In this Maple file, we compute the Casimir coefficients of the Lax matrix L associated to the Painlevé 2 equation in relation with the spectral curve

```
> restart:
P1:=x-> Pinfty01+Pinfty11*x+Pinfty21*x^2;
P2:=x-> Pinfty02+Pinfty12*x+Pinfty22*x^2+Pinfty32*x^3+Pinfty42*
x^4;
SpectralCurve:=unapply( y^2-P1(x)*y+P2(x),y);
```

$$P1 := x \to Pinfty01 + Pinfty11\,x + Pinfty21\,x^2 \quad \textbf{(1)}$$

$$P2 := x \to Pinfty02 + Pinfty12\,x + Pinfty22\,x^2 + Pinfty32\,x^3 + Pinfty42\,x^4$$

$$SpectralCurve := y \to y^2 - \left(x^2\,Pinfty21 + x\,Pinfty11 + Pinfty01\right) y + Pinfty42\,x^4 + Pinfty32\,x^3 + Pinfty22\,x^2 + Pinfty12\,x + Pinfty02$$

```
> DiaginftySheet1:=-tinfty13*x^2-tinfty12*x-tinfty11-tinfty10/x+
Unknown/x^2;
DiaginftySheet2:=-tinfty23*x^2-tinfty22*x-tinfty21-tinfty20/x+
Unknown2/x^2;
```

$$DiaginftySheet1 := -tinfty13\,x^2 - tinfty12\,x - tinfty11 - \frac{tinfty10}{x} + \frac{Unknown}{x^2} \quad \textbf{(2)}$$

$$DiaginftySheet2 := -tinfty23\,x^2 - tinfty22\,x - tinfty21 - \frac{tinfty20}{x} + \frac{Unknown2}{x^2}$$

Expression of P_1 in terms of the diagonal expansion in both sheets

```
> Pinfty01:=-(tinfty11+tinfty21);
Pinfty11:=-(tinfty12+tinfty22);
Pinfty21:=-(tinfty13+tinfty23);
CoherenceEquation1:=residue(DiaginftySheet1+DiaginftySheet2-P1
(x),x=infinity);
tinfty20:=-tinfty10;
```

$$Pinfty01 := -tinfty11 - tinfty21 \quad \textbf{(3)}$$

$$Pinfty11 := -tinfty12 - tinfty22$$

$$Pinfty21 := -tinfty13 - tinfty23$$

$$CoherenceEquation1 := tinfty10 + tinfty20$$

$$tinfty20 := -tinfty10$$

Study at infinity

```
> series(SpectralCurve(DiaginftySheet1),x=infinity,6):
series(SpectralCurve(DiaginftySheet2),x=infinity,6):
EQinfty1:=residue(x^(-5)*SpectralCurve(DiaginftySheet1),x=
infinity);
EQinfty2:=residue(x^(-5)*SpectralCurve(DiaginftySheet2),x=
infinity);
EQinfty3:=residue(x^(-4)*SpectralCurve(DiaginftySheet1),x=
infinity);
EQinfty4:=residue(x^(-4)*SpectralCurve(DiaginftySheet2),x=
infinity);
EQinfty5:=residue(x^(-3)*SpectralCurve(DiaginftySheet1),x=
```

