In this Maple file, we compute the evolution equations for the Painlevé 2 equations using the compatibility equation of the Lax system. We also obtain the expression of the Lax matrices in the geometric gauge without apparent singularities.

Lax matrices in the oper gauge from previous Maple files

Summary of previous files: We have the expression for some coefficients of the Lax matrix L and of A.

The operator is \hbar (alpha13\partial_{t_{\infty^{(1)},3} +alpha23\partial_{t_{\infty^{(2)},3}+ alpha12\partial_{t_{\infty^{(1)},2} +alpha22\partial_{t_{\infty^{(2)},2}+alpha11\partial_{t_{\infty^{(1)},1} +alpha21\partial_{t_{\infty^{(2)},1}))

```
> restart:
  with(LinearAlgebra):
  CoherenceEquation1:=tinfty10+tinfty20;
  tinfty20:=-tinfty10:
  Pinfty42 := tinfty13*tinfty23;
  Pinfty32 := tinfty12*tinfty23+tinfty13*tinfty22;
  Pinfty22 := tinfty12*tinfty22+tinfty13*tinfty21+tinfty11*
  tinfty23;
  Pinfty12 := tinfty20*tinfty13+tinfty12*tinfty21+tinfty10*
  tinfty23+tinfty11*tinfty22;
  Pinfty01 := -tinfty11-tinfty21;
  Pinfty11 := -tinfty12-tinfty22;
  Pinfty21 := -tinfty13-tinfty23;
  P1:=x-> Pinfty01+Pinfty11*x+Pinfty21*x^2:
  P2:=x-> Pinfty02+Pinfty12*x+Pinfty22*x^2+Pinfty32*x^3+Pinfty42*
  x^4:
  tdP2:=unapply(P2(x)-Pinfty02,x):

  dP1dlambda:=unapply(diff(P1(lambda),lambda),lambda):
  dP2dlambda:=unapply(diff(P2(lambda),lambda),lambda):
  L:=Matrix(2,2,0):
  L[1,1]:=0:
  L[1,2]:=1:
  L[2,1]:=-P2(lambda)+Pinfty02 +C -h*lambda*tinfty13 - p*h/
  (lambda-q):
  L[2,2]:= P1(lambda)  +h/(lambda-q)  :

  A:=Matrix(2,2,0):
  A[1,1]:=1/3*(alpha13*tinfty23-alpha23*tinfty13)/(tinfty13-
  tinfty23)*lambda^3+c2*lambda^2+c1*lambda +c0+ rho/(lambda-q):
  A[1,2]:=(alpha13-alpha23)/3/(tinfty13-tinfty23)*lambda+nu+ mu/
  (lambda-q):
```

```
A[2,1]:=AA21(lambda):
A[2,2]:=AA22(lambda):
dAdlambda:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do dAdlambda[i,j]:=diff
(A[i,j],lambda):  od: od:

L;
A;

Q2:=unapply(-p,lambda):
J:=Matrix(2,2,0):
J[1,1]:=1:
J[1,2]:=0:
J[2,1]:=Q2(lambda)/(lambda-q):
J[2,2]:=1/(lambda-q):

dJdlambda:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do dJdlambda[i,j]:=diff
(J[i,j],lambda): od: od:
J:

LJ:=Matrix(2,2,0):
LJ[1,1]:=0:
LJ[1,2]:=0:
LJ[2,2]:=diff(J[2,2],q)*Lq+diff(J[2,2],p)*Lp+h*diff(J[2,2],t):
LJ[2,1]:=diff(J[2,1],q)*Lq+diff(J[2,1],p)*Lp+h*diff(J[2,1],t):
LJ:

checkL:=simplify(Multiply(Multiply(J,L),J^(-1))+h*Multiply
(dJdlambda,J^(-1))):
checkA:=simplify(Multiply(Multiply(J,A),J^(-1))+Multiply(LJ,J^
(-1))):
```

$$CoherenceEquation1 := tinfty10 + tinfty20 \quad \textbf{(1.1)}$$
$$Pinfty42 := tinfty13\, tinfty23$$
$$Pinfty32 := tinfty12\, tinfty23 + tinfty13\, tinfty22$$
$$Pinfty22 := tinfty11\, tinfty23 + tinfty12\, tinfty22 + tinfty13\, tinfty21$$
$$Pinfty12 := -tinfty10\, tinfty13 + tinfty10\, tinfty23 + tinfty11\, tinfty22 + tinfty12\, tinfty21$$
$$Pinfty01 := -tinfty11 - tinfty21$$
$$Pinfty11 := -tinfty12 - tinfty22$$
$$Pinfty21 := -tinfty13 - tinfty23$$

$$\left[\left[0, 1\right], \left[-(-tinfty10\, tinfty13 + tinfty10\, tinfty23 + tinfty11\, tinfty22 + tinfty12\, tinfty21)\,\lambda - (tinfty11\, tinfty23 + tinfty12\, tinfty22 + tinfty13\, tinfty21)\,\lambda^2 - (tinfty12\, tinfty23 + tinfty13\, tinfty22)\,\lambda^3 - tinfty13\, tinfty23\,\lambda^4 + C - h\,\lambda\, tinfty13 - \frac{p\,h}{\lambda - q}, -tinfty11 - tinfty21 + (-tinfty12 - tinfty22)\,\lambda + (-tinfty13 - tinfty23)\,\lambda^2 + \frac{h}{\lambda - q}\right]\right]$$

$$\left[\left[\frac{1}{3}\,\frac{(\alpha 13\, tinfty23 - \alpha 23\, tinfty13)\,\lambda^3}{tinfty13 - tinfty23} + c2\,\lambda^2 + c1\,\lambda + c0 + \frac{\rho}{\lambda - q}, \frac{1}{3}\,\frac{(\alpha 13 - \alpha 23)\,\lambda}{tinfty13 - tinfty23} + \nu + \frac{\mu}{\lambda - q}\right], \left[AA21(\lambda), AA22(\lambda)\right]\right]$$

Solving the compatibility equations to obtain the Hamiltonian evolutions.

The compatibility equation is \mathcal{L}L=h\partial_\lambda A+[A,L]
Since the first line of L is trivial, we may easily obtain A[2,1] et A[2,2] to obtain the full expression for A

```
> LL:=h*dAdlambda+(Multiply(A,L)-Multiply(L,A)):
  Entry11:=LL[1,1]:
  Entry12:=LL[1,2]:
  AA21:=unapply(solve(Entry11=0,AA21(lambda)),lambda):
  AA21bis:=h*dAdlambda[1,1]+A[1,2]*L[2,1]:
  simplify(AA21(lambda)-AA21bis);

  AA22:=unapply(solve(Entry12=0,AA22(lambda)),lambda):
  AA22bis:=h*dAdlambda[1,2]+A[1,1]+A[1,2]*L[2,2]:
  simplify(AA22(lambda)-AA22bis);
  simplify(Entry11);
  simplify(Entry12);
  LL:=h*dAdlambda+(Multiply(A,L)-Multiply(L,A)):
```

$$0 \quad \textbf{(2.1)}$$
$$0$$
$$0$$
$$0$$

We now compute the action of \mathcal{L} on L[2,2] et L[2,1] to obtain the evolution equations

