In this Maple file, we compute the coefficients of the polynomials P_1 and P_2 in terms of the irregular times for the Painlevé 1 case.

```
> restart:
  P1:=x-> Pinfty01+Pinfty11*x;
  P2:=x-> Pinfty02+Pinfty12*x+Pinfty22*x^2+Pinfty32*x^3;
```

$$P1 := x \mapsto Pinfty01 + Pinfty11\,x$$

$$P2 := x \mapsto Pinfty02 + Pinfty12\,x + Pinfty22\,x^2 + Pinfty32\,x^3 \qquad (1)$$

```
> ClassicalSpectralCurve:=unapply(y^2-P1(x)*y+P2(x),y);
```

$$ClassicalSpectralCurve := y \mapsto y^2 - (Pinfty11\,x + Pinfty01)\,y + Pinfty32\,x^3 + Pinfty22\,x^2 + Pinfty12\,x + Pinfty02 \qquad (2)$$

```
> Yinfty:=-1/2*tinfty15*x^(3/2)-1/2*tinfty14*x-1/2*tinfty13*x^
  (1/2)-1/2*tinfty12-1/2*tinfty11*x^(-1/2)-1/2*tinfty10*x^(-1)+
  Unknown/x^(3/2);
  Yinftybis:=-1/2*tinfty25*x^(3/2)-1/2*tinfty24*x-1/2*tinfty23*x^
  (1/2)-1/2*tinfty22-1/2*tinfty21*x^(-1/2)-1/2*tinfty20*x^(-1)+
  Unknownn/x^(3/2);
```

$$Yinfty := -\frac{tinfty15\,x^{3/2}}{2} - \frac{tinfty14\,x}{2} - \frac{tinfty13\,\sqrt{x}}{2} - \frac{tinfty12}{2} - \frac{tinfty11}{2\sqrt{x}} - \frac{tinfty10}{2\,x} + \frac{Unknown}{x^{3/2}}$$

$$Yinftybis := -\frac{tinfty25\,x^{3/2}}{2} - \frac{tinfty24\,x}{2} - \frac{tinfty23\,\sqrt{x}}{2} - \frac{tinfty22}{2} - \frac{tinfty21}{2\sqrt{x}} - \frac{tinfty20}{2\,x} + \frac{Unknownn}{x^{3/2}} \qquad (3)$$

Expression of P1 in terms of both sheets

```
> series(Yinfty+Yinftybis-P1(x),x=infinity);
```

$$\frac{-\frac{tinfty15}{2} - \frac{tinfty25}{2}}{\left(\frac{1}{x}\right)^{3/2}} + \left(-\frac{tinfty14}{2} - \frac{tinfty24}{2} - Pinfty11\right)x + \frac{-\frac{tinfty13}{2} - \frac{tinfty23}{2}}{\sqrt{\frac{1}{x}}} - \frac{tinfty12}{2} - \frac{tinfty22}{2} - Pinfty01 + \left(-\frac{tinfty11}{2} - \frac{tinfty21}{2}\right)\sqrt{\frac{1}{x}} + \frac{-\frac{tinfty10}{2} - \frac{tinfty20}{2}}{x} + (Unknown + Unknownn)\left(\frac{1}{x}\right)^{3/2} \qquad (4)$$

```
> tinfty25:=-tinfty15:
  tinfty23:=-tinfty13:
  tinfty21:=-tinfty11:
  tinfty20:=-tinfty10:
  Pinfty11:=-(tinfty14+tinfty24)/2;
  Pinfty01:=-(tinfty12+tinfty22)/2;
```

```
Unknownn:=-Unknown:
Unknownn2:=Unknown2:
series(Yinfty+Yinftybis-P1(x),x=infinity);
```

$$Pinfty11 := -\frac{tinfty14}{2} - \frac{tinfty24}{2}$$

$$Pinfty01 := -\frac{tinfty12}{2} - \frac{tinfty22}{2}$$

$$0 \quad \textbf{(5)}$$

This implies that coefficients in the second sheet are related to the one in the first sheet

```
> series(simplify(ClassicalSpectralCurve(Yinfty)),x=infinity);
```