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infinity);
EQinfty6:=residue(x^(-3)*SpectralCurve(DiaginftySheet2),x=
infinity);
EQinfty7:=residue(x^(-2)*SpectralCurve(DiaginftySheet1),x=
infinity);
EQinfty8:=residue(x^(-2)*SpectralCurve(DiaginftySheet2),x=
infinity);
EQinfty9:=residue(x^(-1)*SpectralCurve(DiaginftySheet1),x=
infinity);
EQinfty10:=residue(x^(-1)*SpectralCurve(DiaginftySheet2),x=
infinity);
```

$$EQinfty1 := -tinfty13^2 - (-tinfty13 - tinfty23)\ tinfty13 - Pinfty42 \quad \textbf{(4)}$$
$$EQinfty2 := -tinfty23^2 - (-tinfty13 - tinfty23)\ tinfty23 - Pinfty42$$
$$EQinfty3 := -2\ tinfty13\ tinfty12 - (-tinfty13 - tinfty23)\ tinfty12 - (-tinfty12 - tinfty22)\ tinfty13 - Pinfty32$$
$$EQinfty4 := -2\ tinfty23\ tinfty22 - (-tinfty13 - tinfty23)\ tinfty22 - (-tinfty12 - tinfty22)\ tinfty23 - Pinfty32$$
$$EQinfty5 := -2\ tinfty13\ tinfty11 - tinfty12^2 - (-tinfty13 - tinfty23)\ tinfty11 - (-tinfty12 - tinfty22)\ tinfty12 - (-tinfty11 - tinfty21)\ tinfty13 - Pinfty22$$
$$EQinfty6 := -2\ tinfty23\ tinfty21 - tinfty22^2 - (-tinfty13 - tinfty23)\ tinfty21 - (-tinfty12 - tinfty22)\ tinfty22 - (-tinfty11 - tinfty21)\ tinfty23 - Pinfty22$$
$$EQinfty7 := -2\ tinfty13\ tinfty10 - 2\ tinfty12\ tinfty11 - (-tinfty13 - tinfty23)\ tinfty10 - (-tinfty12 - tinfty22)\ tinfty11 - (-tinfty11 - tinfty21)\ tinfty12 - Pinfty12$$
$$EQinfty8 := 2\ tinfty10\ tinfty23 - 2\ tinfty22\ tinfty21 + (-tinfty13 - tinfty23)\ tinfty10 - (-tinfty12 - tinfty22)\ tinfty21 - (-tinfty11 - tinfty21)\ tinfty22 - Pinfty12$$
$$EQinfty9 := 2\ tinfty13\ Unknown - 2\ tinfty12\ tinfty10 - tinfty11^2 + (-tinfty13 - tinfty23)\ Unknown - (-tinfty12 - tinfty22)\ tinfty10 - (-tinfty11 - tinfty21)\ tinfty11 - Pinfty02$$
$$EQinfty10 := 2\ tinfty23\ Unknown2 + 2\ tinfty10\ tinfty22 - tinfty21^2 + (-tinfty13 - tinfty23)\ Unknown2 + (-tinfty12 - tinfty22)\ tinfty10 - (-tinfty11 - tinfty21)\ tinfty21 - Pinfty02$$

```
> Pinfty42:=factor(solve(EQinfty1,Pinfty42));
Pinfty32:=factor(solve(EQinfty3,Pinfty32));
Pinfty22:=factor(solve(EQinfty5,Pinfty22));
Pinfty12:=factor(solve(EQinfty7,Pinfty12));
simplify(EQinfty8);
simplify(EQinfty9);
```

$$Pinfty42 := tinfty13\ tinfty23 \quad \textbf{(5)}$$
$$Pinfty32 := tinfty12\ tinfty23 + tinfty13\ tinfty22$$
$$Pinfty22 := tinfty11\ tinfty23 + tinfty12\ tinfty22 + tinfty13\ tinfty21$$
$$Pinfty12 := -tinfty10\ tinfty13 + tinfty10\ tinfty23 + tinfty11\ tinfty22 + tinfty12\ tinfty21$$
$$0$$
$$(-tinfty12 + tinfty22)\ tinfty10 + (tinfty13 - tinfty23)\ Unknown + tinfty11\ tinfty21 - Pinfty02$$

Summary of the coefficients

```
> Pinfty01:=Pinfty01;
  Pinfty11:=Pinfty11;
  Pinfty21:=Pinfty21;
  Pinfty02:=Pinfty02;
  Pinfty12:=Pinfty12;
  Pinfty22:=Pinfty22;
  Pinfty32:=Pinfty32;
  Pinfty42:=Pinfty42;
  CoherenceEquation1:=tinfty10+tinfty20;
```

$$\begin{aligned}
&Pinfty01 := -tinfty11 - tinfty21\\
&Pinfty11 := -tinfty12 - tinfty22\\
&Pinfty21 := -tinfty13 - tinfty23\\
&Pinfty02 := Pinfty02\\
&Pinfty12 := -tinfty10\, tinfty13 + tinfty10\, tinfty23 + tinfty11\, tinfty22 + tinfty12\, tinfty21\\
&Pinfty22 := tinfty11\, tinfty23 + tinfty12\, tinfty22 + tinfty13\, tinfty21\\
&Pinfty32 := tinfty12\, tinfty23 + tinfty13\, tinfty22\\
&Pinfty42 := tinfty13\, tinfty23\\
&CoherenceEquation1 := 0
\end{aligned} \tag{6}$$

We have one undetermined coefficient Pinfty02, all the other coefficients are expressed in terms of irregular times and monodromies.