In this Maple file, we compute the coefficients of the polynomials P_1 and P_2 in terms of the irregular times for the second element of the Painlevé 1 hierarchy. We check with the theoretical results.

```
> restart:
  rinfty:=4:
  rinfty-2;
  2*rinfty-3;
  P1:=x-> Pinfty01+Pinfty11*x+Pinfty21*x^2;
  P2:=x-> Pinfty02+Pinfty12*x+Pinfty22*x^2+Pinfty32*x^3+Pinfty42*
  x^4+Pinfty52*x^5;
```

$$2$$

$$5$$

$$P1 := x \mapsto Pinfty01 + Pinfty11\,x + Pinfty21\,x^2$$

$$P2 := x \mapsto Pinfty02 + Pinfty12\,x + Pinfty22\,x^2 + Pinfty32\,x^3 + Pinfty42\,x^4 + Pinfty52\,x^5 \qquad \textbf{(1)}$$

```
> ClassicalSpectralCurve:=unapply( y^2-P1(x)*y+P2(x),y);
```

$$ClassicalSpectralCurve := y \mapsto y^2 - \left(Pinfty21\,x^2 + Pinfty11\,x + Pinfty01\right) y + Pinfty52\,x^5 + Pinfty42\,x^4 + Pinfty32\,x^3 + Pinfty22\,x^2 + Pinfty12\,x + Pinfty02 \qquad \textbf{(2)}$$

```
> 2*rinfty-1;
  Yinfty:=-1/2*tinfty17*x^(5/2)-1/2*tinfty16*x^(2)-1/2*tinfty15*x^
  (3/2)-1/2*tinfty14*x-1/2*tinfty13*x^(1/2)-1/2*tinfty12-1/2*
  tinfty11*x^(-1/2)-1/2*tinfty10*x^(-1)+ Unknown/x^(3/2);
  Yinftybis:=-1/2*tinfty27*x^(5/2)-1/2*tinfty26*x^(2)-1/2*tinfty25*
  x^(3/2)-1/2*tinfty24*x-1/2*tinfty23*x^(1/2)-1/2*tinfty22-1/2*
  tinfty21*x^(-1/2)-1/2*tinfty20*x^(-1)+ Unknownn/x^(3/2);
```

$$7$$

$$Yinfty := -\frac{tinfty17\,x^{5/2}}{2} - \frac{tinfty16\,x^2}{2} - \frac{tinfty15\,x^{3/2}}{2} - \frac{tinfty14\,x}{2} - \frac{tinfty13\,\sqrt{x}}{2} - \frac{tinfty12}{2} - \frac{tinfty11}{2\,\sqrt{x}} - \frac{tinfty10}{2\,x} + \frac{Unknown}{x^{3/2}}$$

$$Yinftybis := -\frac{tinfty27\,x^{5/2}}{2} - \frac{tinfty26\,x^2}{2} - \frac{tinfty25\,x^{3/2}}{2} - \frac{tinfty24\,x}{2} - \frac{tinfty23\,\sqrt{x}}{2} - \frac{tinfty22}{2} - \frac{tinfty21}{2\,\sqrt{x}} - \frac{tinfty20}{2\,x} + \frac{Unknownn}{x^{3/2}} \qquad \textbf{(3)}$$

Expression of P1 in terms of both sheets

```
> series(Yinfty+Yinftybis-P1(x),x=infinity);
```

$$\frac{-\frac{tinfty17}{2} - \frac{tinfty27}{2}}{\left(\frac{1}{x}\right)^{5/2}} + \left(-\frac{tinfty16}{2} - \frac{tinfty26}{2} - Pinfty21\right) x^2 + \frac{-\frac{tinfty15}{2} - \frac{tinfty25}{2}}{\left(\frac{1}{x}\right)^{3/2}} + \left(-\frac{tinfty14}{2} - \frac{tinfty24}{2} - Pinfty11\right) x + \frac{-\frac{tinfty13}{2} - \frac{tinfty23}{2}}{\sqrt{\frac{1}{x}}} - \frac{tinfty12}{2} \qquad \textbf{(4)}$$

