In this Maple file, we reduced the general formulas to the canonical set of trivial times in order to obtain the reduced formulas for the Lax matrices and Hamiltonians for the second element of the Painlevé 1 hierarchy. We check with the theoretical formulas.

Let us start by enumerating the symmetric polynomials.

```
> restart:
  with(LinearAlgebra):
  with(ListTools):
  with(combinat):
  with(PolynomialTools):
  with(Groebner):

  chk:=proc()
  local VV,AA,pp,LL,K,N,KK:
  VV:=[seq(args[i],i=2..nargs)];
  AA:=[seq(sigma[i],i=1..nargs-1)];
  pp:=simplify(expand(mul(x_-args[i],i=2..nargs)),x_);
  LL := Reverse([seq((-1)^(r+nargs-1)*coeff(pp, x_, r), r = 0..
  nargs-2)])-AA;
  K:=Basis(LL,tdeg(VV[])):
  N:=NormalForm(args[1],K,tdeg(VV[]));
  KK:=Basis(AA,tdeg(AA[]));
  NormalForm(N,KK,tdeg(AA[]));
  if is(NormalForm(N,KK,tdeg(AA[]))=0) then print("symmetric")else
  print("not symmetric")fi;
  end proc:

  es:=proc()
  local VV, AA, pp, LL, K;
  VV:=[seq(args[i],i=2..nargs)];AA:=[seq(sigma[i],i=1..nargs-1)];
  pp:=simplify(expand(mul(x_-args[i],i=2..nargs)),x_);
  LL := Reverse([seq((-1)^(r+nargs-1)*coeff(pp, x_, r), r = 0..
  nargs-2)])-AA;
  K:=Basis(LL,tdeg(VV[]));
  NormalForm(args[1],K,tdeg(VV[]));
  end proc:

  ss:=proc() local LL, LLL, t, LLLL, H, K;
  LL:=[seq(args[i],i=2..nargs)];
  LLL:=[seq(map(x->x^r,LL),r=1..nargs-1)];
  t:=seq(s[i],i=1..nargs-1);
  LLLL:=[seq(add(i,i in LLL[u]),u=1..nops(LLL))];
  H:=LLLL-[t];
  K:=Basis(H,grlex(LL[]));
```

```
NormalForm(args[1],K,grlex(LL[]));
end proc:

ElementaryS:= proc(k)
local aux,i,Coeff:
aux:=0: for i from 1 to g do aux:=aux+q[i]^k od:
Coeff:=unapply(es(aux,q[1],q[2]),sigma[1],sigma[2]):
return(Coeff(Q[1],Q[2])):
end proc:

rinfty:=5:
g:=rinfty-3:
S[0]:=ElementaryS(0);
S[1]:=ElementaryS(1);
S[2]:=ElementaryS(2);
S[3]:=ElementaryS(3);
S[4]:=ElementaryS(4);
S[5]:=ElementaryS(5);
S[6]:=ElementaryS(6);
S[7]:=ElementaryS(7);
S[8]:=ElementaryS(8);
```

$$S_0 := 2$$
$$S_1 := Q_1$$
$$S_2 := Q_1^2 - 2\,Q_2$$
$$S_3 := Q_1^3 - 3\,Q_1\,Q_2$$
$$S_4 := Q_1^4 - 4\,Q_1^2\,Q_2 + 2\,Q_2^2$$
$$S_5 := Q_1^5 - 5\,Q_1^3\,Q_2 + 5\,Q_1\,Q_2^2$$
$$S_6 := Q_1^6 - 6\,Q_1^4\,Q_2 + 9\,Q_1^2\,Q_2^2 - 2\,Q_2^3$$
$$S_7 := Q_1^7 - 7\,Q_1^5\,Q_2 + 14\,Q_1^3\,Q_2^2 - 7\,Q_1\,Q_2^3$$
$$S_8 := Q_1^8 - 8\,Q_1^6\,Q_2 + 20\,Q_1^4\,Q_2^2 - 16\,Q_1^2\,Q_2^3 + 2\,Q_2^4 \qquad \textbf{(1)}$$

```
> res:=-lambda^(2*rinfty-5):
for k from (rinfty-2) to (2*rinfty-7) do aux:=2*tau[2*rinfty-k-6]
:
for m from (k-rinfty+6) to (rinfty-3) do aux:=aux+tau[rinfty-m-2]
*tau[rinfty-k+m-5]: od:
res:=res-aux*lambda^k: od:
aux2:=2*tau[rinfty-3]:
for m from 3 to (rinfty-3) do aux2:=aux2+tau[rinfty-m-2]*tau[m-2]
: od:
res:=res-aux2*lambda^(rinfty-3):
```

```
res;

tdP2:=unapply(-lambda^5-2*tau[1]*lambda^3-2*tau[2]*lambda^2,
lambda);
for k from rinfty-3 to 2*rinfty-5 do P2[k]:=-residue(tdP2(lambda)
/lambda^(k+1), lambda=infinity): od:

P2[2];
P2[3];
P2[4];
P2[5];
q1:=q[1]:
q2:=q[2]:
p1:=p[1]:
p2:=p[2]:

Elementaryh:= proc(k)
local aux,i,Coeff:
aux:=1: for i from 1 to g do aux:=aux/(1-t*q[i]): od:
Coeff:=unapply(es(residue(aux/t^(k+1),t=0),q[1],q[2]),sigma[1],
sigma[2]):
return(Coeff(Q[1],Q[2])):
end proc:
h[0]:=simplify(Elementaryh(0));
h[1]:=simplify(Elementaryh(1));
h[2]:=simplify(Elementaryh(2));
h[3]:=simplify(Elementaryh(3));
h[4]:=simplify(Elementaryh(4));
h[5]:=simplify(Elementaryh(5));
h[6]:=simplify(Elementaryh(6));
h[7]:=simplify(Elementaryh(7));
h[8]:=simplify(Elementaryh(8));

Q[0]:=1:
Q[1]:=q[1]+q[2]:
Q[2]:=q[1]*q[2]:
SymMatrix:=Matrix(g,g,0):
for i from 1 to g do for j from 1 to g do SymMatrix[i,j]:=diff(Q
[j],q[i]): od: od:
SymMatrix;
Vectorp:=Matrix(g,1,0):
for i from 1 to g do Vectorp[i,1]:=p[i]: od:
```

```
Vectorp:
VectorP:=Multiply(SymMatrix^(-1),Vectorp);
for i from 1 to g do P[i]:=VectorP[i,1]: od:
```