Evolution of entry L_{2,2}

```
> Entry22:=simplify(LL[2,2]):
  Entry22TermLambdaMinusqCube:=factor(residue(Entry22*(lambda-q)
  ^2,lambda=q));
  Entry22TermLambdaMinusqSquare:=factor(residue(Entry22*(lambda-
  q),lambda=q));
  Entry22TermLambdaMinusq:=factor(residue(Entry22,lambda=q));

  Entry22TermLambdaInfty4:=factor(-residue(Entry22/lambda^5,
  lambda=infinity));
  Entry22TermLambdaInfty3:=factor(-residue(Entry22/lambda^4,
  lambda=infinity));
  Entry22TermLambdaInfty2:=factor(-residue(Entry22/lambda^3,
  lambda=infinity));
  Entry22TermLambdaInfty1:=factor(-residue(Entry22/lambda^2,
  lambda=infinity));
  Entry22TermLambdaInfty0:=factor(-residue(Entry22/lambda,lambda=
  infinity));

  simplify( Entry22-(Entry22TermLambdaMinusqSquare/(lambda-q)^2+
  Entry22TermLambdaMinusq/(lambda-q)
  +Entry22TermLambdaInfty0+Entry22TermLambdaInfty1*lambda+
  Entry22TermLambdaInfty2*lambda^2+Entry22TermLambdaInfty3*
  lambda^3+Entry22TermLambdaInfty4*lambda^4) );
```

$$Entry22TermLambdaMinusqCube := 0 \tag{2.2}$$

$$Entry22TermLambdaMinusqSquare := -\frac{1}{3}\,\frac{1}{tinfty13 - tinfty23}\big(h\,(-3\,\mu\,q^2\,tinfty13^2 + 3\,\mu\,q^2\,tinfty23^2 - 3\,\mu\,q\,tinfty12\,tinfty13 + 3\,\mu\,q\,tinfty12\,tinfty23 - 3\,\mu\,q\,tinfty13\,tinfty22 + 3\,\mu\,q\,tinfty22\,tinfty23 + \alpha 13\,h\,q - \alpha 23\,h\,q + 3\,h\,\nu\,tinfty13 - 3\,h\,\nu\,tinfty23 - 3\,\mu\,tinfty11\,tinfty13 + 3\,\mu\,tinfty11\,tinfty23 - 3\,\mu\,tinfty13\,tinfty21 + 3\,\mu\,tinfty21\,tinfty23 + 6\,\rho\,tinfty13 - 6\,\rho\,tinfty23)\big)$$

$$Entry22TermLambdaMinusq := 0$$

$$Entry22TermLambdaInfty4 := 0$$

$$Entry22TermLambdaInfty3 := 0$$

$$Entry22TermLambdaInfty2 := -(\alpha 23 + \alpha 13)\,h$$

$$Entry22TermLambdaInfty1 := -\frac{2}{3}\,\frac{1}{tinfty13 - tinfty23}\big(h\,(3\,\nu\,tinfty13^2 - 3\,\nu\,tinfty23^2 + \alpha 13\,tinfty12 + \alpha 13\,tinfty22 - \alpha 23\,tinfty12 - \alpha 23\,tinfty22 - 6\,c2\,tinfty13 + 6\,c2\,tinfty23)\big)$$

$$Entry22TermLambdaInfty0 := -\frac{1}{3}\,\frac{1}{tinfty13 - tinfty23}\Big(h\,\big(3\,\mu\,tinfty13^2 - 3\,\mu\,tinfty23^2 + 3\,\nu\,tinfty12\,tinfty13 - 3\,\nu\,tinfty12\,tinfty23 + 3\,\nu\,tinfty13\,tinfty22 - 3\,\nu\,tinfty22\,tinfty23 + \alpha 13\,tinfty11 + \alpha 13\,tinfty21 - \alpha 23\,tinfty11 - \alpha 23\,tinfty21 - 6\,c1\,tinfty13 + 6\,c1\,tinfty23\big)\Big)$$

$$0$$

We have:

$$L_{2,\,2} := -tinfty11 - tinfty21 + (-tinfty12 - tinfty22)\,\lambda + (-tinfty13 - tinfty23)\,\lambda^2 + \frac{h}{\lambda - q}$$

Since the operator is \hbar (alpha13\partial_{t_{\infty^{(1)},3} +alpha23\partial_{t_{\infty^{(2)},3}+alpha12*\partial_{t_{\infty^{(1)},2} +alpha22*\partial_{t_{\infty^{(2)},2}+alpha11*\partial_{t_{\infty^{(1)},1} +alpha21*\partial_{t_{\infty^{(2)},1}) we can deduce the action of \mathcal{L} on q

```
> L22OrderLambda2:=-residue(L[2,2]/lambda^3,lambda=infinity):
L22OrderLambda1:=-residue(L[2,2]/lambda^2,lambda=infinity):
L22OrderLambda0:=-residue(L[2,2]/lambda^1,lambda=infinity):
simplify(h*(alpha13*diff(L22OrderLambda2,tinfty13)+alpha23*diff
(L22OrderLambda2,tinfty23)+alpha12*diff(L22OrderLambda2,
tinfty12)+alpha22*diff(L22OrderLambda2,tinfty22)+alpha11*diff
(L22OrderLambda2,tinfty11)+alpha21*diff(L22OrderLambda2,
tinfty21))- Entry22TermLambdaInfty2);
Equation1:=factor(simplify(h*(alpha13*diff(L22OrderLambda1,
tinfty13)+alpha23*diff(L22OrderLambda1,tinfty23)+alpha12*diff
(L22OrderLambda1,tinfty12)+alpha22*diff(L22OrderLambda1,
tinfty22)+alpha11*diff(L22OrderLambda1,tinfty11)+alpha21*diff
(L22OrderLambda1,tinfty21))- Entry22TermLambdaInfty1));
Equation2:=factor(simplify(h*(alpha13*diff(L22OrderLambda0,
tinfty13)+alpha23*diff(L22OrderLambda0,tinfty23)+alpha12*diff
(L22OrderLambda0,tinfty12)+alpha22*diff(L22OrderLambda0,
tinfty22)+alpha11*diff(L22OrderLambda0,tinfty11)+alpha21*diff
(L22OrderLambda0,tinfty21))- Entry22TermLambdaInfty0));
```

$$0 \qquad \textbf{(2.3)}$$

$$Equation1 := -\frac{1}{3}\,\frac{1}{tinfty13 - tinfty23}\Big(h\,\big(-6\,\nu\,tinfty13^2 + 6\,\nu\,tinfty23^2 + 3\,\alpha 12\,tinfty13 - 3\,\alpha 12\,tinfty23 - 2\,\alpha 13\,tinfty12 - 2\,\alpha 13\,tinfty22 + 3\,\alpha 22\,tinfty13 - 3\,\alpha 22\,tinfty23 + 2\,\alpha 23\,tinfty12 + 2\,\alpha 23\,tinfty22 + 12\,c2\,tinfty13 - 12\,c2\,tinfty23\big)\Big)$$

$$Equation2 := -\frac{1}{3}\,\frac{1}{tinfty13 - tinfty23}\Big(h\,\big(-3\,\mu\,tinfty13^2 + 3\,\mu\,tinfty23^2 - 3\,\nu\,tinfty12\,tinfty13 + 3\,\nu\,tinfty12\,tinfty23 - 3\,\nu\,tinfty13\,tinfty22 + 3\,\nu\,tinfty22\,tinfty23 + 3\,\alpha 11\,tinfty13 - 3\,\alpha 11\,tinfty23 - \alpha 13\,tinfty11 - \alpha 13\,tinfty21 + 3\,\alpha 21\,tinfty13 - 3\,\alpha 21\,tinfty23 + \alpha 23\,tinfty11 + \alpha 23\,tinfty21 + 6\,c1\,tinfty13 - 6\,c1\,tinfty23\big)\Big)$$

```
> Lq:=factor(Entry22TermLambdaMinusqSquare/h):
  Lqbis:=-mu*P1(q)+2*rho-h*nu-h*(alpha13-alpha23)/3/(tinfty13-
  tinfty23)*q;
  factor(simplify(series(Lq-Lqbis,q=0)));
```

$$Lqbis := -\mu\left(-tinfty11 - tinfty21 + (-tinfty12 - tinfty22)\,q + (-tinfty13 - tinfty23)\,q^2\right) + 2\,\rho - h\,\nu - \frac{1}{3}\,\frac{h\,(\alpha 13 - \alpha 23)\,q}{tinfty13 - tinfty23} \quad \textbf{(2.4)}$$