$$-\frac{tinfty22}{2}-Pinfty01+\left(-\frac{tinfty11}{2}-\frac{tinfty21}{2}\right)\sqrt{\frac{1}{x}}+\frac{-\frac{tinfty10}{2}-\frac{tinfty20}{2}}{x}$$
$$+(Unknown+Unknownn)\left(\frac{1}{x}\right)^{3/2}$$

We check that the odd irregular times must be opposite and that coefficients of P_1 are reconstructed by only even irregular times

```
> tinfty27:=-tinfty17:
  tinfty25:=-tinfty15:
  tinfty23:=-tinfty13:
  tinfty21:=-tinfty11:
  tinfty20:=-tinfty10:
  Pinfty21:=-(tinfty16+tinfty26)/2;
  Pinfty11:=-(tinfty14+tinfty24)/2;
  Pinfty01:=-(tinfty12+tinfty22)/2;
  Unknownn:=-Unknown:
  Unknownn2:=Unknown2:
  series(Yinfty+Yinftybis-P1(x),x=infinity);
```

$$Pinfty21:=-\frac{tinfty16}{2}-\frac{tinfty26}{2}$$
$$Pinfty11:=-\frac{tinfty14}{2}-\frac{tinfty24}{2}$$
$$Pinfty01:=-\frac{tinfty12}{2}-\frac{tinfty22}{2}$$
$$0 \tag{5}$$

```
> series(simplify(ClassicalSpectralCurve(Yinfty)),x=infinity);
```

$$\left(\frac{tinfty17^2}{4}+Pinfty52\right)x^5+\frac{tinfty17\,(tinfty16-tinfty26)}{4\left(\frac{1}{x}\right)^{9/2}}+\left(\frac{tinfty15\,tinfty17}{2}-\frac{tinfty16\,tinfty26}{4}+Pinfty42\right)x^4 \tag{6}$$
$$+\frac{\frac{(tinfty16-tinfty26)\,tinfty15}{4}+\frac{tinfty17\,(tinfty14-tinfty24)}{4}}{\left(\frac{1}{x}\right)^{7/2}}+\left(\frac{tinfty13\,tinfty17}{2}-\frac{tinfty14\,tinfty26}{4}+\frac{tinfty15^2}{4}-\frac{tinfty16\,tinfty24}{4}+Pinfty32\right)x^3$$
$$+\frac{1}{\left(\frac{1}{x}\right)^{5/2}}\left(\frac{(tinfty16-tinfty26)\,tinfty13}{4}+\frac{(tinfty14-tinfty24)\,tinfty15}{4}+\frac{tinfty17\,(tinfty12-tinfty22)}{4}\right)+\left(\frac{tinfty11\,tinfty17}{2}-\frac{tinfty12\,tinfty26}{4}\right.$$
$$\left.+\frac{tinfty13\,tinfty15}{2}-\frac{tinfty14\,tinfty24}{4}-\frac{tinfty16\,tinfty22}{4}+Pinfty22\right)x^2$$

