In this Maple file, we compute the Casimir coefficients of the Lax matrix L associated to the Painlevé 3 equation in relation with the spectral curve

```
> restart:
  P1:=x-> P021/x^2+P011/x+Pinfty01;
  P2:=x-> P042/x^4+P032/x^3+P022/x^2+P012/x+Pinfty02;
  SpectralCurve:=unapply( y^2-P1(x)*y+P2(x),y);
```

$$P1 := x \to \frac{P021}{x^2} + \frac{P011}{x} + Pinfty01 \tag{1}$$

$$P2 := x \to \frac{P042}{x^4} + \frac{P032}{x^3} + \frac{P022}{x^2} + \frac{P012}{x} + Pinfty02$$

$$SpectralCurve := y \to y^2 - \left(\frac{P021}{x^2} + \frac{P011}{x} + Pinfty01 \right) y + \frac{P042}{x^4} + \frac{P032}{x^3} + \frac{P022}{x^2} + \frac{P012}{x} + Pinfty02$$

```
> DiaginftySheet1:=-tinfty11-tinfty10/x+Unknown/x^2;
  DiaginftySheet2:=-tinfty21-tinfty20/x+Unknown2/x^2;
  Diag0Sheet1:=t011/x^2+t010/x+Unknown3;
  Diag0Sheet2:=t021/x^2+t020/x+Unknown4;
```

$$DiaginftySheet1 := -tinfty11 - \frac{tinfty10}{x} + \frac{Unknown}{x^2} \tag{2}$$

$$DiaginftySheet2 := -tinfty21 - \frac{tinfty20}{x} + \frac{Unknown2}{x^2}$$

$$Diag0Sheet1 := \frac{t011}{x^2} + \frac{t010}{x} + Unknown3$$

$$Diag0Sheet2 := \frac{t021}{x^2} + \frac{t020}{x} + Unknown4$$

Expression of P_1 in terms of the diagonal expansion in both sheets

```
> series(DiaginftySheet1+DiaginftySheet2-P1(x),x=infinity);
  series(Diag0Sheet1+Diag0Sheet2-P1(x),x=0,5);
```

$$-Pinfty01 - tinfty11 - tinfty21 + \frac{-P011 - tinfty10 - tinfty20}{x} + \frac{-P021 + Unknown + Unknown2}{x^2} \tag{3}$$

$$\frac{t011 + t021 - P021}{x^2} + \frac{t010 + t020 - P011}{x} + Unknown3 + Unknown4 - Pinfty01$$

```
> P021:=t011+t021;
  P011:=t010+t020;
  Pinfty01:=-(tinfty11+tinfty21);
  CoherenceEquation1:=residue(DiaginftySheet1+DiaginftySheet2-P1
  (x),x=infinity);
```

$$P021 := t011 + t021 \tag{4}$$

$$P011 := t010 + t020$$

$$Pinfty01 := -tinfty11 - tinfty21$$

$$CoherenceEquation1 := tinfty10 + tinfty20 + t010 + t020$$

Study at x=0

```
> factor(series(SpectralCurve(Diag0Sheet1),x=0)):
  factor(series(SpectralCurve(Diag0Sheet2),x=0)):
  EQ01:=residue(x^3*SpectralCurve(Diag0Sheet1),x=0);
  EQ02:=residue(x^3*SpectralCurve(Diag0Sheet2),x=0);
  EQ03:=residue(x^2*SpectralCurve(Diag0Sheet1),x=0);
  EQ04:=residue(x^2*SpectralCurve(Diag0Sheet2),x=0);
  EQ05:=residue(x*SpectralCurve(Diag0Sheet1),x=0);
  EQ06:=residue(x*SpectralCurve(Diag0Sheet2),x=0);
  EQ07:=residue(x^0*SpectralCurve(Diag0Sheet1),x=0):
  EQ08:=residue(x^0*SpectralCurve(Diag0Sheet2),x=0):
```

$$EQ01 := -t011\,t021 + P042$$
$$EQ02 := -t011\,t021 + P042$$
$$EQ03 := -t010\,t021 - t011\,t020 + P032$$
$$EQ04 := -t010\,t021 - t011\,t020 + P032$$
$$EQ05 := -t010\,t020 + t011\,Unknown3 + t011\,tinfty11 + t011\,tinfty21 - t021\,Unknown3 + P022$$
$$EQ06 := -t010\,t020 - t011\,Unknown4 + t021\,Unknown4 + t021\,tinfty11 + t021\,tinfty21 + P022$$ **(5)**

```
> P042:=solve(EQ01,P042);
  P032:=solve(EQ03,P032);
  simplify(EQ01);
  simplify(EQ02);
  simplify(EQ03);
  simplify(EQ04);
  simplify(EQ05);
  simplify(EQ06);
```

$$P042 := t011\,t021$$
$$P032 := t010\,t021 + t011\,t020$$
$$0$$
$$0$$
$$0$$
$$0$$
$$(Unknown3 + tinfty11 + tinfty21)\,t011 - t010\,t020 - Unknown3\,t021 + P022$$
$$(Unknown4 + tinfty11 + tinfty21)\,t021 - t010\,t020 - Unknown4\,t011 + P022$$ **(6)**