$$-4\,\rho$$

Let us look at \mathcal{L}[L[2,1]]

```
> Entry21:=simplify(LL[2,1]):
  Entry21TermLambdaMinusqCube:=factor(residue(Entry21*(lambda-q)
  ^2,lambda=q));
  Entry21TermLambdaMinusqSquare:=factor(residue(Entry21*(lambda-
  q),lambda=q));
  Entry21TermLambdaMinusq:=factor(residue(Entry21,lambda=q));

  Entry21TermLambdaInfty6:=factor(-residue(Entry21/lambda^7,
  lambda=infinity));
  Entry21TermLambdaInfty5:=factor(-residue(Entry21/lambda^6,
  lambda=infinity));
  Entry21TermLambdaInfty4:=factor(-residue(Entry21/lambda^5,
  lambda=infinity));
  Entry21TermLambdaInfty3:=factor(-residue(Entry21/lambda^4,
  lambda=infinity));
  Entry21TermLambdaInfty2:=factor(-residue(Entry21/lambda^3,
  lambda=infinity));
  Entry21TermLambdaInfty1:=factor(-residue(Entry21/lambda^2,
  lambda=infinity));
  Entry21TermLambdaInfty0:=factor(-residue(Entry21/lambda,lambda=
  infinity));

  simplify( Entry21-(Entry21TermLambdaMinusqCube/(lambda-q)^3+
  Entry21TermLambdaMinusqSquare/(lambda-q)^2+
  Entry21TermLambdaMinusq/(lambda-q)
  +Entry21TermLambdaInfty0+Entry21TermLambdaInfty1*lambda+
  Entry21TermLambdaInfty2*lambda^2+Entry21TermLambdaInfty3*
  lambda^3
  +Entry21TermLambdaInfty4*lambda^4+Entry21TermLambdaInfty5*
  lambda^5+Entry21TermLambdaInfty6*lambda^6
  ) );
  L[2,1];
```

$$Entry21TermLambdaMinusqCube := 3\,(\mu\,p + \rho)\,h^2 \tag{2.5}$$

$$Entry21TermLambdaMinusqSquare := -\frac{1}{3}\,\frac{1}{tinfty13 - tinfty23}\,\big(h\,\big(-6\,\mu\,q^4\,tinfty13^2\,tinfty23 + 6\,\mu\,q^4\,tinfty13\,tinfty23^2 - 6\,\mu\,q^3\,tinfty12\,tinfty13\,tinfty23 + 6\,\mu\,q^3\,tinfty12\,tinfty23^2 - 6\,\mu\,q^3\,tinfty13^2\,tinfty22 + 6\,\mu\,q^3\,tinfty13\,tinfty22\,tinfty23 - 6\,\mu\,q^2\,tinfty11\,tinfty13\,tinfty23 + 6\,\mu\,q^2\,tinfty11\,tinfty23^2 - 6\,\mu\,q^2\,tinfty12\,tinfty13\,tinfty22 + 6\,\mu\,q^2\,tinfty12\,tinfty22\,tinfty23 - 6\,\mu\,q^2\,tinfty13^2\,tinfty21 + 6\,\mu\,q^2\,tinfty13\,tinfty21\,tinfty23 - 6\,h\,\mu\,q\,tinfty13^2 + 6\,h\,\mu\,q\,tinfty13\,tinfty23 + 6\,\mu\,q\,tinfty10\,tinfty13^2 - 12\,\mu\,q\,tinfty10\,tinfty13\,tinfty23 + 6\,\mu\,q\,tinfty10\,tinfty23^2 - 6\,\mu\,q\,tinfty11\,tinfty13\,tinfty22 + 6\,\mu\,q\,tinfty11\,tinfty22\,tinfty23 - 6\,\mu\,q\,tinfty12\,tinfty13\,tinfty21 + 6\,\mu\,q\,tinfty12\,tinfty21\,tinfty23 + 3\,q^2\,\rho\,tinfty13^2 - 3\,q^2\,\rho\,tinfty23^2 - \alpha 13\,h\,p\,q + \alpha 23\,h\,p\,q - 3\,h\,\nu\,p\,tinfty13 + 3\,h\,\nu\,p\,tinfty23 + 3\,q\,\rho\,tinfty12\,tinfty13 - 3\,q\,\rho\,tinfty12\,tinfty23 + 3\,q\,\rho\,tinfty13\,tinfty22 - 3\,q\,\rho\,tinfty22\,tinfty23 + 6\,C\,\mu\,tinfty13 - 6\,C\,\mu\,tinfty23 + 3\,\rho\,tinfty11\,tinfty13 - 3\,\rho\,tinfty11\,tinfty23 + 3\,\rho\,tinfty13\,tinfty21 - 3\,\rho\,tinfty21\,tinfty23\big)\big)$$

$$Entry21TermLambdaMinusq := -\frac{1}{3}\,\frac{1}{tinfty13 - tinfty23}\,\big(h\,\big(-12\,\mu\,q^3\,tinfty13^2\,tinfty23 + 12\,\mu\,q^3\,tinfty13\,tinfty23^2 - 9\,\mu\,q^2\,tinfty12\,tinfty13\,tinfty23 + 9\,\mu\,q^2\,tinfty12\,tinfty23^2 - 9\,\mu\,q^2\,tinfty13^2\,tinfty22 + 9\,\mu\,q^2\,tinfty13\,tinfty22\,tinfty23 + 3\,\alpha 13\,h\,q^2\,tinfty23 - 3\,\alpha 23\,h\,q^2\,tinfty13 - 6\,\mu\,q\,tinfty11\,tinfty13\,tinfty23 + 6\,\mu\,q\,tinfty11\,tinfty23^2 - 6\,\mu\,q\,tinfty12\,tinfty13\,tinfty22 + 6\,\mu\,q\,tinfty12\,tinfty22\,tinfty23 - 6\,\mu\,q\,tinfty13^2\,tinfty21 + 6\,\mu\,q\,tinfty13\,tinfty21\,tinfty23 + 6\,c2\,h\,q\,tinfty13 - 6\,c2\,h\,q\,tinfty23 - 3\,h\,\mu\,tinfty13^2 + 3\,h\,\mu\,tinfty13\,tinfty23 + 3\,\mu\,tinfty10\,tinfty13^2 - 6\,\mu\,tinfty10\,tinfty13\,tinfty23 + 3\,\mu\,tinfty10\,tinfty23^2 - 3\,\mu\,tinfty11\,tinfty13\,tinfty22 + 3\,\mu\,tinfty11\,tinfty22\,tinfty23 - 3\,\mu\,tinfty12\,tinfty13\,tinfty21 + 3\,\mu\,tinfty12\,tinfty21\,tinfty23 + 6\,q\,\rho\,tinfty13^2 - 6\,q\,\rho\,tinfty23^2 + \alpha 13\,h\,p - \alpha 23\,h\,p + 3\,c1\,h\,tinfty13 - 3\,c1\,h\,tinfty23 + 3\,\rho\,tinfty12\,tinfty13 - 3\,\rho\,tinfty12\,tinfty23 + 3\,\rho\,tinfty13\,tinfty22 - 3\,\rho\,tinfty22\,tinfty23\big)\big)$$

$$Entry21TermLambdaInfty6 := 0$$

$$Entry21TermLambdaInfty5 := 0$$

$$Entry21TermLambdaInfty4 := -(\alpha 13\,tinfty23 + \alpha 23\,tinfty13)\,h$$

$$Entry21TermLambdaInfty3 := -\frac{1}{3}\,\frac{1}{tinfty13 - tinfty23}\,\big(h\,\big(12\,\nu\,tinfty13^2\,tinfty23 - 12\,\nu\,tinfty13\,tinfty23^2 + 2\,\alpha 13\,tinfty12\,tinfty23 + 5\,\alpha 13\,tinfty13\,tinfty22 - 3\,\alpha 13\,tinfty22\,tinfty23 + 3\,\alpha 23\,tinfty12\,tinfty13 - 5\,\alpha 23\,tinfty12\,tinfty23 - 2\,\alpha 23\,tinfty13\,tinfty22 - 6\,c2\,tinfty13^2 + 6\,c2\,tinfty23^2\big)\big)$$