$$+\frac{1}{\left(\frac{1}{x}\right)^{3/2}}\left(\frac{tinfty10\,tinfty17}{2}+\frac{(tinfty16-tinfty26)\,tinfty11}{4}\right.$$
$$\left.+\frac{(tinfty14-tinfty24)\,tinfty13}{4}+\frac{tinfty15\,(tinfty12-tinfty22)}{4}\right)$$
$$+\left(\frac{(tinfty16-tinfty26)\,tinfty10}{4}+\frac{tinfty15\,tinfty11}{2}-\frac{tinfty12\,tinfty24}{4}+\frac{tinfty13^2}{4}\right.$$
$$\left.-\frac{tinfty14\,tinfty22}{4}-tinfty17\,Unknown+Pinfty12\right)x+\frac{1}{\sqrt{\frac{1}{x}}}\left(\frac{tinfty10\,tinfty15}{2}\right.$$
$$+\frac{(tinfty14-tinfty24)\,tinfty11}{4}+\frac{(tinfty12-tinfty22)\,tinfty13}{4}$$
$$\left.-\frac{Unknown\,(tinfty16-tinfty26)}{2}\right)+\frac{(tinfty14-tinfty24)\,tinfty10}{4}+\frac{tinfty13\,tinfty11}{2}$$
$$-\frac{tinfty12\,tinfty22}{4}-tinfty15\,Unknown+Pinfty02+\left(\frac{tinfty10\,tinfty13}{2}\right.$$
$$\left.+\frac{(tinfty12-tinfty22)\,tinfty11}{4}-\frac{Unknown\,(tinfty14-tinfty24)}{2}\right)\sqrt{\frac{1}{x}}$$
$$+\frac{\frac{(tinfty12-tinfty22)\,tinfty10}{4}+\frac{tinfty11^2}{4}-tinfty13\,Unknown}{x}+\left(\frac{tinfty10\,tinfty11}{2}\right.$$
$$\left.-\frac{Unknown\,(tinfty12-tinfty22)}{2}\right)\left(\frac{1}{x}\right)^{3/2}+\frac{-Unknown\,tinfty11+\frac{tinfty10^2}{4}}{x^2}$$
$$-Unknown\,tinfty10\left(\frac{1}{x}\right)^{5/2}+\frac{Unknown^2}{x^3}$$

We check here that even irregular times are equal and that there cannot be monodromies otherwise we would get half-power coefficients that should be null

```
> tinfty26:=tinfty16:
  tinfty24:=tinfty14:
  tinfty22:=tinfty12:
  tinfty10:=0:
  series(simplify(ClassicalSpectralCurve(Yinfty)),x=infinity);
```

$$\left(\frac{tinfty17^2}{4}+Pinfty52\right)x^5+\left(\frac{tinfty15\,tinfty17}{2}-\frac{tinfty16^2}{4}+Pinfty42\right)x^4 \quad \textbf{(7)}$$
$$+\left(\frac{tinfty13\,tinfty17}{2}-\frac{tinfty14\,tinfty16}{2}+\frac{tinfty15^2}{4}+Pinfty32\right)x^3$$
$$+\left(\frac{tinfty11\,tinfty17}{2}-\frac{tinfty12\,tinfty16}{2}+\frac{tinfty13\,tinfty15}{2}-\frac{tinfty14^2}{4}+Pinfty22\right)x^2$$
$$+\left(-tinfty17\,Unknown+\frac{tinfty15\,tinfty11}{2}-\frac{tinfty12\,tinfty14}{2}+\frac{tinfty13^2}{4}+Pinfty12\right)x$$
$$-tinfty15\,Unknown+\frac{tinfty13\,tinfty11}{2}-\frac{tinfty12^2}{4}+Pinfty02$$

$$+\frac{-tinfty13\,Unknown+\frac{tinfty11^2}{4}}{x}-\frac{Unknown\,tinfty11}{x^2}+\frac{Unknown^2}{x^3}$$