$$\left(\frac{tinfty15^2}{4} + Pinfty32\right) x^3 + \frac{tinfty15\,(tinfty14 - tinfty24)}{4\left(\frac{1}{x}\right)^{5/2}} + \left(\frac{tinfty13\,tinfty15}{2} - \frac{tinfty14\,tinfty24}{4} + Pinfty22\right) x^2 + \frac{\frac{(tinfty14 - tinfty24)\,tinfty13}{4} - \frac{tinfty15\,(tinfty22 - tinfty12)}{4}}{\left(\frac{1}{x}\right)^{3/2}} + \left(\frac{tinfty11\,tinfty15}{2} - \frac{tinfty12\,tinfty24}{4} + \frac{tinfty13^2}{4} - \frac{tinfty14\,tinfty22}{4} + Pinfty12\right) x + \frac{\frac{tinfty15\,tinfty10}{2} + \frac{(tinfty14 - tinfty24)\,tinfty11}{4} - \frac{tinfty13\,(tinfty22 - tinfty12)}{4}}{\sqrt{\frac{1}{x}}} + \frac{(tinfty14 - tinfty24)\,tinfty10}{4} + \frac{tinfty13\,tinfty11}{2} - tinfty15\,Unknown - \frac{tinfty12\,tinfty22}{4} + Pinfty02 + \left(\frac{tinfty13\,tinfty10}{2} + \frac{(-tinfty22 + tinfty12)\,tinfty11}{4} - \frac{Unknown\,(tinfty14 - tinfty24)}{2}\right)\sqrt{\frac{1}{x}} + \frac{\frac{(-tinfty22 + tinfty12)\,tinfty10}{4} - tinfty13\,Unknown + \frac{tinfty11^2}{4}}{x} + \left(\frac{tinfty10\,tinfty11}{2} + \frac{Unknown\,(tinfty22 - tinfty12)}{2}\right)\left(\frac{1}{x}\right)^{3/2} + \frac{-Unknown\,tinfty11 + \frac{tinfty10^2}{4}}{x^2} - Unknown\,tinfty10\left(\frac{1}{x}\right)^{5/2} + \frac{Unknown^2}{x^3} \quad \textbf{(6)}$$

```
> tinfty24:=tinfty14:
tinfty22:=tinfty12:
tinfty10:=0:
series(simplify(ClassicalSpectralCurve(Yinfty)),x=infinity);
```

(7)

$$\left(\frac{tinfty15^2}{4}+Pinfty32\right)x^3+\left(\frac{tinfty13\,tinfty15}{2}-\frac{tinfty14^2}{4}+Pinfty22\right)x^2+\left(\frac{tinfty11\,tinfty15}{2}-\frac{tinfty12\,tinfty14}{2}+\frac{tinfty13^2}{4}+Pinfty12\right)x-tinfty15\,Unknown+\frac{tinfty13\,tinfty11}{2}-\frac{tinfty12^2}{4}+Pinfty02+\frac{-tinfty13\,Unknown+\frac{tinfty11^2}{4}}{x}-\frac{Unknown\,tinfty11}{x^2}+\frac{Unknown^2}{x^3} \quad (7)$$

Study at infinity

```
> series(ClassicalSpectralCurve(Yinfty),x=infinity,6):
  series(ClassicalSpectralCurve(Yinftybis),x=infinity,6):
  EQinfty1:=residue(simplify(x^(-5)*ClassicalSpectralCurve(Yinfty)
  ),x=infinity);
  EQinfty2:=residue(simplify(x^(-5)*ClassicalSpectralCurve
  (Yinftybis)),x=infinity);
  EQinfty3:=residue(simplify(x^(-4)*ClassicalSpectralCurve(Yinfty)
  ),x=infinity);
  EQinfty4:=residue(simplify(x^(-4)*ClassicalSpectralCurve
  (Yinftybis)),x=infinity);
  EQinfty5:=residue(simplify(x^(-3)*ClassicalSpectralCurve(Yinfty)
  ),x=infinity);
  EQinfty6:=residue(simplify(x^(-3)*ClassicalSpectralCurve
  (Yinftybis)),x=infinity);
  EQinfty7:=residue(simplify(x^(-2)*ClassicalSpectralCurve(Yinfty)
  ),x=infinity);
  EQinfty8:=residue(simplify(x^(-2)*ClassicalSpectralCurve
  (Yinftybis)),x=infinity);
  EQinfty9:=residue(simplify(x^(-1)*ClassicalSpectralCurve(Yinfty)
  ),x=infinity);
  EQinfty10:=residue(simplify(x^(-1)*ClassicalSpectralCurve
  (Yinftybis)),x=infinity);
```