$$Entry21TermLambdaInfty2 := -\frac{1}{3}\,\frac{1}{tinfty13 - tinfty23}\,\big(h\,\big(6\,\mu\,tinfty13^2\,tinfty23 - 6\,\mu\,tinfty13\,tinfty23^2 + 9\,\nu\,tinfty12\,tinfty13\,tinfty23 - 9\,\nu\,tinfty12\,tinfty23^2 + 9\,\nu\,tinfty13^2\,tinfty22 - 9\,\nu\,tinfty13\,tinfty22\,tinfty23 + \alpha 13\,tinfty11\,tinfty23 + 4\,\alpha 13\,tinfty12\,tinfty22 + 4\,\alpha 13\,tinfty13\,tinfty21 - 3\,\alpha 13\,tinfty21\,tinfty23$$

$$+ 3\ \alpha 23\ tinfty11\ tinfty13 - 4\ \alpha 23\ tinfty11\ tinfty23 - 4\ \alpha 23\ tinfty12\ tinfty22 - \alpha 23\ tinfty13\ tinfty21 - 3\ c1\ tinfty13^2 + 3\ c1\ tinfty23^2 - 6\ c2\ tinfty12\ tinfty13 + 6\ c2\ tinfty12\ tinfty23 - 6\ c2\ tinfty13\ tinfty22 + 6\ c2\ tinfty22\ tinfty23))$$

$$Entry21TermLambdaInfty1 := -\frac{1}{tinfty13 - tinfty23} \big(h\ \big(\mu\ tinfty12\ tinfty13\ tinfty23 - \mu\ tinfty12\ tinfty23^2 + \mu\ tinfty13^2\ tinfty22 - \mu\ tinfty13\ tinfty22\ tinfty23 + 2\ \nu\ tinfty11\ tinfty13\ tinfty23 - 2\ \nu\ tinfty11\ tinfty23^2 + 2\ \nu\ tinfty12\ tinfty13\ tinfty22 - 2\ \nu\ tinfty12\ tinfty22\ tinfty23 + 2\ \nu\ tinfty13^2\ tinfty21 - 2\ \nu\ tinfty13\ tinfty21\ tinfty23 + \alpha 13\ h\ tinfty13 - \alpha 13\ h\ tinfty23 - \alpha 13\ tinfty10\ tinfty13 + \alpha 13\ tinfty10\ tinfty23 + \alpha 13\ tinfty11\ tinfty22 + \alpha 13\ tinfty12\ tinfty21 + \alpha 23\ tinfty10\ tinfty13 - \alpha 23\ tinfty10\ tinfty23 - \alpha 23\ tinfty11\ tinfty22 - \alpha 23\ tinfty12\ tinfty21 - c1\ tinfty12\ tinfty13 + c1\ tinfty12\ tinfty23 - c1\ tinfty13\ tinfty22 + c1\ tinfty22\ tinfty23 - 2\ c2\ tinfty11\ tinfty13 + 2\ c2\ tinfty11\ tinfty23 - 2\ c2\ tinfty13\ tinfty21 + 2\ c2\ tinfty21\ tinfty23\big)\big)$$

$$Entry21TermLambdaInfty0 := \frac{1}{3}\ \frac{1}{tinfty13 - tinfty23} \big(h\ \big(6\ \mu\ q^2\ tinfty13^2\ tinfty23 - 6\ \mu\ q^2\ tinfty13\ tinfty23^2 + 3\ \mu\ q\ tinfty12\ tinfty13\ tinfty23 - 3\ \mu\ q\ tinfty12\ tinfty23^2 + 3\ \mu\ q\ tinfty13^2\ tinfty22 - 3\ \mu\ q\ tinfty13\ tinfty22\ tinfty23 - 3\ \alpha 13\ h\ q\ tinfty23 + 3\ \alpha 23\ h\ q\ tinfty13 - 3\ h\ \nu\ tinfty13^2 + 3\ h\ \nu\ tinfty13\ tinfty23 + 3\ \nu\ tinfty10\ tinfty13^2 - 6\ \nu\ tinfty10\ tinfty13\ tinfty23 + 3\ \nu\ tinfty10\ tinfty23^2 - 3\ \nu\ tinfty11\ tinfty13\ tinfty22 + 3\ \nu\ tinfty11\ tinfty22\ tinfty23 - 3\ \nu\ tinfty12\ tinfty13\ tinfty21 + 3\ \nu\ tinfty12\ tinfty21\ tinfty23 + 3\ c1\ tinfty11\ tinfty13 - 3\ c1\ tinfty11\ tinfty23 + 3\ c1\ tinfty13\ tinfty21 - 3\ c1\ tinfty21\ tinfty23 - 3\ \rho\ tinfty13^2 + 3\ \rho\ tinfty23^2 + 2\ C\ \alpha 13 - 2\ C\ \alpha 23\big)\big)$$

$$0$$

$$-(-tinfty10\ tinfty13 + tinfty10\ tinfty23 + tinfty11\ tinfty22 + tinfty12\ tinfty21)\ \lambda - (tinfty11\ tinfty23 + tinfty12\ tinfty22 + tinfty13\ tinfty21)\ \lambda^2 - (tinfty12\ tinfty23 + tinfty13\ tinfty22)\ \lambda^3 - tinfty13\ tinfty23\ \lambda^4 + C - h\ \lambda\ tinfty13 - \frac{p\ h}{\lambda - q}$$

```
> rho:=factor(solve(Entry21TermLambdaMinusqCube,rho));
  simplify(rho-(-p*mu));
  simplify(Entry21TermLambdaMinusqCube);
```

$$\rho := -p\ \mu \qquad (2.6)$$

$$0$$

$$0$$

```
> L21OrderLambda4:=-residue(L[2,1]/lambda^5,lambda=infinity):
  L21OrderLambda3:=-residue(L[2,1]/lambda^4,lambda=infinity):
  L21OrderLambda2:=-residue(L[2,1]/lambda^3,lambda=infinity):
  L21OrderLambda1:=-residue(L[2,1]/lambda^2,lambda=infinity):
  L21OrderLambda0:=-residue(L[2,1]/lambda^1,lambda=infinity):
  simplify(h*(alpha13*diff(L21OrderLambda4,tinfty13)+alpha23*diff
```

```
(L21OrderLambda4,tinfty23)+alpha12*diff(L21OrderLambda4,
tinfty12)+alpha22*diff(L21OrderLambda4,tinfty22)+alpha11*diff
(L21OrderLambda4,tinfty11)+alpha21*diff(L21OrderLambda4,
tinfty21))- Entry21TermLambdaInfty4);
Equation3:=factor(simplify(h*(alpha13*diff(L21OrderLambda3,
tinfty13)+alpha23*diff(L21OrderLambda3,tinfty23)+alpha12*diff
(L21OrderLambda3,tinfty12)+alpha22*diff(L21OrderLambda3,
tinfty22)+alpha11*diff(L21OrderLambda3,tinfty11)+alpha21*diff
(L21OrderLambda3,tinfty21))- Entry21TermLambdaInfty3));
Equation4:=factor(simplify(h*(alpha13*diff(L21OrderLambda2,
tinfty13)+alpha23*diff(L21OrderLambda2,tinfty23)+alpha12*diff
(L21OrderLambda2,tinfty12)+alpha22*diff(L21OrderLambda2,
tinfty22)+alpha11*diff(L21OrderLambda2,tinfty11)+alpha21*diff
(L21OrderLambda2,tinfty21))- Entry21TermLambdaInfty2));
Equation5:=factor(simplify(h*(alpha13*diff(L21OrderLambda1,
tinfty13)+alpha23*diff(L21OrderLambda1,tinfty23)+alpha12*diff
(L21OrderLambda1,tinfty12)+alpha22*diff(L21OrderLambda1,
tinfty22)+alpha11*diff(L21OrderLambda1,tinfty11)+alpha21*diff
(L21OrderLambda1,tinfty21))- Entry21TermLambdaInfty1));
Equation1:=factor(simplify(Equation1));
Equation2:=factor(simplify(Equation2));
```