Study at infinity

```
> series(SpectralCurve(DiaginftySheet1),x=infinity,3):
  series(SpectralCurve(DiaginftySheet2),x=infinity,3):
  EQinfty1:=residue(x^(-2)*SpectralCurve(DiaginftySheet1),x=
  infinity);
```

```
EQinfty2:=residue(x^(-2)*SpectralCurve(DiaginftySheet2),x=
infinity);
EQinfty3:=residue(x^(-1)*SpectralCurve(DiaginftySheet1),x=
infinity);
EQinfty4:=residue(x^(-1)*SpectralCurve(DiaginftySheet2),x=
infinity);
EQinfty5:=residue(x^(0)*SpectralCurve(DiaginftySheet1),x=
infinity);
EQinfty6:=residue(x^(0)*SpectralCurve(DiaginftySheet2),x=
infinity);
EQinfty7:=residue(x^(1)*SpectralCurve(DiaginftySheet1),x=
infinity);
EQinfty8:=residue(x^(1)*SpectralCurve(DiaginftySheet2),x=
infinity);
EQinfty9:=residue(x^(2)*SpectralCurve(DiaginftySheet1),x=
infinity);
EQinfty10:=residue(x^(2)*SpectralCurve(DiaginftySheet2),x=
infinity);
```

$$EQinfty1 := 0$$
$$EQinfty2 := 0$$
$$EQinfty3 := -tinfty11^2 - (-tinfty11 - tinfty21)\, tinfty11 - Pinfty02$$
$$EQinfty4 := -tinfty21^2 - (-tinfty11 - tinfty21)\, tinfty21 - Pinfty02$$
$$EQinfty5 := -2\, tinfty11\, tinfty10 - (-tinfty11 - tinfty21)\, tinfty10 - (t010 + t020)\, tinfty11 - P012$$
$$EQinfty6 := -2\, tinfty21\, tinfty20 - (-tinfty11 - tinfty21)\, tinfty20 - (t010 + t020)\, tinfty21 - P012$$
$$EQinfty7 := 2\, tinfty11\, Unknown - tinfty10^2 + (-tinfty11 - tinfty21)\, Unknown - (t010 + t020)\, tinfty10 - (t011 + t021)\, tinfty11 - P022$$
$$EQinfty8 := 2\, tinfty21\, Unknown2 - tinfty20^2 + (-tinfty11 - tinfty21)\, Unknown2 - (t010 + t020)\, tinfty20 - (t011 + t021)\, tinfty21 - P022$$
$$EQinfty9 := 2\, tinfty10\, Unknown + (t010 + t020)\, Unknown - (t011 + t021)\, tinfty10 - t010\, t021 - t011\, t020$$
$$EQinfty10 := 2\, tinfty20\, Unknown2 + (t010 + t020)\, Unknown2 - (t011 + t021)\, tinfty20 - t010\, t021 - t011\, t020$$

(7)

```
> Pinfty02:=solve(EQinfty3,Pinfty02);
simplify(EQinfty3);
simplify(EQinfty4);
```

$$Pinfty02 := tinfty11\, tinfty21$$
$$0$$
$$0$$

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```
> P012:=factor(solve(EQinfty5+EQinfty6,P012)):
P012bis:=-1/2*(t010+t020)*(tinfty11+tinfty21)-1/2*(tinfty10-
```

```
tinfty20)*(tinfty11-tinfty21):
P012:=-1/2*(t010+t020)*(tinfty11+tinfty21)-1/2*(tinfty10-
tinfty20)*(tinfty11-tinfty21);
simplify(P012-P012bis);
factor(EQinfty5-(tinfty21-tinfty11)/2*CoherenceEquation1);
factor(EQinfty6+(tinfty21-tinfty11)/2*CoherenceEquation1);
```

$$P012 := -\frac{1}{2}\,(t010 + t020)\,(tinfty11 + tinfty21) - \frac{1}{2}\,(tinfty10 - tinfty20)\,(tinfty11 - tinfty21) \tag{9}$$

$$\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$$

Summary of coefficients

```
> P042:=P042;
P032:=P032;
P022:=P022;
P012:=P012;
P011:=P011;
P021:=P021;
Pinfty01:=Pinfty01;
Pinfty02:=Pinfty02;
CoherenceEquation1:=tinfty10+tinfty20+t010+t020;
```

$$P042 := t011\,t021 \tag{10}$$

$$P032 := t010\,t021 + t011\,t020$$

$$P022 := P022$$

$$P012 := -\frac{1}{2}\,(t010 + t020)\,(tinfty11 + tinfty21) - \frac{1}{2}\,(tinfty10 - tinfty20)\,(tinfty11 - tinfty21)$$

$$P011 := t010 + t020$$

$$P021 := t011 + t021$$

$$Pinfty01 := -tinfty11 - tinfty21$$

$$Pinfty02 := tinfty11\,tinfty21$$

$$CoherenceEquation1 := tinfty10 + tinfty20 + t010 + t020$$

We have one unknown coefficient P022.