']
\arrow[Rightarrow, from=3-3, to=M, "yB_1"]
\arrow[Rightarrow, from=1-3, to=B, "-D_1"]
\end{tikzcd}
\quad \Rrightarrow \quad
\begin{tikzcd}
 & -x^2yz \arrow[dr, Rightarrow, bend left, "-x^2\alpha"] & \\
yzyz \arrow[ur, Rightarrow, bend left, "\alpha yz"] \arrow[dr, Rightarrow, bend right, "yz\alpha"'] & \equiv & x^4 \\
 & -yzx^2 \arrow[ur, Rightarrow, bend right, "-\alpha x^2"'] &
\end{tikzcd}$$

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 & -x^2y
\end{tikzcd}
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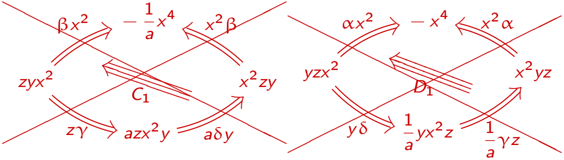
zyz \arrow[ur, Rightarrow, bend left, "\beta z"] \arrow[dr, Rightarrow, bend right, "z\alpha"'] & B_1 \\
 & -zx^2 \arrow[uu, Rightarrow, "-\delta"']
\end{tikzcd}$$



$$\begin{tikzcd}
 & -\frac{1}{a}x^4 & \\
zyx^2 \arrow[ur, Rightarrow, bend left, "\beta x^2"] \arrow[dr, Rightarrow, bend right, "z\gamma"'] & C_1 & x^2zy \arrow[ul, Rightarrow, bend right, "x^2\beta"'] \\
 & azx^2y \arrow[ur, Rightarrow, bend right, "a\delta y"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & -x^4 & \\
yzx^2 \arrow[ur, Rightarrow, bend left, "\alpha x^2"] \arrow[dr, Rightarrow, bend right, "y\delta"'] & D_1 & x^2yz \arrow[ul, Rightarrow, bend right, "x^2\alpha"'] \\
 & \frac{1}{a}yx^2z \arrow[ur, Rightarrow, bend right, "\frac{1}{a}\gamma z"'] &
\end{tikzcd}$$

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$$\begin{tikzcd}
& -\frac{1}{a}yx^2 \\
yzy \arrow[ur, Rightarrow, bend left, "y\beta"] \arrow[dr, Rightarrow, bend right, "\alpha y"'] & \\
& -x^2y
\end{tikzcd}
\qquad
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zyz \arrow[ur, Rightarrow, bend left, "\beta z"] \arrow[dr, Rightarrow, bend right, "z\alpha"'] \arrow[r, phantom, "B_1"] & \\
& -zx^2 \arrow[uu, Rightarrow, "-\delta"']
\end{tikzcd}$$

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Example. (Backelin 1991, Polishchuk-Positselski 2005)

- ▷ A Koszul algebra that has no Poincaré-Birkhoff-Witt basis:

$$\mathbf{A}\langle\, x, y, z \;\mid\; x^2 + yz = 0,\ x^2 + azy = 0 \,\rangle \qquad \text{with } a \neq 0, 1$$

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- ▷ Note that

$$\mathrm{Tor}^{\mathbf{A}}_{0,(0)}(\mathbb{K},\mathbb{K}) \simeq \mathbb{K}, \quad \mathrm{Tor}^{\mathbf{A}}_{1,(1)}(\mathbb{K},\mathbb{K}) \simeq \mathbb{K}^3, \quad \mathrm{Tor}^{\mathbf{A}}_{2,(2)}(\mathbb{K},\mathbb{K}) \simeq \mathbb{K}^2,$$

$$\mathrm{Tor}^{\mathbf{A}}_{k,(i)}(\mathbb{K},\mathbb{K}) = 0 \ \text{ otherwise.}$$

Four families of local branchings in a linear 2-polygraph

- ▷ **Aspherical branchings**

$$\begin{tikzcd}
\lambda u + h \arrow[r, Rightarrow, bend left, "\lambda a + h"] \arrow[r, Rightarrow, bend right, "\lambda a + h"'] & \lambda f + h
\end{tikzcd}$$

with $a : u \Rightarrow f$ 2-monomial, $\lambda \in \mathbb{K} \setminus \{0\}$, $h \in \Lambda_1^\ell$, $u \notin \mathrm{Supp}(h)$.

- ▷ **Additive** branchings,

$$\begin{tikzcd}
 & \lambda f + \mu v + h \\
\lambda u + \mu v + h \arrow[ur, Rightarrow, bend left, "\lambda a + \mu v + h"] \arrow[dr, Rightarrow, bend right, "\lambda u + \mu b + h"'] & \\
 & \lambda u + \mu g + h
\end{tikzcd}$$

with $a : u \Rightarrow f$, $b : v \Rightarrow g$ 2-monomials, $\lambda, \mu \in \mathbb{K} \setminus \{0\}$, $h \in \Lambda_1^\ell$, $u, v \notin \mathrm{Supp}(h)$.

Four families of local branchings in a linear 2-polygraph

- **Peiffer** branchings,

$$\begin{tikzcd}
& \lambda fv + h \\
\lambda uv + h \arrow[ur, Rightarrow, bend left, "\lambda av + h"] \arrow[dr, Rightarrow, bend right, "\lambda ub + h"'] & \\
& \lambda ug + h
\end{tikzcd}$$

with $a : u \Rightarrow f$, $b : v \Rightarrow g$ 2-monomials, $\lambda \in \mathbb{K} \setminus \{0\}$, $h \in \Lambda_1^{\ell}$, $uv \notin \mathrm{Supp}(h)$.

- **Overlapping** branchings,

$$\begin{tikzcd}
& \lambda f + h \\
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& \lambda g + h
\end{tikzcd}$$

with $a : u \Rightarrow f$, $b : u \Rightarrow g$ 2-monomials, such that the branching (a, b) is neither aspherical nor Peiffer, $\lambda \in \mathbb{K} \setminus \{0\}$, $h \in \Lambda_1^{\ell}$, $uv \notin \mathrm{Supp}(h)$.

Linear critical branching Lemma

- Some local branchings can be nonconfluent without termination, even if all critical branchings are confluent.

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Example.

$$\Lambda = \langle\, x, y, z, t \mid xy \overset{\alpha}{\Longrightarrow} xz,\ zt \overset{\beta}{\Longrightarrow} 2yt \,\rangle$$

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$$\begin{tikzcd}
 & & 4xyt \arrow[r, Rightarrow, "4\alpha t"] & 4xzt \arrow[r, Rightarrow, "4x\beta"] & \cdots \\
 & 2xzt \arrow[ur, Rightarrow, bend left, "2x\beta"] \arrow[dr, dotted, Rightarrow, "xzt + x\beta"] & & & \\
xyt + xzt \arrow[ur, Rightarrow, bend left, "\alpha t + xzt"] \arrow[dr, Rightarrow, bend right, "xyt + x\beta"'] \arrow[rr, phantom, "="] & & xzt + 2xyt & & \\
 & 3xyt \arrow[ur, dotted, Rightarrow, "\alpha t + 2xyt"'] \arrow[dr, Rightarrow, bend right, "3\alpha t"'] & & & \\
 & & 3xzt \arrow[r, Rightarrow, "3x\beta"'] & 6xyt \arrow[r, Rightarrow, "6\alpha t"'] & \cdots
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- The critical branching $(\alpha z, x\beta)$ of source xyz is not confluent.