$$0 \tag{2.7}$$

$$Equation3 := -\frac{1}{3}\,\frac{1}{tinfty13-tinfty23}\,\Big(h\,\big(-12\,\nu\,tinfty13^2\,tinfty23+12\,\nu\,tinfty13\,tinfty23^2 +3\,\alpha 12\,tinfty13\,tinfty23-3\,\alpha 12\,tinfty23^2-2\,\alpha 13\,tinfty12\,tinfty23 -2\,\alpha 13\,tinfty13\,tinfty22+3\,\alpha 22\,tinfty13^2-3\,\alpha 22\,tinfty13\,tinfty23 +2\,\alpha 23\,tinfty12\,tinfty23+2\,\alpha 23\,tinfty13\,tinfty22+6\,c2\,tinfty13^2-6\,c2\,tinfty23^2\big)\Big)$$

$$Equation4 := -\frac{1}{3}\,\frac{1}{tinfty13-tinfty23}\,\Big(h\,\big(-6\,\mu\,tinfty13^2\,tinfty23+6\,\mu\,tinfty13\,tinfty23^2 -9\,\nu\,tinfty12\,tinfty13\,tinfty23+9\,\nu\,tinfty12\,tinfty23^2-9\,\nu\,tinfty13^2\,tinfty22 +9\,\nu\,tinfty13\,tinfty22\,tinfty23+3\,\alpha 11\,tinfty13\,tinfty23-3\,\alpha 11\,tinfty23^2 +3\,\alpha 12\,tinfty13\,tinfty22-3\,\alpha 12\,tinfty22\,tinfty23-\alpha 13\,tinfty11\,tinfty23 -4\,\alpha 13\,tinfty12\,tinfty22-\alpha 13\,tinfty13\,tinfty21+3\,\alpha 21\,tinfty13^2 -3\,\alpha 21\,tinfty13\,tinfty23+3\,\alpha 22\,tinfty12\,tinfty13-3\,\alpha 22\,tinfty12\,tinfty23 +\alpha 23\,tinfty11\,tinfty23+4\,\alpha 23\,tinfty12\,tinfty22+\alpha 23\,tinfty13\,tinfty21+3\,c1\,tinfty13^2 -3\,c1\,tinfty23^2+6\,c2\,tinfty12\,tinfty13-6\,c2\,tinfty12\,tinfty23+6\,c2\,tinfty13\,tinfty22 -6\,c2\,tinfty22\,tinfty23\big)\Big)$$

$$Equation5 := -\frac{1}{tinfty13-tinfty23}\,\Big(h\,\big(-\mu\,tinfty12\,tinfty13\,tinfty23+\mu\,tinfty12\,tinfty23^2$$

$$- \mu\, tinfty13^2\, tinfty22 + \mu\, tinfty13\, tinfty22\, tinfty23 - 2\, \nu\, tinfty11\, tinfty13\, tinfty23 + 2\, \nu\, tinfty11\, tinfty23^2 - 2\, \nu\, tinfty12\, tinfty13\, tinfty22 + 2\, \nu\, tinfty12\, tinfty22\, tinfty23 - 2\, \nu\, tinfty13^2\, tinfty21 + 2\, \nu\, tinfty13\, tinfty21\, tinfty23 + \alpha 11\, tinfty13\, tinfty22 - \alpha 11\, tinfty22\, tinfty23 + \alpha 12\, tinfty13\, tinfty21 - \alpha 12\, tinfty21\, tinfty23 - \alpha 13\, tinfty11\, tinfty22 - \alpha 13\, tinfty12\, tinfty21 + \alpha 21\, tinfty12\, tinfty13 - \alpha 21\, tinfty12\, tinfty23 + \alpha 22\, tinfty11\, tinfty13 - \alpha 22\, tinfty11\, tinfty23 + \alpha 23\, tinfty11\, tinfty22 + \alpha 23\, tinfty12\, tinfty21 + c1\, tinfty12\, tinfty13 - c1\, tinfty12\, tinfty23 + c1\, tinfty13\, tinfty22 - c1\, tinfty22\, tinfty23 + 2\, c2\, tinfty11\, tinfty13 - 2\, c2\, tinfty11\, tinfty23 + 2\, c2\, tinfty13\, tinfty21 - 2\, c2\, tinfty21\, tinfty23))$$

$$Equation1 := -\frac{1}{3}\, \frac{1}{tinfty13 - tinfty23} \left(h \left(-6\, \nu\, tinfty13^2 + 6\, \nu\, tinfty23^2 + 3\, \alpha 12\, tinfty13 - 3\, \alpha 12\, tinfty23 - 2\, \alpha 13\, tinfty12 - 2\, \alpha 13\, tinfty22 + 3\, \alpha 22\, tinfty13 - 3\, \alpha 22\, tinfty23 + 2\, \alpha 23\, tinfty12 + 2\, \alpha 23\, tinfty22 + 12\, c2\, tinfty13 - 12\, c2\, tinfty23 \right) \right)$$

$$Equation2 := -\frac{1}{3}\, \frac{1}{tinfty13 - tinfty23} \left(h \left(-3\, \mu\, tinfty13^2 + 3\, \mu\, tinfty23^2 - 3\, \nu\, tinfty12\, tinfty13 + 3\, \nu\, tinfty12\, tinfty23 - 3\, \nu\, tinfty13\, tinfty22 + 3\, \nu\, tinfty22\, tinfty23 + 3\, \alpha 11\, tinfty13 - 3\, \alpha 11\, tinfty23 - \alpha 13\, tinfty11 - \alpha 13\, tinfty21 + 3\, \alpha 21\, tinfty13 - 3\, \alpha 21\, tinfty23 + \alpha 23\, tinfty11 + \alpha 23\, tinfty21 + 6\, c1\, tinfty13 - 6\, c1\, tinfty23 \right) \right)$$

```
> LpFunction:=unapply(-Entry21TermLambdaMinusq/h,C):
  Equation7:=simplify(Entry21TermLambdaMinusqSquare-(-p*h*Lq)):
  Csol:=solve(Equation7,C):
  Csolbis:=p^2- P1(q)*p+P2(q)-Pinfty02+h*q*tinfty13:
  factor(series(Csol-Csolbis,p=0));
```

$$0 \qquad \textbf{(2.8)}$$

```
> Lp:=factor(simplify(LpFunction(Csol))):
  Lpbis:=mu*(p*diff(P1(q),q) -diff(P2(q),q)-h*tinfty13)
  +h/3*(alpha13-alpha23)/(tinfty13-tinfty23)*p+h*(alpha13*
  tinfty23-alpha23*tinfty13)/(tinfty13-tinfty23)*q^2
  +2*h*c2*q+h*c1;
  factor(series(Lp-Lpbis,q=0));

  Lqbis:=-mu*P1(q)+2*p*mu-h*nu-h*(alpha13-alpha23)/3/(tinfty13-
  tinfty23)*q:
  factor(simplify(series(Lq-Lqbis,q=0)))
```

$$Lpbis := \mu \left(p\, (-tinfty12 - tinfty22 + 2\, (-tinfty13 - tinfty23)\, q) + tinfty10\, tinfty13 - tinfty10\, tinfty23 - tinfty11\, tinfty22 - tinfty12\, tinfty21 - 2\, (tinfty11\, tinfty23 + tinfty12\, tinfty22 + tinfty13\, tinfty21)\, q - 3\, (tinfty12\, tinfty23 + tinfty13\, tinfty22)\, q^2 - 4\, q^3\, tinfty13\, tinfty23 - tinfty13\, h \right) + \frac{1}{3}\, \frac{h\, (\alpha 13 - \alpha 23)\, p}{tinfty13 - tinfty23} + \frac{h\, (\alpha 13\, tinfty23 - \alpha 23\, tinfty13)\, q^2}{tinfty13 - tinfty23} + 2\, h\, c2\, q + h\, c1 \qquad \textbf{(2.9)}$$