Study of the asymptotics at infinity

```
> series(ClassicalSpectralCurve(Yinfty),x=infinity,8):
  series(ClassicalSpectralCurve(Yinftybis),x=infinity,8):
  EQinfty0:=residue(simplify(x^(-6)*ClassicalSpectralCurve(Yinfty)
  ),x=infinity);
  EQinfty00:=residue(simplify(x^(-6)*ClassicalSpectralCurve
  (Yinftybis)),x=infinity);
  EQinfty1:=residue(simplify(x^(-5)*ClassicalSpectralCurve(Yinfty)
  ),x=infinity);
  EQinfty2:=residue(simplify(x^(-5)*ClassicalSpectralCurve
  (Yinftybis)),x=infinity);
  EQinfty3:=residue(simplify(x^(-4)*ClassicalSpectralCurve(Yinfty)
  ),x=infinity);
  EQinfty4:=residue(simplify(x^(-4)*ClassicalSpectralCurve
  (Yinftybis)),x=infinity);
  EQinfty5:=residue(simplify(x^(-3)*ClassicalSpectralCurve(Yinfty)
  ),x=infinity);
  EQinfty6:=residue(simplify(x^(-3)*ClassicalSpectralCurve
  (Yinftybis)),x=infinity);
  EQinfty7:=residue(simplify(x^(-2)*ClassicalSpectralCurve(Yinfty)
  ),x=infinity);
  EQinfty8:=residue(simplify(x^(-2)*ClassicalSpectralCurve
  (Yinftybis)),x=infinity);
  EQinfty9:=residue(simplify(x^(-1)*ClassicalSpectralCurve(Yinfty)
  ),x=infinity);
  EQinfty10:=residue(simplify(x^(-1)*ClassicalSpectralCurve
  (Yinftybis)),x=infinity);
```

$$EQinfty0 := -\frac{tinfty17^2}{4}-Pinfty52$$

$$EQinfty00 := -\frac{tinfty17^2}{4}-Pinfty52$$

$$EQinfty1 := -\frac{tinfty15\,tinfty17}{2}+\frac{tinfty16^2}{4}-Pinfty42$$

$$EQinfty2 := -\frac{tinfty15\,tinfty17}{2}+\frac{tinfty16^2}{4}-Pinfty42$$

$$EQinfty3 := -\frac{tinfty13\,tinfty17}{2}+\frac{tinfty14\,tinfty16}{2}-\frac{tinfty15^2}{4}-Pinfty32$$

$$EQinfty4 := -\frac{tinfty13\ tinfty17}{2} + \frac{tinfty14\ tinfty16}{2} - \frac{tinfty15^2}{4} - Pinfty32$$

$$EQinfty5 := -\frac{tinfty11\ tinfty17}{2} + \frac{tinfty12\ tinfty16}{2} - \frac{tinfty13\ tinfty15}{2} + \frac{tinfty14^2}{4} - Pinfty22$$

$$EQinfty6 := -\frac{tinfty11\ tinfty17}{2} + \frac{tinfty12\ tinfty16}{2} - \frac{tinfty13\ tinfty15}{2} + \frac{tinfty14^2}{4} - Pinfty22$$

$$EQinfty7 := tinfty17\ Unknown - \frac{tinfty15\ tinfty11}{2} + \frac{tinfty12\ tinfty14}{2} - \frac{tinfty13^2}{4} - Pinfty12$$

$$EQinfty8 := tinfty17\ Unknown - \frac{tinfty15\ tinfty11}{2} + \frac{tinfty12\ tinfty14}{2} - \frac{tinfty13^2}{4} - Pinfty12$$

$$EQinfty9 := tinfty15\ Unknown - \frac{tinfty13\ tinfty11}{2} + \frac{tinfty12^2}{4} - Pinfty02$$

$$EQinfty10 := tinfty15\ Unknown - \frac{tinfty13\ tinfty11}{2} + \frac{tinfty12^2}{4} - Pinfty02 \quad \textbf{(8)}$$