$$-\lambda^5 - 2\,\tau_1\,\lambda^3 - 2\,\tau_2\,\lambda^2$$

$$tdP2 := \lambda \mapsto -\lambda^5 - 2\,\lambda^3\,\tau_1 - 2\,\lambda^2\,\tau_2$$

$$-2\,\tau_2$$

$$-2\,\tau_1$$

$$0$$

$$-1$$

$$h_0 := 1$$

$$h_1 := Q_1$$

$$h_2 := Q_1^2 - Q_2$$

$$h_3 := Q_1\left(Q_1^2 - 2\,Q_2\right)$$

$$h_4 := Q_1^4 - 3\,Q_1^2\,Q_2 + Q_2^2$$

$$h_5 := Q_1^5 - 4\,Q_1^3\,Q_2 + 3\,Q_1\,Q_2^2$$

$$h_6 := Q_1^6 - 5\,Q_1^4\,Q_2 + 6\,Q_1^2\,Q_2^2 - Q_2^3$$

$$h_7 := Q_1^7 - 6\,Q_1^5\,Q_2 + 10\,Q_1^3\,Q_2^2 - 4\,Q_1\,Q_2^3$$

$$h_8 := Q_1^8 - 7\,Q_1^6\,Q_2 + 15\,Q_1^4\,Q_2^2 - 10\,Q_1^2\,Q_2^3 + Q_2^4$$

$$\begin{bmatrix} 1 & q_2 \\ 1 & q_1 \end{bmatrix}$$

$$VectorP := \begin{bmatrix} \dfrac{q_1\,p_1}{q_1 - q_2} - \dfrac{q_2\,p_2}{q_1 - q_2} \\ -\dfrac{p_1}{q_1 - q_2} + \dfrac{p_2}{q_1 - q_2} \end{bmatrix} \qquad (2)$$

```
> V:=Matrix(2,2,0):
  V[1,1]:=1:
  V[1,2]:=q1:
  V[2,1]:=1:
  V[2,2]:=q2:
  V;
  HVector:=Matrix(2,1,0):
  HVector[1,1]:=H0:
  HVector[2,1]:=H1:
  HVector;
  RHSH:=Matrix(2,1,0):
  RHSH[1,1]:=p1^2+tdP2(q1)+h*(p2-p1)/(q1-q2):
  RHSH[2,1]:=p2^2+tdP2(q2)+h*(p1-p2)/(q2-q1):
```

```
RHSH;
HVector:=simplify(Multiply(V^(-1),RHSH));
```

$$\begin{bmatrix} 1 & q_1 \\ 1 & q_2 \end{bmatrix}$$

$$\begin{bmatrix} H0 \\ H1 \end{bmatrix}$$

$$\begin{bmatrix} p_1^2 - q_1^5 - 2\,q_1^3\tau_1 - 2\,q_1^2\tau_2 + \dfrac{h\,(p_2 - p_1)}{q_1 - q_2} \\ p_2^2 - q_2^5 - 2\,q_2^3\tau_1 - 2\,q_2^2\tau_2 + \dfrac{h\,(p_1 - p_2)}{q_2 - q_1} \end{bmatrix}$$