0
0

The evolutions of the Darboux coordinates are:
L[q]=2*mu*p-mu*P1(q)-h*nu-h*(alpha13-alpha23)/3/(tinfty13-tinfty23)*q;
L[p] = mu*(p*diff(P1(q),q) -diff(P2(q),q)-h*tinfty13)+h/3*(alpha13-alpha23)/(tinfty13-tinfty23)*p+h*(alpha13*tinfty23-alpha23*tinfty13)/(tinfty13-tinfty23)*q^2
+2*h*c2*q+h*c1
We check that it is indeed Hamiltonian

```
> Hamiltonian:= mu*(p^2+tdP2(q)-p*P1(q)+h*tinfty13*q)-h*nu*p-h/3*
  (alpha13-alpha23)/(tinfty13-tinfty23)*p*q-h/3*(alpha13*tinfty23
  -alpha23*tinfty13)/(tinfty13-tinfty23)*q^3-h*c2*q^2-h*c1*q:
  simplify(Lp-(-diff(Hamiltonian,q)));
  simplify(Lq-(diff(Hamiltonian,p)));
```

0 (2.10)
0

Decomposition of the tangent space: shift of Darboux coordinates and non-trivial isomonodromic time

From previous Maple sheet, we have some expressions for the coefficients (c_1,c_2,mu,nu).

```
> nualter:= -1/6*(3*alpha12*tinfty23+3*alpha22*tinfty13-3*
  alpha12*tinfty13+2*alpha23*tinfty22+2*alpha13*tinfty12-2*
  alpha13*tinfty22-3*alpha22*tinfty23-2*alpha23*tinfty12)/(-2*
  tinfty13*tinfty23+tinfty23^2+tinfty13^2):
  mualter := 1/6*(-2*tinfty13*alpha13*tinfty11+2*tinfty23*
  alpha23*tinfty21-3*alpha12*tinfty22*tinfty23-3*alpha22*
  tinfty12*tinfty23+6*alpha11*tinfty23^2-6*alpha21*tinfty13^2+3*
  alpha12*tinfty22*tinfty13-12*alpha11*tinfty23*tinfty13+3*
  alpha22*tinfty12*tinfty13+12*alpha21*tinfty13*tinfty23+6*
  alpha11*tinfty13^2-6*alpha21*tinfty23^2+2*alpha13*tinfty11*
  tinfty23-3*tinfty22*alpha22*tinfty13+3*tinfty22*alpha22*
  tinfty23+4*alpha23*tinfty12*tinfty22-2*alpha23*tinfty13*
  tinfty21-2*alpha23*tinfty11*tinfty23-4*alpha13*tinfty12*
  tinfty22+2*alpha13*tinfty13*tinfty21-2*alpha23*tinfty22^2+2*
  alpha13*tinfty22^2-2*alpha23*tinfty12^2+2*alpha13*tinfty12^2-3*
  tinfty12*alpha12*tinfty13+3*tinfty12*alpha12*tinfty23+2*
  tinfty13*alpha23*tinfty11-2*alpha13*tinfty21*tinfty23)/(3*
  tinfty13*tinfty23^2-3*tinfty13^2*tinfty23+tinfty13^3-
  tinfty23^3):
  c2alter := 1/6*(-3*alpha22*tinfty13^2+3*alpha22*tinfty13*
  tinfty23+2*alpha13*tinfty13*tinfty22+3*alpha12*tinfty23*
  tinfty13-2*alpha23*tinfty13*tinfty22-2*alpha13*tinfty12*
  tinfty23-3*alpha12*tinfty23^2+2*alpha23*tinfty12*tinfty23)/(-2*
  tinfty13*tinfty23+tinfty23^2+tinfty13^2):
```

```
c1alter:=factor(1/6*(-6*alpha21*tinfty13^3+6*alpha11*tinfty23^3
-3*tinfty13*alpha12*tinfty22*tinfty23+3*tinfty13*alpha22*
tinfty12*tinfty23-2*tinfty13*alpha13*tinfty11*tinfty23+3*
tinfty13*tinfty22*alpha22*tinfty23+2*tinfty13*alpha23*tinfty12*
tinfty22+2*tinfty13*alpha23*tinfty11*tinfty23-2*tinfty13*
alpha13*tinfty12*tinfty22-3*tinfty13*tinfty12*alpha12*
tinfty23+2*tinfty23*alpha23*tinfty12*tinfty22+2*tinfty23*
alpha23*tinfty13*tinfty21-2*tinfty23*alpha13*tinfty12*tinfty22
-2*tinfty23*alpha13*tinfty13*tinfty21+2*alpha13*tinfty13^2*
tinfty21+6*alpha11*tinfty23*tinfty13^2+3*alpha12*tinfty22*
tinfty13^2-6*alpha21*tinfty13*tinfty23^2+12*alpha21*tinfty13^2*
tinfty23-3*tinfty22*alpha22*tinfty13^2-3*alpha22*tinfty12*
tinfty23^2+2*alpha13*tinfty11*tinfty23^2-2*alpha23*tinfty13^2*
tinfty21-2*alpha23*tinfty11*tinfty23^2+3*tinfty12*alpha12*
tinfty23^2-12*tinfty13*alpha11*tinfty23^2-2*tinfty13*alpha23*
tinfty22^2+2*tinfty13*alpha13*tinfty22^2-2*tinfty23*alpha23*
tinfty12^2+2*tinfty23*alpha13*tinfty12^2)/(3*tinfty13*
tinfty23^2-3*tinfty13^2*tinfty23+tinfty13^3-tinfty23^3)):
```

Expression of the Lax matrix in the geometric gauge without apparent singularities

```
> C:=Csol:
Q2(lambda);
checkL11bis:=-Q2(lambda);
checkL12bis:=(lambda-q);
checkL22bis:=P1(lambda)+Q2(lambda);
checkL21bis:=h*diff(Q2(lambda)/(lambda-q),lambda)+L[2,1]/
(lambda-q)-P1(lambda)*Q2(lambda)/(lambda-q)
-Q2(lambda)^2/(lambda-q):
simplify(checkL[1,1]-checkL11bis);
simplify(checkL[1,2]-checkL12bis);
simplify(checkL[2,2]-checkL22bis);
simplify(checkL[2,1]-checkL21bis);
simplify(residue(checkL[2,1],lambda=q));
```

$$-p \qquad \textbf{(3.1)}$$

$$checkL11bis := p$$

$$checkL12bis := \lambda - q$$

$$checkL22bis := (-tinfty13 - tinfty23)\,\lambda^2 + (-tinfty12 - tinfty22)\,\lambda - p - tinfty11 - tinfty21$$

$$0$$

$$0$$

$$0$$

$$0$$

$$0$$

Expression of the evolution in the traceless setting and decomposition of the tangent space

```
> p:=tdp+P1(q)/2:
  Ltdp:=simplify( Lp-dP1dlambda(q)/2*Lq
  - 1/2*h*(alpha13*diff(P1(q),tinfty13)+alpha23*diff(P1(q),
  tinfty23)+alpha12*diff(P1(q),tinfty12)+alpha22*diff(P1(q),
  tinfty22)+alpha11*diff(P1(q),tinfty11)+alpha21*diff(P1(q),
  tinfty21)) ):
  Ltdpbis:=mu*(diff(P1(q)^2/4-P2(q),q))
  +h/3*(alpha13-alpha23)/(tinfty13-tinfty23)*tdp
  +( 2*c2+ ((alpha12+alpha22)/2-nu*(tinfty13+tinfty23))-1/3*
  (tinfty12+tinfty22)*(alpha13-alpha23)/(tinfty13-tinfty23))*h*q
  +h*(c1-mu*tinfty13-1/6*(tinfty11+tinfty21)*(alpha13-alpha23)/
  (tinfty13-tinfty23)+1/2*((alpha11+alpha21)-nu*(tinfty12+
  tinfty22))):
  factor(series(Ltdp-Ltdpbis,q=0));
```

$$0 \tag{3.2}$$

```
> Quantiteq:=unapply( 2*c2+ ((alpha12+alpha22)/2-nu*(tinfty13+
  tinfty23))-1/3*(tinfty12+tinfty22)*(alpha13-alpha23)/(tinfty13-
  tinfty23), c2,nu);
  QuantiteConstant:=unapply(c1-mu*tinfty13-1/6*(tinfty11+
  tinfty21)*(alpha13-alpha23)/(tinfty13-tinfty23)+1/2*((alpha11+
  alpha21)-nu*(tinfty12+tinfty22)),c1,nu,mu);
```