We compute the value of some coefficients of P_2

```
> Pinfty52:=factor(solve(EQinfty0,Pinfty52));
  Pinfty42:=factor(solve(EQinfty1,Pinfty42));
  Pinfty32:=factor(solve(EQinfty3,Pinfty32));
  Pinfty22:=factor(solve(EQinfty5,Pinfty22));
  simplify(EQinfty00);
  simplify(EQinfty2);
  simplify(EQinfty4);
  simplify(EQinfty6);
  simplify(EQinfty8);
```

$$Pinfty52 := -\frac{tinfty17^2}{4}$$

$$Pinfty42 := -\frac{tinfty15\ tinfty17}{2} + \frac{tinfty16^2}{4}$$

$$Pinfty32 := -\frac{tinfty13\ tinfty17}{2} + \frac{tinfty14\ tinfty16}{2} - \frac{tinfty15^2}{4}$$

$$Pinfty22 := -\frac{tinfty11\ tinfty17}{2} + \frac{tinfty12\ tinfty16}{2} - \frac{tinfty13\ tinfty15}{2} + \frac{tinfty14^2}{4}$$

$$0$$

$$0$$

$$0$$

$$0$$

$$tinfty17\ Unknown - \frac{tinfty15\ tinfty11}{2} + \frac{tinfty12\ tinfty14}{2} - \frac{tinfty13^2}{4} - Pinfty12 \quad \textbf{(9)}$$

Summary of coefficients

```
> Pinfty01:=Pinfty01;
  Pinfty11:=Pinfty11;
  Pinfty21:=Pinfty21;
```

```
Pinfty02:=Pinfty02;
Pinfty12:=Pinfty12;
Pinfty22:=Pinfty22;
Pinfty32:=Pinfty32;
Pinfty42:=Pinfty42;
Pinfty52:=Pinfty52;
```

$$Pinfty01 := -tinfty12$$
$$Pinfty11 := -tinfty14$$
$$Pinfty21 := -tinfty16$$
$$Pinfty02 := Pinfty02$$
$$Pinfty12 := Pinfty12$$
$$Pinfty22 := -\frac{tinfty11\,tinfty17}{2} + \frac{tinfty12\,tinfty16}{2} - \frac{tinfty13\,tinfty15}{2} + \frac{tinfty14^2}{4}$$
$$Pinfty32 := -\frac{tinfty13\,tinfty17}{2} + \frac{tinfty14\,tinfty16}{2} - \frac{tinfty15^2}{4}$$
$$Pinfty42 := -\frac{tinfty15\,tinfty17}{2} + \frac{tinfty16^2}{4}$$
$$Pinfty52 := -\frac{tinfty17^2}{4} \quad (10)$$

We have 2 unknown coefficients: Pinfty02 and Pinfty12

We check the formula for the coefficients of P_2

```
> Pinfty22theo:= 1/4*sum((-1)^i*tinfty[i]*tinfty[2*(rinfty-2)-i+4],
i=2*(rinfty-2)-2*rinfty+5..2*rinfty-1);
Pinfty32theo:= 1/4*sum((-1)^i*tinfty[i]*tinfty[2*(rinfty-1)-i+4],
i=2*(rinfty-1)-2*rinfty+5..2*rinfty-1);
Pinfty42theo:= 1/4*sum((-1)^i*tinfty[i]*tinfty[2*(rinfty)-i+4],i=
2*(rinfty)-2*rinfty+5..2*rinfty-1);
Pinfty52theo:= 1/4*sum((-1)^i*tinfty[i]*tinfty[2*(rinfty+1)-i+4],
i=2*(rinfty+1)-2*rinfty+5..2*rinfty-1);
```

$$Pinfty22theo := -\frac{tinfty_1\,tinfty_7}{2} + \frac{tinfty_2\,tinfty_6}{2} - \frac{tinfty_3\,tinfty_5}{2} + \frac{tinfty_4^2}{4}$$
$$Pinfty32theo := -\frac{tinfty_3\,tinfty_7}{2} + \frac{tinfty_4\,tinfty_6}{2} - \frac{tinfty_5^2}{4}$$
$$Pinfty42theo := -\frac{tinfty_5\,tinfty_7}{2} + \frac{tinfty_6^2}{4}$$
$$Pinfty52theo := -\frac{tinfty_7^2}{4} \quad (11)$$