$$HVector := \left[\left[\frac{1}{q_1 - q_2}\left(h\,(p_2 - p_1) + q_1^5 q_2 + 2\,q_1^3\tau_1 q_2 + 2\,q_1^2\tau_2 q_2 + \left(-q_2^5 - 2\,q_2^3\tau_1 - 2\,q_2^2\tau_2 + p_2^2\right) q_1 - p_1^2 q_2\right)\right], \left[\frac{-q_1^5 + q_2^5 - 2\,q_1^3\tau_1 + 2\,q_2^3\tau_1 - 2\,q_1^2\tau_2 + 2\,q_2^2\tau_2 + p_1^2 - p_2^2}{q_1 - q_2}\right]\right] \quad (3)$$

```
> H0:=-(p1-p2)*h/(q1-q2)+(p2^2*q1-p1^2*q2)/(q1-q2)+(q1+q2)*q1*q2*
  (q1^2+q2^2+2*tau[1])+2*q1*q2*tau[2];
  H1:=(p1^2-p2^2)/(q1-q2) -2*tau[2]*(q1+q2)-2*(q1^2+q1*q2+q2^2)*tau
  [1]-q1^4-q1^3*q2-q1^2*q2^2-q1*q2^3-q2^4;
  factor(HVector[1,1]-H0);
  factor(HVector[2,1]-H1);

  nu1tau1:=1/(2*rinfty-2*1-5)*1;
  nu2tau1:=0;
  nu1tau2:=0;
  nu2tau2:=1/(2*rinfty-2*2-5)*1;

  nutau1Vector:=Matrix(2,1,0):
  nutau1Vector[1,1]:=nu1tau1:
  nutau1Vector[2,1]:=nu2tau1:
  nutau1Vector;

  mutau1Vector:=Multiply((LinearAlgebra[Transpose](V))^
  (-1),nutau1Vector);
  mu1tau1:=mutau1Vector[1,1]:
  mu2tau1:=mutau1Vector[2,1]:
  mutau1Vector;
```

```
nutau2Vector:=Matrix(2,1,0):
nutau2Vector[1,1]:=nu1tau2:
nutau2Vector[2,1]:=nu2tau2:
nutau2Vector;

mutau2Vector:=Multiply((LinearAlgebra[Transpose](V))^(-1),
nutau2Vector);
mu1tau2:=mutau2Vector[1,1]:
mu2tau2:=mutau2Vector[2,1]:

Hamtau1:= nu1tau1*H0+nu2tau1*H1;
Hamtau2:= nu1tau2*H0+nu2tau2*H1;

QQ:=unapply(-p1*(lambda-q2)/(q1-q2)-p2*(lambda-q1)/(q2-q1),
lambda);

J:=Matrix(2,2,0):
J[1,1]:=1:
J[1,2]:=0:
J[2,1]:=QQ(lambda)/(lambda-q1)/(lambda-q2):
J[2,2]:=1/(lambda-q1)/(lambda-q2):
J;
dJdlambda:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do
dJdlambda[i,j]:=diff(J[i,j],lambda): od: od:
dJdlambda;

L:=Matrix(2,2,0):
L[1,1]:=0:
L[1,2]:=1:
L[2,1]:=-tdP2(lambda)+H1*lambda+H0-h*p1/(lambda-q1)-h*p2/(lambda-
q2):
L[2,2]:=h/(lambda-q1)+h/(lambda-q2):
L;

dq1dtau1:=factor(1/h*diff(Hamtau1,p1));
dp1dtau1:=factor(-1/h*diff(Hamtau1,q1));
dq2dtau1:=factor(1/h*diff(Hamtau1,p2));
dp2dtau1:=factor(-1/h*diff(Hamtau1,q2));

dq1dtau2:=factor(1/h*diff(Hamtau2,p1));
dp1dtau2:=factor(-1/h*diff(Hamtau2,q1));
```

```
dq2dtau2:=factor(1/h*diff(Hamtau2,p2));
dp2dtau2:=factor(-1/h*diff(Hamtau2,q2));

Atau1:=Matrix(2,2,0):
Atau1[1,1]:=-p1*mu1tau1/(lambda-q1)-p2*mu2tau1/(lambda-q2):
Atau1[1,2]:= mu1tau1/(lambda-q1)+mu2tau1/(lambda-q2):
Atau1[2,1]:=h*diff(Atau1[1,1],lambda)+Atau1[1,2]*L[2,1]:
Atau1[2,2]:=h*diff(Atau1[1,2],lambda)+Atau1[1,1]+Atau1[1,2]*L[2,
2]:
Atau1:

Atau2:=Matrix(2,2,0):
Atau2[1,1]:=-p1*mu1tau2/(lambda-q1)-p2*mu2tau2/(lambda-q2):
Atau2[1,2]:=mu1tau2/(lambda-q1)+mu2tau2/(lambda-q2):
Atau2[2,1]:=h*diff(Atau2[1,1],lambda)+Atau2[1,2]*L[2,1]:
Atau2[2,2]:=h*diff(Atau2[1,2],lambda)+Atau2[1,1]+Atau2[1,2]*L[2,
2]:
Atau2:


dJdtau1:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do
dJdtau1[i,j]:=diff(J[i,j],tau1)+diff(J[i,j],q1)*dq1dtau1+diff(J
[i,j],p1)*dp1dtau1+diff(J[i,j],q2)*dq2dtau1+diff(J[i,j],p2)*
dp2dtau1: od: od:
dJdtau1:

dJdtau2:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do
dJdtau2[i,j]:=diff(J[i,j],tau2)+diff(J[i,j],q1)*dq1dtau2+diff(J
[i,j],p1)*dp1dtau2+diff(J[i,j],q2)*dq2dtau2+diff(J[i,j],p2)*
dp2dtau2: od: od:
dJdtau2:

CheckL:=simplify(Multiply(Multiply(J,L),J^(-1))+h*Multiply
(dJdlambda,J^(-1)));
CheckAtau1:=simplify(Multiply(Multiply(J,Atau1),J^(-1))+h*
Multiply(dJdtau1,J^(-1)));
CheckAtau2:=simplify(Multiply(Multiply(J,Atau2),J^(-1))+h*
Multiply(dJdtau2,J^(-1)));
```

$$H0 := -\frac{(p_1 - p_2)\,h}{q_1 - q_2} + \frac{-p_1^2 q_2 + p_2^2 q_1}{q_1 - q_2} + (q_1 + q_2)\,q_1\,q_2\left(q_1^2 + q_2^2 + 2\,\tau_1\right) + 2\,q_1\,q_2\,\tau_2$$

$$H1 := \frac{p_1^2 - p_2^2}{q_1 - q_2} - 2\,\tau_2\,(q_1 + q_2) - 2\left(q_1^2 + q_1\,q_2 + q_2^2\right)\tau_1 - q_1^4 - q_1^3\,q_2 - q_1^2\,q_2^2 - q_1\,q_2^3 - q_2^4$$

$$\begin{matrix} 0 \\ 0 \end{matrix}$$

$$nu1tau1 := \frac{1}{3}$$

$$nu2tau1 := 0$$

$$nu1tau2 := 0$$

$$nu2tau2 := 1$$

$$\begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}$$

$$mutau1Vector := \begin{bmatrix} -\frac{q_2}{3\,(q_1 - q_2)} \\ \frac{q_1}{3\,(q_1 - q_2)} \end{bmatrix}$$

$$\begin{bmatrix} -\frac{q_2}{3\,(q_1 - q_2)} \\ \frac{q_1}{3\,(q_1 - q_2)} \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$mutau2Vector := \begin{bmatrix} \frac{1}{q_1 - q_2} \\ -\frac{1}{q_1 - q_2} \end{bmatrix}$$