$$Quantiteq := (c2, \nu) \rightarrow 2\, c2 + \frac{1}{2}\, \alpha 12 + \frac{1}{2}\, \alpha 22 - \nu\, (tinfty13 + tinfty23) - \frac{1}{3}\, \frac{(tinfty12 + tinfty22)\, (\alpha 13 - \alpha 23)}{tinfty13 - tinfty23} \tag{3.3}$$

$$QuantiteConstant := (c1, \nu, \mu) \rightarrow c1 - \mu\, tinfty13 - \frac{1}{6}\, \frac{(tinfty21 + tinfty11)\, (\alpha 13 - \alpha 23)}{tinfty13 - tinfty23} + \frac{1}{2}\, \alpha 11 + \frac{1}{2}\, \alpha 21 - \frac{1}{2}\, (tinfty12 + tinfty22)\, \nu$$

```
> mu:=mualter:
  nu:=nualter:
  c1:=c1alter:
  c2:=c2alter:
```

```
> Lpfunction:=unapply(simplify(Lp),alpha13,alpha23,alpha12,
  alpha22,alpha11,alpha21):
  Ltdpfunction:=unapply(simplify(Ltdp),alpha13,alpha23,alpha12,
  alpha22,alpha11,alpha21):
  Lqfunction:=unapply(simplify(Lq),alpha13,alpha23,alpha12,
  alpha22,alpha11,alpha21):
  c1function:=unapply(c1alter,alpha13,alpha23,alpha12,alpha22,
  alpha11,alpha21):
  c2function:=unapply(c2alter,alpha13,alpha23,alpha12,alpha22,
```

```
alpha11,alpha21):
nufunction:=unapply(nualter,alpha13,alpha23,alpha12,alpha22,
alpha11,alpha21):
mufunction:=unapply(mualter,alpha13,alpha23,alpha12,alpha22,
alpha11,alpha21):
```

```
> factor(Ltdpfunction(1,1,0,0,0,0));
factor(Lqfunction(1,1,0,0,0,0));
factor(c1function(1,1,0,0,0,0));
factor(c2function(1,1,0,0,0,0));
factor(nufunction(1,1,0,0,0,0));


factor(Lqfunction(0,0,1,1,0,0));
factor(Ltdpfunction(0,0,1,1,0,0));
factor(c1function(0,0,1,1,0,0));
factor(c2function(0,0,1,1,0,0));
factor(nufunction(0,0,1,1,0,0));


factor(Lqfunction(0,0,0,0,1,1));
factor(Ltdpfunction(0,0,0,0,1,1));
factor(c1function(0,0,0,0,1,1));
factor(c2function(0,0,0,0,1,1));
factor(nufunction(0,0,0,0,1,1));
```

$$0 \tag{3.4}$$
$$0$$
$$0$$
$$0$$
$$0$$
$$0$$
$$0$$
$$0$$
$$-\frac{1}{2}$$
$$0$$
$$0$$
$$0$$
$$-1$$
$$0$$
$$0$$

```
> simplify(mufunction(1,1,0,0,0,0));
simplify(mufunction(0,0,1,1,0,0));
simplify(mufunction(0,0,0,0,1,1));
```

```
simplify(mufunction(0,0,2*tinfty13,2*tinfty23,tinfty12,
tinfty22));
simplify(mufunction(3*tinfty13,3*tinfty23,2*tinfty12,2*
tinfty22,tinfty11,tinfty21));

simplify(nufunction(1,1,0,0,0,0));
simplify(nufunction(0,0,1,1,0,0));
simplify(nufunction(0,0,0,0,1,1));
simplify(nufunction(0,0,2*tinfty13,2*tinfty23,tinfty12,
tinfty22));
simplify(nufunction(3*tinfty13,3*tinfty23,2*tinfty12,2*
tinfty22,tinfty11,tinfty21));

simplify(mufunction(0,0,0,0,tinfty13,tinfty23));
simplify(nufunction(0,0,0,0,tinfty13,tinfty23));
```

$$0 \tag{3.5}$$
$$0$$
$$0$$
$$0$$
$$0$$
$$0$$
$$0$$
$$0$$
$$1$$
$$0$$
$$1$$
$$0$$

```
> simplify(mufunction(0,0,2*tinfty13,2*tinfty23,tinfty12,
tinfty22));
factor(Ltdpfunction(0,0,2*tinfty13,2*tinfty23,tinfty12,
tinfty22));
factor(Lpfunction(0,0,2*tinfty13,2*tinfty23,tinfty12,tinfty22))
;
factor(Lqfunction(0,0,2*tinfty13,2*tinfty23,tinfty12,tinfty22)
- (-h));
factor(c1function(0,0,2*tinfty13,2*tinfty23,tinfty12,tinfty22))
;
factor(c2function(0,0,2*tinfty13,2*tinfty23,tinfty12,tinfty22))
;
factor(nufunction(0,0,2*tinfty13,2*tinfty23,tinfty12,tinfty22))
;
```

$$0 \tag{3.6}$$
$$0$$
$$0$$
$$0$$

$$0$$
$$0$$
$$1$$

```
> simplify(mufunction(3*tinfty13,3*tinfty23,2*tinfty12,2*
  tinfty22,tinfty11,tinfty21));
  factor(Lpfunction(3*tinfty13,3*tinfty23,2*tinfty12,2*tinfty22,
  tinfty11,tinfty21)-h*p);
  factor(Ltdpfunction(3*tinfty13,3*tinfty23,2*tinfty12,2*
  tinfty22,tinfty11,tinfty21)- h*tdp);
  factor(Lqfunction(3*tinfty13,3*tinfty23,2*tinfty12,2*tinfty22,
  tinfty11,tinfty21) -(-h*q));
  simplify(c1function(3*tinfty13,3*tinfty23,2*tinfty12,2*
  tinfty22,tinfty11,tinfty21));
  simplify(c2function(3*tinfty13,3*tinfty23,2*tinfty12,2*
  tinfty22,tinfty11,tinfty21));
  simplify(nufunction(3*tinfty13,3*tinfty23,2*tinfty12,2*
  tinfty22,tinfty11,tinfty21));
```

$$0 \tag{3.7}$$
$$0$$
$$0$$
$$0$$
$$0$$
$$0$$
$$0$$

```
> factor(simplify(mufunction(0,0,0,0,tinfty13,tinfty23)));
  factor(Ltdpfunction(0,0,0,0,tinfty13,tinfty23)- (diff((P1(q)
  ^2/4-P2(q)),q) -h/2*(tinfty13-tinfty23)));
  factor(Lqfunction(0,0,0,0,tinfty13,tinfty23)- (2*tdp) );
  simplify(c1function(0,0,0,0,tinfty13,tinfty23));
  simplify(c2function(0,0,0,0,tinfty13,tinfty23));
  simplify(nufunction(0,0,0,0,tinfty13,tinfty23));
```

$$1 \tag{3.8}$$
$$0$$
$$0$$
$$0$$
$$0$$
$$0$$

```
> factor(Lqfunction(1,1,0,0,0,0));
  factor(Lqfunction(0,0,1,1,0,0));
  factor(Lqfunction(0,0,0,0,1,1));
  factor(Lqfunction(0,0,2*tinfty13,2*tinfty23,tinfty12,tinfty22)
  - (-h));
  factor(Lqfunction(3*tinfty13,3*tinfty23,2*tinfty12,2*tinfty22,
  tinfty11,tinfty21) -(-h*q));
```