$$EQinfty1 := 0$$

$$EQinfty2 := 0$$

$$EQinfty3 := -\frac{tinfty15^2}{4}-Pinfty32$$

$$EQinfty4 := -\frac{tinfty15^2}{4}-Pinfty32$$

$$EQinfty5 := -\frac{tinfty13\,tinfty15}{2}+\frac{tinfty14^2}{4}-Pinfty22$$

$$EQinfty6 := -\frac{tinfty13\ tinfty15}{2} + \frac{tinfty14^2}{4} - Pinfty22$$
$$EQinfty7 := -\frac{tinfty11\ tinfty15}{2} + \frac{tinfty12\ tinfty14}{2} - \frac{tinfty13^2}{4} - Pinfty12$$
$$EQinfty8 := -\frac{tinfty11\ tinfty15}{2} + \frac{tinfty12\ tinfty14}{2} - \frac{tinfty13^2}{4} - Pinfty12$$
$$EQinfty9 := tinfty15\ Unknown - \frac{tinfty13\ tinfty11}{2} + \frac{tinfty12^2}{4} - Pinfty02$$
$$EQinfty10 := tinfty15\ Unknown - \frac{tinfty13\ tinfty11}{2} + \frac{tinfty12^2}{4} - Pinfty02 \quad \textbf{(8)}$$

```
> Pinfty32:=factor(solve(EQinfty3,Pinfty32));
  Pinfty22:=factor(solve(EQinfty5,Pinfty22));
  Pinfty12:=factor(solve(EQinfty7,Pinfty12));
  simplify(EQinfty4);
  simplify(EQinfty6);
  simplify(EQinfty8);
```

$$Pinfty32 := -\frac{tinfty15^2}{4}$$
$$Pinfty22 := -\frac{tinfty13\ tinfty15}{2} + \frac{tinfty14^2}{4}$$
$$Pinfty12 := -\frac{tinfty11\ tinfty15}{2} + \frac{tinfty12\ tinfty14}{2} - \frac{tinfty13^2}{4}$$
$$0$$
$$0$$
$$0 \quad \textbf{(9)}$$

Summary of the coefficients

```
> Pinfty01:=Pinfty01;
  Pinfty11:=Pinfty11;

  Pinfty02:=Pinfty02;
  Pinfty12:=Pinfty12;
  Pinfty22:=Pinfty22;
  Pinfty32:=Pinfty32;
```

$$Pinfty01 := -tinfty12$$
$$Pinfty11 := -tinfty14$$
$$Pinfty02 := Pinfty02$$
$$Pinfty12 := -\frac{tinfty11\ tinfty15}{2} + \frac{tinfty12\ tinfty14}{2} - \frac{tinfty13^2}{4}$$

$$Pinfty22 := -\frac{tinfty13\ tinfty15}{2} + \frac{tinfty14^2}{4}$$

$$Pinfty32 := -\frac{tinfty15^2}{4} \tag{10}$$

We have one undetermined coefficients: Pinfty02

```
> P1(lambda);
  P2(lambda);
```

$$-tinfty14\,\lambda - tinfty12$$

$$Pinfty02 + \left(-\frac{tinfty11\ tinfty15}{2} + \frac{tinfty12\ tinfty14}{2} - \frac{tinfty13^2}{4}\right)\lambda + \left(-\frac{tinfty13\ tinfty15}{2} + \frac{tinfty14^2}{4}\right)\lambda^2 - \frac{tinfty15^2\,\lambda^3}{4} \tag{11}$$