$$Hamtau1 := -\frac{(p_1 - p_2)\,h}{3\,(q_1 - q_2)} + \frac{-p_1^2\,q_2 + p_2^2\,q_1}{3\,(q_1 - q_2)} + \frac{(q_1 + q_2)\,q_1\,q_2\left(q_1^2 + q_2^2 + 2\,\tau_1\right)}{3} + \frac{2\,q_1\,q_2\,\tau_2}{3}$$

$$Hamtau2 := \frac{p_1^2 - p_2^2}{q_1 - q_2} - 2\,\tau_2\,(q_1 + q_2) - 2\left(q_1^2 + q_1\,q_2 + q_2^2\right)\tau_1 - q_1^4 - q_1^3\,q_2 - q_1^2\,q_2^2 - q_1\,q_2^3 - q_2^4$$

$$QQ := \lambda \mapsto -\frac{p_1\,(\lambda - q_2)}{q_1 - q_2} - \frac{p_2\,(\lambda - q_1)}{q_2 - q_1}$$

$$\begin{bmatrix} 1 & 0 \\ -\frac{p_1(\lambda-q_2)}{q_1-q_2} - \frac{p_2(\lambda-q_1)}{q_2-q_1} & \frac{1}{(\lambda-q_1)(\lambda-q_2)} \\ \frac{}{(\lambda-q_1)(\lambda-q_2)} & \end{bmatrix}$$

$$\left[\left[0,0\right],\left[\begin{matrix} -\frac{p_1}{q_1-q_2} - \frac{p_2}{q_2-q_1} & -\frac{p_1(\lambda-q_2)}{q_1-q_2} - \frac{p_2(\lambda-q_1)}{q_2-q_1} \\ \frac{}{(\lambda-q_1)(\lambda-q_2)} & -\frac{}{(\lambda-q_1)^2(\lambda-q_2)} \end{matrix}\right.\right.$$

$$\left.\left.\begin{matrix} -\frac{p_1(\lambda-q_2)}{q_1-q_2} - \frac{p_2(\lambda-q_1)}{q_2-q_1} \\ -\frac{}{(\lambda-q_1)(\lambda-q_2)^2} \end{matrix}, \begin{matrix} -\frac{1}{(\lambda-q_1)^2(\lambda-q_2)} \\ -\frac{1}{(\lambda-q_1)(\lambda-q_2)^2} \end{matrix}\right]\right]$$

$$\left[\left[0,1\right],\left[\begin{matrix} \lambda^5 + 2\tau_1\lambda^3 + 2\tau_2\lambda^2 + \left(\frac{p_1^2-p_2^2}{q_1-q_2} - 2\tau_2(q_1+q_2) - 2(q_1^2+q_1q_2+q_2^2)\tau_1 - q_1^4 - q_1^3q_2\right. \\ \left. -q_1^2q_2^2 - q_1q_2^3 - q_2^4\right)\lambda - \frac{(p_1-p_2)h}{q_1-q_2} + \frac{-p_1^2q_2+p_2^2q_1}{q_1-q_2} + (q_1+q_2)q_1q_2(q_1^2+q_2^2 \\ +2\tau_1) + 2q_1q_2\tau_2 - \frac{hp_1}{\lambda-q_1} - \frac{hp_2}{\lambda-q_2} \end{matrix}, \begin{matrix} \\ \frac{h}{\lambda-q_1} + \frac{h}{\lambda-q_2} \end{matrix}\right]\right]$$

,

$$dq1dtau1 := -\frac{2p_1q_2+h}{3h(q_1-q_2)}$$

$$dp1dtau1 := -\frac{1}{3h(q_1-q_2)^2}\left(4q_1^5q_2 - 5q_1^4q_2^2 + q_2^6 + 4q_1^3\tau_1q_2 - 6q_1^2q_2^2\tau_1 + 2q_2^4\tau_1 + 2\right.$$
$$\left.q_1^2\tau_2q_2 - 4q_2^2\tau_2q_1 + 2q_2^3\tau_2 + p_1^2q_2 - p_2^2q_2 + hp_1 - hp_2\right)$$

$$dq2dtau1 := \frac{2p_2q_1+h}{3h(q_1-q_2)}$$

$$dp2dtau1 := \frac{1}{3h(q_1-q_2)^2}\left(-q_1^6 + 5q_1^2q_2^4 - 4q_1q_2^5 - 2q_1^4\tau_1 + 6q_1^2q_2^2\tau_1 - 4q_2^3\tau_1q_1 - 2\right.$$
$$\left.q_1^3\tau_2 + 4q_1^2\tau_2q_2 - 2q_2^2\tau_2q_1 + p_1^2q_1 - p_2^2q_1 + hp_1 - hp_2\right)$$

$$dq1dtau2 := \frac{2p_1}{h(q_1-q_2)}$$

$$dp1dtau2 :=$$

$$\frac{1}{h\left(q_1-q_2\right)^2}\left(4\,q_1^5-5\,q_1^4\,q_2+q_2^5+4\,q_1^3\,\tau_1-6\,q_1^2\,q_2\,\tau_1+2\,q_2^3\,\tau_1+2\,q_1^2\,\tau_2-4\,q_1\,q_2\,\tau_2\right.$$
$$\left.+2\,q_2^2\,\tau_2+p_1^2-p_2^2\right)$$

$$dq2dtau2 := -\frac{2\,p_2}{h\left(q_1-q_2\right)}$$

$$dp2dtau2 :=$$
$$-\frac{1}{h\left(q_1-q_2\right)^2}\left(-q_1^5+5\,q_1\,q_2^4-4\,q_2^5-2\,q_1^3\,\tau_1+6\,q_1\,q_2^2\,\tau_1-4\,q_2^3\,\tau_1-2\,q_1^2\,\tau_2\right.$$
$$\left.+4\,q_1\,q_2\,\tau_2-2\,q_2^2\,\tau_2+p_1^2-p_2^2\right)$$