$$0 \tag{3.9}$$

$$0$$
$$0$$
$$0$$
$$0$$

```
> pdsolve({
diff(Q(t1,t2,t3,t4,t5,t6),t1)+diff(Q(t1,t2,t3,t4,t5,t6),t2)=0,
diff(Q(t1,t2,t3,t4,t5,t6),t3)+diff(Q(t1,t2,t3,t4,t5,t6),t4)=0,
diff(Q(t1,t2,t3,t4,t5,t6),t5)+diff(Q(t1,t2,t3,t4,t5,t6),t6)=0,
2*t1*diff(Q(t1,t2,t3,t4,t5,t6),t3)+2*t2*diff(Q(t1,t2,t3,t4,t5,
t6),t4)+t3*diff(Q(t1,t2,t3,t4,t5,t6),t5)+t4*diff(Q(t1,t2,t3,t4,
t5,t6),t6)=-1,
3*t1*diff(Q(t1,t2,t3,t4,t5,t6),t1)+3*t2*diff(Q(t1,t2,t3,t4,t5,
t6),t2)+2*t3*diff(Q(t1,t2,t3,t4,t5,t6),t3)+2*t4*diff(Q(t1,t2,
t3,t4,t5,t6),t4)+t5*diff(Q(t1,t2,t3,t4,t5,t6),t5)+t6*diff(Q(t1,
t2,t3,t4,t5,t6),t6)=-Q(t1,t2,t3,t4,t5,t6)
},Q(t1,t2,t3,t4,t5,t6));
```

$$\left\{Q(t1, t2, t3, t4, t5, t6) = -\frac{1}{(-t1+t2)^{4/3}}\left((t1 - t2)\ \_F1\left(\frac{(t5-t6)\ t1 + t2\ (-t5+t6) - \frac{1}{4}\ (t3-t4)^2}{(-t1+t2)^{4/3}}\right) - \frac{1}{2}\ (t3-t4)\ (-t1+t2)^{1/3}\right)\right\} \tag{3.10}$$

```
> factor(Ltdpfunction(1,1,0,0,0,0));
factor(Ltdpfunction(0,0,1,1,0,0));
factor(Ltdpfunction(0,0,0,0,1,1));
factor(Ltdpfunction(0,0,2*tinfty13,2*tinfty23,tinfty12,
tinfty22));
factor(Ltdpfunction(3*tinfty13,3*tinfty23,2*tinfty12,2*
tinfty22,tinfty11,tinfty21) -(h*tdp));
```

$$0 \tag{3.11}$$
$$0$$
$$0$$
$$0$$
$$0$$

```
> pdsolve({
diff(tdP(t1,t2,t3,t4,t5,t6),t1)+diff(tdP(t1,t2,t3,t4,t5,t6),t2)
=0,
diff(tdP(t1,t2,t3,t4,t5,t6),t3)+diff(tdP(t1,t2,t3,t4,t5,t6),t4)
=0,
diff(tdP(t1,t2,t3,t4,t5,t6),t5)+diff(tdP(t1,t2,t3,t4,t5,t6),t6)
=0,
```

```
2*t1*diff(tdP(t1,t2,t3,t4,t5,t6),t3)+2*t2*diff(tdP(t1,t2,t3,t4,
t5,t6),t4)+t3*diff(tdP(t1,t2,t3,t4,t5,t6),t5)+t4*diff(tdP(t1,
t2,t3,t4,t5,t6),t6)=0,
3*t1*diff(tdP(t1,t2,t3,t4,t5,t6),t1)+3*t2*diff(tdP(t1,t2,t3,t4,
t5,t6),t2)+2*t3*diff(tdP(t1,t2,t3,t4,t5,t6),t3)+2*t4*diff(tdP
(t1,t2,t3,t4,t5,t6),t4)+t5*diff(tdP(t1,t2,t3,t4,t5,t6),t5)+t6*
diff(tdP(t1,t2,t3,t4,t5,t6),t6)=tdP(t1,t2,t3,t4,t5,t6)
},tdP(t1,t2,t3,t4,t5,t6));
```

$$\left\{ tdP(t1,t2,t3,t4,t5,t6) = \_F1\left(\frac{(t5-t6)\, t1 + t2\,(-t5+t6) - \frac{1}{4}\,(t3-t4)^2}{(-t1+t2)^{4/3}} \right) (-t1 + t2)^{1/3} \right\} \quad (3.12)$$

This gives the shift of the Darboux coordinates and the non-trivial isomonodromic times.

Expression of the Lax matrices in the geometric gauge after the symplectic reduction and the Painlevé 2 equation

Simplification of the formulas after the reduction and expression of the Lax matrices in the geometric gauge after reduction. In this case, we have \check{q}=q and \check{p}=p=\td{p}.

```
> tinfty23:=-tinfty13:
  tinfty22:=-tinfty12:
  tinfty21:=-tinfty11:
  tinfty20:=-tinfty10:
  tinfty11:=tau/2:
  tinfty13:=1:
  tinfty12:=0:
  q:=tdq:
```

```
> c2:=c2alter;
  c1:=c1alter;
  c0:=0:
  nu:=nualter;
  mu:=mualter;
  alpha11:=1/2:
  alpha21:=-1/2:
  alpha13:=0:
  alpha23:=0:
  alpha12:=0:
  alpha22:=0:
  checkL:=simplify(checkL);
  checkA:=simplify(checkA);
```

(4.1)

$$c2 := -\frac{1}{4}\,\alpha 12 - \frac{1}{4}\,\alpha 22$$ **(4.1)**

$$c1 := -\frac{1}{2}\,\alpha 11 - \frac{1}{2}\,\alpha 21$$

$$\nu := \frac{1}{4}\,\alpha 12 - \frac{1}{4}\,\alpha 22$$

$$\mu := -\frac{1}{12}\,\alpha 13\,\tau + \frac{1}{12}\,\alpha 23\,\tau + \frac{1}{2}\,\alpha 11 - \frac{1}{2}\,\alpha 21$$

$$\begin{bmatrix} tdp & \lambda - tdq \\ \lambda^3 + tdq\,\lambda^2 + (tdq^2 + \tau)\,\lambda + tdq^3 + \tau\,tdq - h + 2\,tinfty10 & -tdp \end{bmatrix}$$

$$\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2}\,\lambda^2 + \lambda\,tdq + \frac{3}{2}\,tdq^2 + \frac{1}{2}\,\tau & 0 \end{bmatrix}$$

```
> G1:=Matrix(2,2,0):
G1[1,1]:=1:
G1[2,2]:=1:
G1[1,2]:=0:
G1[2,1]:=g1*lambda+g0:
g1:=tinfty13:
g0:=tinfty13*q+tinfty12:

dG1dlambda:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do dG1dlambda[i,j]:=diff
(G1[i,j],lambda): od: od:

dG1dtau:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do dG1dtau[i,j]:=diff(G1
[i,j],tau)+diff(G1[i,j],q)*dqdt+diff(G1[i,j],p)*dpdt : od: od:


dqdt:=Lq/h:
dpdt:=Lp/h:
tdp:=checkp:
tdq:=checkq:
dcheckqdt:=dqdt:
dcheckqdt:=dpdt:

tdL:=simplify(Multiply(Multiply(G1,checkL),G1^(-1))+h*Multiply
(dG1dlambda,G1^(-1))):
tdA:=simplify(Multiply(Multiply(G1,checkA),G1^(-1))+h*Multiply
(dG1dtau,G1^(-1))):
```

```
simplify(tdL);
tdA;
```

$$\left[\left[checkq^2 - \lambda^2 + checkp, \lambda - checkq\right], \left[2\, checkq^3 + 2\, checkq^2 \lambda + (\tau + 2\, checkp)\, checkq + (\tau + 2\, checkp)\, \lambda + 2\, tinfty10, -checkq^2 + \lambda^2 - checkp\right]\right] \quad (4.2)$$

$$\begin{bmatrix} -\frac{1}{2}\, checkq - \frac{1}{2}\, \lambda & \frac{1}{2} \\ checkq^2 + \frac{1}{2}\, \tau + checkp & \frac{1}{2}\, checkq + \frac{1}{2}\, \lambda \end{bmatrix}$$

```
> h*dcheckqdt;
h*dcheckpdt;
```

$$2\, checkq^3 + \tau\, checkq - \frac{1}{2}\, h + tinfty10 \quad (4.3)$$

$$h\, dcheckpdt$$