$$CheckL := \left[\left[\frac{p_1\left(\lambda-q_2\right)-p_2\left(\lambda-q_1\right)}{q_1-q_2},\ \left(\lambda-q_1\right)\left(\lambda-q_2\right)\right],\right.$$
$$\left[\frac{1}{\left(q_1-q_2\right)^2}\left(q_1^5+\left(\lambda-q_2\right)q_1^4+\left(\lambda^2-\lambda\,q_2+2\,\tau_1\right)q_1^3+\left(\left(-\lambda^2-2\,\tau_1\right)q_2+\lambda^3\right.\right.\right.$$
$$\left.+2\,\tau_1\,\lambda+2\,\tau_2\right)q_1^2-2\,q_2\left(\frac{q_2^3}{2}+\frac{\lambda\,q_2^2}{2}+\left(\frac{\lambda^2}{2}+\tau_1\right)q_2+\lambda^3+2\,\tau_1\,\lambda+2\,\tau_2\right)q_1+q_2^5$$
$$\left.+\lambda\,q_2^4+\left(\lambda^2+2\,\tau_1\right)q_2^3+\left(\lambda^3+2\,\tau_1\,\lambda+2\,\tau_2\right)q_2^2-\left(p_1-p_2\right)^2\right),$$
$$\left.\left.\frac{\left(-\lambda+q_2\right)p_1+p_2\left(\lambda-q_1\right)}{q_1-q_2}\right]\right]$$

$$CheckAtau1 := \begin{bmatrix} \dfrac{p_1-p_2}{3\,q_1-3\,q_2} & -\dfrac{q_1}{3}+\dfrac{\lambda}{3}-\dfrac{q_2}{3} \\ \dfrac{\lambda^2}{3}+\dfrac{\left(q_1+q_2\right)\lambda}{3}+\dfrac{q_1^2}{3}+\dfrac{q_2^2}{3}+\dfrac{2\,\tau_1}{3} & \dfrac{p_2-p_1}{3\,q_1-3\,q_2} \end{bmatrix}$$

$$CheckAtau2 := \begin{bmatrix} 0 & 1 \\ \lambda+2\,q_1+2\,q_2 & 0 \end{bmatrix} \qquad \textbf{(4)}$$

```
> CheckL21:=CheckL[2,1]:
  CheckL21bis:=lambda^3+(q1+q2)*lambda^2+(q1^2+q1*q2+q2^2+2*tau[1])
  *lambda-(p1-p2)^2/(q1-q2)^2 + (q1^2+q2^2+2*tau[1])*(q1+q2)+2*tau
  [2]:
  factor(series(CheckL21-CheckL21bis,p2));
```

$$0 \qquad \textbf{(5)}$$

```
> tdL11theo:=0:
  for j from 0 to g-1 do aux:=0: for i from j+1 to g do aux:=aux+P
  [i]*Q[i-j-1]: od: tdL11theo:=tdL11theo-(-1)^(j-1)*aux*lambda^j:
  od:
```

```
tdL11theo:=simplify(tdL11theo);
simplify(tdL11theo-CheckL[1,1]);
```

$$tdL11theo := \frac{p_1\left(\lambda - q_2\right) - p_2\left(\lambda - q_1\right)}{q_1 - q_2}$$

$$0 \tag{6}$$

```
> tdL12theo:=0:
  for m from 0 to g do tdL12theo:=tdL12theo+(-1)^(g-m)*Q[g-m]*
  lambda^m: od:
  tdL12theo:=simplify(tdL12theo);
  simplify(tdL12theo-CheckL[1,2]);
```

$$tdL12theo := \left(\lambda - q_1\right)\left(\lambda - q_2\right)$$

$$0 \tag{7}$$

```
> Term1:=0:
  for i from 0 to rinfty-2 do for j from g+i to 2*rinfty-5 do
  Term1:=Term1- P2[j]*h[j-g-i]*lambda^i: od: od:
  Term1:
  Term2:=0:
  for i from 0 to g-2 do for j1 from i+1 to g-1 do for j2 from g+i-
  j1 to g-1 do for i1 from j1+1 to g do for i2 from j2+1 to g do
  Term2:=Term2- (-1)^(j1+j2)*P[i1]*Q[i1-j1-1]*P[i2]*Q[i2-j2-1]*h
  [j1+j2-g-i]*lambda^i:
  od: od: od: od: od:
  Term2:
  tdL21theo:=simplify(Term1+Term2):
  simplify(tdL21theo-CheckL[2,1]);
```

$$0 \tag{8}$$

We have verified the \td{L} formula. Let us do A^{tau1}.

```
> nu[1]:=nu1tau1;
  nu[2]:=nu2tau1;

  nu[rinfty-2]:=0:
  for k from 1 to g do nu[rinfty-2]:=nu[rinfty-2]+(-1)^(g-k)*nu[k]*
  Q[g+1-k]: od:
  nu[rinfty-2];
```

$$\nu_1 := \frac{1}{3}$$

$$\nu_2 := 0$$

$$-\frac{q_1 q_2}{3} \tag{9}$$

```
> tdA11theo:=0:
  for i from 0 to g-2 do for m from 1 to g-1-i do for r from i+m+1
  to g do tdA11theo:=tdA11theo-(-1)^(i+m-1)*nu[m]*P[r]*Q[r-i-m-1]*
```

```
lambda^i od: od: od:
tdA11theo:=simplify(tdA11theo);
factor(simplify(tdA11theo-CheckAtau1[1,1]));
```

$$tdA11theo := \frac{p_1 - p_2}{3\,q_1 - 3\,q_2}$$

$$0 \tag{10}$$

```
> tdA12theo:=0:
for j from 0 to g-1 do for m from 1 to g-j do tdA12theo:=
tdA12theo+(-1)^(g-j-m)*nu[m]*Q[g-j-m]*lambda^j: od: od:
tdA12theo:=simplify(tdA12theo);
factor(simplify(tdA12theo-CheckAtau1[1,2]));
```

$$tdA12theo := -\frac{q_1}{3} + \frac{\lambda}{3} - \frac{q_2}{3}$$

$$0 \tag{11}$$

```
> tdA21theoTerm1:=0:
for i from 0 to g do lowerpoint:=max(g,g+i-1): for j from
lowerpoint to 2*rinfty-5 do for m from 1 to j-g-i do
 tdA21theoTerm1:=tdA21theoTerm1-nu[m]*h[j-g-m-i]*P2[j]*lambda^i:
od: od: od:
tdA21theoTerm1;
tdA21theoTerm2:=0:
for i from 0 to g do for j1 from 0 to g-1 do for j2 from 1 to g-1
do for m from 1 to j1+j2-g-i do for r1 from j1+1 to g do for r2
from j2+1 to g do
tdA21theoTerm2:=tdA21theoTerm2-(-1)^(j1+j2)*nu[m]*h[j1+j2-g-i-m]*
P[r1]*P[r2]*Q[r1-j1-1]*Q[r2-j2-1]*lambda^i:
od: od: od: od: od: od:
tdA21theoTerm2;
tdA21theo:=simplify(tdA21theoTerm1+tdA21theoTerm2);
factor(simplify(tdA21theo-CheckAtau1[2,1]));
```

$$\frac{2\,\tau_1}{3} + \frac{(q_1+q_2)^2}{3} - \frac{2\,q_1\,q_2}{3} + \frac{(q_1+q_2)\,\lambda}{3} + \frac{\lambda^2}{3}$$

$$0$$

$$tdA21theo := \frac{\lambda^2}{3} + \frac{(q_1+q_2)\,\lambda}{3} + \frac{q_1^2}{3} + \frac{q_2^2}{3} + \frac{2\,\tau_1}{3}$$

$$0 \tag{12}$$

```
> Term1Hamtheo:=0:
for i from 1 to g do for k from i+1 to g do Term1Hamtheo:=
Term1Hamtheo-h*nu[i]*(-1)^i*(g-i)*P[k]*Q[k-1-i]: od: od:
for i from 1 to g do for k from i+1 to g do for m from i+1 to k-1
do Term1Hamtheo:=Term1Hamtheo-h*nu[i]*(-1)^m*P[k]*Q[k-1-m]*S[m-i]
: od: od: od:
```

```
Term2Hamtheo:=0:
for i from 1 to g do for k1 from 1 to g do for k2 from 1 to g do
for r1 from max(0,i-k2) to min(k1-1,i-1) do
Term2Hamtheo:=Term2Hamtheo+nu[i]*P[k1]*P[k2]*(-1)^(i-1)*Q[k1-1-
r1]*Q[k2-i+r1]: od: od: od: od:

for i from 1 to g do for k1 from 1 to g do for k2 from 1 to g do
for r1 from 0 to k1-1 do for r2 from 0 to k2-1 do for m from i to
g do if r1+r2>=g then  Term2Hamtheo:=Term2Hamtheo+nu[i]*P[k1]*P
[k2]*(-1)^(r1+r2)*Q[k1-1-r1]*Q[k2-1-r2]*(-1)^(g-m)*Q[g-m]*h[r1+
r2+m-i-g+1]: fi: od: od: od: od: od: od:

Term3Hamtheo:=0:
for i from 1 to g do for r from g to 2*rinfty-5 do for m from i
to g do Term3Hamtheo:=Term3Hamtheo+nu[i]*(-1)^(g-m)*P2[r]*Q[g-m]*
h[r+m-i-g+1]: od: od: od:

Hamilton:=simplify(Term1Hamtheo+Term2Hamtheo+Term3Hamtheo):
simplify(Hamilton-Hamtau1);


Term1Hamtheo:=0:
for i from 1 to g do for k from i+1 to g do Term1Hamtheo:=
Term1Hamtheo-h*nu[i]*(-1)^i*(g-i)*P[k]*Q[k-1-i]: od: od:
for i from 1 to g do for k from i+1 to g do for m from i+1 to k-1
do Term1Hamtheo:=Term1Hamtheo-h*nu[i]*(-1)^m*P[k]*Q[k-1-m]*S[m-i]
: od: od: od:

Term2Hamtheo:=0:
for i from 1 to g do for k1 from 1 to g do for k2 from 1 to g do
for r1 from 0 to k1-1 do for r2 from 0 to k2-1 do if r1+r2<=g-1
then if r1+r2=i-1 then
Term2Hamtheo:=Term2Hamtheo+nu[i]*P[k1]*P[k2]*(-1)^(r1+r2)*Q[k1-1-
r1]*Q[k2-1-r2]: fi: fi: od: od: od: od: od:

for i from 1 to g do for k1 from 1 to g do for k2 from 1 to g do
for r1 from 0 to k1-1 do for r2 from 0 to k2-1 do for m from i to
g do if r1+r2>=g then  Term2Hamtheo:=Term2Hamtheo+nu[i]*P[k1]*P
[k2]*(-1)^(r1+r2)*Q[k1-1-r1]*Q[k2-1-r2]*(-1)^(g-m)*Q[g-m]*h[r1+
r2+m-i-g+1]: fi: od: od: od: od: od: od:

Term3Hamtheo:=0:
```

```
for i from 1 to g do for r from g to 2*rinfty-5 do for m from i
to g do Term3Hamtheo:=Term3Hamtheo+nu[i]*(-1)^(g-m)*P2[r]*Q[g-m]*
h[r+m-i-g+1]: od: od: od:

Hamilton:=simplify(Term1Hamtheo+Term2Hamtheo+Term3Hamtheo):
simplify(Hamilton-Hamtau1);
```

$$\begin{matrix} 0 \\ 0 \end{matrix} \tag{13}$$

```
> simplify(Hamtau1);
```

$$\frac{h\left(p_2-p_1\right)+q_1^5 q_2+2\,q_1^3\tau_1\,q_2+2\,q_1^2\tau_2\,q_2+\left(-q_2^5-2\,q_2^3\tau_1-2\,q_2^2\tau_2+p_2^2\right)q_1-p_1^2\,q_2}{3\,q_1-3\,q_2} \tag{14}$$

```
> factor(P[1]^2+2*P[1]*P[2]*Q[1]+P[2]^2*(Q[1]^2-Q[2]));
```

$$-\frac{p_1^2\,q_2-p_2^2\,q_1}{q_1-q_2} \tag{15}$$

```
> simplify(Hamtau1-(h*P[2]/3 +1/3*(P[1]^2+2*P[1]*P[2]*Q[1]+P[2]^2*
(Q[1]^2-Q[2]))));
f1:=unapply(es(simplify(Hamtau1-h*P[2]/3-1/3*(P[1]^2+2*P[1]*P[2]*
Q[1]+P[2]^2*(Q[1]^2-Q[2]))),q[1],q[2]),sigma[1],sigma[2]):
f1(Q1,Q2);

simplify(Hamtau1- 1/3*(h*P[2]+P[1]^2+2*P[1]*P[2]*Q[1]+P[2]^2*(Q
[1]^2-Q[2])+Q[1]^3*Q[2]-2*Q[1]*Q[2]^2+2*Q[1]*Q[2]*tau[1]+2*Q[2]*
tau[2]));
```

$$\begin{matrix} \dfrac{\left(q_1^3+q_1^2\,q_2+\left(q_2^2+2\,\tau_1\right)q_1+q_2^3+2\,q_2\,\tau_1+2\,\tau_2\right)q_1\,q_2}{3} \\ \dfrac{1}{3}\,Q1^3\,Q2-\dfrac{2}{3}\,Q1\,Q2^2+\dfrac{2}{3}\,Q1\,Q2\,\tau_1+\dfrac{2}{3}\,Q2\,\tau_2 \\ 0 \end{matrix} \tag{16}$$

```
> simplify(CheckAtau1[1,1]-(-1/3*P[2]));
simplify(CheckAtau1[1,2]-1/3*(lambda-Q[1]));
simplify(CheckAtau1[2,1]-1/3*(lambda^2+Q[1]*lambda+Q[1]^2-2*Q[2]
+2*tau[1]));
```

$$\begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \tag{17}$$

We have verified the \td{A}^{tau1} formula. Let us do A^{tau2}.

```
> nu[1]:=nu1tau2;
nu[2]:=nu2tau2;

nu[rinfty-2]:=0:
```

```
for k from 1 to g do nu[rinfty-2]:=nu[rinfty-2]+(-1)^(g-k)*nu[k]*
Q[g+1-k]: od:
nu[rinfty-2];
```

$$\nu_1 := 0$$

$$\nu_2 := 1$$

$$q_1 + q_2 \tag{18}$$

```
> tdA11theo:=0:
for i from 0 to g-2 do for m from 1 to g-1-i do for r from i+m+1
to g do tdA11theo:=tdA11theo-(-1)^(i+m-1)*nu[m]*P[r]*Q[r-i-m-1]*
lambda^i od: od: od:
tdA11theo:=simplify(tdA11theo);
factor(simplify(tdA11theo-CheckAtau2[1,1]));
```

$$tdA11theo := 0$$

$$0 \tag{19}$$

```
> tdA12theo:=0:
for j from 0 to g-1 do for m from 1 to g-j do tdA12theo:=
tdA12theo+(-1)^(g-j-m)*nu[m]*Q[g-j-m]*lambda^j: od: od:
tdA12theo:=simplify(tdA12theo);
factor(simplify(tdA12theo-CheckAtau2[1,2]));
```

$$tdA12theo := 1$$

$$0 \tag{20}$$

```
> tdA21theoTerm1:=0:
for i from 0 to g do lowerpoint:=max(g,g+i-1): for j from
lowerpoint to 2*rinfty-5 do for m from 1 to j-g-i do
 tdA21theoTerm1:=tdA21theoTerm1-nu[m]*h[j-g-m-i]*P2[j]*lambda^i:
od: od: od:
tdA21theoTerm1;
tdA21theoTerm2:=0:
for i from 0 to g do for j1 from 0 to g-1 do for j1 from 1 to g-1
do for m from 1 to j1+j2-g-i do for r1 from j1+1 to g do for r2
from j2+1 to g do
tdA21theoTerm2:=tdA21theoTerm2-(-1)^(j1+j2)*nu[m]*h[j1+j2-g-i-m]*
P[r1]*P[r2]*Q[r1-j1-1]*Q[r2-j2-1]*lambda^i:
od: od: od: od: od: od:
tdA21theoTerm2;
tdA21theo:=simplify(tdA21theoTerm1+tdA21theoTerm2);
factor(simplify(tdA21theo-CheckAtau2[2,1]));
```

$$\lambda + 2\,q_1 + 2\,q_2$$

$$0$$

$$tdA21theo := \lambda + 2\,q_1 + 2\,q_2$$

$$0 \tag{21}$$

```
> Term1Hamtheo:=0:
```

```
for i from 1 to g do for k from i+1 to g do Term1Hamtheo:=
Term1Hamtheo-h*nu[i]*(-1)^i*(g-i)*P[k]*Q[k-1-i]: od: od:
for i from 1 to g do for k from i+1 to g do for m from i+1 to k-1
do Term1Hamtheo:=Term1Hamtheo-h*nu[i]*(-1)^m*P[k]*Q[k-1-m]*S[m-i]
: od: od: od:

Term2Hamtheo:=0:
for i from 1 to g do for k1 from 1 to g do for k2 from 1 to g do
for r1 from 0 to k1-1 do for r2 from 0 to k2-1 do if r1+r2<=g-1
then if r1+r2=i-1 then
Term2Hamtheo:=Term2Hamtheo+nu[i]*P[k1]*P[k2]*(-1)^(r1+r2)*Q[k1-1-
r1]*Q[k2-1-r2]: fi: fi: od: od: od: od: od:

for i from 1 to g do for k1 from 1 to g do for k2 from 1 to g do
for r1 from 0 to k1-1 do for r2 from 0 to k2-1 do for m from i to
g do if r1+r2>=g then  Term2Hamtheo:=Term2Hamtheo+nu[i]*P[k1]*P
[k2]*(-1)^(r1+r2)*Q[k1-1-r1]*Q[k2-1-r2]*(-1)^(g-m)*Q[g-m]*h[r1+
r2+m-i-g+1]: fi: od: od: od: od: od: od:

Term3Hamtheo:=0:
for i from 1 to g do for r from g to 2*rinfty-5 do for m from i
to g do Term3Hamtheo:=Term3Hamtheo+nu[i]*(-1)^(g-m)*P2[r]*Q[g-m]*
h[r+m-i-g+1]: od: od: od:

Hamilton:=simplify(Term1Hamtheo+Term2Hamtheo+Term3Hamtheo):
simplify(Hamilton-Hamtau2);
```

$$0 \qquad (22)$$

```
> Term1Hamtheo:=0:
for i from 1 to g do for k from i+1 to g do Term1Hamtheo:=
Term1Hamtheo-h*nu[i]*(-1)^i*(g-i)*P[k]*Q[k-1-i]: od: od:
for i from 1 to g do for k from i+1 to g do for m from i+1 to k-1
do Term1Hamtheo:=Term1Hamtheo-h*nu[i]*(-1)^m*P[k]*Q[k-1-m]*S[m-i]
: od: od: od:

Term2Hamtheo:=0:
for i from 1 to g do for k1 from 1 to g do for k2 from 1 to g do
for r1 from max(0,i-k2) to min(k1-1,i-1) do
Term2Hamtheo:=Term2Hamtheo+nu[i]*P[k1]*P[k2]*(-1)^(i-1)*Q[k1-1-
r1]*Q[k2-i+r1]: od: od: od: od:
```

```
for i from 1 to g do for k1 from 1 to g do for k2 from 1 to g do
for r1 from 0 to k1-1 do for r2 from 0 to k2-1 do for m from i to
g do if r1+r2>=g then  Term2Hamtheo:=Term2Hamtheo+nu[i]*P[k1]*P
[k2]*(-1)^(r1+r2)*Q[k1-1-r1]*Q[k2-1-r2]*(-1)^(g-m)*Q[g-m]*h[r1+
r2+m-i-g+1]: fi: od: od: od: od: od: od:

Term3Hamtheo:=0:
for i from 1 to g do for r from g to 2*rinfty-5 do for m from i
to g do Term3Hamtheo:=Term3Hamtheo+nu[i]*(-1)^(g-m)*P2[r]*Q[g-m]*
h[r+m-i-g+1]: od: od: od:

Hamilton:=simplify(Term1Hamtheo+Term2Hamtheo+Term3Hamtheo):
simplify(Hamilton-Hamtau2);
```

$$0 \tag{23}$$

```
> simplify(Hamtau2-factor(-2*P[1]*P[2]-P[2]^2*Q[1]));
f2:=unapply(es(simplify(Hamtau2-factor(-2*P[1]*P[2]-P[2]^2*Q[1])
),q[1],q[2]),sigma[1],sigma[2]):
f2(Q1,Q2);

simplify(Hamtau2- (-2*P[1]*P[2]-P[2]^2*Q[1]-Q[1]^4+3*Q[1]^2*Q[2]
-2*Q[1]^2*tau[1]-2*Q[1]*tau[2]-Q[2]^2+2*Q[2]*tau[1]));
```

$$-q_1^4 - q_1^3 q_2 + \left(-q_2^2 - 2\,\tau_1\right) q_1^2 + \left(-q_2^3 - 2\,q_2\,\tau_1 - 2\,\tau_2\right) q_1 - \left(q_2^3 + 2\,q_2\,\tau_1 + 2\,\tau_2\right) q_2$$

$$-Q1^4 + 3\,Q1^2\,Q2 - 2\,Q1^2\,\tau_1 - 2\,Q1\,\tau_2 - Q2^2 + 2\,Q2\,\tau_1$$

$$0 \tag{24}$$

```
> factor(P[1]);
factor(P[2]);
factor(Q[1]);
factor(Q[2]);
```

$$\frac{q_1 p_1 - q_2 p_2}{q_1 - q_2}$$

$$-\frac{p_1 - p_2}{q_1 - q_2}$$

$$q_1 + q_2$$

$$q_1\,q_2 \tag{25}$$

```
> factor(CheckL[1,1]-(-P[2]*lambda)-Q[1]*P[2]-P[1]);
factor(CheckL[1,2]-(lambda^2-Q[1]*lambda+Q[2]));
factor(CheckL[2,2]-(P[2]*lambda)+Q[1]*P[2]+P[1]);
factor(series(CheckL[2,1]-(lambda^3+Q[1]*lambda^2+(Q[1]^2-Q[2]+2*
tau[1])*lambda -P[2]^2+Q[1]*(Q[1]^2-2*Q[2])+2*Q[1]*tau[1]+2*tau
[2]),p[2]=0));
```

$$0$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \textbf{(26)}$$